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Samy Skander Bahoura

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SUP-INF INEQUALITY ON MANIFOLD OF DIMENSION 3

SAMY SKANDER BAHOURA

ABSTRACT. We give an estimate of type $\sup \times \inf$ on Riemannian manifold of Dimension 3 for prescribed curvature equation.

INTRODUCTION AND RESULTS.

We are on Riemannian manifold (M, g) of dimension 3 not necessary compact. In this paper we denote $\Delta = -\nabla^j(\nabla_j)$ the geometric laplacian.

The scalar curvature equation is:

$$8\Delta u + R_g u = V u^{N-1}, \quad u > 0. \quad (E)$$

Where R_g is the scalar curvature and V is a function (prescribed scalar curvature).

We consider three positive real number a, b, A and we suppose V lipschitzian:

$$0 < a \leq V(x) \leq b < +\infty \text{ and } \|\nabla V\|_{L^\infty(M)} \leq A. \quad (C)$$

The equation (E) was studied lot of when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [B], [CL1], [L1]. In this case we have a $\sup \times \inf$ inequality.

The corresponding equation in two dimensions on open set Ω of \mathbb{R}^2 , is:

$$\Delta u = V e^u, \quad (E')$$

The equation (E') was studed lot of and we can find very important result about a priori estimates in [BM], [BLS], [CL2], [L2], and [S].

In the case $V \equiv 1$ and M compact, the equation (E) is Yamabe equation. It was studed lot of, T.Aubin and R.Schoen have proved the existence of solution in this cas, see for example [Au] and [L-P].

When M is a compact Riemannian manifold, it exist some compactness result for equation (E) see [L-Zh]. Li and Zhu [L-Zh], proved that the energy is bounded and if we suppose M not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose M Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [L-Z] proved that the product $\sup \times \inf$ is bounded.

Here, we give an equality of type $\sup \times \inf$ for the equation (E) with general conditions (C) . We have:

Theorem. For all compact set K of M and all positive numbers a, b, A , it exists a positive constant c , which depends only on, K, a, b, A, M, g such that:

$$(\sup_K u)^{1/3} \times \inf_M u \leq c,$$

for all u solution of (E) with conditions (C) .

Note that in our work, we have not assumption on energy or boundary condition if we suppose the manifold M with boundary.

Next, in the proof of the previous theorem, we can replace the scalar curvature by any smooth function f , but here we do the proof with R_g the scalar curvature.

Proof of the theorem.

Part I: The metric and the laplacian in polar coordinates.

Let (M, g) a Riemannian manifold. We note $g_{x,ij}$ the local expression of the metric g in the exponential map centred in x .

We are concerning by the polar coordinates expression of the metric. By using Gauss lemma, we can write:

$$g = ds^2 = dt^2 + g_{ij}^k(r, \theta) d\theta^i d\theta^j = dt^2 + r^2 \tilde{g}_{ij}^k(r, \theta) d\theta^i d\theta^j = g_{x,ij} dx^i dx^j,$$

in a polar chart with origin x , $]0, \epsilon_0[\times U^k$, with (U^k, ψ) a chart of \mathbb{S}_{n-1} . We can write the element volume:

$$dV_g = r^{n-1} \sqrt{|\tilde{g}^k|} dr d\theta^1 \dots d\theta^{n-1} = \sqrt{[\det(g_{x,ij})]} dx^1 \dots dx^n,$$

then,

$$dV_g = r^{n-1} \sqrt{[\det(g_{x,ij})]} [\exp_x(r\theta)] \alpha^k(\theta) dr d\theta^1 \dots d\theta^{n-1},$$

where, α^k is such that, $d\sigma_{\mathbb{S}_{n-1}} = \alpha^k(\theta) d\theta^1 \dots d\theta^{n-1}$. (Riemannian volume element of the la sphere in the chart (U^k, ψ)).

Then,

$$\sqrt{|\tilde{g}^k|} = \alpha^k(\theta) \sqrt{[\det(g_{x,ij})]},$$

Clearly, we have the following proposition:

Proposition 1: Let $x_0 \in M$, there exist $\epsilon_1 > 0$ and if we reduce U^k , we have:

$$|\partial_r \tilde{g}_{ij}^k(x, r, \theta)| + |\partial_r \partial_{\theta^m} \tilde{g}_{ij}^k(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

and,

$$|\partial_r |\tilde{g}^k|(x, r, \theta)| + \partial_r \partial_{\theta^m} |\tilde{g}^k|(x, r, \theta)| \leq Cr, \quad \forall x \in B(x_0, \epsilon_1) \quad \forall r \in [0, \epsilon_1], \quad \forall \theta \in U^k.$$

Remark:

$\partial_r [\log \sqrt{|\tilde{g}^k|}]$ is a local function of θ , and the restriction of the global function on the sphere \mathbb{S}_{n-1} , $\partial_r [\log \sqrt{\det(g_{x,ij})}]$. We will note, $J(x, r, \theta) = \sqrt{\det(g_{x,ij})}$.

Let's write the laplacian in $[0, \epsilon_1] \times U^k$,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log \sqrt{|\tilde{g}^k|}] \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We have,

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r \log J(x, r, \theta) \partial_r + \frac{1}{r^2 \sqrt{|\tilde{g}^k|}} \partial_{\theta^i} (\tilde{g}^{\theta^i \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}).$$

We write the laplacian (radial and angular decomposition),

$$-\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [\log J(x, r, \theta)] \partial_r - \Delta_{S_r(x)},$$

where $\Delta_{S_r(x)}$ is the laplacian on the sphere $S_r(x)$.

We set $L_\theta(x, r)(\dots) = r^2 \Delta_{S_r(x)}(\dots) [\exp_x(r\theta)]$, clearly, this operator is a laplacian on \mathbb{S}_{n-1} for particular metric. We write,

$$L_\theta(x, r) = \Delta_{g_{x,r,\mathbb{S}_{n-1}}},$$

and,

$$\Delta = \partial_{rr} + \frac{n-1}{r} \partial_r + \partial_r [J(x, r, \theta)] \partial_r - \frac{1}{r^2} L_\theta(x, r).$$

If, u is function on M , then, $\bar{u}(r, \theta) = u \circ \exp_x(r\theta)$ is the corresponding function in polar coordinates centred in x . We have,

$$-\Delta u = \partial_{rr} \bar{u} + \frac{n-1}{r} \partial_r \bar{u} + \partial_r [J(x, r, \theta)] \partial_r \bar{u} - \frac{1}{r^2} L_\theta(x, r) \bar{u}.$$

Part II: "Blow-up" and "Moving-plane" methods

The "blow-up" technic

Let, $(u_i)_i$ a sequence of functions on M such that,

$$\Delta u_i + R_g u_i = V_i u_i^5, \quad u_i > 0, \quad (E')$$

We argue by contradiction and we suppose that $\sup^{1/3} \times \inf$ is not bounded.

We assume that:

$\forall c, R > 0 \exists u_{c,R}$ solution of (E') such that:

$$R \left[\sup_{B(x_0, R)} u_{c,R} \right]^{1/3} \times \inf_M u_{c,R} \geq c, \quad (H)$$

Proposition 2:

There exist a sequence of points $(y_i)_i$, $y_i \rightarrow x_0$ and two sequences of positive real number $(l_i)_i, (L_i)_i$, $l_i \rightarrow 0, L_i \rightarrow +\infty$, such that if we consider $v_i(y) = \frac{u_i \circ \exp_{y_i}(y)}{u_i(y_i)}$, we have:

$$0 < v_i(y) \leq \beta_i \leq 2^{1/2}, \quad \beta_i \rightarrow 1.$$

$$v_i(y) \rightarrow \left(\frac{1}{1 + |y|^2} \right)^{1/2}, \quad \text{uniformly on every compact set of } \mathbb{R}^3.$$

$$l_i^{1/2} [u_i(y_i)]^{1/3} \times \inf_M u_i \rightarrow +\infty$$

Proof:

We use the hypothesis (H) , we can take two sequences $R_i > 0, R_i \rightarrow 0$ and $c_i \rightarrow +\infty$, such that,

$$R_i \left[\sup_{B(x_0, R_i)} u_i \right]^{1/3} \times \inf_M u_i \geq c_i \rightarrow +\infty,$$

Let, $x_i \in B(x_0, R_i)$, such that $\sup_{B(x_0, R_i)} u_i = u_i(x_i)$ and $s_i(x) = [R_i - d(x, x_i)]^{1/2} u_i(x)$, $x \in B(x_i, R_i)$. Then, $x_i \rightarrow x_0$.

We have,

$$\max_{B(x_i, R_i)} s_i(x) = s_i(y_i) \geq s_i(x_i) = R_i^{1/2} u_i(x_i) \geq \sqrt{c_i} \rightarrow +\infty.$$

Set :

$$l_i = R_i - d(y_i, x_i), \quad \bar{u}_i(y) = u_i \circ \exp_{y_i}(y), \quad v_i(z) = \frac{u_i \circ \exp_{y_i}(z / [u_i(y_i)]^2)}{u_i(y_i)}.$$

Clearly, $y_i \rightarrow x_0$. We obtain:

$$L_i = \frac{l_i}{(c_i)^{1/2}} [u_i(y_i)]^2 = \frac{[s_i(y_i)]^2}{c_i^{1/2}} \geq \frac{c_i^1}{c_i^{1/2}} = c_i^{1/2} \rightarrow +\infty.$$

If $|z| \leq L_i$, then $y = \exp_{y_i}[z/[u_i(y_i)]^2] \in B(y_i, \delta_i l_i)$ with $\delta_i = \frac{1}{(c_i)^{1/2}}$ and $d(y, y_i) < R_i - d(y_i, x_i)$, thus, $d(y, x_i) < R_i$ and, $s_i(y) \leq s_i(y_i)$, we can write,

$$u_i(y)[R_i - d(y, y_i)]^{1/2} \leq u_i(y_i)(l_i)^{1/2}.$$

But, $d(y, y_i) \leq \delta_i l_i$, $R_i > l_i$ and $R_i - d(y, y_i) \geq R_i - \delta_i l_i > l_i - \delta_i l_i = l_i(1 - \delta_i)$, we obtain,

$$0 < v_i(z) = \frac{u_i(y)}{u_i(y_i)} \leq \left[\frac{l_i}{l_i(1 - \delta_i)} \right]^{1/2} \leq 2^{1/2}.$$

We set, $\beta_i = \left(\frac{1}{1 - \delta_i} \right)^{1/2}$, clearly $\beta_i \rightarrow 1$.

The function v_i is solution of:

$$-g^{jk}[\exp_{y_i}(y)]\partial_{jk}v_i - \partial_k \left[g^{jk} \sqrt{|g|} \right] [\exp_{y_i}(y)]\partial_j v_i + \frac{R_g[\exp_{y_i}(y)]}{[u_i(y_i)]^4} v_i = V_i v_i^5,$$

By elliptic estimates and Ascoli, Ladyzenskaya theorems, $(v_i)_i$ converge uniformly on each compact to the function v solution on \mathbb{R}^3 of,

$$\Delta v = V(x_0)v^5, \quad v(0) = 1, \quad 0 \leq v \leq 1 \leq 2^{1/2},$$

Without loss of generality, we can suppose $V(x_0) = 3$.

By using maximum principle, we have $v > 0$ on \mathbb{R}^3 , the result of Caffarelli-Gidas-Spruck (see [C-G-S]) give, $v(y) = \left(\frac{1}{1 + |y|^2} \right)^{1/2}$. We have the same properties for v_i in the previous paper [B2].

Polar coordinates and "moving-plane" method

Let,

$$w_i(t, \theta) = e^{1/2} \bar{u}_i(e^t, \theta) = e^{t/2} u_i \circ \exp_{y_i}(e^t \theta), \quad \text{et } a(y_i, t, \theta) = \log J(y_i, e^t, \theta).$$

Lemma 1:

The function w_i is solution of:

$$-\partial_{tt} w_i - \partial_t a \partial_t w_i - L_\theta(y_i, e^t) + c w_i = V_i w_i^5,$$

with,

$$c = c(y_i, t, \theta) = \left(\frac{1}{2} \right)^2 + \frac{1}{2} \partial_t a - \lambda e^{2t},$$

Proof:

We write:

$$\partial_t w_i = e^{3t/2} \partial_r \bar{u}_i + \frac{1}{2} w_i, \quad \partial_{tt} w_i = e^{5t/2} \left[\partial_{rr} \bar{u}_i + \frac{2}{e^t} \partial_r \bar{u}_i \right] + \left(\frac{1}{2} \right)^2 w_i.$$

$$\partial_t a = e^t \partial_r \log J(y_i, e^t, \theta), \quad \partial_t a \partial_t w_i = e^{5t/2} [\partial_r \log J \partial_r \bar{u}_i] + \frac{1}{2} \partial_t a w_i.$$

the lemma is proved.

Now we have, $\partial_t a = \frac{\partial_t b_1}{b_1}$, $b_1(y_i, t, \theta) = J(y_i, e^t, \theta) > 0$,

We can write,

$$-\frac{1}{\sqrt{b_1}} \partial_{tt} (\sqrt{b_1} w_i) - L_\theta(y_i, e^t) w_i + [c(t) + b_1^{-1/2} b_2(t, \theta)] w_i = V_i w_i^{N-1},$$

where, $b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}}\partial_{tt}b_1 - \frac{1}{4(b_1)^{3/2}}(\partial_t b_1)^2$.

Let,

$$\tilde{w}_i = \sqrt{b_1}w_i.$$

Lemma 2:

The function \tilde{w}_i is solution of:

$$\begin{aligned} -\partial_{tt}\tilde{w}_i + \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)\tilde{w}_i = \\ = V_i \left(\frac{1}{b_1} \right)^2 \tilde{w}_i^5, \end{aligned}$$

where, $c_2 = \left[\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right]$.

Proof:

We have:

$$-\partial_{tt}\tilde{w}_i - \sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}w_i + (c + b_2)\tilde{w}_i = V_i \left(\frac{1}{b_1} \right)^2 \tilde{w}_i^5,$$

But,

$$\Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\sqrt{b_1}w_i) = \sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}w_i - 2\nabla_\theta w_i \cdot \nabla_\theta \sqrt{b_1} + w_i\Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\sqrt{b_1}),$$

and,

$$\nabla_\theta(\sqrt{b_1}w_i) = w_i\nabla_\theta\sqrt{b_1} + \sqrt{b_1}\nabla_\theta w_i,$$

we deduce,

$$\sqrt{b_1}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}w_i = \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\tilde{w}_i) + 2\nabla_\theta(\tilde{w}_i) \cdot \nabla_\theta \log(\sqrt{b_1}) - c_2\tilde{w}_i,$$

with $c_2 = \left[\frac{1}{\sqrt{b_1}}\Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\sqrt{b_1}) + |\nabla_\theta \log(\sqrt{b_1})|^2 \right]$. The lemma is proved.

The "moving-plane" method:

Let ξ_i a real number, and suppose $\xi_i \leq t$, we set $t^{\xi_i} = 2\xi_i - t$ and $\tilde{w}_i^{\xi_i}(t, \theta) = \tilde{w}_i(t^{\xi_i}, \theta)$.

We have,

$$\begin{aligned} -\partial_{tt}\tilde{w}_i^{\xi_i} + \Delta_{g_{y_i, e^{t^{\xi_i}}, \mathbb{S}_2}}(\tilde{w}_i^{\xi_i}) + 2\nabla_\theta(\tilde{w}_i^{\xi_i}) \cdot \nabla_\theta \log(\sqrt{b_1})\tilde{w}_i^{\xi_i} + [c(t^{\xi_i}) + b_1^{-1/2}(t^{\xi_i}, \cdot)b_2(t^{\xi_i}) - c_2^{\xi_i}]\tilde{w}_i^{\xi_i} = \\ = V_i^{\xi_i} \left(\frac{1}{b_1^{\xi_i}} \right)^2 (\tilde{w}_i^{\xi_i})^5. \end{aligned}$$

By using the same arguments than in [B], we have:

Proposition 3:

We have:

$$1) \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) \geq \tilde{k} > 0, \forall \theta \in \mathbb{S}_2.$$

For all $\beta > 0$, there exist $c_\beta > 0$ such than:

$$2) \frac{1}{c_\beta}e^{t/2} \leq \tilde{w}_i(\lambda_i + t, \theta) \leq c_\beta e^{t/2}, \forall t \leq \beta, \forall \theta \in \mathbb{S}_2.$$

We set,

$$\bar{Z}_i = -\partial_{tt}(\dots) + \Delta_{g_{y_i, e^t, \mathbb{S}_2}}(\dots) + 2\nabla_{\theta}(\dots) \cdot \nabla_{\theta} \log(\sqrt{b_1}) + (c + b_1^{-1/2}b_2 - c_2)(\dots)$$

Remark: In the operator \bar{Z}_i , by using the proposition 3, the coefficient $c + b_1^{-1/2}b_2 - c_2$ verify:

$$c + b_1^{-1/2}b_2 - c_2 \geq k' > 0, \text{ pour } t \ll 0,$$

it is fundamental if we want to apply Hopf maximum principle.

Goal:

Like in [B], we have elliptic second order operator, here it's \bar{Z}_i , the goal is to use the "moving-plane" method to have a contradiction. For this, we must have:

$$\bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) \leq 0, \text{ if } \tilde{w}_i^{\xi_i} - \tilde{w}_i \leq 0.$$

We write:

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &= (\Delta_{g_{y_i, e^t, \xi_i, \mathbb{S}_2}} - \Delta_{g_{y_i, e^t, \mathbb{S}_2}})(\tilde{w}_i^{\xi_i}) + \\ &+ 2(\nabla_{\theta, e^t \xi_i} - \nabla_{\theta, e^t})(\tilde{w}_i^{\xi_i}) \cdot \nabla_{\theta, e^t \xi_i} \log(\sqrt{b_1^{\xi_i}}) + 2\nabla_{\theta, e^t}(\tilde{w}_i^{\xi_i}) \cdot \nabla_{\theta, e^t \xi_i} [\log(\sqrt{b_1^{\xi_i}}) - \log \sqrt{b_1}] + \\ &+ 2\nabla_{\theta, e^t} \tilde{w}_i^{\xi_i} \cdot (\nabla_{\theta, e^t \xi_i} - \nabla_{\theta, e^t}) \log \sqrt{b_1} - [(c + b_1^{-1/2}b_2 - c_2)^{\xi_i} - (c + b_1^{-1/2}b_2 - c_2)] \tilde{w}_i^{\xi_i} + \\ &+ V_i^{\xi_i} \left(\frac{1}{b_1^{\xi_i}} \right)^2 (\tilde{w}_i^{\xi_i})^5 - V_i \left(\frac{1}{b_1} \right)^2 \tilde{w}_i^5. \quad (** * 1) \end{aligned}$$

Clearly, we have:

Lemma 3 :

$$b_1(y_i, t, \theta) = 1 - \frac{1}{3} \text{Ricci}_{y_i}(\theta, \theta) e^{2t} + \dots,$$

$$R_g(e^t \theta) = R_g(y_i) + \langle \nabla R_g(y_i) | \theta \rangle e^t + \dots$$

According to proposition 1 and lemma 3,

Proposition 4 :

$$\begin{aligned} \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) &\leq V_i b_1^{(-2)} [(\tilde{w}_i^{\xi_i})^5 - \tilde{w}_i^5] + 2(\tilde{w}_i^{\xi_i})^5 |V_i^{\xi_i} - V_i| + \\ &+ C' |e^{2t} - e^{2t \xi_i}| \left[|\nabla_{\theta} \tilde{w}_i^{\xi_i}| + |\nabla_{\theta}^2(\tilde{w}_i^{\xi_i})| + |\text{Ricci}_{y_i}| [|\tilde{w}_i^{\xi_i} + (\tilde{w}_i^{\xi_i})^5| + |R_g(y_i)| \tilde{w}_i^{\xi_i}] + C' |e^{3t \xi_i} - e^{3t}| \right]. \end{aligned}$$

Proof:

We use proposition 1, we have:

$$a(y_i, t, \theta) = \log J(y_i, e^t, \theta) = \log b_1, |\partial_t b_1(t)| + |\partial_{tt} b_1(t)| + |\partial_{tt} a(t)| \leq C e^{2t},$$

and,

$$|\partial_{\theta_j} b_1| + |\partial_{\theta_j, \theta_k} b_1| + |\partial_{t, \theta_j} b_1| + |\partial_{t, \theta_j, \theta_k} b_1| \leq C e^{2t},$$

then,

$$|\partial_t b_1(t^{\xi_i}) - \partial_t b_1(t)| \leq C' |e^{2t} - e^{2t \xi_i}|, \text{ sur }]-\infty, \log \epsilon_1] \times \mathbb{S}_2, \forall x \in B(x_0, \epsilon_1)$$

Locally,

$$\Delta_{g_{y_i, e^t, \mathbb{S}_2}} = L_{\theta}(y_i, e^t) = -\frac{1}{\sqrt{|\tilde{g}^k(e^t, \theta)|}} \partial_{\theta^i} [\tilde{g}^{\theta^i \theta^j}(e^t, \theta) \sqrt{|\tilde{g}^k(e^t, \theta)|} \partial_{\theta^j}].$$

Thus, in $[0, \epsilon_1] \times U^k$, we have,

$$A_i = \left[\left[\frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^l} (\tilde{g}^{\theta^l \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}) \right]^{\xi_i} - \frac{1}{\sqrt{|\tilde{g}^k|}} \partial_{\theta^l} (\tilde{g}^{\theta^l \theta^j} \sqrt{|\tilde{g}^k|} \partial_{\theta^j}) \right] (\tilde{w}_i^{\xi_i})$$

then, $A_i = B_i + D_i$ with,

$$B_i = \left[\tilde{g}^{\theta^l \theta^j} (e^{t \xi_i}, \theta) - \tilde{g}^{\theta^l \theta^j} (e^t, \theta) \right] \partial_{\theta^l \theta^j} \tilde{w}_i^{\xi_i}(t, \theta),$$

and,

$$D_i = \left[\frac{1}{\sqrt{|\tilde{g}^k|(e^{t \xi_i}, \theta)}} \partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j} (e^{t \xi_i}, \theta) \sqrt{|\tilde{g}^k|(e^{t \xi_i}, \theta)}] - \frac{1}{\sqrt{|\tilde{g}^k|(e^t, \theta)}} \partial_{\theta^l} [\tilde{g}^{\theta^l \theta^j} (e^t, \theta) \sqrt{|\tilde{g}^k|(e^t, \theta)}] \right] \partial_{\theta^j} \tilde{w}_i^{\xi_i}(t, \theta),$$

we deduce,

$$A_i \leq C_k |e^{2t} - e^{2t \xi_i}| \left[|\nabla_{\theta} \tilde{w}_i^{\xi_i}| + |\nabla_{\theta}^2 (\tilde{w}_i^{\xi_i})| \right],$$

If we take $C = \max\{C_i, 1 \leq i \leq q\}$ and if we use (***) , we obtain proposition 4.

We have,

$$c(y_i, t, \theta) = \left(\frac{n-2}{2} \right)^2 + \frac{n-2}{2} \partial_t a + R_g e^{2t}, \quad (\alpha_1)$$

$$b_2(t, \theta) = \partial_{tt}(\sqrt{b_1}) = \frac{1}{2\sqrt{b_1}} \partial_{tt} b_1 - \frac{1}{4(b_1)^{3/2}} (\partial_t b_1)^2, \quad (\alpha_2)$$

$$c_2 = \left[\frac{1}{\sqrt{b_1}} \Delta_{g_{y_i, e^t, \mathbb{S}_{n-1}}}(\sqrt{b_1}) + |\nabla_{\theta} \log(\sqrt{b_1})|^2 \right], \quad (\alpha_3)$$

Then,

$$\partial_t c(y_i, t, \theta) = \frac{1}{2} \partial_{tt} a + 2e^{2t} R_g(e^t \theta) + e^{3t} \langle \nabla R_g(e^t \theta) | \theta \rangle,$$

by proposition 1,

$$|\partial_t c_2| + |\partial_t b_1| + |\partial_t b_2| + |\partial_t c| \leq K_1 e^{2t},$$

Now, we consider the function, $\bar{w}_i(t, \theta) = \tilde{w}_i(t, \theta) - \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2} e^t$, and $\lambda > 2 > 0$.

For $t \leq t_i = -(2/3) \log u_i(y_i)$, we have:

$$\begin{aligned} \bar{w}_i(t, \theta) &= e^t \left[b_1(t, \theta) e^{-t/2} u_i \circ \exp_{y_i}(e^t \theta) - \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2} \right] \geq \\ &\geq e^t \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2} > 0, \end{aligned}$$

We set, $\mu_i = \frac{[u_i(y_i)]^{1/3} \times \min_M u_i}{2}$.

We use proposition 3 and the same arguments than in [B], we obtain:

Lemma 4:

There exists $\nu < 0$ such that for $\mu \leq \nu$:

$$\bar{w}_i^{\mu}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \quad \forall (t, \theta) \in [\mu, t_i] \times \mathbb{S}_2,$$

We set, $\lambda_i = -2 \log u_i(y_i)$, then,

Lemma 5:

$$\bar{w}_i(\lambda_i, \theta) - \bar{w}_i(\lambda_i + 4, \theta) > 0.$$

Proof of lemma 5:

Clearly:

$$\bar{w}_i(\lambda_i, \theta) - \bar{w}_i(\lambda_i + 4, \theta) = \tilde{w}_i(\lambda_i, \theta) - \tilde{w}_i(\lambda_i + 4, \theta) + \mu_i(e^4 - 1),$$

we deduce lemma 6 from proposition 3.

Let, $\xi_i = \sup\{\mu \leq \lambda_i + 2, \bar{w}_i^{\xi_i}(t, \theta) - \bar{w}_i(t, \theta) \leq 0, \forall (t, \theta) \in [\xi_i, t_i] \times \mathbb{S}_2\}$.

The real ξ_i exists (see [B]), if we use (**2), we have:

$$\tilde{w}_i^{\xi_i}(t, \theta) + |\nabla_{\theta} \tilde{w}_i^{\xi_i}(t, \theta)| + |\nabla_{\theta}^2 \tilde{w}_i^{\xi_i}(t, \theta)| \leq C(R), \quad \forall (t, \theta) \in]-\infty, \log R] \times \mathbb{S}_2,$$

We can write:

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) = \bar{Z}_i(\tilde{w}_i^{\xi_i} - \tilde{w}_i) - \mu_i \bar{Z}_i(e^{2t\xi_i} - e^{2t}),$$

$$-\bar{Z}_i(e^{2t\xi_i} - e^{2t}) = [1 - \frac{1}{4} - \frac{3}{2}\partial_t a - R_g e^{2t} + b_1^{-1/2} b_2 - c_2](e^{2t\xi_i} - e^{2t}) \leq c_1(e^{t\xi_i} - e^t),$$

with $c_1 > 0$, $\text{car } |\partial_t a| + |\partial_t b_1| + |\partial_{tt} b_1| + |\partial_{t, \theta_j} b_1| + |\partial_{t, \theta_j, \theta_k} b_1| \leq C' e^{2t} < 1$, for t very small.

We use proposition 4, to obtain on, $[\xi_i, t_i] \times \mathbb{S}_2$,

$$\bar{Z}_i(\bar{w}_i^{\xi_i} - \bar{w}_i) \leq c_2[V_i(\tilde{w}_i^{\xi_i})^5 - \tilde{w}_i^5] + |V_i^{\xi_i} - V_i|(w_i^{\xi_i})^5 + [\mu_i c_1 - C'(R)](e^{t\xi_i} - e^t) \leq 0,$$

Like in [B], after using Hopf maximum principle, we have,

$$\sup_{\theta \in \mathbb{S}_2} [\bar{w}_i^{\xi_i}(t_i, \theta) - \bar{w}_i(t_i, \theta)] = 0.$$

We have:

$$\bar{w}_i^{\xi_i}(t_i, \theta_i) - \bar{w}_i(t_i, \theta_i) = 0, \quad \forall i.$$

We deduce,

$$[u(y_i)]^{1/3} \times \inf_M u_i \leq c, \quad \forall i.$$

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DEPARTMENT OF MATHEMATICS, PATRAS UNIVERSITY, 26500 PATRAS , GREECE
E-mail address: samybahoura@yahoo.fr , bahoura@ccr.jussieu.fr