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Giovanni Peccati, Murad Taqqu

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Abstract

We prove sufficient conditions, ensuring that a sequence of multiple Wiener-Itô integrals (with respect to a general Gaussian process) converges stably to a mixture of normal distributions. Our key tool is an asymptotic decomposition of contraction kernels, realized by means of increasing families of projection operators. We also use an infinite-dimensional Clark-Ocone formula, as well as a version of the correspondence between "abstract" and "concrete" filtered Wiener spaces, in a spirit similar to Üstünel and Zakai (1997).

Key Words – Stable Convergence; Multiple Wiener-Itô Integrals; Projection Operators; Gaussian Processes.

AMS Subject Classification – 60G60, 60G57, 60F05, 60H05, 60H07

1 Introduction

Let X be a centered Gaussian process and, for $d \geq 2$ and $n \geq 1$, let $I_d^X(f_n)$ be a multiple Wiener-Itô stochastic integral, of order d, of some symmetric and square-integrable kernel f_n with respect to X. The aim of this paper is to establish general sufficient conditions on the kernels f_n, ensuring that the sequence $I_d^X(f_n)$ converges stably to a mixture of Gaussian probability laws. The reader is referred e.g. to [10, Chapter 4], [21] and Section 2.3 below, for an exhaustive characterization of stable convergence. Here, we shall recall that such a convergence is stronger than the convergence in law, and can be used in particular to explain several non-central limit results for functionals of independently scattered random measures; see for instance [21]. Our starting point is the following Central Limit Theorem (CLT).

Theorem 1 (see [19, Theorem 1 and Proposition 3]) *If the variance of $I_d^X(f_n)$ converges to 1 ($n \to +\infty$) the following three conditions are equivalent: (i) $I_d^X(f_n)$ converges in law to a standard Gaussian random variable $N(0,1)$, (ii) $E\left[I_d^X(f_n)^4\right] \to 3$, (iii) for every $r = 1, ..., d-1$, the contraction kernel $f_n \otimes_{d-r} f_n$ converges to 0.*

Although the implication (ii) $\Rightarrow$ (i) is rather striking, several recent applications of Theorem 1 (see [22], [9], [4] or [3]) have shown that condition (iii) is easier to verify than (ii), since in general there is no manageable formula for the fourth moment of a non-trivial multiple Wiener-Itô integral. Also, the implication (iii) $\Rightarrow$ (i) (which can be regarded as a simplification of the *method of diagrams*—see e.g. [28]) suggests that the asymptotic study of the contraction kernels associated to the sequence f_n may lead to more general convergence results. In particular, in this paper we address the following problem. Let $Y \geq 0$ be a non-constant random variable having the (finite) chaotic representation $Y = 1 + I_2^X(g_2) + \cdots + I_{2(d-1)}^X\left(g_{2(d-1)}\right)$, let $N \sim N(0,1)$ be independent of Y, and suppose that the sequence $I_d^X(f_n)$ satisfies adequate normalization conditions; is it possible to associate to each f_n and

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each $r = 1, ..., d-1$, two *generalized contraction kernels*, say $f_n \otimes^*_{d-r} f_n$ and $f_n \otimes^{**}_{d-r} f_n$, in such a way that the two relations

$$f_n \otimes^*_{d-r} f_n \underset{n\to+\infty}{\to} g_{2r} \quad \text{and} \quad f_n \otimes^{**}_{d-r} f_n \underset{n\to+\infty}{\to} 0, \quad \forall r = 1, ..., d-1, \tag{1}$$

imply that $I^X_d(f_n)$ converges stably to $\sqrt{Y} \times N$? This kind of non-central phenomena (convergence towards non-trivial mixtures of Gaussian laws) appears regularly, for instance in the analysis of the power variations of fractional processes (see e.g. [3]), or in the study of non linear functionals of stationary Gaussian sequences (see e.g. [7, Theorems 7-9]). Although there exists a panoply of results characterizing the stable convergence in a semi-martingale setting (see [14], [6] or [10, Ch. 4]), none of them can be directly applied to the case of a Gaussian process for which there is no explicit (semi)martingale structure (this is true, in particular, for fractional processes). In this paper, we aim at providing results in this direction for multiple integrals with respect to general Gaussian processes, by using some ancillary devices borrowed from continuous-time martingale calculus (in a spirit similar to [19]), as well as a part of the theory of filtrations on general Wiener spaces, as developed e.g. in [32] and [30] (see also [21] for some related results in a non-Gaussian framework).

Now let $\mathfrak{H}$ be a separable Hilbert space, and suppose that the process $X = X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$ is a centered Gaussian measure (also called an isonormal Gaussian process) over $\mathfrak{H}$ (see e.g. [17, Ch. 1], or Section 2.2 below). Then, f_n is a symmetric element of $\mathfrak{H}^{\otimes d}$ (i.e., the dth tensor product of $\mathfrak{H}$) for every n, and $f_n \otimes_{d-r} f_n \in \mathfrak{H}^{\otimes 2r}$, $\forall r = 1, ..., d-1$. In what follows (see Theorem 11 and formulae (34) and (35) below) we construct the two kernels $f_n \otimes^*_{d-r} f_n$ and $f_n \otimes^{**}_{d-r} f_n$ appearing in (1), by using *resolutions of the identity*. These objects are defined as continuous and non-decreasing families of orthogonal projections $\pi = \{\pi_t : t \in [0,1]\}$ over $\mathfrak{H}$, indexed by $[0,1]$ and such that $\pi_0 = 0$ and $\pi_1 = \text{Id}_{\mathfrak{H}}$. Each resolution π induces a time structure on the Gaussian field $X(\mathfrak{H})$, and generates the canonical filtration $\mathcal{F}^\pi_t = \sigma\{X(\pi_t h) : h \in \mathfrak{H}\}$, $t \in [0,1]$ (note that $\mathcal{F}^\pi_1 = \sigma(X)$ for every π). In particular, the infinite dimensional process $t \mapsto \{X(\pi_t h) : h \in \mathfrak{H}\} \triangleq X(\pi_t \mathfrak{H})$ can be seen as an infinite collection of possibly correlated Gaussian $\mathcal{F}^\pi_t$-martingales. As proved e.g. in [32] and [30] in the framework of abstract Wiener spaces, each $\sigma(X)$-measurable and square integrable random variable (as Skorohod and multiple Wiener-Itô integrals with respect to X) is therefore the terminal value of a $\mathcal{F}^\pi_t$-martingale, which is in turn a "generalized adapted stochastic integral" with respect to the infinite dimensional process $t \mapsto X(\pi_t \mathfrak{H})$. Since every real-valued $\mathcal{F}^\pi_t$-martingale can be shown to be continuous, it follows that the stable convergence of $\sigma(X)$-measurable random variables can be studied by means of the theory of stable convergence for continuous local martingales (see e.g. [10, Ch. 4]). In particular, our starting point in the construction of the two contraction operators appearing in (1) is a stable convergence result, proved in Proposition 9 below, involving the quadratic variation of continuous local martingales, as well as a stochastic time-change result known as the Dambis-Dubins-Schwarz Theorem (DDS Theorem) (see e.g. [25, Ch. V]). Observe that our Proposition 9 is reminiscent of the stable convergence results proved by Feigin in [6]. See [31] for similar results involving the stable convergence of multi-dimensional martingales, and [21] for an alternative approach based on a decoupling technique, known as the "principle of conditioning".

We recall that the use of the DDS Theorem has already been crucial in the proof of Theorem 1 and its generalizations, as stated in [19] and [22]. However, we shall stress that the proofs of the main results of the present paper (in particular, Theorem 11 and Theorem 12 below) are considerably more complicated. Indeed, when no resolution of the identity is involved – as it is the case for Theorem 1 – all infinite dimensional Gaussian spaces are trivially isomorphic. It follows that every relevant element of the proof of Theorem 1 is contained in the case of $X(\mathfrak{H})$ being the Gaussian space generated by a standard one-dimensional Brownian motion on $[0,1]$ (that is, $\mathfrak{H} = L^2([0,1])$), and the extension to general Gaussian measures can be achieved by elementary considerations (see for instance [19, Section 2.2]). However, in the present paper the filtrations $\mathcal{F}^\pi_t = \sigma\{X(\pi_t h) : h \in \mathfrak{H}\}$ play a prominent role, and the complexity of these objects may considerably vary, depending on the structure of the resolution π

(in particular, depending on the *rank* of π–see Section 2.1 below). We shall therefore use a notion of *equivalence* between pairs $(\mathfrak{H}, \pi)$, where $\mathfrak{H}$ is a Hilbert space and π is a resolution, instead of the usual notion of isomorphism between Hilbert spaces. The use of this equivalence relation implies that, if the rank of π equals q ($q = 1, ..., +\infty$), then $\mathcal{F}_t^\pi$ has roughly the structure of the filtration generated by a q-dimensional Brownian motion. As a consequence, our first step will be the proof of our main results in the framework of an infinite-dimensional Brownian motion, and the extension to the general case will be realized by means of rather delicate arguments involving the previously described equivalence relation (see Lemma 16 below). As will become clear later on, our techniques can be regarded as a ramification of the theory of *concrete representations* for abstract Wiener spaces, a concept introduced in [30, Section 5]. The reader is also referred to [21] for some related results in a non-Gaussian context.

The remainder of the paper is organized as follows. In Section 2.1, we formally introduce the notion of resolution of the identity and discuss some of its basic properties. In Section 2.2 some notions from stochastic analysis and Skorohod integration are recalled. Sections 2.3-2.5 contain the statements and the proofs of some useful stable convergence result for Skorohod integrals. Section 3 is devoted to the proof of our main convergence results. We also discuss some relations with the theory of abstract Wiener spaces. An Appendix contains the proof of a technical lemma.

2 Preliminary definitions and results

Throughout the paper, the following conventions are in order: all random objects are supposed to be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$; all σ-fields are assumed to be complete; the symbol $\xrightarrow{\mathbb{P}}$ stands for convergence in probability; $\mathbb{R}$ is the set of real numbers.

2.1 Hilbert spaces and resolutions of the identity

Let $\mathfrak{H}$ be a real separable Hilbert space. The symbol $(\cdot, \cdot\cdot)_{\mathfrak{H}}$ indicates the inner product on $\mathfrak{H}$, and $\|\cdot\|_{\mathfrak{H}} = (\cdot, \cdot)_{\mathfrak{H}}^{1/2}$ as usual. The space $\mathfrak{H}$ is always endowed with the Borel σ-field generated by the open sets of the canonical distance associated to $\|\cdot\|_{\mathfrak{H}}$. As already done in [21], we first study the convergence of Skorohod integrals by means of increasing families of orthogonal projections, known as *resolutions of the identity*.

Definition I – A *continuous resolution of the identity*, is a family $\pi = \{\pi_t : t \in [0,1]\}$ of orthogonal projections satisfying:

(I-a) $\pi_0 = 0$, and $\pi_1 = \text{Id}.$;

(I-b) $\forall 0 \leq s < t \leq 1$, $\pi_s \mathfrak{H} \subseteq \pi_t \mathfrak{H}$;

(I-c) $\forall t_0 \in [0,1]$, $\forall h \in \mathfrak{H}$, $\lim_{t \to t_0} \|(\pi_t - \pi_{t_0})\, h\|_{\mathfrak{H}} = 0$.

A subset F of $\mathfrak{H}$ is said to be π-*reproducing* if the linear span of the set $\{\pi_t f : f \in F,\, t \in [0,1]\}$ is dense in $\mathfrak{H}$. The *rank* of π is the smallest of the dimensions of all the subspaces generated by the π-reproducing subsets of $\mathfrak{H}$. A π-reproducing subset F of $\mathfrak{H}$ is *fully orthogonal* if $(\pi_t f, g)_{\mathfrak{H}} = 0$ for every $t \in [0,1]$ and every $f, g \in F$. The collection of all π verifying properties (**I-a**)-(**I-c**) is noted $\mathcal{R}(\mathfrak{H})$.

The reader is referred to [2] or [34] for further properties and characterizations of the class $\mathcal{R}(\mathfrak{H})$. In particular, we shall use the following consequence of [2, Lemma 23.2], that can be proved by a standard Gram-Schmidt orthogonalization.

Lemma 2 *Let $\pi \in \mathcal{R}(\mathfrak{H})$ and let F be a π-reproducing subset of $\mathfrak{H}$ such that* $\dim\left(\overline{F}\right) = \mathit{rank}\,(\pi)$, *where $\overline{A}$ stands for the closure of the vector space generated by a given set A. Then, there exists a π-reproducing* and fully orthogonal *subset F' of $\mathfrak{H}$, such that* $\dim\left(\overline{F'}\right) = \dim\left(\overline{F}\right)$.

We will sometimes need to work with elements of $\mathcal{R}(\mathfrak{H})$ that are not only continuous, but also absolutely continuous.

Definition II – A resolution $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}(\mathfrak{H})$ is said to be *absolutely continuous* if, for every $f, g \in \mathfrak{H}$, the function $t \mapsto (\pi_t g, f)_{\mathfrak{H}}$, $t \in [0,1]$, is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. The class of absolutely continuous resolutions in $\mathcal{R}(\mathfrak{H})$ is noted $\mathcal{R}_{AC}(\mathfrak{H})$.

The elements of $\mathcal{R}_{AC}(\mathfrak{H})$ are used in [30, Section 5] to prove a remarkable bijection between *abstract* and *concrete* filtered Wiener spaces. More details will be given in Section 3, were we establish a similar result for isonormal Gaussian processes as a step to prove stable convergence criteria for multiple integrals. With the next result we point out that, up to a "change of time", every $\pi \in \mathcal{R}(\mathfrak{H})$ can be represented in terms of some element of $\mathcal{R}_{AC}(\mathfrak{H})$.

Lemma 3 *For any $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}(\mathfrak{H})$, there exists a non decreasing function*

$$\psi = \{\psi(t) : t \in [0,1]\}$$

such that $\psi(0) = 0$ and the monotone family of projections

$$\widetilde{\pi}_t \triangleq \pi_{\psi(t)}, \quad t \in [0,1],$$

is an element of $\mathcal{R}_{AC}(\mathfrak{H})$.

Proof. Let $q = rank(\pi)$ (q is possibly infinite) and let $F_\pi = \{f_j : 1 \leq j \leq q\}$ be a π-reproducing subset of $\mathfrak{H}$, normalized in such a way that $\sum_{j=1}^q \|f_j\|_{\mathfrak{H}}^2 = 1$. Define moreover the increasing function $\phi(t) = \sum_{j=1}^q \|\pi_t f_j\|_{\mathfrak{H}}^2$, $t \in [0,1]$, and set $\psi(t) = \inf\{a : \phi(a) = t\}$. Then, ψ is non decreasing, $\psi(0) = 0$, and the family of projections

$$\widetilde{\pi}_t \triangleq \pi_{\psi(t)}, \quad t \in [0,1],$$

is a resolution of the identity verifying $\sum_{j=1}^q \|\widetilde{\pi}_t f_j\|_{\mathfrak{H}}^2 = t$, for every $t \in [0,1]$. Since F_π is also $\widetilde{\pi}$-reproducing, we deduce from [2, Lemma 23.1] that $\widetilde{\pi}$ is absolutely continuous. ■

2.2 Gaussian processes, Malliavin operators and representation theorems

Throughout the following, we write

$$X = X(\mathfrak{H}) = \{X(f) : f \in \mathfrak{H}\}$$

to indicate an *isonormal Gaussian process*, or a *Gaussian measure*, over the Hilbert space $\mathfrak{H}$. This means that X is a centered Gaussian family, indexed by the elements of $\mathfrak{H}$ and satisfying the isomorphic relation

$$\mathbb{E}[X(f)X(g)] = (f,g)_{\mathfrak{H}}, \quad \text{for every } f, g \in \mathfrak{H} \tag{2}$$

(the notation $X(\mathfrak{H})$ is adopted exclusively when the role of $\mathfrak{H}$ is relevant to the discussion).

As in [32] or [30], to every $\pi \in \mathcal{R}(\mathfrak{H})$ we associate the collection of σ-fields

$$\mathcal{F}_t^\pi(X) = \sigma\{X(\pi_t f) : f \in \mathfrak{H}\}, \quad t \in [0,1], \tag{3}$$

and we observe that, for every $\pi \in \mathcal{R}(\mathfrak{H})$, $t \mapsto \mathcal{F}_t^\pi(X)$ defines a continuous filtration (see [30, p. 14]). Also, for every $f \in \mathfrak{H}$, the process $t \mapsto X(\pi_t f)$, $t \in [0,1]$, is a centered and continuous $\mathcal{F}_t^\pi(X)$-martingale such that, for every $\eta > 0$, the increment $X((\pi_{t+\eta} - \pi_t) f) = X(\pi_{t+\eta} f) - X(\pi_t f)$ is independent of $\mathcal{F}_t^\pi(X)$ (see e.g. [30, Corollary 2.1]).

As in [21], we write $L^2(\mathbb{P}, \mathfrak{H}, X) = L^2(\mathfrak{H}, X)$ to indicate the set of $\sigma(X)$-measurable and $\mathfrak{H}$-valued random variables Y such that $\mathbb{E}\left[\|Y\|_{\mathfrak{H}}^2\right] < +\infty$. The class $L^2(\mathfrak{H}, X)$ is a Hilbert space, with inner product given by $(Y, Z)_{L^2(\mathfrak{H},X)} = \mathbb{E}\left[(Y, Z)_{\mathfrak{H}}\right]$. Following [30], we associate to every $\pi \in \mathcal{R}(\mathfrak{H})$ the subspace $L_\pi^2(\mathfrak{H}, X)$ of π-*adapted* elements of $L^2(\mathfrak{H}, X)$, that is: $Y \in L_\pi^2(\mathfrak{H}, X)$ if, and only if, $Y \in L^2(\mathfrak{H}, X)$ and, for every $t \in [0, 1]$ and every $h \in \mathfrak{H}$,

$$(Y, \pi_t h)_{\mathfrak{H}} \in \mathcal{F}_t^\pi(X). \tag{4}$$

For any resolution $\pi \in \mathcal{R}(\mathfrak{H})$, $L_\pi^2(\mathfrak{H}, X)$ is a closed subspace of $L^2(\mathfrak{H}, X)$. We may occasionally write $(u, z)_{L_\pi^2(\mathfrak{H})}$ instead of $(u, z)_{L^2(\mathfrak{H})}$, when both u and z are in $L_\pi^2(\mathfrak{H}, X)$. Now, for $\pi \in \mathcal{R}(\mathfrak{H})$, define $\mathcal{E}_\pi(\mathfrak{H}, X)$ to be the space of *elementary elements* of $L_\pi^2(\mathfrak{H}, X)$, that is, $\mathcal{E}_\pi(\mathfrak{H}, X)$ is the collection of those elements of $L_\pi^2(\mathfrak{H}, X)$ that are linear combinations of $\mathfrak{H}$-valued random variables of the type

$$h = \Phi(t_1)(\pi_{t_2} - \pi_{t_1}) f, \tag{5}$$

where $t_2 > t_1$, $f \in \mathfrak{H}$ and $\Phi(t_1)$ is a $\mathcal{F}_{t_1}^\pi(X)$-measurable, real-valued and square-integrable random variable. A proof of the following useful result can be found in [21, Lemma 3] or [30, Lemma 2.2].

Lemma 4 *For every $\pi \in \mathcal{R}(\mathfrak{H})$, the span of the set $\mathcal{E}_\pi(\mathfrak{H}, X)$ of adapted elementary elements is dense in $L_\pi^2(\mathfrak{H}, X)$.*

In what follows, we shall apply to the Gaussian measure X some standard notions and results from Malliavin calculus (the reader is again referred to [17] and [18] for any unexplained notation or definition). For instance, $D = D_X$ and $\delta = \delta_X$ stand, respectively, for the usual Malliavin derivative and Skorohod integral with respect to the Gaussian measure X (the dependence on X will be dropped, when there is no risk of confusion); for $k \geq 1$, $\mathbb{D}_X^{k,2}$ is the space of k times differentiable functionals of X, endowed with the norm $\|\cdot\|_{k,2}$ (see [17, Chapter 1] for a definition of this norm); $dom(\delta_X)$ is the domain of the operator δ_X. Note that D_X is an operator from $\mathbb{D}_X^{k,2}$ to $L^2(\mathfrak{H}, X)$, and also that $dom(\delta_X) \subset L^2(\mathfrak{H}, X)$. For every $d \geq 1$, we define $\mathfrak{H}^{\otimes d}$ and $\mathfrak{H}^{\odot d}$ to be, respectively, the dth tensor product and the dth *symmetric* tensor product of $\mathfrak{H}$. For $d \geq 1$ we will denote by I_d^X the isometry between $\mathfrak{H}^{\odot d}$ equipped with the norm $\sqrt{d!}\,\|\cdot\|_{\mathfrak{H}^{\otimes d}}$ and the dth Wiener chaos of X. Given $g \in \mathfrak{H}^{\otimes d}$, we note $(g)_s$ the symmetrization of g, and

$$I_d^X(g) = I_d^X((g)_s)$$

Plainly, for $f, g \in \mathfrak{H}^{\otimes d}$, $I_d^X(f + g) = I_d^X((f)_s + (g)_s) = I_d^X(f) + I_d^X(g)$. Recall that, when $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, $(Z, \mathcal{Z})$ is a measurable space, and ν is a σ-finite measure with no atoms, then $\mathfrak{H}^{\odot d} = L_s^2\left(Z^d, \mathcal{Z}^{\otimes d}, \nu^{\otimes d}\right)$, where $L_s^2\left(Z^d, \mathcal{Z}^{\otimes d}, \nu^{\otimes d}\right)$ is the space of symmetric and square integrable functions on Z^d. Moreover, for $f \in \mathfrak{H}^{\odot d}$, $I_d^X(f)$ coincides with the *multiple Wiener-Itô integral* (of order d) of f with respect to X, as defined e.g. in [17, Section 1.1.2].

To establish the announced stable convergence results, we use the elements of $\mathcal{R}(\mathfrak{H})$ to represent random variables of the type $\delta_X(u)$, $u \in dom(\delta_X)$, in terms of continuous-time martingales. In particular, we will use the fact that (i) for any $\pi \in \mathcal{R}(\mathfrak{H})$, $L_\pi^2(\mathfrak{H}, X) \subseteq dom(\delta_X)$, and (ii) for any $u \in L_\pi^2(\mathfrak{H}, X)$ the random variable $\delta_X(u)$ can be regarded as the terminal value of a real-valued $\mathcal{F}_t^\pi$-martingale, where $\mathcal{F}_t^\pi$ is given by (3). A proof of the following result can be found in [32, Lemme 1] and [30, Corollary 2.1]

Proposition 5 *Let the assumptions of this section prevail. Then:*

1. *$L_\pi^2(\mathfrak{H}, X) \subseteq dom(\delta_X)$, and for every $h_1, h_2 \in L_\pi^2(\mathfrak{H}, X)$*

$$\mathbb{E}\left[\delta_X(h_1)\,\delta_X(h_2)\right] = (h_1, h_2)_{L_\pi^2(\mathfrak{H},X)}. \tag{6}$$

2. *If* $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ *has the form* $h = \sum_{i=1}^n h_i$, *where* $n \geq 1$, *and* $h_i \in \mathcal{E}_\pi(\mathfrak{H}, X)$ *is s.t.*

$$h_i = \Phi_i \times \left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}}\right) f_i, \quad f_i \in \mathfrak{H}, \quad i = 1, ..., n,$$

with $t_2^{(i)} > t_1^{(i)}$ *and* Φ_i *square integrable and* $\mathcal{F}^\pi_{t_1^{(i)}}(X)$*-measurable, then*

$$\delta_X(h) = \sum_{i=1}^n \Phi_i \times \left[X\left(\pi_{t_2^{(i)}} f_i\right) - X\left(\pi_{t_1^{(i)}} f_i\right)\right]. \tag{7}$$

3. *For every* $u \in L^2_\pi(\mathfrak{H}, X)$, *the process*

$$t \mapsto \delta_X(\pi_t u), \quad t \in [0,1],$$

is a continuous $\mathcal{F}^\pi_t(X)$*-martingale initialized at zero, with quadratic variation equal to*

$$\left\{\|\pi_t u\|^2_{\mathfrak{H}} : t \in [0,1]\right\}.$$

In the terminology of [32], relation (6) implies that $L^2_\pi(\mathfrak{H}, X)$ is a closed subspace of the *isometric subset of* $dom(\delta_X)$, defined as the collection of those $h \in dom(\delta_X)$ such that

$$\mathbb{E}\left(\delta_X(h)^2\right) = \|h\|^2_{L^2(\mathfrak{H},X)}. \tag{8}$$

Note that, in general, this isometric subset is not a vector space – see e.g. [32, p. 170]. The next result is partly a consequence of the continuity of π. It is an abstract version of the *Clark-Ocone formula* (see [17]), and can be proved along the lines of [32, Théorème 1, formula (2.4) and Théorème 3]. Observe that, in [32], such a result is proved in the context of abstract Wiener spaces. However, such a proof uses exclusively isometric properties such as (8), and the role of the underlying probability space is immaterial. It follows that the extension to general isonormal Gaussian processes is standard: see e.g. [18, Section 1.1]. The reader is also referred to [15] for a general Clark-Ocone formula concerning Banach space valued Wiener functionals.

Proposition 6 (Abstract Clark-Ocone formula) *Under the above notation and assumptions (in particular,* $\pi \in \mathcal{R}(\mathfrak{H})$*), for every* $F \in \mathbb{D}^{1,2}_X$,

$$F = \mathbb{E}(F) + \delta\left(proj\left\{D_X F \mid L^2_\pi(\mathfrak{H}, X)\right\}\right), \tag{9}$$

where $D_X F$ *is the Malliavin derivative of* F, *and* $proj\left\{\cdot \mid L^2_\pi(\mathfrak{H}, X)\right\}$ *is the orthogonal projection operator on* $L^2_\pi(\mathfrak{H}, X)$.

Remarks – (a) The right-hand side of (9) is well defined, since $D_X F \in L^2(\mathfrak{H}, X)$ by definition, and therefore

$$proj\left\{D_X F \mid L^2_\pi(\mathfrak{H}, X)\right\} \in L^2_\pi(\mathfrak{H}, X) \subseteq dom(\delta_X),$$

where the last inclusion is stated in Proposition 5.

(b) Since $\mathbb{D}^{1,2}_X$ is dense in $L^2(\mathbb{P})$ and $\delta_X\left(L^2_\pi(\mathfrak{H}, X)\right)$ is an isometry (due to (6)), formula (9) yields that every $F \in L^2(\mathbb{P}, \sigma(X))$ admits a unique "predictable representation" of the form

$$F = \mathbb{E}(F) + \delta_X(u), \quad u \in L^2_\pi(\mathfrak{H}, X); \tag{10}$$

see also [32, Remarque 2, p. 172].

In the next section, we present a general criterion (Theorem 7), ensuring the stable convergence of a sequence of Skorohod integrals towards a mixture of Gaussian distributions. The result has been proved in [21], by using a general convergence criteria for functionals of independently scattered random measures. Here we present an alternative proof (partly inspired by some arguments contained in [31]), which is based on a time-change technique for continuous-time martingales.

2.3 Stable convergence of Skorohod integrals

We first present a standard definition of the classes $\mathbf{M}$ and $\widehat{\mathbf{M}}$ of random probability measures and random fourier transform.

Definition III – Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-field on $\mathbb{R}$.

(III-a) A map $\mu(\cdot,\cdot)$, from $\mathcal{B}(\mathbb{R})\times\Omega$ to $\mathbb{R}$ is called a *random probability* (on $\mathbb{R}$) if, for every $C\in\mathcal{B}(\mathbb{R})$, $\mu(C,\cdot)$ is a random variable and, for $\mathbb{P}$-a.e. ω, the map $C\mapsto\mu(C,\omega)$, $C\in\mathcal{B}(\mathbb{R})$, defines a probability measure on $\mathbb{R}$. The class of all random probabilities is noted $\mathbf{M}$, and, for $\mu\in\mathbf{M}$, we write $\mathbb{E}\mu(\cdot)$ to indicate the (deterministic) probability measure

$$\mathbb{E}\mu(C)\triangleq\mathbb{E}[\mu(C,\cdot)],\quad C\in\mathcal{B}(\mathbb{R}). \tag{11}$$

(III-b) For a measurable map $\phi(\cdot,\cdot)$, from $\mathbb{R}\times\Omega$ to $\mathbb{C}$, we write $\phi\in\widehat{\mathbf{M}}$ if there exists $\mu\in\mathbf{M}$ such that

$$\phi(\lambda,\omega)=\widehat{\mu}(\lambda)(\omega),\quad\forall\lambda\in\mathbb{R},\text{ for }\mathbb{P}\text{-a.e. }\omega, \tag{12}$$

where $\widehat{\mu}(\cdot)$ is defined as

$$\widehat{\mu}(\lambda)(\omega)=\begin{cases}\int\exp(i\lambda x)\,\mu(dx,\omega) & \text{if }\mu(\cdot,\omega)\text{ is a probability measure}\\ 1 & \text{otherwise.}\end{cases},\quad\lambda\in\mathbb{R}. \tag{13}$$

For every $\omega\in\Omega$, $\widehat{\mu}(\lambda)(\omega)$ is of course a continuous function of λ, and the probability $\mathbb{E}\mu(\cdot)=\int_\Omega\mu(\cdot,\omega)\,d\mathbb{P}(\omega)$ defined in (11) is often called a *mixture* of probability measures. The notion of *stable convergence*, which is the content of the next definition, extends the usual notion of convergence in law.

Definition IV (see e.g. [10, Chapter 4]) – Let $\mathcal{F}^*\subseteq\mathcal{F}$ be a σ-field, and let $\mu\in\mathbf{M}$. A sequence of real valued r.v.'s $\{Z_n:n\geq1\}$ is said to *converge $\mathcal{F}^*$-stably* to $\mathbb{E}\mu(\cdot)$, written $X_n\to_{(s,\mathcal{F}^*)}\mathbb{E}\mu(\cdot)$, if, for every $\lambda,\gamma\in\mathbb{R}$ and every $\mathcal{F}^*$-measurable r.v. Z,

$$\lim_{n\to+\infty}\mathbb{E}[\exp(i\gamma Z)\times\exp(i\lambda X_n)]=\mathbb{E}[\exp(i\gamma Z)\times\widehat{\mu}(\lambda)], \tag{14}$$

$\widehat{\mu}\in\widehat{\mathbf{M}}$ is given by (13).

If X_n converges $\mathcal{F}^*$-stably, then the conditional distributions $\mathcal{L}(X_n\mid A)$ converge for any $A\in\mathcal{F}^*$ such that $\mathbb{P}(A)>0$ (the reader is referred e.g. to [10, Proposition 5.33] for an exhaustive characterization of stable convergence). By setting $Z=0$, we obtain that if $X_n\to_{(s,\mathcal{F}^*)}\mathbb{E}\mu(\cdot)$, then the law of the X_n's converges weakly to $\mathbb{E}\mu(\cdot)$. Observe also that, if a sequence of random variables $\{U_n:n\geq0\}$ is such that $(U_n-Z_n)\to0$ in $L^1(\mathbb{P})$ and $X_n\to_{(s,\mathcal{F}^*)}\mathbb{E}\mu(\cdot)$, then $U_n\to_{(s,\mathcal{F}^*)}\mathbb{E}\mu(\cdot)$.

In what follows, $\mathfrak{H}_n$, $n\geq1$, is a sequence of real separable Hilbert spaces, whereas, for each $n\geq1$, $X_n=X_n(\mathfrak{H}_n)=\{X_n(g):g\in\mathfrak{H}_n\}$, is an isonormal Gaussian process over $\mathfrak{H}$. The following theorem already appears in [21], where it is proved by using a decoupling technique known as the "principle of conditioning". In Section 2.4 we shall present an alternative proof based exclusively on continuous-time martingale arguments.

Theorem 7 *Under the previous notation and assumptions, for $n\geq1$, let $\pi_n=\{\pi_{n,t}:t\in[0,1]\}\in\mathcal{R}(\mathfrak{H}_n)$ and $u_n\in L^2_{\pi_n}(\mathfrak{H}_n,X_n)$. Suppose that there exists a sequence $\{t_n:n\geq1\}\subset[0,1]$ and σ-fields $\{\mathcal{U}_n:n\geq1\}$, such that*

$$\|\pi_{n,t_n}u_n\|^2_{\mathfrak{H}_n}\xrightarrow{\mathbb{P}}0 \tag{15}$$

and

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}_{t_n}^{\pi_n}(X_n). \tag{16}$$

If

$$\|u_n\|_{\mathfrak{H}}^2 \xrightarrow{\mathbb{P}} Y, \tag{17}$$

for some $Y \in L^2(\mathbb{P})$ *such that* $Y \neq 0$, $Y \geq 0$ *and* $Y \in \mathcal{U}^* \triangleq \vee_n \mathcal{U}_n$, *then, as* $n \to +\infty$,

$$\delta_{X_n}(u_n) \to_{(s,\mathcal{U}^*)} \mathbb{E}\mu(\cdot),$$

where $\mu \in \mathbf{M}$ *verifies* $\widehat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2}Y\right)$.

Remark – Condition (16), that already appears in the statement of the main results of [21], can be seen as a weak version of the *nesting condition* used e.g. in [6] to establish sufficient conditions for the stable convergence of semimartingales.

By using the Clark-Ocone formula stated in Proposition 6, we deduce from Theorem 7 a criterion for the stable convergence of (Malliavin) differentiable functionals. It is the key to prove the main results of the paper.

Corollary 8 *Let* $\mathfrak{H}_n$, $X_n(\mathfrak{H}_n)$, π_n, t_n *and* $\mathcal{U}_n$, $n \geq 1$, *satisfy the assumptions of Theorem 7, and consider a sequence of random variables* $\{F_n : n \geq 1\}$, *such that* $\mathbb{E}(F_n) = 0$ *and* $F_n \in \mathbb{D}_{X_n}^{1,2}$ *for every* n. *Then, a sufficient condition to have that*

$$F_n \to_{(s,\mathcal{U}^*)} \mathbb{E}\mu(\cdot)$$

where $\mathcal{U}^* \triangleq \vee_n \mathcal{U}_n$, $\widehat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2}Y\right)$, $\forall \lambda \in \mathbb{R}$, *and* $Y \geq 0$ *is such that* $Y \in \mathcal{U}^*$, *is*

$$\left\|\pi_{n,t_n} proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right\|^2_{\mathfrak{H}_n} \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \left\|proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right\|^2_{\mathfrak{H}_n} \underset{n\to+\infty}{\xrightarrow{\mathbb{P}}} Y. \tag{18}$$

2.4 Martingale proof of Theorem 7

In this section, we provide a proof of Theorem 7 involving exclusively continuous martingale arguments. It is based on the following general result.

Proposition 9 *Fix* $T \leq +\infty$. *For* $n \geq 1$, *let* $\{W_t^n : t \in [0,T)\}$ *be a Brownian motion with respect to a filtration* $\mathcal{H}^n = \{\mathcal{H}_t^n : t \in [0,T)\}$ *of the space* $(\Omega, \mathcal{F}, \mathbb{P})$ *(satisfying the usual conditions) and suppose that there exists a sequence of random variables* $\{\tau_n : n \geq 1\}$ *such that* τ_n *is a* $\mathcal{H}^n$*-stopping time with values in* $[0,T)$ *and* $\tau_n \xrightarrow{\mathbb{P}} 0$ *as* $n \to +\infty$. *Set moreover, for* $n \geq 1$, $V_t^n = W^n_{\tau_n+t} - W^n_{\tau_n}$, $t \in [0,T)$. *Then,*

1. $V^n - W^n \overset{law}{\Rightarrow} 0$;
2. *if there exists a sequence of* σ*-fields* $\{\mathcal{U}_n : n \geq 1\}$ *such that*

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{H}^n_{\tau_n}$$

then for every random element X *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$, *with values in some Polish space* $(S, \mathcal{S})$ *and measurable with respect to* $\mathcal{U}^* \triangleq \vee_n \mathcal{U}_n$,

$$(W^n, X) \overset{law}{\Rightarrow} (W, X) \quad \text{and} \quad (V^n, X) \overset{law}{\Rightarrow} (W, X) \tag{19}$$

where W *is a standard Brownian motion independent of* X.

Proof. The proof is partly inspired by that of [31, Theorem 3.1]. Since $\tau_n \stackrel{\mathbb{P}}{\to} 0$ by assumption, Point 1 in the statement is a direct consequence of the Continuous Mapping Theorem (see e.g. [1]). Moreover, (19) is proved, once it is shown that $(V^n, X) \stackrel{\text{law}}{\Rightarrow} (W, X)$. To do this, observe first that, for every n, V_n is a standard Brownian motion, started from zero and independent of $\mathcal{H}^n_{\tau_n}$. We shall now show that, as $n \to +\infty$, for every $A \in \mathcal{B}(C[0,T))$ ($\mathcal{B}(C[0,T))$ is the Borel σ-field of the class $C[0,T)$ of the continuous functions on $[0,T)$) and every $B \in \mathcal{S}$,

$$\mathbb{P}[V^n \in A, X \in B] - \mathbb{P}[W \in A]\,\mathbb{P}[X \in B] \to 0. \tag{20}$$

As a matter of fact, since $\mathcal{U}_n \subseteq \mathcal{H}^n_{\tau_n}$, and thanks to the martingale convergence theorem and the fact that $X \in \vee_n \mathcal{U}_n$,

$$\begin{aligned}\mathbb{E}\left|\mathbb{P}\left[X \in B \mid \mathcal{H}^n_{\tau_n}\right] - \mathbf{1}_{X\in B}\right| &\leq \mathbb{E}\left|\mathbb{P}\left[X \in B \mid \mathcal{H}^n_{\tau_n}\right] - \mathbb{P}[X \in B \mid \mathcal{U}_n]\right| \\ &\quad + \mathbb{E}\left|\mathbf{1}_{X\in B} - \mathbb{P}[X \in B \mid \mathcal{U}_n]\right| \\ &\leq 2\mathbb{E}\left|\mathbf{1}_{X\in B} - \mathbb{P}[X \in B \mid \mathcal{U}_n]\right| \underset{n\to+\infty}{\to} 0,\end{aligned}$$

thus implying that $\mathbb{P}\left[X \in B \mid \mathcal{H}^n_{\tau_n}\right] \stackrel{L^1}{\to} \mathbf{1}_{X\in B}$, and therefore

$$\begin{aligned}\mathbb{P}[V^n \in A, X \in B] - \mathbb{P}[W \in A]\,\mathbb{P}[X \in B] &= \mathbb{P}[V^n \in A, X \in B] - \mathbb{E}\left[\mathbf{1}_{V_n\in A}\mathbb{P}\left[X \in B \mid \mathcal{H}^n_{\tau_n}\right]\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{V_n\in A}\mathbb{P}\left[X \in B \mid \mathcal{H}^n_{\tau_n}\right]\right] - \mathbb{P}[W \in A]\,\mathbb{P}[X \in B] \\ &= \mathbb{P}[V^n \in A, X \in B] - \mathbb{E}\left[\mathbf{1}_{V_n\in A}\mathbb{P}\left[X \in B \mid \mathcal{H}^n_{\tau_n}\right]\right] \underset{n\to+\infty}{\to} 0,\end{aligned}$$

where the last equality follows from the independence of V_n and $\mathcal{H}^n_{\tau_n}$. Since (20) implies that $(V^n, X) \stackrel{\text{law}}{\Rightarrow} (W, X)$, with W and X independent, the proof is concluded. ■

Proof of Theorem 7 – According to Proposition 5-3, for each n the process $t \mapsto \delta_{X_n}(\pi_{n,t}u_n)$ is a continuous square-integrable $\mathcal{F}^{\pi_n}_t(X_n)$-martingale with quadratic variation $t \mapsto \|\pi_{n,t}u_n\|^2_{\mathfrak{H}_n} \triangleq \psi_n(t)$, $t \in [0,1]$. Now define

$$\mathcal{G}^n_s \triangleq \mathcal{F}^{\pi_n}_{\rho_{n,s}}, \quad s \geq 0, \quad \text{where} \quad \rho_{n,s} = \inf\{x \in [0,1] : \psi_n(x) > s\},$$

with $\inf \varnothing = 1$, and observe that the above definition is well given and also that, for every $n \geq 1$, every $t \in [0,1]$ and $s \geq 0$, $\psi_n(t) = \|\pi_{n,t}u_n\|^2_{\mathfrak{H}_n}$ is a $\mathcal{G}^n$-stopping time and $\rho_{n,s}$ is a $\mathcal{F}^{\pi_n}$-stopping time. In particular, for every $x \geq 0$ and $t \in [0,1]$,

$$\{\psi_n(t) > x\} = \{\rho_{n,x} < t\} \in \mathcal{F}^{\pi_n}_t.$$

According to the well known Dambis-Dubins-Schwarz Theorem (see e.g. [25, Ch. V]), the underlying probability space can be suitably enlarged in order to support a sequence of stochastic processes W^n such that, for each fixed n, W^n is a $\mathcal{G}^n$-Brownian motion started from zero, and also

$$\delta_{X_n}(\pi_{n,t}u_n) = W^{(n)}_{\psi_n(t)}, \quad t \in [0,1]. \tag{21}$$

Since, in general, $\rho_{n,\psi_n(t)} \geq t$, $\mathcal{G}^n_{\psi_n(t)} \supseteq \mathcal{F}^{\pi_n}_t$ for every $t \in [0,1]$. It follows that, for the sequence t_n appearing in the statement of Theorem 7,

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}^{\pi_n}_{t_n} \subseteq \mathcal{U}_{n+1} \cap \mathcal{G}^n_{\psi_n(t_n)}.$$

Thus, all conditions of Proposition 9 are verified, with $\mathcal{H}^n = \mathcal{G}^n$ and $\tau_n = \psi_n(t_n)$, and therefore, for every $\mathcal{U}^* = \vee_n \mathcal{U}_n$-measurable and real-valued random variable Z,

$$(W^n, Z) \stackrel{\text{law}}{\Rightarrow} (W, Z),$$

where W is a Brownian motion independent of Z. Moreover, since $\psi_n(1) = \|\pi_{n,1}u_n\|^2_{\mathfrak{H}_n} \xrightarrow{\mathbb{P}} Y \in \mathcal{U}^*$ by assumption, we conclude that, for every $Z \in \mathcal{U}^*$,

$$(W^n, Z, \psi_n(1)) \overset{\text{law}}{\Rightarrow} (W, Z, Y).$$

Now observe that $\delta_{X_n}(u_n) = W^n_{\psi_n(1)}$ by (21) and also that, thanks to a further application of the Continuous Mapping Theorem,

$$\left(W^n_{\psi_n(1)}, Z\right) = (\delta_{X_n}(u_n), Z) \overset{\text{law}}{\Rightarrow} (W_Y, Z),$$

implying that, for every $\gamma, \lambda \in \mathbb{R}$,

$$\mathbb{E}\left[e^{i\gamma Z} e^{i\lambda \delta_{X_n}(u_n)}\right] \to \mathbb{E}\left[e^{i\gamma Z} e^{i\lambda W_Y}\right] = \mathbb{E}\left[e^{i\gamma Z} e^{-\frac{\lambda^2}{2}Y}\right],$$

which yields the desired conclusion. ■

2.5 Further refinements

The following result is a refinement of Theorem 7 and Corollary 8. It will be used in the next section to characterize the stable convergence of double Wiener-Itô integrals. It is proved in [21, Proposition 10, Theorem 22 and formula (123)]. The setting is that of Theorem 7: $\mathfrak{H}_n$, $n \geq 1$, is a real separable Hilbert space; $X_n = X_n(\mathfrak{H}_n)$, $n \geq 1$, is an isonormal Gaussian process over $\mathfrak{H}_n$.

Theorem 10 *Keep the assumptions of Theorem 7 (in particular, $u_n \in L^2_{\pi_n}(\mathfrak{H}_n, X_n)$ for every n, and (15), (16) and (17) are verified). Then, as $n \to +\infty$,*

$$\mathbb{E}\left[\exp\left(i\lambda\delta_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi_n}_{t_n}(X_n)\right] \xrightarrow{\mathbb{P}} \exp\left(-\frac{\lambda^2}{2}Y\right), \quad \forall \lambda \in \mathbb{R}.$$

Moreover, if there exists a finite random variable $C(\omega) > 0$ such that, for some $\eta > 0$,

$$\mathbb{E}\left[|\delta_{X_n}(u_n)|^\eta \mid \mathcal{F}^{\pi_n}_{t_n}\right] < C(\omega), \ \ \forall n \geq 1, \ \ a.s.\text{-}\mathbb{P},$$

then, there is a subsequence $\{n(k) : k \geq 1\}$ such that, a.s. - $\mathbb{P}$,

$$\mathbb{E}\left[\exp\left(i\lambda\delta_{X_n}(u_n)\right) \mid \mathcal{F}^{\pi_{n(k)}}_{t_{n(k)}}\right] \underset{k\to+\infty}{\to} \exp\left(-\frac{\lambda^2}{2}Y\right), \quad \forall \lambda \in \mathbb{R}.$$

3 Main results

Although Corollary 8 is quite general, the explicit computation of the projections

$$proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}, X_n)\right\}, \quad n \geq 1,$$

may be rather difficult. In this section, we prove simpler sufficient conditions ensuring that the second asymptotic relation in (18) is satisfied, when (F_n) is a sequence of multiple Wiener-Itô integrals of a fixed order. In particular, these conditions do not involve any projection on the spaces $L^2_{\pi_n}(\mathfrak{H}, X_n)$. The techniques developed below can be suitably extended to study the joint convergence of vectors of multiple Wiener-Itô integrals. This issue will be studied in a separate paper.

3.1 Statements

To start, fix a real separable Hilbert space $\mathfrak{H}$ and let $\{e_k : k \geq 1\}$ be a complete orthonormal system in $\mathfrak{H}$. For every $d \geq 1$, every $p = 0, ..., d$ and $f \in \mathfrak{H}^{\odot d}$, we define the *contraction* of f of order p to be the element of $\mathfrak{H}^{\otimes 2(d-p)}$ given by

$$f \otimes_p f = \sum_{i_1,...,i_p=1}^{\infty} \left\langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathfrak{H}^{\otimes p}} \otimes \left\langle f, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathfrak{H}^{\otimes p}}, \tag{22}$$

and we denote by $(f \otimes_p f)_s$ its symmetrization. As shown in [19] and [22], the asymptotic behavior of the contractions $f \otimes_p f$, $p = 1, ..., n-1$, plays a crucial role in the proof of CLTs for multiple Wiener-Itô integrals. To obtain analogous results in the case of stable convergence, we need to define a further class of contraction operators, constructed by means of resolutions of the identity. To this end, fix $\pi \in \mathcal{R}(\mathfrak{H})$, $t \in [0,1]$ and $d \geq 1$, and define $\pi_t^{\otimes d} : \mathfrak{H}^{\otimes d} \mapsto \mathfrak{H}^{\otimes d}$ to be the nth tensor product of π_t, that is, $\pi_t^{\otimes d}$ is the projection operator, from $\mathfrak{H}^{\otimes d}$ to itself, given by

$$\pi_t^{\otimes d} = \underbrace{\pi_t \otimes \pi_t \otimes \cdots \otimes \pi_t}_{d \text{ times}}. \tag{23}$$

For every $d \geq 1$, $p = 0, ..., n$, $t \in [0,1]$ and $f \in \mathfrak{H}^{\odot d}$, we write $f \otimes_p^{\pi,t} f$ to indicate the element of $\mathfrak{H}^{\otimes 2(d-p)}$ given by

$$f \otimes_p^{\pi,t} f = \sum_{i_1,...,i_p=1}^{\infty} \left\langle f, \left(\pi_1^{\otimes p} - \pi_t^{\otimes p}\right) e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathfrak{H}^{\otimes p}} \otimes \left\langle f, \left(\pi_1^{\otimes p} - \pi_t^{\otimes p}\right) e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{\mathfrak{H}^{\otimes p}}, \tag{24}$$

and, as before, we denote by $\left(f \otimes_p^{\pi,t} f\right)_s$ its symmetrization. We define $f \otimes_p^{\pi,t} f$ to be the *generalized contraction kernel* of order p, associated to π and t. For instance, for $f \in \mathfrak{H}^{\odot d}$,

$$f \otimes_p^{\pi,0} f = f \otimes_p f, \quad f \otimes_p^{\pi,1} f = 0, \text{ and } f \otimes_d^{\pi,t} f = \left\| \left(\pi_1^{\otimes d} - \pi_t^{\otimes d}\right) f \right\|_{\mathfrak{H}^{\otimes d}}^2. \tag{25}$$

Remark – When $\mathfrak{H} = L^2(Z, \mathcal{Z}, \nu)$, where ν is σ-finite and non-atomic, and $\pi \in \mathcal{R}(\mathfrak{H})$ has the form

$$\pi_t f(z) = f(z) \mathbf{1}_{Z_t}(z), \quad z \in Z,$$

where Z_t is an increasing sequence in $\mathcal{Z}$ such that $Z_0 = \varnothing$ and $Z_1 = Z$, we have the following elementary relation: for every $d \geq 1$, $p = 0, ..., d$, $t \in [0,1]$ and $f \in \mathfrak{H}^{\odot d} = L_s^2\left(Z^{\otimes d}, \mathcal{Z}^{\otimes d}, \nu^{\otimes d}\right)$,

$$f \otimes_p^{\pi,t} f\left(z_1, ..., z_{2(d-p)}\right) = \int_{Z^p \setminus Z_t^p} f(z_1, ..., z_{d-p}, \mathbf{x}_p) f\left(z_{d-p+1}, ..., z_{2(d-p)}, \mathbf{x}_p\right) \nu^{\otimes p}(d\mathbf{x}_p),$$

since, for every $h \in L_s^2\left(Z^{\otimes p}, \mathcal{Z}^{\otimes p}, \nu^{\otimes p}\right)$,

$$\left(\pi_1^{\otimes p} - \pi_t^{\otimes p}\right) h(\mathbf{x}_p) = \left(\mathbf{1}_{Z^p}(\mathbf{x}_p) - \mathbf{1}_{Z_t^p}(\mathbf{x}_p)\right) h(\mathbf{x}_p) = \mathbf{1}_{Z^p \setminus Z_t^p}(\mathbf{x}_p) h(\mathbf{x}_p).$$

The next result, which is the main achievement of the paper, generalizes the crucial part of the CLT stated in [19, Theorem 1] to the case of the stable convergence. Its proof is postponed to the next section.

Theorem 11 *Let $\mathfrak{H}_n$, $X_n(\mathfrak{H}_n)$, π_n, t_n and $\mathcal{U}_n$, $n \geq 1$, satisfy the assumptions of Corollary 8. Fix $d \geq 2$, and consider a sequence of random variables $\{F_n : n \geq 1\}$, such that, for every n,*

$$F_n = I_d^{X_n}(f_n), \tag{26}$$

for a certain $f_n \in \mathfrak{H}_n^{\odot d}$*, and moreover*

$$\mathbb{E}\left[F_n \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] = I_d^{X_n}\left(\pi_{n,t_n}^{\otimes d} f_n\right) \overset{L^2(\mathbb{P})}{\to} 0 \tag{27}$$

and

$$\mathbb{E}\left[F_n^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \overset{\mathbb{P}}{\to} Y \in \mathcal{U}^*. \tag{28}$$

Then, the following hold:

1. *for every* $n \geq 1$,

$$\begin{aligned}\mathbb{E}\left[F_n^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] &= d!\left\|f_n\right\|_{\mathfrak{H}_n^{\otimes d}}^2 \\ &\quad + \sum_{r=1}^{d-1}(d-r)!\binom{d}{r}^2 I_{2r}^{X_n}\left[\pi_{n,t_n}^{\otimes 2r}\left(f_n \otimes_{d-r}^{\pi_n,t_n} f_n\right)\right] + o_{\mathbb{P}}(1),\end{aligned} \tag{29}$$

   where $o_{\mathbb{P}}(1)$ *stands for a sequence of random variables converging to zero in probability;*

2. *if* $\pi_n \in \mathcal{R}_{AC}(\mathfrak{H}_n)$ *for every* $n \geq 1$ *and, for every* $r = 1, ..., d-1$,

$$\left\|\left(\pi_{n,1}^{\otimes 2r} - \pi_{n,t_n}^{\otimes 2r}\right)\left(f_n \otimes_{d-r}^{\pi_n,t_n} f_n\right)\right\|_{\mathfrak{H}_n^{\otimes 2r}}^2 \underset{n\to+\infty}{\to} 0, \tag{30}$$

   then

$$\left\|proj\left\{D_{X_n}F_n \mid L_{\pi_n}^2(\mathfrak{H}_n, X_n)\right\}\right\|_{\mathfrak{H}_n}^2 \underset{n\to+\infty}{\overset{\mathbb{P}}{\to}} Y, \tag{31}$$

   and therefore

$$F_n \to_{(s,\mathcal{U}^*)} \mathbb{E}\mu(\cdot) \quad \text{and} \quad \mathbb{E}\left[\exp(i\lambda F_n) \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \overset{\mathbb{P}}{\to} \exp\left(-\frac{\lambda^2}{2}Y\right), \quad \forall \lambda \in \mathbb{R},$$

   where $\widehat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2}Y\right)$, $\forall \lambda \in \mathbb{R}$.

Remarks – (a) Since, due to Lemma 3 and for any continuous $\pi \in \mathcal{R}(\mathfrak{H})$, there exists a non decreasing function ϕ such that $\widetilde{\pi} \triangleq \pi_{\phi(\cdot)}$ is absolutely continuous, Theorem 11 applies *de facto* to any sequence $\pi_n \in \mathcal{R}(\mathfrak{H}_n)$, $n \geq 1$.

(b) Suppose that $X_n(\mathfrak{H}_n) = X(\mathfrak{H})$ for every $n \geq 1$. Then, the random variables F_n, $n \geq 1$, appearing in (26) all belong to the same Wiener chaos, and, due to (29), the sequence $\mathbb{E}\left[F_n^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right]$, $n \geq 1$, belongs to the same finite sum of d Wiener chaoses. Recall also that, on a finite sum of Wiener chaoses, the topology induced by convergence in probability is equivalent to the L^p topology, for every $p \geq 1$ (see e.g. [27]). When (27) and (28) are verified, we therefore deduce from (29) that Y has necessarily the form

$$Y = \mathbb{E}(Y) + \sum_{r=1}^{d-1} I_{2r}^X(g_r), \tag{32}$$

for some $g_r \in \mathfrak{H}^{\odot r}$ and $r = 1, ..., d-1$. Moreover, (28) is equivalent to the condition: as $n \to +\infty$, $d!\left\|f_n\right\|_{\mathfrak{H}^{\otimes d}}^2 \to \mathbb{E}(Y)$ and, for $r = 1, ..., d-1$,

$$(d-r)!\binom{d}{r}^2 \times \left(\pi_{n,t_n}^{\otimes 2r}\left(f_n \otimes_{d-r}^{\pi_n,t_n} f_n\right)\right)_s \to g_r \tag{33}$$

in $\mathfrak{H}^{\odot 2r}$. It follows that, for $r = 1, ..., d-1$, the two operators $f_n \otimes_{d-r}^* f_n$ and $f_n \otimes_{d-r}^{**} f_n$, from $\mathfrak{H}$ to $\mathfrak{H}^{\otimes 2r}$, defined as

$$f_n \mapsto (d-r)!\binom{d}{r}^2 \left(\pi_{n,t_n}^{\otimes 2r}\left(f_n \otimes_{d-r}^{\pi_n,t_n} f_n\right)\right)_s \triangleq f_n \otimes_{d-r}^* f_n \tag{34}$$

$$f_n \mapsto \left(\pi_{n,1}^{\otimes 2r} - \pi_{n,t_n}^{\otimes 2r}\right)\left(f_n \otimes_{d-r}^{\pi_n,t_n} f_n\right) \triangleq f_n \otimes_{d-r}^{**} f_n, \tag{35}$$

solve the problem raised in the Introduction. Indeed, under (27) and the normalization condition $d!\left\|f_n\right\|_{\mathfrak{H}^{\otimes d}}^2 \to \mathbb{E}(Y)$, due to Theorem 11 and (33), the asymptotic relation (1) implies that $I_d^X(f_n)$ converges stably to $\sqrt{Y} \times N$, where N is a centered standard Gaussian random variable independent of Y.

We now show that the conclusions of Theorem 11 may be strengthened in the case of a sequence of double Wiener-Itô integrals, i.e. in the case $d = 2$. The proof of the next result is deferred to the Section 3.3.

Theorem 12 *Under the assumptions and notation of Theorem 11 (in particular, (27) and (28) are in order), suppose that $d = 2$ and that the following implication holds:*

$$\mathbb{E}\left(\mathbb{E}\left[F_n^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] - 3Y^2\right)^2 \underset{n\to+\infty}{\to} 0 \quad \textit{if, and only if,} \quad \mathbb{E}\left[F_n^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \overset{\mathbb{P}}{\to} 3Y^2. \tag{36}$$

Then, the following are equivalent

(i) $\left\|proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right\|_{\mathfrak{H}_n}^2 \underset{n\to+\infty}{\overset{\mathbb{P}}{\to}} Y$;

(ii) $\mathbb{E}\left[\exp(i\lambda F_n) \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \overset{\mathbb{P}}{\to} \exp\left(-\frac{\lambda^2}{2}Y\right), \quad \forall \lambda \in \mathbb{R}$;

(iii) $\mathbb{E}\left[F_n^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \overset{\mathbb{P}}{\to} 3Y^2$;

(iv) $\left\|\left(\pi_{n,1}^{\otimes 2} - \pi_{n,t_n}^{\otimes 2}\right)\left(f_n \otimes_1^{\pi_n,t_n} f_n\right)\right\|_{\mathfrak{H}_n^{\odot 2}}^2 \underset{n\to+\infty}{\to} 0.$

Moreover, if either one of conditions (i)-(iv) is satisfied, $F_n \to_{(s,\mathcal{U}^*)} \mathbb{E}\mu(\cdot)$, where $\widehat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2}Y\right)$.

Remark – (a) Due again to the equivalence of the L^0 and L^2 topology on a finite sum of Wiener chaoses (see [27]), condition (36) is verified in the case $\mathfrak{H}_n = \mathfrak{H}$ and $X_n(\mathfrak{H}) = X(\mathfrak{H})$, for every $n \geq 1$.

(b) When $d = 2$, the second part of Theorem 11 corresponds to the implications (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) of Theorem 12.

The next consequence of Theorem 11 is a central limit theorem, generalizing Theorem 1.

Corollary 13 *Let $\mathfrak{H}_n$, $X_n(\mathfrak{H}_n)$, $n \geq 1$, be defined as above, and suppose that $\pi_n \in \mathcal{R}_{AC}(\mathfrak{H}_n)$ for each n. For $d \geq 2$, consider a sequence of multiple Wiener-Itô integrals $\left\{I_d^{X_n}(f_n) : n \geq 1\right\}$ s.t. $f_n \in \mathfrak{H}_n^{\odot d}$, and*

$$\mathbb{E}\left[I_d^{X_n}(f_n)^2\right] = d!\left\|f_n\right\|_{\mathfrak{H}_n^{\otimes d}}^2 \underset{n\to+\infty}{\to} 1.$$

Then, the following are equivalent

(i) $\left\|proj\left\{D_{X_n}I_d^{X_n}(f_n) \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right\|_{\mathfrak{H}_n}^2 \underset{n\to+\infty}{\overset{\mathbb{P}}{\to}} 1$;

(ii) $\mathbb{E}\left[\exp\left(i\lambda I_d^{X_n}(f_n)\right)\right] \underset{n\to+\infty}{\to} \exp\left(-\frac{\lambda^2}{2}\right)$, $\forall \lambda \in \mathbb{R}$, *that is,* $I_d^{X_n}(f_n) \overset{law}{\to} N(0,1)$;

(iii) $\mathbb{E}\left[I_d^{X_n}(f_n)^4\right] \underset{n\to+\infty}{\to} 3$;

(iv) $\left\|f_n \otimes_{d-r} f_n\right\|_{\mathfrak{H}_n^{\odot 2r}}^2 \underset{n\to+\infty}{\to} 0$, $\forall r = 1, ..., d-1$.

Proof. The equivalence of the three conditions (ii)-(iv) is the object of Theorem 1. That (i) implies (ii) follows from Corollary 8, in the case $t_n = 0$ for every n. Finally, (iv) implies (i) thanks to Theorem 11-2, again in the case $t_n = 0$. ■

3.2 Proof of Theorem 11

We start by proving an auxiliary analytic result. Let $(A, \mathcal{A})$ be a measurable space. For $m \geq 1$, let ξ_m be shorthand for a vector $\xi_m = ((a_1, x_1); ..., (a_m; x_m)) \in (A \times [0,1])^m$ and, for such ξ_m, note $\overline{\xi}_m$ the maximum of ξ_m in the variables $x_1, ..., x_m$, i.e. $\overline{\xi}_m = \overline{((a_1, x_1); ..., (a_m; x_m))} = \max_{i=1,...,n}(x_i)$. In what follows, $l(d\xi)$ stands for a σ-finite positive measure on $A \times [0,1]$, such that, for every fixed $x^* \in [0,1]$, $l\{(a,x) : x = x^*\} = 0$ (note that this implies that l is non-atomic). For $m \geq 1$, $l^m(d\xi_m)$ is the canonical product measure on $(A \times [0,1])^m$ (with $l^1 = l$ by convention).

Lemma 14 *Let $m, r \geq 1$, $C \subseteq [0,1]^m$ and $D \subseteq [0,1]^r$. Then, for every symmetric function $f \in L^2_s\left((A \times [0,1])^{m+r}, l^{m+r}\right) \triangleq L^2_s(l^{m+r})$*

$$\begin{aligned}
&\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) l^r(d\alpha_r)\right]^2 l^m(d\xi_m) l^m(d\gamma_m) \\
=\ &\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) \mathbf{1}_{\left(\overline{\alpha}_r < \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]^2 l^m(d\xi_m) l^m(d\gamma_m) \qquad (37)\\
&+\int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r) f(\xi_m,\beta_r) \mathbf{1}_{\left(\overline{\xi}_m < \max\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]^2 l^r(d\alpha_r) l^r(d\beta_r).
\end{aligned}$$

Proof. Start by writing

$$\begin{aligned}
&\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) l^r(d\alpha_r)\right]^2 l^m(d\xi_m) l^m(d\gamma_m) \\
=\ &\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) \times\right. \qquad (38)\\
&\left.\times\left(\mathbf{1}_{\left(\overline{\alpha}_r < \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} + \mathbf{1}_{\left(\overline{\alpha}_r > \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)}\right) l^r(d\alpha_r)\right]^2 l^m(d\xi_m) l^m(d\gamma_m) \\
=\ &\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) \mathbf{1}_{\left(\overline{\alpha}_r < \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]^2 l^m(d\xi_m) l^m(d\gamma_m) \\
&+\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) \mathbf{1}_{\left(\overline{\alpha}_r > \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]^2 l^m(d\xi_m) l^m(d\gamma_m) \\
&+2\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r) f(\gamma_m,\alpha_r) \mathbf{1}_{\left(\overline{\alpha}_r < \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right] \times \\
&\qquad\qquad \times\left[\int_{A\times D} f(\xi_m,\beta_r) f(\gamma_m,\beta_r) \mathbf{1}_{\left(\overline{\beta}_r > \max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\beta_r)\right] l^m(d\xi_m) l^m(d\gamma_m) \\
\triangleq\ &L(1) + L(2) + L(3)
\end{aligned}$$

(note that the equality (38) holds because of the assumption: $l\{(a,x) : x = x^*\} = 0, \forall x^*$). Now, by using a standard Fubini theorem,

$$L(2) = \int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\gamma_m,\alpha_r) f(\gamma_m,\beta_r) \mathbf{1}_{\left(\overline{\gamma}_m < \min\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\gamma_m)\right]^2 l^r(d\alpha_r) l^r(d\beta_r),$$

and also

$$\begin{aligned}
L(3) &= \int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r)\, f(\gamma_m,\alpha_r)\, \mathbf{1}_{\left(\min\left(\overline{\alpha}_r,\overline{\beta}_r\right)<\max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]\times \\
&\qquad \times\left[\int_{A\times D} f(\xi_m,\beta_r)\, f(\gamma_m,\beta_r)\, \mathbf{1}_{\left(\max\left(\overline{\alpha}_r,\overline{\beta}_r\right)>\max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\beta_r)\right] l^m(d\xi_m)\, l^m(d\gamma_m) \\
&= \int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r)\, f(\xi_m,\beta_r)\ \times\right. \\
&\qquad \left.\times\ \mathbf{1}_{\left(\min\left(\overline{\alpha}_r,\overline{\beta}_r\right)<\overline{\xi}_m<\max\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]^2 l^r(d\alpha_r)\, l^r(d\beta_r) + \\
&\quad +2\int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r)\, f(\xi_m,\beta_r)\, \mathbf{1}_{\left(\min\left(\overline{\alpha}_r,\overline{\beta}_r\right)<\overline{\xi}_m<\max\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]\times \\
&\qquad\qquad \left[\int_{A\times C} f(\gamma_m,\alpha_r)\, f(\gamma_m,\beta_r)\, \mathbf{1}_{\left(\overline{\gamma}_m<\min\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\gamma_m)\right] l^r(d\alpha_r)\, l^r(d\beta_r)\,.
\end{aligned}$$

The last relation implies that

$$L(2)+L(3)=\int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r)\, f(\xi_m,\beta_r)\, \mathbf{1}_{\left(\overline{\xi}_m<\max\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]^2 l^r(d\alpha_r)\, l^r(d\beta_r)\,,$$

hence proving (37). ■

Remark – With the notation of Lemma 14, suppose that the sequence $f_n \in L_s^2(l^{m+r})$, $n\geq 1$, is such that $\left\{\|f_n\|_{L_s^2(\zeta^{m+r})} : n\geq 1\right\}$ is bounded and, as $n\to+\infty$,

$$\begin{aligned}
&\int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f_n(\xi_m,\alpha_r)\, f_n(\gamma_m,\alpha_r)\, l^r(d\alpha_r)\right]^2 l^m(d\xi_m)\, l^m(d\gamma_m) \\
&= \int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f_n(\xi_m,\alpha_r)\, f_n(\xi_m,\beta_r)\, l^m(d\xi_m)\right]^2 l^r(d\alpha_r)\, l^r(d\beta_r) \qquad (39)\\
&\to 0
\end{aligned}$$

(note that the equality in (39) derives from a standard Fubini theorem). Then, by (38) and (37), Lemma 14 implies that the sequence $Q_i(n)$, defined for $i=1,2,3,4$ by

$$\begin{aligned}
Q_1(n) &= \int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r)\, f(\gamma_m,\alpha_r)\, \mathbf{1}_{\left(\overline{\alpha}_r<\max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]^2 l^m(d\xi_m)\, l^m(d\gamma_m)\,, \\
Q_2(n) &= \int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r)\, f(\xi_m,\beta_r)\, \mathbf{1}_{\left(\overline{\xi}_m<\max\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]^2 l^r(d\alpha_r)\, l^r(d\beta_r)\,, \\
Q_3(n) &= \int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r)\, f(\gamma_m,\alpha_r)\, \mathbf{1}_{\left(\overline{\alpha}_r>\max\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]^2 l^m(d\xi_m)\, l^m(d\gamma_m) \\
&= \int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r)\, f(\xi_m,\beta_r)\, \mathbf{1}_{\left(\overline{\xi}_m<\min\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]^2 l^r(d\alpha_r)\, l^r(d\beta_r)\,, \\
Q_4(n) &= \int_{A\times D}\int_{A\times D}\left[\int_{A\times C} f(\xi_m,\alpha_r)\, f(\xi_m,\beta_r)\, \mathbf{1}_{\left(\overline{\xi}_m>\max\left(\overline{\alpha}_r,\overline{\beta}_r\right)\right)} l^m(d\xi_m)\right]^2 l^r(d\alpha_r)\, l^r(d\beta_r) \\
&= \int_{A\times C}\int_{A\times C}\left[\int_{A\times D} f(\xi_m,\alpha_r)\, f(\gamma_m,\alpha_r)\, \mathbf{1}_{\left(\overline{\alpha}_r<\min\left(\overline{\xi}_m,\overline{\gamma}_m\right)\right)} l^r(d\alpha_r)\right]^2 l^m(d\xi_m)\, l^m(d\gamma_m)
\end{aligned}$$

(the equalities after the definitions of $Q_3(n)$ and $Q_4(n)$ are again a consequence of the Fubini theorem) converges to 0 as $n\to+\infty$. This fact will be used in the proof of Theorem 11-2.

(*Proof of Theorem 11-1*) By using a standard multiplication formula for multiple stochastic integrals (see e.g. [17, Proposition 1.5.1]), we obtain that

$$F_n^2 = d!\left\|f_n\right\|_{\mathfrak{H}_n^{\otimes d}}^2 + \sum_{r=1}^{d}(d-r)!\binom{d}{r}^2 I_{2r}^{X_n}\left[f_n \otimes_{d-r} f_n\right], \quad n \geq 1,$$

and consequently, for $n \geq 1$,

$$\begin{aligned}\mathbb{E}\left[F_n^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] &= d!\left\|f_n\right\|_{\mathfrak{H}_n^{\otimes d}}^2 \\ &\quad + \sum_{r=1}^{d}(d-r)!\binom{d}{r}^2 I_{2r}^{X_n}\left[\pi_{n,t_n}^{\otimes 2r}\left(f_n \otimes_{d-r} f_n\right)\right].\end{aligned} \tag{40}$$

Now observe that, for $r = 1, ..., d$,

$$\begin{aligned}I_{2r}^{X_n}\left[\pi_{n,t_n}^{\otimes 2r}\left(f_n \otimes_{d-r} f_n\right)\right] &= I_{2r}^{X_n}\left[\pi_{n,t_n}^{\otimes 2r}\left(f_n \otimes_{d-r}^{\pi_n,t_n} f_n\right)\right] \\ &\quad + I_{2r}^{X_n}\left[\left(\pi_{n,t_n}^{\otimes d} f_n\right) \otimes_{d-r} \left(\pi_{n,t_n}^{\otimes d} f_n\right)\right]\end{aligned} \tag{41}$$

and, in particular, for $r = d$

$$I_{2d}^{X_n}\left[\pi_{n,t_n}^{\otimes 2d}\left(f_n \otimes_0 f_n\right)\right] = I_{2d}^{X_n}\left[\left(\pi_{n,t_n}^{\otimes d} f_n\right) \otimes_0 \left(\pi_{n,t_n}^{\otimes d} f_n\right)\right]. \tag{42}$$

It follows from formulae (40), (41) and (42), that Theorem 11-1 is proved, once it is shown that $I_{2r}^{X_n}\left[\left(\pi_{n,t_n}^{\otimes d} f_n\right) \otimes_{d-r} \left(\pi_{n,t_n}^{\otimes d} f_n\right)\right] \to 0$, in $L^2(\mathbb{P})$, for every $r = 1, ..., d$. But

$$\mathbb{E}\left\{I_{2r}^{X_n}\left[\left(\pi_{n,t_n}^{\otimes d} f_n\right) \otimes_{d-r} \left(\pi_{n,t_n}^{\otimes d} f_n\right)\right]^2\right\} \leq (d-r)!\left\|\pi_{n,t_n}^{\otimes d} f_n\right\|_{\mathfrak{H}^{\otimes d}}^4 \underset{n\to+\infty}{\to} 0$$

due to assumption (27), hence yielding the desired conclusion.

(*Proof of Theorem 11-2*) For $m \geq 1$, we write $\mathbf{x}_m$ to indicate a vector $\mathbf{x}_m = (x_1, ..., x_m) \in [0,1]^m$, and also $\widehat{\mathbf{x}}_m = \max_{i=1,...,n}(x_i)$. Moreover, $d\mathbf{x}_m$ indicates the restriction of the Lebesgue measure to $[0,1]^m$. We first prove Theorem 11-2 when the following assumptions (a) and (b) are verified: (a) for every $n \geq 1$,

$$\mathfrak{H}_n = L^2\left(A_n \times [0,1], \mu_n, \mathcal{A}_n \otimes \mathcal{B}([0,1])\right), \tag{43}$$

where $(A_n, \mathcal{A}_n)$ is a measurable space, ν_n is a σ-finite (positive) measure on $(A_n, \mathcal{A}_n)$, and

$$\mu_n(da, dx) = k_n(a,x)\left\{\nu_n(da) \otimes dx\right\}, \tag{44}$$

where $k_n \in L^1\left(A_n \times [0,1], \nu_n, \mathcal{A}_n \otimes \mathcal{B}([0,1])\right)$ and $k_n \geq 0$; (b) for every n, for every $(a,x) \in A_n \times [0,1]$ and for every $h \in \mathfrak{H}_n$,

$$\pi_{n,t}h(a,x) = h(a,x)\,\mathbf{1}_{[0,t]}(x), \quad \forall t \in [0,1]. \tag{45}$$

Note that μ_n is non-atomic, and also that, in this setting, $\mathfrak{H}_n^{\otimes d} = L^2\left(\left(A_n \times [0,1]\right)^d, \mu_n^{\otimes d}\right)$ for every $d \geq 2$ and

$$\mathfrak{H}_n^{\odot d} = L_s^2\left(\left(A_n \times [0,1]\right)^d, \mu_n^{\otimes d}\right).$$

It follows that every $f \in \mathfrak{H}_n^{\odot d}$ can be identified with a (square integrable) function

$$f(a_1, ..., a_d; x_1, ..., x_d) = f(\mathbf{a}_d; \mathbf{x}_d), \quad \mathbf{a}_d \in A_n^d, \ \ \mathbf{x}_d \in [0,1]^d,$$

which is symmetric in the variables $(a_1, x_1), ..., (a_d, x_d)$. Moreover, by using the notation introduced in formulae (23)-(25), for every $f \in \mathfrak{H}_n^{\otimes d}$ and every $t \in [0,1]$,

$$\pi_{n,t}^{\otimes d} f(\mathbf{a}_d; \mathbf{x}_d) = f(\mathbf{a}_d; \mathbf{x}_d) \mathbf{1}_{[0,t]^d}(\mathbf{x}_d) \quad \text{and} \tag{46}$$

$$\left(\pi_{n,1}^{\otimes d} - \pi_{n,t}^{\otimes d}\right) f(\mathbf{a}_d; \mathbf{x}_d) = f(\mathbf{a}_d; \mathbf{x}_d) \mathbf{1}_{[0,1]^d \setminus [0,t]^d}(\mathbf{x}_d), \quad \mathbf{a}_d \in A_n^d, \ \mathbf{x}_d \in [0,1]^d. \tag{47}$$

Finally, we observe that (by using the notation introduced before the statement of Lemma 14), for every $m \geq 1$, every $\mathbf{x}_m = (x_1, ..., x_m) \in [0,1]^m$ and every $\xi_m \in (A_n \times [0,1])^m$ with the form $\xi_m = ((a_1, x_1), ..., (a_m, x_m))$,

$$\overline{\xi}_m = \overline{((a_1, x_1), ..., (a_m, x_m))} = \widehat{(x_1, ..., x_m)} = \widehat{\mathbf{x}}_m. \tag{48}$$

For $\mathfrak{H}_n$ and $\pi_n \in \mathcal{R}(\mathfrak{H}_n)$ $(n \geq 1)$ as in (43), (44) and (45), consider the sequence of isonormal Gaussian processes $X_n = X_n(\mathfrak{H}_n)$, $n \geq 1$, appearing in the statement of Theorem 11. Since, according to (26), $F_n = I_d^{X_n}(f_n)$, we obtain immediately that, for $n \geq 1$,

$$D_{X_n} F_n(a, x) = d \times I_{d-1}^{X_n}(f_n(a, \cdot; x, \cdot)), \tag{49}$$

where, for every fixed $(a, x) \in A \times [0,1]$, $f_n(a, \cdot; x, \cdot)$ stands for the (symmetric) function, from $(A \times [0,1])^{d-1}$ to $\mathbb{R}$,

$$(a_1, ..., a_{d-1}; x_1, ..., x_{d-1}) \mapsto f_n(a, a_1, ..., a_{d-1}; x, x_1, ..., x_{d-1}).$$

In this framework, the sequence $proj\left\{D_{X_n} F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}$, $n \geq 1$, can be easily made explicit by means of the following result.

Lemma 15 *If $\mathfrak{H}_n$ and $\pi_n \in \mathcal{R}(\mathfrak{H}_n)$, $n \geq 1$, satisfy relations (43), (44) and (45), for every $u = u(a, x) \in L^2(\mathfrak{H}_n, X_n)$, $\mathbb{P}$-a.s.,*

$$proj\left\{u \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}(a, x) = \mathbb{E}\left[u(a, x) \mid \mathcal{F}_x^{\pi_n}(X_n)\right], \tag{50}$$

for μ_n-a.e. (a, x), where the filtration $\mathcal{F}_x^{\pi_n}(X_n)$, $x \in [0,1]$, is defined according to (3).

Proof. Denote by u^* the process appearing on the right hand side of (50). To show that u^* is an element of $L^2_{\pi_n}(\mathfrak{H}_n, X_n)$ we need to show that it is a π_n-adapted element of $L^2(\mathfrak{H}_n, X_n)$. Since u belongs to $L^2(\mathfrak{H}_n, X_n)$, so does u^*. Moreover, u^* is π_n-adapted because, for every $h \in \mathfrak{H}_n$ and every $t \in [0,1]$,

$$\begin{aligned}
\left(u^*, \pi_{n,t} h\right)_{\mathfrak{H}_n} &= \int_{A_n \times [0,1]} u^*(a, x) \pi_{n,t} h(a, x) \mu_n(da, dx) \\
&= \int_{A_n \times [0,t]} \mathbb{E}\left[u(a, x) \mid \mathcal{F}_x^{\pi_n}(X_n)\right] h(a, x) \mu_n(da, dx) \in \mathcal{F}_t^{\pi_n}(X_n),
\end{aligned}$$

by (45). Now consider an element of $\mathcal{E}_{\pi_n}(\mathfrak{H}_n, X_n)$ with the form $g = \Phi(t_1)\left(\pi_{n,t_2} - \pi_{n,t_1}\right) f$ where $t_2 > t_1$, $f \in \mathfrak{H}_n$ and $\Phi(t_1) \in \mathcal{F}_{t_1}^{\pi_n}(X_n)$ is square-integrable. Then,

$$\begin{aligned}
(u, g)_{L^2(\mathfrak{H}_n, X_n)} &= \mathbb{E} \int_{A_n \times [0,1]} u(a, t) g(a, x) \mu_n(da, dx) \\
&= \int_{A_n \times (t_1, t_2]} \mathbb{E}\left(\Phi(t_1) u(a, x) f(a, x)\right) \mu_n(da, dx) \\
&= \int_{A_n \times (t_1, t_2]} \mathbb{E}\left(\Phi(t_1) \mathbb{E}\left[u(a, x) \mid \mathcal{F}_x^{\pi_n}(X_n)\right]\right) f(a, x) \mu_n(da, dx) \\
&= \left(u^*, g\right)_{L^2(\mathfrak{H}_n, X_n)},
\end{aligned}$$

where we have used a Fubini theorem and the fact that $\Phi(t_1) \in \mathcal{F}_{t_1}^{\pi_n}(X_n)$. Since $\mathcal{E}_{\pi_n}(\mathfrak{H}_n, X_n)$ is total in $L^2_{\pi_n}(\mathfrak{H}_n, X_n)$, we deduce that $(u,g)_{L^2(\mathfrak{H}_n,X_n)} = (u^*,g)_{L^2(\mathfrak{H}_n,X_n)}$, for every $g \in L^2_{\pi_n}(\mathfrak{H}_n, X_n)$, hence $u^* = proj\left\{u \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}$ as required. ■

In particular, thanks to the classic properties of multiple Wiener-Itô integral and conditional expectations (see e.g. [18]), we deduce from (46) and (49) that, for $F_n = I_d^{X_n}(f_n)$ as in (26) and $x \in [0,1]$,

$$\begin{aligned} proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}(a,x) &= d\mathbb{E}\left[I_{d-1}^{X_n}\left(f_n(a,\cdot;x,\cdot)\right) \mid \mathcal{F}_x^{\pi_n}(X_n)\right] \\ &= dI_{d-1}^{X_n}\left(f_n(a,\cdot;x,\cdot)\,\mathbf{1}_{A_n^{d-1}\times[0,x]^{d-1}}(\cdot,\cdot)\right), \end{aligned} \tag{51}$$

and consequently

$$\left\|proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right\|^2_{\mathfrak{H}_n} = d^2\int_{A_n\times[0,1]} I_{d-1}^{X_n}\left(f_n(a,\cdot;x,\cdot)\,\mathbf{1}_{A_n^{d-1}\times[0,x]^{d-1}}\right)^2\mu_n(da,dx). \tag{52}$$

Now note that, thanks to (6), (9) and the fact that $\mathbb{E}(F_n) = 0$,

$$\begin{aligned} \mathbb{E}\left\|proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right\|^2_{\mathfrak{H}_n} &= \mathbb{E}\left[\delta\left(proj\left\{D_{X_n}F_n \mid L^2_{\pi_n}(\mathfrak{H}_n, X_n)\right\}\right)^2\right] \\ &= \mathbb{E}\left[F_n^2\right] = d!\left\|f_n\right\|^2_{\mathfrak{H}_n^{\otimes d}}. \end{aligned}$$

Moreover, the chaotic expansion of the right hand side of (52) can be made explicit thanks to the standard multiplication formula (see again [17, Proposition 1.5.1])

$$I_{d-1}^{X_n}(g)^2 = (d-1)!\left\|g\right\|^2_{\mathfrak{H}^{\otimes d-1}} + \sum_{q=0}^{d-1} q!\binom{d-1}{q}^2 I^{X_n}_{2(d-1-q)}(g\otimes_q g),$$

applied to $g = f_n(a,\cdot;x,\cdot)\,\mathbf{1}_{A_n^{d-1}\times[0,x]^{d-1}}$ (for every fixed (a,x)), from which we obtain

$$\begin{aligned} & d^2\int_{A_n\times[0,1]} I_{d-1}^{X_n}\left(f_n(a,\cdot;x,\cdot)\,\mathbf{1}_{A_n^{d-1}\times[0,x]^{d-1}}\right)^2\mu_n(da,dx) \\ = \;& d!\left\|f_n\right\|^2_{\mathfrak{H}_n^{\otimes d}} + d^2\sum_{q=0}^{d-2} q!\binom{d-1}{q}^2 \times \\ & \times\int_{A_n\times[0,1]} I^{X_n}_{2(d-1-q)}\left(\int_{(A_n\times[0,x])^q} f_n(a,\mathbf{a}_q,\cdot;x,\mathbf{x}_q,\cdot)\times\right. \\ & \left.\times\, f_n(a,\mathbf{a}_q,\cdot\cdot;x,\mathbf{x}_q,\cdot\cdot)\,\mathbf{1}_{(A_n\times[0,x])^{2(d-1-q)}}(\cdot;\cdot\cdot)\,\mu_n^{\otimes q}(d\mathbf{a}_q,d\mathbf{x}_q)\right)\mu_n(da,dx) \\ = \;& d!\left\|f_n\right\|^2_{\mathfrak{H}_n^{\otimes d}} + \sum_{r=1}^{d-1}(d-r)!\binom{d}{r}^2(d-r)\times \\ & \times I^{X_n}_{2r}\left(\int_{A_n\times[0,1]}\int_{(A_n\times[0,x])^{d-1-r}} f_n(a,\mathbf{a}_{d-1-r},\cdot;x,\mathbf{x}_{d-1-r},\cdot)\times\right. \\ & \left.\times\, f_n(a,\mathbf{a}_{d-1-r},\cdot\cdot;x,\mathbf{x}_{d-1-r},\cdot\cdot)\,\mathbf{1}_{(A_n\times[0,x])^{2r}}(\cdot,\cdot)\,\mu_n^{\otimes d-1-r}(d\mathbf{a}_{d-1-r},d\mathbf{x}_{d-1-r})\,\mu_n(da,dx)\right) \end{aligned} \tag{53}$$

where the last term is obtained by putting $r = d-q-1$, and by using the identity

$$d^2q!\binom{d-1}{d-q-1}^2 = d^2(d-1-r)!\binom{d-1}{r}^2 = (d-r)!\binom{d}{r}^2(d-r),$$

and where we also applied, to obtain (53), a standard stochastic Fubini theorem (which is a consequence of the linearity of multiple stochastic integrals–see e.g. [17, Chapter 1]). We shall now use the symmetry of the function f_n in its first $d-r$ variables, as well as the relation $[0,1]^{d-r} \overset{a.s.}{=} \cup_{i=1,...,d-r}\{(a_1,...,a_{d-r}) : a_i > a_j, \quad \forall j \neq i\}$, where the union is disjoint, and the symbol ' $\overset{a.s.}{=}$ ' means that the equality is true up to sets of zero Lebesgue measure. Thus, for $r = 1,...,d-1$, and for any pair $(\mathbf{b}_r, \mathbf{z}_r), (\mathbf{b}'_r, \mathbf{z}'_r) \in (A_n \times [0,1])^r$

$$
\begin{aligned}
& (d-r) \int_{A_n\times[0,1]} \int_{(A_n\times[0,x])^{d-1-r}} f_n\left(a, \mathbf{a}_{d-1-r}, \mathbf{b}_r; x, \mathbf{x}_{d-1-r}, \mathbf{z}_r\right) f_n\left(a, \mathbf{a}_{d-1-r}, \mathbf{b}'_r; x, \mathbf{x}_{d-1-r}, \mathbf{z}'_r\right) \times \\
& \times \mathbf{1}_{(A_n\times[0,x])^r}\left(\mathbf{b}_r, \mathbf{z}_r\right) \mathbf{1}_{(A_n\times[0,x])^r}\left(\mathbf{b}'_r, \mathbf{z}'_r\right) \mu_n^{\otimes d-1-r}\left(d\mathbf{a}_{d-1-r}, d\mathbf{x}_{d-1-r}\right) \mu_n\left(da, dx\right) \\
= & \int_{(A_n\times[0,1])^{d-r}} f_n\left(\mathbf{a}_{d-r}, \mathbf{b}_r; \mathbf{x}_{d-r}, \mathbf{z}_r\right) f_n\left(\mathbf{a}_{d-r}, \mathbf{b}'_r; \mathbf{x}_{d-r}, \mathbf{z}'_r\right) \times \\
& \times \mathbf{1}_{(A_n\times[0,\widehat{\mathbf{x}}_{d-r}])^r}\left(\mathbf{b}_r, \mathbf{z}_r\right) \mathbf{1}_{(A_n\times[0,\widehat{\mathbf{x}}_{d-r}])^r}\left(\mathbf{b}'_r, \mathbf{z}'_r\right) \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right) \\
= & \int_{(A_n\times[0,1])^{d-r}} f_n\left(\mathbf{a}_{d-r}, \mathbf{b}_r; \mathbf{x}_{d-r}, \mathbf{z}_r\right) f_n\left(\mathbf{a}_{d-r}, \mathbf{b}'_r; \mathbf{x}_{d-r}, \mathbf{z}'_r\right) \mathbf{1}_{\{\max(\widehat{\mathbf{z}}_r, \widehat{\mathbf{z}}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right);
\end{aligned}
$$

(recall that $\widehat{\mathbf{x}}_{d-r} = \max_{i=1,...,d-r} x_i$). Observe that the last integral would be the contraction $f_n \otimes_{d-r} f_n$, if there was no indicator functions inside the integral. Now denote by

$$
\int_{(A_n\times[0,1])^{d-r}} f_n\left(\mathbf{a}_{d-r}, **_r; \mathbf{x}_{d-r}, *_r\right) f_n\left(\mathbf{a}_{d-r}, **'_r; \mathbf{x}_{d-r}, *'_r\right) \mathbf{1}_{\{\max(\widehat{*}_r, \widehat{*}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right)
$$

the application, from $(A_n \times [0,x])^r \times (A_n \times [0,x])^r$ to $\mathbb{R}$, given by

$$
\begin{aligned}
\left(\left(\mathbf{b}_r, \mathbf{z}_r\right), \left(\mathbf{b}'_r, \mathbf{z}'_r\right)\right) \quad \mapsto \quad & \int_{(A_n\times[0,1])^{d-r}} f_n\left(\mathbf{a}_{d-r}, \mathbf{b}_r; \mathbf{x}_{d-r}, \mathbf{z}_r\right) \times \\
& \times f_n\left(\mathbf{a}_{d-r}, \mathbf{b}'_r; \mathbf{x}_{d-r}, \mathbf{z}'_r\right) \mathbf{1}_{\{\max(\widehat{\mathbf{z}}_r, \widehat{\mathbf{z}}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right).
\end{aligned}
$$

Relation (52) and the preceding computation imply that

$$
\begin{aligned}
& \left\| proj\left\{D_{X_n} F_n \mid L^2_{\pi^{(n)}}\left(\mathfrak{H}_n, X_n\right)\right\}\right\|^2_{\mathfrak{H}_n} \\
= \quad & d! \left\|f_n\right\|^2_{\mathfrak{H}_n^{\odot d}} + \sum_{r=1}^{d-1} (d-r)! \binom{d}{r}^2 I^{X_n}_{2r}\left(\int_{(A_n\times[0,1])^{d-r}} f_n\left(\mathbf{a}_{d-r}, **_r; x, \mathbf{x}_{d-r}, *_r\right) \times \right. \qquad (54) \\
& \left. \times f_n\left(\mathbf{a}_{d-r}, **'_r; \mathbf{x}_{d-r}, *'_r\right) \mathbf{1}_{\{\max(\widehat{*}_r, \widehat{*}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right)\right).
\end{aligned}
$$

Now, for $r = 1,...,d-1$ and $t \in [0,1]$,

$$
\begin{aligned}
& \int_{(A_n\times[0,1])^{d-r}} f_n\left(\mathbf{a}_{d-r}, **_r; \mathbf{x}_{d-r}, *_r\right) f_n\left(\mathbf{a}_{d-r}, **'_r; \mathbf{x}_{d-r}, *'_r\right) \mathbf{1}_{\{\max(\widehat{*}_r, \widehat{*}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right) \\
= & \int_{(A_n\times[0,t])^{d-r}} f_n\left(\mathbf{a}_{d-r}, **_r; \mathbf{x}_{d-r}, *_r\right) f_n\left(\mathbf{a}_{d-r}, **'_r; \mathbf{x}_{d-r}, *'_r\right) \mathbf{1}_{\{\max(\widehat{*}_r, \widehat{*}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right) \\
& + \int_{A_n^{d-r}\times\left([0,1]^{d-r}\setminus[0,t]^{d-r}\right)} f_n\left(\mathbf{a}_{d-r}, **_r; \mathbf{x}_{d-r}, *_r\right) \times \\
& \qquad \times f_n\left(\mathbf{a}_{d-r}, **'_r; \mathbf{x}_{d-r}, *'_r\right) \mathbf{1}_{\{\max(\widehat{*}_r, \widehat{*}'_r) \leq \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}\right) \\
\triangleq \quad & G^r_{n,t}(1) + G^r_{n,t}(2),
\end{aligned}
$$

(plainly, $G^r_{n,t}(1), G^r_{n,t}(2) \in \mathfrak{H}_n^{\otimes 2r}$) and observe that, by bounding the indicator function by 1 and using the Cauchy-Schwarz inequality,

$$
\mathbb{E}\left[I^{X_n}_{2r}\left(G^r_{n,t}(1)\right)^2\right] = (2r)! \left\|\left(G^r_{n,t}(1)\right)_s\right\|^2_{\mathfrak{H}_n^{\otimes 2r}} \leq (2r)! \left\|f_n \mathbf{1}_{[0,t]^d}\right\|^4_{\mathfrak{H}_n^{\otimes d}}.
$$

Now, if t_n is the sequence in the statement of Theorem 11, one has

$$d!\left\|f_n\mathbf{1}_{[0,t_n]^d}\right\|^2_{\mathfrak{H}_n^{\otimes d}}=\mathbb{E}\left(I_d^{X_n}\left(\pi^{\otimes d}_{n,t_n}f_n\right)^2\right)\to 0.$$

Thus, (27) implies

$$\lim_{n\to+\infty}\mathbb{E}\left[I_{2r}^{X_n}\left(G^r_{n,t}\left(1\right)\right)^2\right]\leq(2r)!\lim_{n\to+\infty}\left\|f_n\mathbf{1}_{[0,t_n]^d}\right\|^4_{\mathfrak{H}_n^{\otimes d}}=0.$$

We now deal with $G^r_{n,t}\left(2\right)$. For every $t\in[0,1]$, we may write

$$G^r_{n,t}\left(2\right)=G^r_{n,t}\left(2\right)\mathbf{1}_{A_n^{2r}\times[0,t]^{2r}}+G^r_{n,t}\left(2\right)\mathbf{1}_{A_n^{2r}\times\left([0,1]^{2r}\setminus[0,t]^{2r}\right)}\triangleq H^r_{n,t}\left(1\right)+H^r_{n,t}\left(2\right). \tag{55}$$

Consider first $H^r_{n,t}\left(1\right)$. Because of the presence of the indicator function $\mathbf{1}_{A_n^{2r}\times[0,t]^{2r}}$, the indicator function in the integral defining $G^r_{n,t}\left(2\right)$ is always equal to 1, and one gets, for every $r=1,...,d-1$,

$$\begin{aligned}H^r_{n,t}\left(1\right) &= \left\{\int_{A_n^{d-r}\times\left([0,1]^{d-r}\setminus[0,t]^{d-r}\right)}f_n\left(\mathbf{a}_{d-r},{**}_r;\mathbf{x}_{d-r},{*}_r\right)\times\right.\\ &\quad\left.\times\, f_n\left(\mathbf{a}_{d-r},{**}'_r;\mathbf{x}_{d-r},{*}'_r\right)\mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r},d\mathbf{x}_{d-r}\right)\right\}\times\mathbf{1}_{A_n^{2r}\times[0,t]^{2r}}\\ &= \pi^{\otimes 2r}_{n,t_n}\left(f_n\otimes^{\pi_n,t_n}_{d-r}f_n\right),\end{aligned}$$

which appears in (29). Then, in view of (28), (29) and (54), we have that

$$\left\|proj\left\{D_{X_n}F_n\mid L^2_{\pi_n}\left(\mathfrak{H}_n,X_n\right)\right\}\right\|^2_{\mathfrak{H}_n}=Y+o_{\mathbb{P}}\left(1\right)+\sum_{r=1}^{d-1}(d-r)!\binom{d}{r}^2I_{2r}^{X_n}\left(H^r_{n,t_n}\left(2\right)\right),$$

where $H^r_{n,t_n}\left(2\right)\in\mathfrak{H}_n^{\otimes 2r}$ is defined by (55). We shall now show that (30) implies $H^r_{n,t_n}\left(2\right)\to 0$ in $\mathfrak{H}_n^{\otimes 2r}$, for every $r=1,...,d-1$. Now observe that, because of (43), (44) and (45), condition (30) can be rewritten as follows: for every $r=1,...,d-1$, the sequence $Z_r\left(n\right)\in\mathfrak{H}_n^{\otimes 2r}$, $n\geq 1$, defined as

$$\begin{aligned}Z_r\left(n\right) &= \left(\int_{A_n^{d-r}\times\left([0,1]^{d-r}\setminus[0,t_n]^{d-r}\right)}f_n\left(\mathbf{a}_{d-r},{**}_r;\mathbf{x}_{d-r},{*}_r\right)\times\right.\\ &\quad\left.\times\, f_n\left(\mathbf{a}_{d-r},{**}'_r;\mathbf{x}_{d-r},{*}'_r\right)\mu_n^{\otimes d-r}\left(d\mathbf{a}_{d-r},d\mathbf{x}_{d-r}\right)\right)\mathbf{1}_{A_n^{2r}\times\left([0,1]^{2r}\setminus[0,t_n]^{2r}\right)}.\end{aligned} \tag{56}$$

is such that

$$\lim_{n\to+\infty}\left\|Z_r\left(n\right)\right\|^2_{\mathfrak{H}_n^{\otimes 2r}}=0. \tag{57}$$

As a consequence, in this case the statement is proved once it is shown that, for $r=1,...,d-1$, (57) implies necessarily that $\lim_{n\to+\infty}\left\|H^r_{n,t_n}\left(2\right)\right\|^2_{\mathfrak{H}_n^{\otimes 2r}}=0$ (recall that $H^r_{n,t_n}\left(2\right)$ is given by (55)). To this end, introduce the notation: for every $q\geq 1$, every $p=0,...,q$ and $t\in[0,1]$

$$T^t_n\left(q,p\right)\triangleq\left\{\left(\mathbf{a}_q,\mathbf{x}_q\right)\in A_n^q\times[0,1]^q:\ \text{there are exactly } p \text{ indices } i \text{ such that } x_i\leq t\right\}$$

and note that, for $q\geq 1$, up to sets of zero μ_n – measure,

$$S^t_n\left(q\right) \triangleq A_n^q\times\left([0,1]^q\setminus[0,t]^q\right)=\bigcup_{p=0}^{q-1}T^t_n\left(q,p\right), \tag{58}$$

$$S^t_n\left(2q\right) = A_n^{2q}\times\left([0,1]^{2q}\setminus[0,t]^{2q}\right)=\bigcup_{\substack{p,s\geq 0\\ p\wedge s\leq q-1}}T^t_n\left(q,p\right)\times T^t_n\left(q,s\right), \tag{59}$$

where the unions are disjoint. With this notation, by (58), (59) and the fact that μ_n is non-atomic (so that we can write $<$ instead of $\leq$ in the indicator function defining $H^r_{n,t_n}(2)$), we therefore obtain that, for each n,

$$\begin{aligned}
&\left\|H^r_{n,t_n}(2)\right\|^2_{\mathfrak{H}_n^{\otimes 2r}} \\
&= \int_{A_n^{2r}\times\left([0,1]^{2r}\setminus[0,t_n]^{2r}\right)} \left(\int_{A_n^{d-r}\times\left([0,1]^{d-r}\setminus[0,t_n]^{d-r}\right)} f_n(\mathbf{a}_{d-r},\mathbf{b}_r;\mathbf{x}_{d-r},\mathbf{z}_r)\, f_n(\mathbf{a}_{d-r},\mathbf{b}'_r;\mathbf{x}_{d-r},\mathbf{z}'_r) \times \right. \\
&\qquad \left. \times\, \mathbf{1}_{\{\max(\widehat{\mathbf{z}}_r,\widehat{\mathbf{z}}'_r)<\widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}(d\mathbf{a}_{d-r},d\mathbf{x}_{d-r}) \right)^2 \mu_n^{\otimes 2r}(d\mathbf{b}_r,d\mathbf{z}_r)\, \mu_n^{\otimes 2r}(d\mathbf{b}'_r,d\mathbf{z}'_r) \qquad (60) \\
&= \sum_{p,s\geq 0,\ \ p\wedge s\leq r-1} \int_{T_n^{t_n}(r,p)\times T_n^{t_n}(r,s)} \left(\sum_{q=0}^{d-r-1} \int_{T_n^{t_n}(d-r,q)} f_n(\mathbf{a}_{d-r},\mathbf{b}_r;\mathbf{x}_{d-r},\mathbf{z}_r) \times \right. \qquad (61) \\
&\quad \left. \times\, f_n(\mathbf{a}_{d-r},\mathbf{b}'_r;\mathbf{x}_{d-r},\mathbf{z}'_r)\, \mathbf{1}_{\{\max(\widehat{\mathbf{z}}_r,\widehat{\mathbf{z}}'_r)<\widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}(d\mathbf{a}_{d-r},d\mathbf{x}_{d-r}) \right)^2 \mu_n^{\otimes 2r}(d\mathbf{b}_r,d\mathbf{z}_r)\, \mu_n^{\otimes 2r}(d\mathbf{b}'_r,d\mathbf{z}'_r)
\end{aligned}$$

To prove that $\left\|H^r_{n,t_n}(2)\right\|^2_{\mathfrak{H}_n^{\otimes 2r}} \to 0$, it is therefore sufficient to show that, for every $r=1,...,d-1$, $q=0,...,d-r-1$ and every $p,s\geq 0$ with $p\wedge s\leq r-1$, the sequence

$$\begin{aligned}
&\int_{T_n^{t_n}(r,p)} \int_{T_n^{t_n}(r,s)} \left(\int_{T_n^{t_n}(d-r,q)} f_n(\mathbf{a}_{d-r},\mathbf{b}_r;\mathbf{x}_{d-r},\mathbf{z}_r)\, f_n(\mathbf{a}_{d-r},\mathbf{b}'_r;\mathbf{x}_{d-r},\mathbf{z}'_r) \right. \qquad (62) \\
&\qquad \left. \times\, \mathbf{1}_{\{\max(\widehat{\mathbf{z}}_r,\widehat{\mathbf{z}}'_r)<\widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}(d\mathbf{a}_{d-r},d\mathbf{x}_{d-r}) \right)^2 \mu_n^{\otimes 2r}(d\mathbf{b}_r,d\mathbf{z}_r)\, \mu_n^{\otimes 2r}(d\mathbf{b}'_r,d\mathbf{z}'_r)
\end{aligned}$$

converges to zero, as $n\to+\infty$. To prove this result, write (for $n\geq 1$ and $r=1,...,d-1$) $\|Z_r(n)\|^2_{\mathfrak{H}_n^{\otimes 2r}}$ by means of (56), decompose the set $S_n^{t_n}(2r)$ according to (59), and apply a standard Fubini argument to obtain that (57) implies that, for every $r=1,...,d-1$ and $q=0,...,r-1$, the quantity

$$\begin{aligned}
&\int_{S_n^{t_n}(d-r)} \int_{S_n^{t_n}(d-r)} \left[\int_{T_n^{t_n}(r,q)} f_n(\mathbf{a}_r,\mathbf{b}_{d-r};\mathbf{x}_r,\mathbf{z}_{d-r}) \right. \qquad (63) \\
&\qquad \left. \times\, f_n\left(\mathbf{a}_r,\mathbf{b}'_{d-r};\mathbf{x}_r,\mathbf{z}'_{d-r}\right) \mu_n^{\otimes r}(d\mathbf{a}_r,d\mathbf{x}_r) \right]^2 \mu_n^{\otimes d-r}(d\mathbf{b}_{d-r},d\mathbf{z}_{d-r})\, \mu_n^{\otimes d-r}\left(d\mathbf{b}'_{d-r},d\mathbf{z}'_{d-r}\right) \\
&= \sum_{p=0}^{d-r-1} \sum_{s=0}^{d-r-1} \int_{T_n^{t_n}(d-r,p)} \int_{T_n^{t_n}(d-r,s)} \left[\int_{T_n^{t_n}(r,q)} f_n(\mathbf{a}_r,\mathbf{b}_{d-r};\mathbf{x}_r,\mathbf{z}_{d-r}) \right. \qquad (64) \\
&\qquad \left. \times\, f_n\left(\mathbf{a}_r,\mathbf{b}'_{d-r};\mathbf{x}_r,\mathbf{z}'_{d-r}\right) \mu_n^{\otimes r}(d\mathbf{a}_r,d\mathbf{x}_r) \right]^2 \mu_n^{\otimes d-r}(d\mathbf{b}_{d-r},d\mathbf{z}_{d-r})\, \mu_n^{\otimes d-r}\left(d\mathbf{b}'_{d-r},d\mathbf{z}'_{d-r}\right) \\
&\to 0, \quad \text{as } n\to+\infty, \qquad (65)
\end{aligned}$$

where the equality (64) is a consequence of (59). Now fix $p\in\{0,...,d-r-1\}$ and $q\in\{0,...,r-1\}$. For every $n\geq 1$, we can apply Lemma 14 in the case $A=A_n$, $l=\mu_n$, $f_n=f$, $m=d-r$ and $C\subseteq[0,1]^{d-r}$

and $D \subseteq [0,1]^r$ such that $A_n \times C = T_n^{t_n}(d-r,p)$ and $A_n \times D = T_n^{t_n}(r,q)$ to obtain that

$$\int_{T_n^{t_n}(d-r,p)} \int_{T_n^{t_n}(d-r,p)} \left[\int_{T_n^{t_n}(r,q)} f_n(\mathbf{a}_r, \mathbf{b}_{d-r}; \mathbf{x}_r, \mathbf{z}_{d-r}) \right. \tag{66}$$

$$\left. \times f_n\left(\mathbf{a}_r, \mathbf{b}'_{d-r}; \mathbf{x}_r, \mathbf{z}'_{d-r}\right) \mu_n^{\otimes r}(d\mathbf{a}_r, d\mathbf{x}_r) \right]^2 \mu_n^{\otimes d-r}(d\mathbf{b}_{d-r}, d\mathbf{z}_{d-r}) \, \mu_n^{\otimes d-r}\left(d\mathbf{b}'_{d-r}, d\mathbf{z}'_{d-r}\right)$$

$$= \int_{T_n^{t_n}(d-r,p)} \int_{T_n^{t_n}(d-r,p)} \left[\int_{T_n^{t_n}(r,q)} f_n(\mathbf{a}_r, \mathbf{b}_{d-r}; \mathbf{x}_r, \mathbf{z}_{d-r}) \, f_n\left(\mathbf{a}_r, \mathbf{b}'_{d-r}; \mathbf{x}_r, \mathbf{z}'_{d-r}\right) \right. \tag{67}$$

$$\left. \times \mathbf{1}_{\left(\widehat{\mathbf{x}}_r < \max\left(\widehat{\mathbf{z}}_{d-r}, \widehat{\mathbf{z}}'_{d-r}\right)\right)} \mu_n^{\otimes r}(d\mathbf{a}_r, d\mathbf{x}_r) \right]^2 \mu_n^{\otimes d-r}(d\mathbf{b}_{d-r}, d\mathbf{z}_{d-r}) \, \mu_n^{\otimes d-r}\left(d\mathbf{b}'_{d-r}, d\mathbf{z}'_{d-r}\right)$$

$$+ \int_{T_n^{t_n}(r,q)} \int_{T_n^{t_n}(r,q)} \left[\int_{T_n^{t_n}(d-r,p)} f_n(\mathbf{a}_r, \mathbf{b}_{d-r}; \mathbf{x}_r, \mathbf{z}_{d-r}) \, f_n(\mathbf{a}'_r, \mathbf{b}_{d-r}; \mathbf{x}'_r, \mathbf{z}_{d-r}) \right.$$

$$\left. \times \mathbf{1}_{(\widehat{\mathbf{z}}_{d-r} < \max(\widehat{\mathbf{x}}_r, \widehat{\mathbf{x}}'_r))} \mu_n^{\otimes r}(d\mathbf{b}_{d-r}, d\mathbf{z}_{d-r}) \right]^2 \mu_n^{\otimes r}(d\mathbf{a}_r, d\mathbf{x}_r) \, \mu_n^{\otimes r}(d\mathbf{a}'_r, d\mathbf{x}'_r).$$

Now observe that the sequence $\left\{ \|f_n\|_{\mathfrak{H}_n^{\otimes d}} : n \geq 1 \right\}$ is bounded by assumption (indeed, relation (28) holds). We can therefore argue as in the remark following the proof of Lemma 14, and deduce that, since for every $p \in \{0, ..., d-r-1\}$ and $q \in \{0, ..., r-1\}$ the sequence in (66) converges to 0 (by (65)), then the sequence in (62) converges to 0 for every $r = 1, ..., d-1$, whenever $p = s \leq r-1$ and $q = 0, ..., d-r-1$. To prove that (62) converges to 0 for every $r = 1, ..., d-1$, $q = 0, ..., d-r-1$ and every $p, s \geq 0$ such that $p \wedge s \leq r-1$, thus concluding the proof of Theorem 7 in this special setting, observe that, due to the Fubini theorem, the quantity

$$\int_{T_n^{t_n}(r,p)} \int_{T_n^{t_n}(r,s)} \left(\int_{T_n^{t_n}(d-r,q)} f_n(\mathbf{a}_{d-r}, \mathbf{b}_r; \mathbf{x}_{d-r}, \mathbf{z}_r) \, f_n(\mathbf{a}_{d-r}, \mathbf{b}'_r; \mathbf{x}_{d-r}, \mathbf{z}'_r) \right.$$

$$\left. \times \mathbf{1}_{\{\max(\widehat{\mathbf{z}}_r, \widehat{\mathbf{z}}'_r) < \widehat{\mathbf{x}}_{d-r}\}} \mu_n^{\otimes d-r}(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}) \right)^2 \mu_n^{\otimes 2r}(d\mathbf{b}_r, d\mathbf{z}_r) \, \mu_n^{\otimes 2r}(d\mathbf{b}'_r, d\mathbf{z}'_r)$$

can be rewritten as

$$\int_{T_n^{t_n}(d-r,q)} \int_{T_n^{t_n}(d-r,q)} \left(\int_{T_n^{t_n}(r,p)} f_n(\mathbf{a}_{d-r}, \mathbf{b}_r; \mathbf{x}_{d-r}, \mathbf{z}_r) \, f_n\left(\mathbf{a}'_{d-r}, \mathbf{b}; \mathbf{x}'_{d-r}, \mathbf{z}_r\right) \right.$$

$$\left. \times \mathbf{1}_{\left\{\widehat{\mathbf{z}}_r < \min\left(\widehat{\mathbf{x}}_{d-r}, \widehat{\mathbf{x}}'_{d-r}\right)\right\}} \mu_n^{\otimes r}(d\mathbf{b}_r, d\mathbf{z}_r) \right) \times$$

$$\left(\int_{T_n^{t_n}(r,s)} f_n(\mathbf{a}_{d-r}, \mathbf{b}_r; \mathbf{x}_{d-r}, \mathbf{z}_r) \, f_n\left(\mathbf{a}'_{d-r}, \mathbf{b}; \mathbf{x}'_{d-r}, \mathbf{z}\right) \times \right.$$

$$\left. \times \mathbf{1}_{\left\{\widehat{\mathbf{z}}_r < \min\left(\widehat{\mathbf{x}}_{d-r}, \widehat{\mathbf{x}}'_{d-r}\right)\right\}} \mu_n^{\otimes r}(d\mathbf{b}_r, d\mathbf{z}_r) \right) \mu_n^{\otimes d-r}(d\mathbf{a}_{d-r}, d\mathbf{x}_{d-r}) \, \mu_n^{\otimes d-r}\left(d\mathbf{a}'_{d-r}, d\mathbf{x}'_{d-r}\right),$$

so that the conclusion is obtained by a further application of (67), as well as a standard version of the Cauchy-Schwarz inequality.

To prove Theorem 11 in the general case, we start by showing that, for every real separable Hilbert space $\mathfrak{H}$, and for every absolutely continuous resolution of the identity $\pi = \{\pi_t : t \in [0,1]\} \in \mathcal{R}_{AC}(\mathfrak{H})$, there exists a Hilbert space $\mathfrak{H}^\sharp$ with the form (43) and (44) (the dependence on n has been momentarily dropped), and a resolution $\pi^\sharp = \left\{ \pi_t^\sharp : t \in [0,1] \right\}$ on $\mathfrak{H}^\sharp$ as in (45), such that the following property is verified: there exists a unitary transformation

$$T : \mathfrak{H}^\sharp \mapsto \mathfrak{H}, \tag{68}$$

from $\mathfrak{H}^\sharp$ onto $\mathfrak{H}$, such that, for every $t \in [0,1]$,

$$\pi_t T = T\pi_t^\sharp. \tag{69}$$

In the language of [30, Definition 5.1], (69) implies that the two pairs $(\mathfrak{H}, \pi)$ and $\left(\mathfrak{H}^\sharp, \pi^\sharp\right)$ are *equivalent*. Note that (69) holds if, and only if, the following condition is verified: for every $t \in [0,1]$, $\pi_t = T\pi_t^\sharp T^{-1}$. Moreover, since T is a unitary transformaton, $T^{-1} = T^*$. To prove the existence of such a T, let π be absolutely continuous on $\mathfrak{H}$, set $q = rank\,(\pi)$, and consider a fully orthogonal π-reproducing subset

$$S = \{g_j : 1 \leq j \leq q\} \subseteq \mathfrak{H}. \tag{70}$$

Note that the full orthogonality of S implies that, for every $s, t \in [0,1]$ and for every $i \neq j$, $(\pi_s g_i, \pi_t g_j)_{\mathfrak{H}} = 0$. Moreover, since π is absolutely continuous, for every $j \geq 1$ there exists a function $m_j(t)$, $t \in [0,1]$, such that $m_j(\cdot) \geq 0$, and

$$\|\pi_t g_j\|_{\mathfrak{H}}^2 = \int_0^t m_j(x)\,dx, \quad t \in [0,1]. \tag{71}$$

Note that (71) implies that $m_j(\cdot) \in L^1([0,1], dx)$, and that we can always define the set S in (70) to be such that

$$\sum_{j=1}^q \|g_j\|_{\mathfrak{H}}^2 = \sum_{j=1}^q \int_0^1 m_j(x)\,dx < +\infty. \tag{72}$$

Now define $A = \mathbb{N} = \{1, 2, ...\}$, set ν equal to the counting measure on A, and μ equal to the measure on $A \times [0,1]$ given by $\mu(da, dx) = k(a,x)\{\nu(da) \otimes dx\}$, where

$$k(a,x) = \sum_{j=1}^q \mathbf{1}_{\{j\}}(a)\, m_j(x), \quad (a,x) \in A \times [0,1],$$

$\mathbf{1}_{\{j\}}$ stands for the indicator of the singleton $\{j\}$, and dx is once again Lebesgue measure. Finally, we define

$$\begin{aligned} \mathfrak{H}^\sharp &= L^2(A \times [0,1], \mu, \mathcal{A} \otimes \mathcal{B}[0,1]) \\ \pi_t^\sharp h^\sharp(a,x) &= h^\sharp(a,x)\,\mathbf{1}_{[0,t]}(x), \end{aligned} \tag{73}$$

for every $h^\sharp \in \mathfrak{H}^\sharp$, every $t \in [0,1]$, and every $(a,x) \in A \times [0,1]$. We now introduce a transformation T defined on a dense subset of $\mathfrak{H}^\sharp$: for every $h^\sharp \in \mathfrak{H}^\sharp$ with the form

$$h^\sharp(a,x) = \sum_{j=1}^M c_j \mathbf{1}_{\{k_j\}}(a)\,\mathbf{1}_{[0,u_j]}(x) \tag{74}$$

($(a,x) \in A \times [0,1]$), where $M, k_j \geq 1$, $u_j \in [0,1]$ and $c_j \in \mathbb{R}$ ($j = 1, ..., M$),

$$Th^\sharp = \sum_{j=1}^M c_j \pi_{u_j} g_{k_j},$$

where the g_k's are the elements of the full orthogonal set S, as defined in (70). By using the relation,

$$\left(Th^\sharp, Th^\sharp\right)_{\mathfrak{H}} = \left(h^\sharp, h^\sharp\right)_{\mathfrak{H}^\sharp},$$

which is verified for every $h^\sharp$ as in (74), one immediately sees that T can be extended by density to a unitary transformation, from $\mathfrak{H}^\sharp$ onto $\mathfrak{H}$, and moreover, since, for every $t \in [0,1]$,

$$\pi_t T h^\sharp = \sum_{j=1}^M c_j \pi_{u_j \wedge t} g_{k_j} = T\pi_t^\sharp h^\sharp,$$

condition (69) is verified. We note again T this extended isomorphism, and, for $d \geq 2$, we write $T^d \triangleq T^{\otimes d}$, and also $T^1 \triangleq T^{\otimes 1} = T$. Observe that T^d is an isomorphism from $\left(\mathfrak{H}^\sharp\right)^{\otimes d}$ onto $\mathfrak{H}^{\otimes d}$, $\left(T^{-1}\right)^d = \left(T^d\right)^{-1}$. Also, for $t \in [0,1]$ and due to (69),

$$T^d \left(\pi_t^\sharp\right)^{\otimes d} = \left(T\pi_t^\sharp\right)^{\otimes d} = (\pi_t T)^{\otimes d} = \pi_t^{\otimes d} T^d. \tag{75}$$

Now, for an absolutely continuous resolution π on $\mathfrak{H}$, and for $h^\sharp$ and $\mathfrak{H}^\sharp$ as in (73), we define $X = \{X(f) : f \in \mathfrak{H}\}$ to be an isonormal Gaussian process over $\mathfrak{H}$, and set

$$X_T = X_T\left(\mathfrak{H}^\sharp\right) = \left\{X_T\left(h^\sharp\right) : h^\sharp \in \mathfrak{H}^\sharp\right\}, \tag{76}$$

where $X_T\left(h^\sharp\right) \triangleq X\left(Th^\sharp\right)$, $\forall h^\sharp \in \mathfrak{H}^\sharp$. It is clear that, due to the isometric property of T, X_T is an isonormal Gaussian process over $\mathfrak{H}^\sharp$. The proof of the following useful lemma is deferred to the Appendix.

Lemma 16 *Under the above notation,*

1. *For every $d \geq 1$, $f \in \mathfrak{H}^{\otimes d}$, $I_d^X(f) = I_d^{X_T}\left(\left(T^d\right)^{-1} f\right)$;*
2. *$\mathbb{D}_X^{1,2} = \mathbb{D}_{X_T}^{1,2}$, and, for every $F \in \mathbb{D}_X^{1,2}$,*
$$D_X F = T\left(D_{X_T} F\right); \tag{77}$$
3. *For every $t \in [0,1]$,*
$$\mathcal{F}_t^\pi(X) = \sigma\{X(\pi_t f) : f \in \mathfrak{H}\} = \sigma\left\{X_T\left(\pi_t^\sharp h^\sharp\right) : h^\sharp \in \mathfrak{H}^\sharp\right\} = \mathcal{F}_t^{\pi^\sharp}(X_T);$$
4. *For every $u \in L^2\left(\mathfrak{H}^\sharp, X_T\right)$, $u \in L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)$ if, and only if, $Tu \in L^2_\pi(\mathfrak{H}, X)$;*
5. *For every $F \in \mathbb{D}_X^{1,2}$, a.s.-$\mathbb{P}$,*
$$\begin{aligned} proj\left\{D_X F \mid L^2_\pi(\mathfrak{H}, X)\right\} &= T \circ proj\left\{D_{X_T} F \mid L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)\right\} &(78)\\ &= T \circ proj\left\{T^{-1} D_X F \mid L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)\right\}; &(79)\end{aligned}$$
6. *For every $d \geq 2$, $f \in \mathfrak{H}^{\otimes d}$ (therefore, f need not be a symmetric tensor), $r = 1, ..., d-1$ and $t \in [0,1]$,*
$$\begin{aligned} &\left\|\left(\pi_1^{\otimes 2r} - \pi_t^{\otimes 2r}\right)\left(f \otimes_{d-r}^{\pi,t} f\right)\right\|_{\mathfrak{H}^{\otimes 2r}} &(80)\\ = &\left\|\left(\left(\pi_1^\sharp\right)^{\otimes 2r} - \left(\pi_t^\sharp\right)^{\otimes 2r}\right)\left(\left(T^d\right)^{-1} f \otimes_{d-r}^{\pi^\sharp,t} \left(T^d\right)^{-1} f\right)\right\|_{\left(\mathfrak{H}^\sharp\right)^{\otimes 2r}}. \end{aligned}$$

Now adopt the assumptions and notation of Theorem 11. If π_n is absolutely continuous on $\mathfrak{H}_n$, for every $n \geq 1$ there exists an isomorphism T_n, from $\mathfrak{H}_n$ onto some space $\mathfrak{H}_n^\sharp$, endowed with a resolution $\pi_n^\sharp$ as in (73) and such that properties (69) and (75) (with T_n substituting T) are verified. We also note $X_{T_n}\left(h^\sharp\right) = X_n\left(T_n h^\sharp\right)$, for every $h^\sharp \in \mathfrak{H}_n^\sharp$. It follows from Lemma 16-6 above that, if for every $r = 1, ..., d-1$, relation (30) is verified, then

$$\left\|\left(\left(\pi_{n,1}^\sharp\right)^{\otimes 2r} - \left(\pi_{n,t}^\sharp\right)^{\otimes 2r}\right)\left(\left(T_n^d\right)^{-1} f_n \otimes_{d-r}^{\pi_n^\sharp,t} \left(T_n^d\right)^{-1} f_n\right)\right\|_{\left(\mathfrak{H}_n^\sharp\right)^{\otimes 2r}} \underset{n \to +\infty}{\to} 0.$$

Moreover, thanks to Points 1 and 3 of Lemma 16,

$$\mathbb{E}\left[I_d^{X_{T_n}}\left(\left(T_n^d\right)^{-1} f_n\right) \mid \mathcal{F}_{t_n}^{\pi_n^\sharp}(X_{T_n})\right] = \mathbb{E}\left[F_n \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \overset{\mathbb{P}}{\to} 0$$

and

$$\mathbb{E}\left[I_d^{X_{T_n}}\left(\left(T_n^d\right)^{-1}f_n\right)^2 \mid \mathcal{F}_{t_n}^{\pi_n^\sharp}\left(X_{T_n}\right)\right] = \mathbb{E}\left[F_n^2 \mid \mathcal{F}_{t_n}^{\pi_n}\left(X_n\right)\right] \xrightarrow{\mathbb{P}} Y \in \mathcal{F}_*,$$

from which, by using the first part of the proof, we deduce that

$$\left\|proj\left\{D_{X_{T_n}}F_n \mid L^2_{\pi_n^\sharp}\left(\mathfrak{H}_n^\sharp, X_{T_n}\right)\right\}\right\|^2_{\mathfrak{H}_n^\sharp} \xrightarrow[n\to+\infty]{\mathbb{P}} Y. \tag{81}$$

The proof of Theorem 11 is now concluded by using (81) and Theorem 7 since, due to Lemma 16-5 above and the fact that T is an isomorphism,

$$\begin{aligned}\left\|proj\left\{D_{X_{T_n}}F_n \mid L^2_{\pi_n^\sharp}\left(\mathfrak{H}_n^\sharp, X_{T_n}\right)\right\}\right\|^2_{\mathfrak{H}_n^\sharp} &= \left\|T_n \circ proj\left\{D_{X_{T_n}}F_n \mid L^2_{\pi_n^\sharp}\left(\mathfrak{H}_n^\sharp, X_{T_n}\right)\right\}\right\|^2_{\mathfrak{H}_n} \\ &= \left\|proj\left\{D_X F_n \mid L^2_{\pi_n}\left(\mathfrak{H}_n, X_n\right)\right\}\right\|^2_{\mathfrak{H}_n}.\end{aligned}$$

Remark (*Concrete realizations of Wiener spaces*) – For the sake of completeness, we establish some connections between the unitary transformation $T : \mathfrak{H}^\sharp \mapsto \mathfrak{H}$ used in the last part of the preceding proof (see (68)) and the concept of *concrete (filtered) Wiener space* introduced in [30, Section 5]. In particular, we point out that every "filtered" isonormal Gaussian process such as the pair $\left(X_T\left(\mathfrak{H}^\sharp\right), \pi^\sharp\right)$ introduced in (73) and (76), is equivalent (in a sense analogous to [30, Definition 5.1]) to a concrete Wiener space whose dimension equals the rank of $\pi^\sharp$. To do this, fix $q \in \{1, 2, ..., +\infty\}$ and define $C_0\left([0,1]\right)$ to be set of continuous functions on $[0,1]$ that are initialized at zero. We define $\mathbb{W}_{(q)}$ to be the set of all q-dimensional vectors of the type $\mathbf{w}_{(q)} = (w_1, w_2, ..., w_q)$ (plainly, if $q = +\infty$, $\mathbf{w}_{(q)}$ is an infinite sequence) where $\forall i$, $w_i \in C_0\left([0,1]\right)$. The set $\mathbb{W}_{(q)}$ is endowed with the norm $\left\|\mathbf{w}_{(q)}\right\|_{(q)} = \sup_{i\leq q}|w_i|$, where $|w_i| = \sup_{t\in[0,1]}|w_i(t)|$. Under $\|\cdot\|_{(q)}$, $\mathbb{W}_{(q)}$ is a Banach space. Now consider an Hilbert space $\mathfrak{H}$, as well as a resolution $\pi \in \mathcal{R}_{AC}\left(\mathfrak{H}\right)$ such that $rank\left(\pi\right) = q$. We define $S = \{g_j : 1 \leq j \leq q\}$ to be the fully orthogonal π-reproducing subset of $\mathfrak{H}$ appearing in formula (70), and associate to each $g_j \in S$ the function $m_j \in L^1\left([0,1], dx\right)$ satisfying (71), in such a way that (72) is verified. To the pair $(\mathfrak{H}, \pi)$ we associate the Hilbert space $\mathbb{H}_{(q)}$ and a resolution of the identity $\pi^{(q)} = \left\{\pi_s^{(q)} : s \in [0,1]\right\} \in \mathcal{R}\left(\mathbb{H}_{(q)}\right)$ defined as follows: (i) $\mathbb{H}_{(q)}$ is the collection of all vectors of the kind $\mathbf{h}_{(q)} = (h_1, h_2..., h_q)$, where, for each $j \leq q$, h_j is a function of the form $h_j(t) = \int_0^t h_j'(x)\,dx$, for some $h_j' \in L^2\left([0,1], m_j(x)\,dx\right)$, and also

$$\sum_{j=1}^q \int_0^1 \left(h_j'(x)\right)^2 m_j(x)\,dx < +\infty; \tag{82}$$

(ii) $\mathbb{H}_{(q)}$ is endowed with the inner product

$$\left(\mathbf{h}_{(q)}, \mathbf{k}_{(q)}\right)_{(q)} = \sum_{j=1}^q \int_0^1 h_j'(x)\,k_j'(x)\,m_j(x)\,dx, \tag{83}$$

whereas $|\cdot|_{(q)} = (\cdot,\cdot)_{(q)}^{1/2}$ is the corresponding norm; (iii) for every $s \in [0,1]$ and every $\mathbf{h}_{(q)} = (h_1, h_2..., h_q) \in \mathbb{H}_{(q)}$,

$$\pi_s^{(q)}\mathbf{h}_{(q)} = \left(h_1^s, ..., h_q^s\right), \text{ where } \; h_j^s(t) \triangleq \int_0^{t\wedge s} h_j'(x)\,dx. \tag{84}$$

Note that $\mathbb{H}_{(q)} \subset \mathbb{W}_{(q)}$, and therefore $\mathbb{W}_{(q)}^* \subset \mathbb{H}_{(q)}^* = \mathbb{H}_{(q)}$. Moreover, from relation (82) it follows that the restriction of $\|\cdot\|_{(q)}$ to $\mathbb{H}_{(q)}$ is a measurable seminorm, in the sense of [13, Definition 4.4]. Also, $\mathbb{W}_{(q)}$ is the completion of $\mathbb{H}_{(q)}$ with respect to $\|\cdot\|_{(q)}$, and $\mathbb{W}_{(q)}^*$ is dense in $\mathbb{H}_{(q)}$ with respect to the norm $|\cdot|_{(q)}$. As a consequence (see again [13, Theorem 4.1]), there exists a canonical Gaussian measure $\mu_{(q)}$ on $\left(\mathbb{W}_{(q)}, \mathcal{B}\left(\mathbb{W}_{(q)}\right)\right)$, such that, for every $(\mathbf{l}_1, ..., \mathbf{l}_m) \in \left(\mathbb{W}_{(q)}^*\right)^m$, the mapping $\mathbf{w}_{(q)} \mapsto$

$\left(\mathbf{l}_1\left(\mathbf{w}_{(q)}\right),...,\mathbf{l}_m\left(\mathbf{w}_{(q)}\right)\right)$ defines a centered Gaussian vector such that, for every $j=1,...,m$ and every $\lambda\in\mathbb{R}$,

$$\mathbb{E}_{\mu_{(q)}}\left[\exp\left(i\lambda\mathbf{l}_j\right)\right]\triangleq\int_{\mathbb{W}^*_{(q)}}\exp\left(i\lambda\mathbf{l}_j\left(\mathbf{w}_{(q)}\right)\right)d\mu_{(q)}\left(\mathbf{w}_{(q)}\right)=\exp\left(-\frac{\lambda^2}{2}\left|\mathbf{l}_j\right|^2_{(q)}\right). \tag{85}$$

Following [30, p. 26], the triple $\left(\mathbb{W}_{(q)},\mathbb{H}_{(q)},\mu_{(q)}\right)$ (endowed with the resolution $\pi^{(q)}$ defined in (84)) is called a *concrete Wiener space* of dimension q. Note that, since $\mathbb{W}^*_{(q)}$ is dense in $\mathbb{H}_{(q)}$, there exists a unique collection of centered Gaussian random variables defined on $\left(\mathbb{W}_{(q)},\mathcal{B}\left(\mathbb{W}_{(q)}\right)\right)$, denoted

$$X_{(q)}=X_{(q)}\left(\mathbb{H}_{(q)}\right)=\left\{X_{(q)}\left(\mathbf{h}_{(q)}\right):\mathbf{h}_{(q)}\in\mathbb{H}_{(q)}\right\}, \tag{86}$$

such that $X_{(q)}\left(\mathbf{l}\right)\left(\mathbf{w}_{(q)}\right)=\mathbf{l}\left(\mathbf{w}_{(q)}\right)$ for every $\mathbf{l}\in\mathbb{W}^*_{(q)}$ and $\mathbb{E}_{\mu_{(q)}}\left[\exp\left(i\lambda X_{(q)}\left(\mathbf{h}_{(q)}\right)\right)\right]=\exp\left(-\frac{\lambda^2}{2}\left|\mathbf{h}_{(q)}\right|^2_{(q)}\right)$, $\forall\mathbf{h}_{(q)}\in\mathbb{H}_{(q)}$. In particular, $X_{(q)}\left(\mathbb{H}_{(q)}\right)$ is an isonormal Gaussian process over $\mathbb{H}_{(q)}$. Now consider the Hilbert space $\mathfrak{H}^\sharp$ and the resolution $\pi^\sharp$ defined in (73), and define the application $T_\circ:\mathfrak{H}^\sharp\mapsto\mathbb{H}_{(q)}$ as follows: for every $h^\sharp(a,x)\in\mathfrak{H}^\sharp$,

$$T_\circ h^\sharp=\left(\int_0^{\cdot}h^\sharp(1,x)\,dx,...,\int_0^{\cdot}h^\sharp(q,x)\,dx\right).$$

It is easily seen that $T_\circ$ is a unitary transformation such that $T_\circ\pi^\sharp_t=\pi^{(q)}_tT_\circ$ for every t, thus implying that the two pairs $\left(\mathfrak{H}^\sharp,\pi^\sharp\right)$ and $\left(\mathbb{H}_{(q)},\pi^{(q)}\right)$, and hence the two filtered isonormal processes $\left(X_{(q)}\left(\mathbb{H}_{(q)}\right),\pi^{(q)}\right)$ and $\left(X_T\left(\mathfrak{H}^\sharp\right),\pi^\sharp\right)$, are equivalent in the sense of [30, Definition 5.1].

3.3 Proof of Theorem 12

The implications (i) $\Longrightarrow$ (ii) and (iv) $\Longrightarrow$ (i) (in which assumption (36) is immaterial) are consequences, respectively, of Theorem 7 and Theorem 11. Now suppose (ii) is verified. Since $\mathbb{E}\left[F_n^2\mid\mathcal{F}^{\pi_n}_{t_n}(X_n)\right]\xrightarrow{\mathbb{P}}Y$ by assumption, we may use the second part of Theorem 10 to deduce that for every sequence $n(k)$, there exists a subsequence $n(k_r)$, $r\geq1$, s.t., a.s.-$\mathbb{P}$,

$$\mathbb{E}\left[\exp\left(i\lambda F_{n(k_r)}\right)\mid\mathcal{F}^{\pi_{n(k_r)}}_{t_{n(k_r)}}\left(X_{n(k_r)}\right)\right]\underset{r\to+\infty}{\to}\exp\left(-\frac{\lambda^2}{2}Y\right),\quad\forall\lambda\in\mathbb{R}.$$

Moreover, since the usual properties of multiple Wiener-Itô integrals (see e.g. [12, Chapter VI]) imply that, a.s.-$\mathbb{P}$ and due to (28),

$$\sup_{r\geq1}\mathbb{E}\left[\left|F_{n(k_r)}\right|^M\mid\mathcal{F}^{\pi_{n(k_r)}}_{t_{n(k_r)}}\left(X_{n(k_r)}\right)\right]<+\infty,\quad\forall M\geq1,$$

we conclude that, a.s.-$\mathbb{P}$,

$$\mathbb{E}\left[\left(F_{n(k_r)}\right)^4\mid\mathcal{F}^{\pi_{n(k_r)}}_{t_{n(k_r)}}\left(X_{n(k_r)}\right)\right]\underset{r\to+\infty}{\to}3Y^2,$$

and therefore that (iii) holds. To conclude, assume that the two conditions (iii) and (36) are verified, and write

$$F_n=I_2^{X_n}\left(\pi^{\otimes2}_{n,t_n}f_n\right)+2I_2^{X_n}\left(\pi_{n,t_n}\otimes\left(\pi_{n,1}-\pi_{n,t_n}\right)f_n\right)+I_2^{X_n}\left(\left(\pi_{n,1}-\pi_{n,t_n}\right)^{\otimes2}f_n\right)\triangleq F_{n,0}+F_{n,1}+F_{n,2}. \tag{87}$$

Due to (27), $F_{n,0}\to0$ in L^2. Also, for every $n\geq1$, $F_{n,2}$ is independent of $\mathcal{F}^{\pi_n}_{t_n}(X_n)$ and, conditionally on $\mathcal{F}^{\pi_n}_{t_n}(X_n)$, $F_{n,1}$ is a centered Gaussian random variable. Moreover

$$\mathbb{E}\left[\left(F_n\right)^2\mid\mathcal{F}^{\pi_n}_{t_n}(X_n)\right]=\mathbb{E}\left[\left(F_{n,0}\right)^2+\left(F_{n,1}\right)^2+\left(F_{n,2}\right)^2\mid\mathcal{F}^{\pi_n}_{t_n}(X_n)\right].$$

By writing $A_n \sim B_n$ to indicate that $A_n - B_n \xrightarrow{\mathbb{P}} 0$, we have therefore

$$\begin{aligned}\mathbb{E}\left[(F_n)^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] &\sim \mathbb{E}\left[(F_{n,1})^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] + \mathbb{E}\left[(F_{n,2})^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \\ &\quad +6\mathbb{E}\left[(F_{n,1}F_{n,2})^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] \\ &= 3\mathbb{E}\left[(F_{n,1})^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right]^2 + \mathbb{E}\left[(F_{n,2})^4\right] \\ &\quad +6\mathbb{E}\left[(F_{n,1}F_{n,2})^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right].\end{aligned}$$

By reasoning as in [19, pp. 182-183], and noting $f_{n,0} = (\pi_{n,1} - \pi_{n,t_n})^{\otimes 2} f_n \in \mathfrak{H}^{\odot 2}$,

$$\begin{aligned}\mathbb{E}\left[(F_{n,2})^4\right] &= 3\left\|f_{n,0}\right\|_{\mathfrak{H}^{\odot 2}}^4 + 48\left\|f_{n,0} \otimes_1 f_{n,0}\right\|_{\mathfrak{H}^{\otimes 2}}^2 \\ &= 3\mathbb{E}\left[(F_{n,2})^2 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right]^2 + 48\left\|(\pi_{n,1} - \pi_{n,t_n})^{\otimes 2} f_n \otimes_1^{\pi_n,t_n} f_n\right\|_{\mathfrak{H}^{\otimes 2}}^2.\end{aligned}$$

Standard calculations yield finally that, since (27) and (28) hold, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}\mathbb{E}\left[\left(\mathbb{E}\left[(F_n)^4 \mid \mathcal{F}_{t_n}^{\pi_n}(X_n)\right] - 3Y^2\right)^2\right] &= c_1\left\|(\pi_{n,1} - \pi_{n,t_n})^{\otimes 2} f_n \otimes_1^{\pi_n,t_n} f_n\right\|_{\mathfrak{H}^{\otimes 2}}^2 \\ &\quad +c_2\left\|(\pi_{n,1} - \pi_{n,t_n}) \otimes \pi_{n,t_n}\left(f_n \otimes_1^{\pi_n,t_n} f_n\right)\right\|_{\mathfrak{H}^{\otimes 2}}^2,\end{aligned}$$

and, since (36) is verified and

$$(\pi_{n,1} - \pi_{n,t_n})^{\otimes 2} + (\pi_{n,1} - \pi_{n,t_n}) \otimes \pi_{n,t_n} + \pi_{n,t_n} \otimes (\pi_{n,1} - \pi_{n,t_n}) = \pi_{n,1}^{\otimes 2} - \pi_{n,t_n}^{\otimes 2},$$

we obtain immediately the desired implication (iii)$\Longrightarrow$(iv).■

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4 Appendix

Proof of Lemma 16 – (*Point 1*) Let $\{e_j : j \geq 1\}$ be an orthonormal basis of $\mathfrak{H}$, and define, for $d \geq 1$, $\mathbb{A}[d]$ to be the set of sequences $(a_1, a_2, ...)$ with values in $\mathbb{N}$, and such that $\sum_{j\geq 1} a_j = d$ (note that this implies that there are only finitely many a_j that are different from zero). Then, a total set in $\mathfrak{H}^{\odot d}$ is given by

$$\mathcal{A}_d = \left\{ (\bigotimes_{j=1}^{\infty} e_j^{\otimes a_j})_s : (a_1, a_2, ...) \in \mathbb{A}[d] \right\},$$

where $e^{\otimes 0} = 1$ by definition, and $(\cdot)_s$ indicates symmetrization. Moreover, a classic characterization of multiple stochastic integrals (see [17, Ch. 1]) as well as the fact that $X(e_i) = X\left(TT^{-1}e_i\right) = X_T\left(T^{-1}e_i\right)$ by definition, imply the following relations: for every $(a_1, a_2, ...) \in \mathbb{A}[d]$,

$$\begin{aligned} I_d^X\left((\bigotimes_{j=1}^{\infty} e_j^{\otimes a_j})_s\right) &= d! \prod_{j=1}^{\infty} H_{a_j}(X(e_j)) = d! \prod_{j=1}^{\infty} H_{a_j}\left(X_T\left(T^{-1}e_j\right)\right) \\ &= I_d^{X_T}\left((\bigotimes_{j=1}^{\infty} \left(T^{-1}e_j\right)^{\otimes a_j})_s\right) = I_d^{X_T}\left(\left(T^d\right)^{-1}(\bigotimes_{j=1}^{\infty} (e_j)^{\otimes a_j})_s\right), \end{aligned}$$

where $\{H_a : a \geq 1\}$ is the family of Hermite polynomials defined e.g. in [17, p. 4]. It is therefore clear that $I_d^X(f) = I_d^{X_T}\left(\left(T^d\right)^{-1} f\right)$ is true for every f that is a linear combination of elements of $\mathcal{A}_d$, and the general result is achieved by a standard density argument. (*Point 2*) For $m \geq 1$, let $C_b^{\infty}(\mathbb{R}^m)$ denote the class of bounded and infinitely differentiable functions on $\mathbb{R}^m$, whose derivatives are also bounded. We start by observing that, since T is a one-to-one unitary transformation, random variables of the type

$$F = f(X(h_1), ..., X(h_m)) = f\left(X_T\left(T^{-1}h_1\right), ..., X_T\left(T^{-1}h_m\right)\right) \tag{88}$$

(the equality is again a consequence of the relation $X(h_i) = X\left(TT^{-1}e_i\right) = X_T\left(T^{-1}h_i\right)$), where $m \geq 1$, $f \in C_b^{\infty}(\mathbb{R}^m)$ and $h_1, ..., h_m \in C_b^{\infty}(\mathbb{R}^m)$, are dense both in $\mathbb{D}_X^{1,2}$ and $\mathbb{D}_{X_T}^{1,2}$. To conclude, use a density argument, as well as the fact that, for F as in (88),

$$\begin{aligned} D_X F &= \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f(X(h_1), ..., X(h_m)) h_j \\ &= \sum_{j=1}^{m} \frac{\partial}{\partial x_j} f\left(X_T\left(T^{-1}h_1\right), ..., X_T\left(T^{-1}h_m\right)\right) TT^{-1}h_j = TD_{X_T}F, \end{aligned}$$

hence proving (77). (*Point 3*) This is a consequence of the relations $X(\pi_t h) = X_T\left(T^{-1}\pi_t h\right) = X_T\left(T^{-1}\pi_t T T^{-1} h\right) = X_T\left(\pi_t^\sharp T^{-1} h\right)$, that are verified for every $t \in [0,1]$, since $T^{-1}\pi_t T = \pi_t^\sharp$, due to (69). (*Point 4*) Suppose $u \in L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)$. Then, since T is an isometry, $\mathbb{E}\left[\|Tu\|^2_{\mathfrak{H}}\right] = \mathbb{E}\left[\|u\|^2_{\mathfrak{H}^\sharp}\right] < +\infty$, and therefore $Tu \in L^2_\pi(\mathfrak{H}, X)$. To prove that Tu is also π-adapted, use the fact that, since T is an isometry and (69) holds, for every $t \in [0,1]$ and every $h \in \mathfrak{H}$,

$$\begin{aligned}(Tu, \pi_t h)_{\mathfrak{H}} &= \left(Tu, TT^{-1}\pi_t h\right)_{\mathfrak{H}} = \left(u, T^{-1}\pi_t h\right)_{\mathfrak{H}^\sharp} \\ &= \left(u, \pi_t^\sharp T^{-1} h\right)_{\mathfrak{H}^\sharp} \in \mathcal{F}_t^{\pi^\sharp}(X_T) = \mathcal{F}_t^\pi(X),\end{aligned}$$

due to Point 3, thus yielding $u \in L^2_\pi(\mathfrak{H}, X)$. The opposite implication is obtained analogously. (*Point 5*) Consider first an elementary random variable $\eta^\sharp \in \mathcal{E}_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)$ with the form $\eta^\sharp = \Phi(t)\left(\pi^\sharp_{t+s} - \pi^\sharp_t\right) h^\sharp$, where $\Phi(t) \in \mathcal{F}_t^{\pi^\sharp}(X_T)$ $(= \mathcal{F}_t^\pi(X))$, $h^\sharp \in \mathfrak{H}^\sharp$ and $s, t \geq 0$. Then, due to (69), $T\eta^\sharp = \Phi(t) T\left(\pi^\sharp_{t+s} - \pi^\sharp_t\right) h^\sharp$ $= \Phi(t)(\pi_{t+s} - \pi_t) T h^\sharp$, and therefore $Th^\sharp \in \mathcal{E}_\pi(\mathfrak{H}, X)$. Now, for $F \in \mathbb{D}^{1,2}_{X_T}$, observe that a variable $P \in L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)$ is equal to $proj\left\{D_{X_T}F \mid L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)\right\}$ if, and only if, for every $\eta^\sharp \in \mathcal{E}_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)$ as before

$$\mathbb{E}\left[\left(P, \eta^\sharp\right)_{\mathfrak{H}^\sharp}\right] = \mathbb{E}\left[\left(D_{X_T}F, \eta^\sharp\right)_{\mathfrak{H}^\sharp}\right]. \tag{89}$$

But, since T is an isometry, (89) and (77) imply also that

$$\mathbb{E}\left[\left(TP, T\eta^\sharp\right)_{\mathfrak{H}}\right] = \mathbb{E}\left[\left(TD_{X_T}F, T\eta^\sharp\right)_{\mathfrak{H}}\right] = \mathbb{E}\left[\left(D_X F, T\eta^\sharp\right)_{\mathfrak{H}}\right].$$

Hence, since $TP \in L^2_\pi(\mathfrak{H}, X)$ due to Point 4,

$$TP = T \circ proj\left\{D_{X_T}F \mid L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)\right\} = proj\left\{D_X F \mid L^2_{\pi^\sharp}\left(\mathfrak{H}^\sharp, X_T\right)\right\},$$

thus proving (78). To prove (79), just observe that (77) implies that $D_{X_T}F = T^{-1}D_X F$. (*Point 6*) Let again $\{e_j : j \geq 1\}$ be an ONB of $\mathfrak{H}$. Note first that, for every $d \geq 2$, $f \in \mathfrak{H}^{\otimes d}$, $r = 1, ..., d-1$, $t \in [0,1]$, and $i_1, ..., i_{d-r} \geq 1$

$$\begin{aligned}&\left(\left(\pi_1^{\otimes d-r} - \pi_t^{\otimes d-r}\right) f, e_{i_1} \otimes \cdots \otimes e_{i_{d-r}}\right)_{\mathfrak{H}^{\otimes d-r}} \\ =\ &\left(\left(T^{d-r}\right)^{-1}\left(\pi_1^{\otimes d-r} - \pi_t^{\otimes d-r}\right) f, T^{-1}e_{i_1} \otimes \cdots \otimes T^{-1}e_{i_{d-r}}\right)_{(\mathfrak{H}^\sharp)^{\otimes d-r}} \\ =\ &\left(\left(\left(\pi_1^\sharp\right)^{\otimes d-r} - \left(\pi_t^\sharp\right)^{\otimes d-r}\right)\left(T^{d-r}\right)^{-1} f, T^{-1}e_{i_1} \otimes \cdots \otimes T^{-1}e_{i_{d-r}}\right)_{(\mathfrak{H}^\sharp)^{\otimes d-r}}.\end{aligned}$$

Thanks to (24), it follows that

$$\begin{aligned}&\left(T^{2r}\right)^{-1} f \otimes^{\pi,t}_{d-r} f \\ =\ &\sum_{i_1,...,i_{d-r}=1}^{\infty} \left\langle\left(\left(T^d\right)^{-1} f\right), \left(\left(\pi_1^\sharp\right)^{\otimes d-r} - \left(\pi_t^\sharp\right)^{\otimes d-r}\right) T^{-1}e_{i_1} \otimes \cdots \otimes T^{-1}e_{i_{d-r}}\right\rangle_{(\mathfrak{H}^\sharp)^{\otimes d-r}} \\ &\qquad \left\langle\left(\left(T^d\right)^{-1} f\right), \left(\left(\pi_1^\sharp\right)^{\otimes d-r} - \left(\pi_t^\sharp\right)^{\otimes d-r}\right) T^{-1}e_{i_1} \otimes \cdots \otimes T^{-1}e_{i_{d-r}}\right\rangle_{(\mathfrak{H}^\sharp)^{\otimes d-r}} \\ =\ &\left(\left(T^d\right)^{-1} f \otimes^{\pi^\sharp,t}_{d-r} \left(T^d\right)^{-1} f\right)\end{aligned}$$

As a consequence, by using (75) and the fact that T^d and $\left(T^d\right)^{-1}$ are isometries,

$$\begin{aligned}
& \left\| \left(\left(\pi_1^\sharp\right)^{\otimes 2r} - \left(\pi_t^\sharp\right)^{\otimes 2r} \right) \left(\left(T^d\right)^{-1} f \otimes_{d-r}^{\pi^\sharp,t} \left(T^d\right)^{-1} f \right) \right\|_{(\mathfrak{H}^\sharp)^{\otimes 2r}} \\
= & \left\| \left(\left(\pi_1^\sharp\right)^{\otimes 2r} - \left(\pi_t^\sharp\right)^{\otimes 2r} \right) \left(T^{2r}\right)^{-1} f \otimes_{d-r}^{\pi,t} f \right\|_{(\mathfrak{H}^\sharp)^{\otimes 2r}} \\
= & \left\| \left(T^{2r}\right)^{-1} \left(\pi_1^{\otimes 2r} - \pi_t^{\otimes 2r}\right) f \otimes_{d-r}^{\pi,t} f \right\|_{(\mathfrak{H}^\sharp)^{\otimes 2r}} \\
= & \left\| \left(\pi_1^{\otimes 2r} - \pi_t^{\otimes 2r}\right) \left(f \otimes_{d-r}^{\pi,t} f\right) \right\|_{\mathfrak{H}^{\otimes 2r}},
\end{aligned}$$

which proves (80). ■