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Stable convergence of generalized stochastic integrals and the principle of conditioning: L^2 theory

Giovanni PECCATI* and Murad S. TAQQU† ‡

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Abstract

Consider generalized adapted stochastic integrals with respect to independently scattered random measures with second moments. We use a decoupling technique, known as the “principle of conditioning”, to study their stable convergence towards mixtures of infinitely divisible distributions. Our results apply, in particular, to multiple integrals with respect to independently scattered and square integrable random measures, as well as to Skorohod integrals on abstract Wiener spaces. As a specific application, we establish a Central Limit Theorem for sequences of double integrals with respect to a general Poisson measure, thus extending the results contained in Nualart and Peccati (2005) and Peccati and Tudor (2004) to a non-Gaussian context.

Key Words – Generalized stochastic integrals; Independently scattered measures; Decoupling; Principle of conditioning; Resolutions of the identity; Stable convergence; Weak convergence; multiple Poisson integrals; Skorohod integrals.

AMS Subject Classification – 60G60, 60G57, 60F05

1 Introduction

In this paper we establish several criteria, ensuring the stable convergence of sequences of “generalized integrals” with respect to independently scattered random measures over abstract Hilbert spaces. The notion of generalized integral is understood in a very wide sense, and includes for example Skorohod integrals with respect to isonormal Gaussian processes (see e.g. [17]), multiple Wiener-Itô integrals associated to general Poisson measures (see [21], or [13]), or the class of iterated integrals with respect to orthogonalized Teugels martingales introduced in [20]. All these random objects can be represented as appropriate generalized “adapted stochastic integrals” with respect to a (possibly infinite) family of Lévy processes, constructed by means of a well-chosen increasing family of orthogonal projections. These adapted integrals are also the limit of sums of arrays of random variables with a special dependence structure. We shall show, in particular, that their asymptotic behavior can be naturally studied by means of a decoupling technique, known as the “principle of conditioning” (see e.g. [12] and [37]), that we use in the framework of stable convergence (see [11, Chapter 4]).

Our setup is roughly the following. We shall consider a centered and square integrable random field $X = \{X(h) : h \in \mathfrak{H}\}$, indexed by a separable Hilbert space \mathfrak{H} , and verifying the isomorphic relation $\mathbb{E}[X(h)X(h')] = (h, h')_{\mathfrak{H}}$, where $(\cdot, \cdot)_{\mathfrak{H}}$ is the inner product on \mathfrak{H} . There is no time involved. To introduce time, endow the space \mathfrak{H} with an increasing family of orthogonal projections, say π_t , $t \in [0, 1]$, such that $\pi_0 = 0$ and $\pi_1 = \text{id.}$. Such projections operators induce the (canonical) filtration $\mathcal{F}^\pi = \{\mathcal{F}_t^\pi : t \in [0, 1]\}$, where each \mathcal{F}_t^π is generated by random variables of the type $X(\pi_t h)$, and one can define (e.g., as in [35] for Gaussian processes) a class of \mathcal{F}^π -adapted and \mathfrak{H} -valued random variables. If for every $h \in \mathfrak{H}$ the

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application $t \mapsto X(\pi_t h)$ is also a \mathcal{F}^π -Lévy process, then there exists a natural Itô type stochastic integral, of adapted and \mathfrak{H} -valued variables, with respect to the infinite dimensional process $t \mapsto \{X(\pi_t h) : h \in \mathfrak{H}\}$. Denote by $J_X(u)$ the integral of an adapted random variable u with respect to X . As will be made clear in the subsequent discussion, several random objects appearing in stochastic analysis (such as Skorohod integrals, or the multiple Poisson integrals quoted above) are in fact generalized adapted integrals of the type $J_X(u)$, for some well chosen random field X . Moreover, the definition of $J_X(u)$ mimics in many instances the usual construction of adapted stochastic integrals with respect to real-valued martingales. In particular: (i) each stochastic integral $J_X(u)$ is associated to a \mathcal{F}^π -martingale, namely the process $t \mapsto J_X(\pi_t u)$ and (ii) $J_X(u)$ is the limit (in L^2) of finite “adapted Riemann sums” of the kind $S(u) = \sum_{j=1, \dots, n} F_j X((\pi_{t_{j+1}} - \pi_{t_j}) h_j)$, where $h_j \in \mathfrak{H}$, $t_n > t_{n-1} > \dots > t_1$ and $F_j \in \mathcal{F}_{t_j}^\pi$. We show that, by using a decoupling result known as “principle of conditioning” (see Theorem 1 in [37], and Section 2 below, for a very general form of such principle), the stable and, in particular, the weak convergence of sequences of sums such as $S(u)$ is completely determined by the asymptotic behavior of random variables of the type

$$\tilde{S}(u) = \sum_{j=1, \dots, n} F_j \tilde{X}((\pi_{t_{j+1}} - \pi_{t_j}) h_j),$$

where \tilde{X} is an independent copy of X . Note that the vector

$$\tilde{V} = \left(F_1 \tilde{X}((\pi_{t_2} - \pi_{t_1}) h_1), \dots, F_n \tilde{X}((\pi_{t_{n+1}} - \pi_{t_n}) h_n) \right),$$

enjoys the specific property of being *decoupled* (i.e., conditionally on the F_j 's, its components are independent) and *tangent* to the “original” vector

$$V = \left(F_1 X((\pi_{t_2} - \pi_{t_1}) h_1), \dots, F_n X((\pi_{t_{n+1}} - \pi_{t_n}) h_n) \right),$$

in the sense that for every j , and conditionally on the r.v.'s F_k , $k \leq j$, $F_j X((\pi_{t_{j+1}} - \pi_{t_j}) h_j)$ and $F_j \tilde{X}((\pi_{t_{j+1}} - \pi_{t_j}) h_j)$ have the same law (the reader is referred to [10] or [14] for a discussion of the general theory of tangent processes). The convergence of sequences such as $J_X(u_n)$, $n \geq 1$, where each u_n is adapted, can therefore be studied by means of simpler random variables $\tilde{J}_X(u_n)$, obtained from a decoupled and tangent version of the martingale $t \mapsto J_X(\pi_t u_n)$. In particular (see Theorem 7 below, as well as its consequences) we shall prove that, since such decoupled processes can be shown to have conditionally independent increments, the problem of the stable convergence of $J_X(u_n)$ can be reduced to the study of the convergence in probability of sequences of random Lévy-Khinchine exponents. This represents an extension of the techniques initiated in [19] and [24] where, in a purely Gaussian context, the CLTs for multiple Wiener-Itô integrals are characterized by means of the convergence in probability of the quadratic variation of Brownian martingales. We remark that the extensions of [19] and [24] achieved in this paper go in two directions: (a) we consider general (not necessarily Gaussian) square integrable and independently scattered random measures, (b) we study stable convergence, instead of weak convergence, so that, for instance, our results can be used in the Gaussian case to obtain non-central limit theorems (see e.g. Section 6 below, as well as [23]).

When studying the stable convergence of random variables that are terminal values of continuous-time martingales, one could alternatively use the general criteria for the stable convergence of semimartingales, as developed e.g. in [16], [5] or [11, Chapter 4], instead of the above decoupling techniques. However, the principle of conditioning (which is in some sense the discrete-time skeleton of the general semimartingale results), as formulated in the present paper, often requires less stringent assumptions. For instance, conditions (7) and (37) below are weak versions of the *nesting condition* introduced by Feigin in the classic reference [5].

The paper is organized as follows. In Section 2, we discuss a general version of the principle of conditioning. In Section 3 we present a general setup to which such decoupling techniques can be applied, and in Section 4 the above mentioned convergence results are established. In Section 5.1 and 5.2, we

apply our techniques to sequences of multiple stochastic integrals with respect to independently scattered random measures with second moments, whereas in Section 3.3 we give a specific application to Central Limit Theorems for double Poisson integrals. Finally, in Section 6 our results are applied to study the stable convergence of Skorohod integrals with respect to a general isonormal Gaussian process.

2 The principle of conditioning

We shall present a general version of the *principle of conditioning* (POC in the sequel) for arrays of real valued random variables. Our discussion is mainly inspired by a remarkable paper by X.-H. Xue [37], generalizing the classic results by Jakubowski [12] to the framework of stable convergence. Note that the results discussed below refer to a discrete time setting. However, thanks to some density arguments, we will be able to apply most of the POC techniques to general stochastic measures on abstract Hilbert spaces.

Instead of adopting the formalism of [37] we choose, for the sake of clarity, to rely in part on the slightly different language of [6, Ch. 6 and 7]. To this end, we shall recall some notions concerning stable convergence, conditional independence and decoupled sequences of random variables. From now on, all random objects are supposed to be defined on an adequate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and all σ -fields introduced below will be tacitly assumed to be complete; $\xrightarrow{\mathbb{P}}$ means convergence in probability; \mathbb{R} stands for the set of real numbers; \triangleq denotes a new definition.

We start by defining the class \mathbf{M} of random probability measures, and the class $\widehat{\mathbf{M}}$ (resp. $\widehat{\mathbf{M}}_0$) of random (resp. non-vanishing and random) characteristic functions.

Definition A (see e.g. [37]) – Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ -field on \mathbb{R} .

(A-i) A map $\mu(\cdot, \cdot)$, from $\mathcal{B}(\mathbb{R}) \times \Omega$ to \mathbb{R} is called a *random probability* (on \mathbb{R}) if, for every $C \in \mathcal{B}(\mathbb{R})$, $\mu(C, \cdot)$ is a random variable and, for \mathbb{P} -a.e. ω , the map $C \mapsto \mu(C, \omega)$, $C \in \mathcal{B}(\mathbb{R})$, defines a probability measure on \mathbb{R} . The class of all random probabilities is noted \mathbf{M} , and, for $\mu \in \mathbf{M}$, we write $\mathbb{E}\mu(\cdot)$ to indicate the (deterministic) probability measure

$$\mathbb{E}\mu(C) \triangleq \mathbb{E}[\mu(C, \cdot)], \quad C \in \mathcal{B}(\mathbb{R}). \quad (1)$$

(A-ii) For a measurable map $\phi(\cdot, \cdot)$, from $\mathbb{R} \times \Omega$ to \mathbb{C} , we write $\phi \in \widehat{\mathbf{M}}$ whenever there exists $\mu \in \mathbf{M}$ such that

$$\phi(\lambda, \omega) = \widehat{\mu}(\lambda)(\omega), \quad \forall \lambda \in \mathbb{R}, \text{ for } \mathbb{P}\text{-a.e. } \omega, \quad (2)$$

where $\widehat{\mu}(\cdot)$ is defined as

$$\widehat{\mu}(\lambda)(\omega) = \begin{cases} \int \exp(i\lambda x) \mu(dx, \omega) & \text{if } \mu(\cdot, \omega) \text{ is a probability measure} \\ 1 & \text{otherwise.} \end{cases}, \quad \lambda \in \mathbb{R}. \quad (3)$$

(A-iii) For a given $\phi \in \widehat{\mathbf{M}}$, we write $\phi \in \widehat{\mathbf{M}}_0$ whenever $\mathbb{P}\{\omega : \phi(\lambda, \omega) \neq 0 \quad \forall \lambda \in \mathbb{R}\} = 1$.

Observe that, for every $\omega \in \Omega$, $\widehat{\mu}(\lambda)(\omega)$ is a continuous function of λ . The probability $\mathbb{E}\mu(\cdot) = \int_{\Omega} \mu(\cdot, \omega) d\mathbb{P}(\omega)$ defined in (1) is often called a *mixture* of probability measures. The following definition of *stable convergence* extends the usual notion of convergence in law.

Definition B (see e.g. [11, Chapter 4] or [37]) – Let $\mathcal{F}^* \subseteq \mathcal{F}$ be a σ -field, and let $\mu \in \mathbf{M}$. A sequence of real valued r.v.'s $\{X_n : n \geq 1\}$ is said to *converge \mathcal{F}^* -stably* to $\mathbb{E}\mu(\cdot)$, written $X_n \rightarrow_{(s, \mathcal{F}^*)} \mathbb{E}\mu(\cdot)$, if, for every $\lambda \in \mathbb{R}$ and every bounded \mathcal{F}^* -measurable r.v. Z ,

$$\lim_{n \rightarrow +\infty} \mathbb{E}[Z \times \exp(i\lambda X_n)] = \mathbb{E}[Z \times \widehat{\mu}(\lambda)], \quad (4)$$

where the notation is the same as in (3).

If X_n converges \mathcal{F}^* -stably, then the conditional distributions $\mathcal{L}(X_n | A)$ converge for any $A \in \mathcal{F}^*$ such that $\mathbb{P}(A) > 0$. (see e.g. [11, Section 5, §5c] for further characterizations of stable convergence). Note that, by setting $Z = 1$, we obtain that if $X_n \rightarrow_{(s, \mathcal{F}^*)} \mathbb{E}\mu(\cdot)$, then the law of the X_n 's converges weakly to $\mathbb{E}\mu(\cdot)$. Moreover, by a monotone class argument, $X_n \rightarrow_{(s, \mathcal{F}^*)} \mathbb{E}\mu(\cdot)$ if, and only if, (4) holds for random variables with the form $Z = \exp(i\gamma Y)$, where $\gamma \in \mathbb{R}$ and Y is \mathcal{F}^* -measurable. Eventually, we note that, if a sequence of random variables $\{U_n : n \geq 0\}$ is such that $(U_n - X_n) \rightarrow 0$ in $L^1(\mathbb{P})$ and $X_n \rightarrow_{(s, \mathcal{F}^*)} \mathbb{E}\mu(\cdot)$, then $U_n \rightarrow_{(s, \mathcal{F}^*)} \mathbb{E}\mu(\cdot)$. The following definition shows how to replace an array $X^{(1)}$ of real-valued random variables by a simpler, *decoupled* array $X^{(2)}$.

Definition C (see [6, Chapter 7]) – Let $\{N_n : n \geq 1\}$ be a sequence of positive natural numbers, and let

$$X^{(i)} \triangleq \left\{ X_{n,j}^{(i)} : 0 \leq j \leq N_n, n \geq 1 \right\}, \text{ with } X_{n,0}^{(i)} = 0,$$

$i = 1, 2$, be two arrays of real valued r.v.'s, such that, for $i = 1, 2$ and for each n , the sequence

$$X_n^{(i)} \triangleq \left\{ X_{n,j}^{(i)} : 0 \leq j \leq N_n \right\}$$

is adapted to a discrete filtration $\{\mathcal{F}_{n,j} : 0 \leq j \leq N_n\}$ (of course, $\mathcal{F}_{n,j} \subseteq \mathcal{F}$). For a given $n \geq 1$, we say that $X_n^{(2)}$ is a *decoupled tangent sequence* to $X_n^{(1)}$ if the following two conditions are verified:

★ (*Tangency*) for each $j = 1, \dots, N_n$

$$\mathbb{E} \left[\exp \left(i\lambda X_{n,j}^{(1)} \right) \mid \mathcal{F}_{n,j-1} \right] = \mathbb{E} \left[\exp \left(i\lambda X_{n,j}^{(2)} \right) \mid \mathcal{F}_{n,j-1} \right] \quad (5)$$

for each $\lambda \in \mathbb{R}$, a.s.- \mathbb{P} ;

★ (*Conditional independence*) there exists a σ -field $\mathcal{G}_n \subseteq \mathcal{F}$ such that, for each $j = 1, \dots, N_n$,

$$\mathbb{E} \left[\exp \left(i\lambda X_{n,j}^{(2)} \right) \mid \mathcal{F}_{n,j-1} \right] = \mathbb{E} \left[\exp \left(i\lambda X_{n,j}^{(2)} \right) \mid \mathcal{G}_n \right] \quad (6)$$

for each $\lambda \in \mathbb{R}$, a.s.- \mathbb{P} , and the random variables $X_{n,1}^{(2)}, \dots, X_{n,N_n}^{(2)}$ are conditionally independent given \mathcal{G}_n .

Observe that, in (5), $\mathcal{F}_{n,j-1}$ depends on j , but \mathcal{G}_n does not. The array $X^{(2)}$ is said to be a *decoupled tangent array* to $X^{(1)}$ if $X_n^{(2)}$ is a decoupled tangent sequence to $X_n^{(1)}$ for each $n \geq 1$.

Remark – In general, given $X^{(1)}$ as above, there exists a canonical way to construct an array $X^{(2)}$, which is decoupled and tangent to $X^{(1)}$. The reader is referred to [14, Section 2 and 3] for a detailed discussion of this point, as well as other relevant properties of decoupled tangent sequences.

The following result is essentially a translation of Theorem 2.1 in [37] into the language of this section. It is a “stable convergence generalization” of the results obtained by Jakubowski in [12].

Theorem 1 (Xue, 1991) *Let $X^{(2)}$ be a decoupled tangent array to $X^{(1)}$, and let the notation of Definition C prevail (in particular, the collection of σ -fields $\{\mathcal{F}_{n,j}, \mathcal{G}_n : 0 \leq j \leq N_n, n \geq 1\}$ satisfies (5) and (6)). We write, for every n and every $k = 0, \dots, N_n$, $S_{n,k}^{(i)} \triangleq \sum_{j=0, \dots, k} X_{n,j}^{(i)}$, $i = 1, 2$. Suppose that there exists a sequence $\{r_n : n \geq 1\} \subset \mathbb{N}$, and a sequence of σ -fields $\{\mathcal{V}_n : n \geq 1\}$ such that*

$$\mathcal{V}_n \subseteq \mathcal{F} \quad \text{and} \quad \mathcal{V}_n \subseteq \mathcal{V}_{n+1} \cap \mathcal{F}_{n,r_n}, \quad n \geq 1, \quad (7)$$

and, as $n \rightarrow +\infty$,

$$S_{n,r_n \wedge N_n}^{(1)} \xrightarrow{\mathbb{P}} 0, \quad \mathbb{E} \left[\exp \left(i\lambda S_{n,r_n \wedge N_n}^{(2)} \right) \mid \mathcal{G}_n \right] \xrightarrow{\mathbb{P}} 1. \quad (8)$$

If moreover

$$\mathbb{E} \left[\exp \left(i\lambda S_{n,N_n}^{(2)} \right) \mid \mathcal{G}_n \right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R}, \quad (9)$$

where $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall \lambda \in \mathbb{R}$, $\phi(\lambda) \in \vee_n \mathcal{V}_n$, then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp \left(i\lambda S_{n,N_n}^{(1)} \right) \mid \mathcal{F}_{n,r_n} \right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \quad (10)$$

and

$$S_{n,N_n}^{(1)} \xrightarrow{(s,\nu)} \mathbb{E}\mu(\cdot), \quad (11)$$

where $\mathcal{V} \triangleq \vee_n \mathcal{V}_n$, and $\mu \in \mathbf{M}$ verifies (2).

Remarks – (a) Condition (7) says that \mathcal{V}_n , $n \geq 1$, must be an increasing sequence of σ -fields, whose n th term is contained in \mathcal{F}_{n,r_n} , for every $n \geq 1$. Condition (8) ensures that, for $i = 1, 2$, the sum of the first r_n terms of the vector $X_n^{(i)}$ is asymptotically negligible.

(b) There are some differences between the statement of Theorem 1 above, and the original result presented in [37]. On the one hand, in [37] the sequence $\{N_n : n \geq 1\}$ is such that each N_n is a $\mathcal{F}_{n,\cdot}$ -stopping time (but we do not need such a generality). On the other hand, in [37] one considers only the case of the family of σ -fields $\mathcal{V}_n^* = \bigcap_{j \geq n} \mathcal{F}_{j,r_n}$, $n \geq 1$, where r_n is non decreasing (note that, due to the monotonicity of r_n , the \mathcal{V}_n^* 's satisfy automatically (7)). However, by inspection of the proof of [37, Theorem 2.1 and Lemma 2.1], one sees immediately that all is needed to prove Theorem 1 is that the \mathcal{V}_n 's verify condition (7). For instance, if r_n is a general sequence of natural numbers such that $\mathcal{F}_{n,r_n} \subseteq \mathcal{F}_{n+1,r_{n+1}}$ for each $n \geq 1$, then the sequence $\mathcal{V}_n = \mathcal{F}_{n,r_n}$, $n \geq 1$, trivially satisfies (7), even if it does not fit Xue's original assumptions.

(c) The main theorem in the paper by Jakubowski [12, Theorem 1.1] (which, to our knowledge, is the first systematic account of the POC) corresponds to the special case $\mathcal{F}_{n,0} = \{\emptyset, \Omega\}$ and $r_n = 0$, $n \geq 1$. Under such assumptions, necessarily $\mathcal{V}_n = \mathcal{F}_{n,0}$, $S_{n,r_n \wedge N_n}^{(i)} = 0$, $i = 1, 2$, and $\phi(\lambda)$, which is $\vee_n \mathcal{V}_n = \{\emptyset, \Omega\}$ -measurable, is deterministic for every λ . In particular, relations (7) and (8) become immaterial. See also [15, Theorem 5.8.3] and [6, Theorem 7.1.4] for some detailed discussions of the POC in this setting.

(d) For the case $r_n = 0$ and $\mathcal{F}_{n,0} = \mathcal{A}$ ($n \geq 1$), where \mathcal{A} is not trivial, see also [9, Section (1.c)].

The next proposition will be used in Section 5.

Proposition 2 *Let the notation of Theorem 1 prevail, suppose that the sequence $S_{n,N_n}^{(1)}$ verifies (10) for some $\phi \in \widehat{\mathbf{M}}_0$, and assume moreover that there exists a finite random variable $C(\omega) > 0$ such that, for some $\eta > 0$,*

$$\mathbb{E} \left[\left| S_{n,N_n}^{(1)} \right|^\eta \mid \mathcal{F}_{n,r_n} \right] < C(\omega), \quad \forall n \geq 1, \quad a.s.-\mathbb{P}. \quad (12)$$

Then, there exists a subsequence $\{n(k) : k \geq 1\}$ such that, a.s. - \mathbb{P} ,

$$\mathbb{E} \left[\exp \left(i\lambda S_{n(k),N_{n(k)}}^{(1)} \right) \mid \mathcal{F}_{n(k),r_{n(k)}} \right] \xrightarrow{k \rightarrow +\infty} \phi(\lambda) \quad (13)$$

for every real λ .

Proof. Combining (10) and (12), we deduce the existence of a set Ω^* of probability one, as well as of a subsequence $n(k)$, such that, for every $\omega \in \Omega^*$, relation (12) is satisfied and (13) holds for every rational λ . We now fix $\omega \in \Omega^*$, and show that (13) holds for all real λ . Relations (10) and (12) imply that

$$\mathbb{P}_k^\omega[\cdot] = \mathbb{P}\left[S_{n(k), N_{n(k)}}^{(1)} \in \cdot \mid \mathcal{F}_{n(k), r_{n(k)}}\right](\omega), \quad k \geq 1,$$

is tight and hence relatively compact: every sequence of $n(k)$ has a further subsequence $\{n(k_r) : r \geq 1\}$ such that $\mathbb{P}_{k_r}^\omega[\cdot]$ is weakly convergent, so that the corresponding characteristic function converges. In view of (13), such characteristic function must also satisfy the asymptotic relation

$$\mathbb{E}\left[\exp\left(i\lambda S_{n(k_r), N_{n(k_r)}}^{(1)}\right) \mid \mathcal{F}_{n(k_r), r_{n(k_r)}}\right](\omega) \xrightarrow{r \rightarrow +\infty} \phi(\lambda)(\omega)$$

for every rational λ , hence for every real λ , because $\phi(\lambda)(\omega)$ is continuous in λ . ■

3 General framework for applications of the POC

We now present a general framework in which the POC techniques discussed in the previous paragraph can be applied. The main result of this section turns out to be the key tool to obtain stable convergence results for multiple stochastic integrals with respect to independently scattered random measures.

Our first goal is to define an Itô type stochastic integral with respect to a real valued and square integrable stochastic process X (not necessarily Gaussian) verifying the following three conditions: (i) X is indexed by the elements f of a real separable Hilbert space \mathfrak{H} , (ii) X satisfies the isomorphic relation

$$\mathbb{E}[X(f)X(g)] = (f, g)_{\mathfrak{H}}, \quad \forall f, g \in \mathfrak{H}, \quad (14)$$

and (iii) X has independent increments (the notion of “increment”, in this context, is defined through orthogonal projections—see below). We shall then show that the asymptotic behavior of such integrals can be studied by means of arrays of random variables, to which the POC applies quite naturally. Note that the elements of \mathfrak{H} need not be functions – they may be e.g. distributions on \mathbb{R}^d , $d \geq 1$. Our construction is inspired by the theory initiated by L. Wu (see [36]) and A.S. Üstünel and M. Zakai (see [35]), concerning Skorohod integrals and filtrations on abstract Wiener spaces. These authors have introduced the notion of time in the context of abstract Wiener spaces by using resolutions of the identity.

Definition D (see e.g. [2], [38] and [35]) – Let \mathfrak{H} be a separable real Hilbert space, endowed with an inner product $(\cdot, \cdot)_{\mathfrak{H}}$ ($\|\cdot\|_{\mathfrak{H}}$ is the corresponding norm). A (continuous) *resolution of the identity*, is a family $\pi = \{\pi_t : t \in [0, 1]\}$ of orthogonal projections satisfying:

(D-i) $\pi_0 = 0$, and $\pi_1 = id.$;

(D-ii) $\forall 0 \leq s < t \leq 1$, $\pi_s \mathfrak{H} \subseteq \pi_t \mathfrak{H}$;

(D-iii) $\forall 0 \leq t_0 \leq 1$, $\forall h \in \mathfrak{H}$, $\lim_{t \rightarrow t_0} \|\pi_t h - \pi_{t_0} h\|_{\mathfrak{H}} = 0$.

A subset F (not necessarily closed, nor linear) of \mathfrak{H} is said to be π -reproducing, and is denoted F_π , if the linear span of the set $\{\pi_t f : f \in F_\pi, t \in [0, 1]\}$ is dense in \mathfrak{H} (in which case we say that such a set is *total* in \mathfrak{H}). The *rank* of π is the smallest of the dimensions of all the closed subspaces generated by the π -reproducing subsets of \mathfrak{H} . A π -reproducing subset F_π of \mathfrak{H} is called *fully orthogonal* if $(\pi_t f, g)_{\mathfrak{H}} = 0$ for every $t \in [0, 1]$ and every $f, g \in F_\pi$. The class of all resolutions of the identity satisfying conditions **(D-i)**–**(D-iii)** is denoted $\mathcal{R}(\mathfrak{H})$.

Remarks – (a) Since \mathfrak{H} is separable, for every resolution of the identity π there always exists a countable π -reproducing subset of \mathfrak{H} .

(b) Let π be a resolution of the identity, and note $\overline{\text{v.s.}}(A)$ the closure of the vector space generated by some $A \subseteq \mathfrak{H}$. By a standard Gram-Schmidt orthogonalization procedure, it is easy to prove that for every π -reproducing subset F_π of \mathfrak{H} such that $\dim(\overline{\text{v.s.}}(F_\pi)) = \text{rank}(\pi)$, there exists a π -reproducing and fully orthogonal subset F'_π of \mathfrak{H} , such that $\dim(\overline{\text{v.s.}}(F'_\pi)) = \dim(\overline{\text{v.s.}}(F_\pi))$ (see e.g. [2, Lemma 23.2], or [35, p. 27]).

Examples – The following examples are related to the content of Section 5 and Section 6.

(a) Take $\mathfrak{H} = L^2([0, 1], dx)$, i.e. the space of square integrable functions on $[0, 1]$. Then, a family of projection operators naturally associated to \mathfrak{H} can be as follows: for every $t \in [0, 1]$ and every $f \in \mathfrak{H}$,

$$\pi_t f(x) = f(x) \mathbf{1}_{[0, t]}(x). \quad (15)$$

It is easily seen that this family $\pi = \{\pi_t : t \in [0, 1]\}$ is a resolution of the identity verifying conditions **(Di)**–**(Diii)** in Definition D. Also, $\text{rank}(\pi) = 1$, since the linear span of the projections of the function $f(x) \equiv 1$ generates \mathfrak{H} .

(b) If $\mathfrak{H} = L^2([0, 1]^2, dxdy)$, we define: for every $t \in [0, 1]$ and every $f \in \mathfrak{H}$,

$$\pi_t f(x, y) = f(x, y) \mathbf{1}_{[0, t]^2}(x, y). \quad (16)$$

The family $\pi = \{\pi_t : t \in [0, 1]\}$ appearing in (16) is a resolution of the identity as in Definition D. However, in this case $\text{rank}(\pi) = +\infty$. Other choices of π_t are also possible, for instance

$$\pi_t f(x, y) = f(x, y) \mathbf{1}_{[\frac{1}{2}-\frac{t}{2}, \frac{1}{2}+\frac{t}{2}]^2}(x, y),$$

which expands from the center of the square $[0, 1]^2$.

Now fix a real separable Hilbert space \mathfrak{H} , as well as a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows, we will write

$$X = X(\mathfrak{H}) = \{X(f) : f \in \mathfrak{H}\} \quad (17)$$

to denote a collection of centered random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by the elements of \mathfrak{H} and satisfying the isomorphic relation (14) (we use the notation $X(\mathfrak{H})$ when the role of the space \mathfrak{H} is relevant to the discussion). Note that relation (14) implies that, for every $f, g \in \mathfrak{H}$, $X(f+g) = X(f) + X(g)$, a.s.- \mathbb{P} .

Let $X(\mathfrak{H})$ be defined as in (17). Then, for every resolution $\pi = \{\pi_t : t \in [0, 1]\} \in \mathcal{R}(\mathfrak{H})$, the following property is verified: $\forall m \geq 2, \forall h_1, \dots, h_m \in \mathfrak{H}$ and $\forall 0 \leq t_0 < t_1 < \dots < t_m \leq 1$, the vector

$$(X((\pi_{t_1} - \pi_{t_0})h_1), X((\pi_{t_2} - \pi_{t_1})h_2), \dots, X((\pi_{t_m} - \pi_{t_{m-1}})h_m)) \quad (18)$$

is composed of uncorrelated random variables, because the π_t 's are orthogonal projections. We stress that the class $\mathcal{R}(\mathfrak{H})$ depends only on the Hilbert space \mathfrak{H} , and not on X . Now define $\mathcal{R}_X(\mathfrak{H})$ to be the subset of $\mathcal{R}(\mathfrak{H})$ containing those π such that the vector (18) is composed of jointly independent random variables, for any choice of $m \geq 2, h_1, \dots, h_m \in \mathfrak{H}$ and $0 \leq t_0 < t_1 < \dots < t_m \leq 1$. The set $\mathcal{R}_X(\mathfrak{H})$ depends in general of X . Note that, if $X(\mathfrak{H})$ is a Gaussian family, then $\mathcal{R}_X(\mathfrak{H}) = \mathcal{R}(\mathfrak{H})$ (see Section 3 below). To every $\pi \in \mathcal{R}_X(\mathfrak{H})$ we associate the filtration

$$\mathcal{F}_t^\pi(X) = \sigma\{X(\pi_t f) : f \in \mathfrak{H}\}, \quad t \in [0, 1], \quad (19)$$

so that, for instance, $\mathcal{F}_1^\pi(X) = \sigma(X)$.

Remark – Note that, for every $h \in \mathfrak{H}$ and every $\pi \in \mathcal{R}_X(\mathfrak{H})$, the stochastic process $t \mapsto X(\pi_t h)$ is a centered, square integrable $\mathcal{F}_t^\pi(X)$ -martingale with independent increments. Moreover, since π is continuous and (14) holds, $X(\pi_s h) \xrightarrow{\mathbb{P}} X(\pi_t h)$ whenever $s \rightarrow t$. In the terminology of [31, p. 3], this

implies that $\{X(\pi_t h) : t \in [0, 1]\}$ is an *additive process in law*. In particular, if $\mathcal{R}_X(\mathfrak{H})$ is not empty, for every $h \in \mathfrak{H}$ the law of $X(h)$ is infinitely divisible (see e.g. [31, Theorem 9.1]). As a consequence (see [31, Theorem 8.1 and formula (8.8), p. 39]), for every $h \in \mathfrak{H}$ there exists a unique pair $(c^2(h), \nu_h)$ such that $c^2(h) \in [0, +\infty)$ and ν_h is a measure on \mathbb{R} satisfying

$$\nu_h(\{0\}) = 0, \quad \int_{\mathbb{R}} (x^2 \wedge 1) \nu_h(dx) < +\infty \quad \text{and} \quad \int_{|x| \geq 1} x^2 \nu_h(dx) < +\infty \quad (20)$$

(the last relation follows from the fact that $X(h)$ is square integrable (see [31, Section 5.25])), and moreover, for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(i\lambda X(h))] = \exp\left[-\frac{c^2(h)\lambda^2}{2} + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x) \nu_h(dx)\right]. \quad (21)$$

Observe that, since the Lévy-Khintchine representation of an infinitely divisible distribution is unique, the pair $(c^2(h), \nu_h)$ does not depend on the choice of $\pi \in \mathcal{R}_X(\mathfrak{H})$. In what follows, when $\mathcal{R}_X(\mathfrak{H}) \neq \emptyset$, we will use the notation: for every $\lambda \in \mathbb{R}$ and every $h \in \mathfrak{H}$,

$$\psi_{\mathfrak{H}}(h; \lambda) \triangleq -\frac{c^2(h)\lambda^2}{2} + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x) \nu_h(dx), \quad (22)$$

where the pair $(c^2(h), \nu_h)$, characterizing the law of the random variable $X(h)$, is given by (21). Note that, if $h_n \rightarrow h$ in \mathfrak{H} , then $X(h_n) \rightarrow X(h)$ in $L^2(\mathbb{P})$, and therefore $\psi_{\mathfrak{H}}(h_n; \lambda) \rightarrow \psi_{\mathfrak{H}}(h; \lambda)$ for every $\lambda \in \mathbb{R}$ (uniformly on compacts). We shall always endow \mathfrak{H} with the σ -field $\mathcal{B}(\mathfrak{H})$, generated by the open sets with respect to the distance induced by the norm $\|\cdot\|_{\mathfrak{H}}$. Since, for every real λ , the complex-valued application $h \mapsto \psi_{\mathfrak{H}}(h; \lambda)$ is continuous, it is also $\mathcal{B}(\mathfrak{H})$ -measurable.

Examples – (a) Take $\mathfrak{H} = L^2([0, 1], dx)$, suppose that $X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$ is a centered Gaussian family verifying (14), and define the resolution of the identity $\pi = \{\pi_t : t \in [0, 1]\}$ according to (15). Then, if $\mathbf{1}$ indicates the function which is constantly equal to one, the process

$$W_t \triangleq X(\pi_t \mathbf{1}), \quad t \in [0, 1], \quad (23)$$

is a standard Brownian motion started from zero,

$$\mathcal{F}_t^\pi(X) = \sigma\{W_s : s \leq t\}, \quad \forall t \in [0, 1],$$

and, for every $f \in \mathfrak{H}$,

$$X(\pi_t f) = \int_0^t f(s) dW_s,$$

where the stochastic integration is in the usual Wiener-Itô sense. Of course, $X(\pi_t f)$ is a Gaussian $\mathcal{F}_t^\pi(X)$ -martingale with independent increments, and also, by using the notation (22), for every $f \in L^2([0, 1], dx)$ and $\lambda \in \mathbb{R}$, $\psi_{\mathfrak{H}}(f; \lambda) = -(\lambda^2/2) \int_0^1 f(x)^2 dx$.

(b) Take $\mathfrak{H} = L^2([0, 1]^2, dxdy)$ and define the resolution $\pi = \{\pi_t : t \in [0, 1]\}$ as in (16). We consider a compensated Poisson measure $\widehat{N} = \{\widehat{N}(C) : C \in \mathcal{B}([0, 1]^2)\}$ over $[0, 1]^2$. This means that (1) for every $C \in \mathcal{B}([0, 1]^2)$,

$$\widehat{N}(C) \stackrel{\text{law}}{=} N(C) - \mathbb{E}(N(C))$$

where $N(C)$ is a Poisson random variable with parameter $Leb(C)$ (i.e., the Lebesgue measure of C), and (2) $\widehat{N}(C_1)$ and $\widehat{N}(C_2)$ are stochastically independent whenever $C_1 \cap C_2 = \emptyset$. Then, the family $X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$, defined by

$$X(h) = \int_{[0, 1]^2} h(x, y) \widehat{N}(dx, dy), \quad h \in \mathfrak{H},$$

satisfies the isomorphic relation (14). Moreover

$$\mathcal{F}_t^\pi(X) = \sigma \left\{ \widehat{N}([0, s] \times [0, u]) : s \vee u \leq t \right\}, \quad \forall t \in [0, 1],$$

and for every $h \in \mathfrak{H}$, the process

$$X(\pi_t h) = \int_{[0, t]^2} h(x, y) \widehat{N}(dx, dy), \quad t \in [0, 1],$$

is a $\mathcal{F}_t^\pi(X)$ - martingale with independent increments, and hence $\pi \in \mathcal{R}_X(\mathfrak{H})$. Moreover, for every $h \in L^2([0, 1]^2, dxdy)$ and $\lambda \in \mathbb{R}$ the exponent $\psi_{\mathfrak{H}}(h; \lambda)$ in (22) verifies the relation (see e.g. [31, Proposition 19.5])

$$\psi_{\mathfrak{H}}(h; \lambda) = \int_0^1 \int_0^1 [\exp(i\lambda h(x, y)) - 1 - i\lambda h(x, y)] dxdy.$$

We now want to consider random variables with values in \mathfrak{H} , and define an Itô type stochastic integral with respect to X . To do so, we let $L^2(\mathbb{P}, \mathfrak{H}, X) = L^2(\mathfrak{H}, X)$ be the space of $\sigma(X)$ -measurable and \mathfrak{H} -valued random variables Y satisfying $\mathbb{E}[\|Y\|_{\mathfrak{H}}^2] < +\infty$ (note that $L^2(\mathfrak{H}, X)$ is a Hilbert space, with inner product $(Y, Z)_{L^2(\mathfrak{H}, X)} = \mathbb{E}[(Y, Z)_{\mathfrak{H}}]$). Following for instance [35] (which concerns uniquely the Gaussian case), we associate to every $\pi \in \mathcal{R}_X(\mathfrak{H})$ the subspace $L_\pi^2(\mathfrak{H}, X)$ of the π -adapted elements of $L^2(\mathfrak{H}, X)$, that is: $Y \in L_\pi^2(\mathfrak{H}, X)$ if, and only if, $Y \in L^2(\mathfrak{H}, X)$ and, for every $t \in [0, 1]$ and every $h \in \mathfrak{H}$,

$$(Y, \pi_t h)_{\mathfrak{H}} \in \mathcal{F}_t^\pi(X). \quad (24)$$

For any resolution $\pi \in \mathcal{R}_X(\mathfrak{H})$, $L_\pi^2(\mathfrak{H}, X)$ is a closed subspace of $L^2(\mathfrak{H}, X)$. Indeed, if $Y_n \in L_\pi^2(\mathfrak{H}, X)$ and $Y_n \rightarrow Y$ in $L^2(\mathfrak{H}, X)$, then necessarily $(Y_n, \pi_t h)_{\mathfrak{H}} \xrightarrow{\mathbb{P}} (Y, \pi_t h)_{\mathfrak{H}} \forall t \in [0, 1]$ and every $h \in \mathfrak{H}$, thus yielding $Y \in L_\pi^2(\mathfrak{H}, X)$. We will occasionally write $(u, z)_{L_\pi^2(\mathfrak{H})}$ instead of $(u, z)_{L^2(\mathfrak{H})}$, when both u and z are in $L_\pi^2(\mathfrak{H}, X)$. Now define, for $\pi \in \mathcal{R}_X(\mathfrak{H})$, $\mathcal{E}_\pi(\mathfrak{H}, X)$ to be the space of (π -adapted) elementary elements of $L_\pi^2(\mathfrak{H}, X)$, that is, $\mathcal{E}_\pi(\mathfrak{H}, X)$ is the collection of those elements of $L_\pi^2(\mathfrak{H}, X)$ that are linear combinations of \mathfrak{H} -valued random variables of the type

$$h = \Phi(t_1) (\pi_{t_2} - \pi_{t_1}) f, \quad (25)$$

where $t_2 > t_1$, $f \in \mathfrak{H}$ and $\Phi(t_1)$ is a random variable which is square-integrable and $\mathcal{F}_{t_1}^\pi(X)$ - measurable.

Lemma 3 *For every $\pi \in \mathcal{R}_X(\mathfrak{H})$, the set $\mathcal{E}_\pi(\mathfrak{H}, X)$, of adapted elementary elements, is total (i.e., its span is dense) in $L_\pi^2(\mathfrak{H}, X)$.*

Proof. The proof is similar to [35, Lemma 2.2]. Suppose $u \in L_\pi^2(\mathfrak{H}, X)$ and $(u, g)_{L^2(\mathfrak{H}, X)} = 0$ for every $g \in \mathcal{E}_\pi(\mathfrak{H}, X)$. We shall show that $u = 0$, a.s. - \mathbb{P} . For every $t_{i+1} > t_i$, every bounded and $\mathcal{F}_{t_i}^\pi(X)$ -measurable r.v. $\Phi(t_i)$, and every $f \in \mathfrak{H}$

$$\mathbb{E} \left[(\Phi(t_i) (\pi_{t_{i+1}} - \pi_{t_i}) f, u)_{\mathfrak{H}} \right] = 0,$$

and therefore $t \mapsto (\pi_t f, u)_{\mathfrak{H}}$ is a continuous (since π is continuous) $\mathcal{F}_t^\pi(X)$ - martingale starting from zero. Moreover, for every $0 = t_0 < \dots < t_n = 1$

$$\sum_{i=0}^{n-1} \left| (f, (\pi_{t_{i+1}} - \pi_{t_i}) u)_{\mathfrak{H}} \right| \leq \|u\|_{\mathfrak{H}} \|f\|_{\mathfrak{H}} \underset{\text{a.s.-}\mathbb{P}}{<} +\infty,$$

which implies that the continuous martingale $t \mapsto (\pi_t f, u)_{\mathfrak{H}}$ has also (a.s.- \mathbb{P}) bounded variation. It is therefore constant and hence equal to zero (see e.g. [28, Proposition 1.2]). It follows that, a.s.- \mathbb{P} , $(f, u)_{\mathfrak{H}} = (\pi_1 f, u)_{\mathfrak{H}} = 0$ for every $f \in \mathfrak{H}$, and consequently $u = 0$, a.s.- \mathbb{P} . ■

We now want to introduce, for every $\pi \in \mathcal{R}_X(\mathfrak{H})$, an Itô type stochastic integral with respect to X . To this end, we fix $\pi \in \mathcal{R}_X(\mathfrak{H})$ and first consider simple integrands of the form $h = \sum_{i=1}^n \lambda_i h_i \in \mathcal{E}_\pi(\mathfrak{H}, X)$, where $\lambda_i \in \mathbb{R}$, $n \geq 1$, and h_i is as in (25), i.e.

$$h_i = \Phi_i \left(t_1^{(i)} \right) \left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}} \right) f_i, \quad f_i \in \mathfrak{H}, \quad i = 1, \dots, n, \quad (26)$$

with $t_2^{(i)} > t_1^{(i)}$, and $\Phi_i \left(t_1^{(i)} \right) \in \mathcal{F}_{t_1^{(i)}}^\pi(X)$ and square integrable. Then, the stochastic integral of such a h with respect to X and π , is defined as

$$J_X^\pi(h) = \sum_{i=1}^n \lambda_i J_X^\pi(h_i) = \sum_{i=1}^n \lambda_i \Phi_i \left(t_1^{(i)} \right) X \left(\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}} \right) f_i \right). \quad (27)$$

Observe that the $\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}} \right) f_i$ in (26) becomes the argument of X in (27). Note also that, although X has π -independent increments, there may be a very complex dependence structure between the random variables

$$J_X^\pi(h_i) = \Phi_i \left(t_1^{(i)} \right) X \left(\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}} \right) f_i \right), \quad i = 1, \dots, n,$$

since the Φ_i 's are non-trivial functionals of X . We therefore introduce a “decoupled” version of the integral $J_X^\pi(h)$, by considering an independent copy of X , noted \tilde{X} , and by substituting X with \tilde{X} in formula (27). That is, for every $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ as in (26) we define

$$J_{\tilde{X}}^\pi(h) = \sum_{i=1}^n \lambda_i \Phi_i \left(t_1^{(i)} \right) \tilde{X} \left(\left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}} \right) f_i \right). \quad (28)$$

Note that if $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ is non random, i.e. $h(\omega) = h^* \in \mathfrak{H}$, a.s.- $\mathbb{P}(d\omega)$, then the integrals $J_X^\pi(h) = X(h^*)$ and $J_{\tilde{X}}^\pi(h) = \tilde{X}(h^*)$ are independent copies of each other.

Proposition 4 Fix $\pi \in \mathcal{R}_X(\mathfrak{H})$. Then, for every $h, h' \in \mathcal{E}_\pi(\mathfrak{H}, X)$,

$$\begin{aligned} \mathbb{E}(J_X^\pi(h) J_X^\pi(h')) &= (h, h')_{L_\pi^2(\mathfrak{H})} \\ \mathbb{E}(J_{\tilde{X}}^\pi(h) J_{\tilde{X}}^\pi(h')) &= (h, h')_{L_\pi^2(\mathfrak{H})}. \end{aligned} \quad (29)$$

As a consequence, there exist two linear extensions of J_X^π and $J_{\tilde{X}}^\pi$ to $L_\pi^2(\mathfrak{H}, X)$ satisfying the following two conditions:

1. if h_n converges to h in $L_\pi^2(\mathfrak{H}, X)$, then

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[(J_X^\pi(h_n) - J_X^\pi(h))^2 \right] = \lim_{n \rightarrow +\infty} \mathbb{E} \left[(J_{\tilde{X}}^\pi(h_n) - J_{\tilde{X}}^\pi(h))^2 \right] = 0;$$

2. for every $h, h' \in L_\pi^2(\mathfrak{H}, X)$

$$\mathbb{E}(J_X^\pi(h) J_X^\pi(h')) = \mathbb{E}(J_{\tilde{X}}^\pi(h) J_{\tilde{X}}^\pi(h')) = (h, h')_{L_\pi^2(\mathfrak{H})}. \quad (30)$$

The two extensions J_X^π and $J_{\tilde{X}}^\pi$ are unique, in the sense that if \widehat{J}_X^π and $\widehat{J}_{\tilde{X}}^\pi$ are two other extensions satisfying properties 1 and 2 above, then necessarily, a.s.- \mathbb{P} ,

$$J_X^\pi(h) = \widehat{J}_X^\pi(h) \quad \text{and} \quad J_{\tilde{X}}^\pi(h) = \widehat{J}_{\tilde{X}}^\pi(h)$$

for every $h \in L_\pi^2(\mathfrak{H}, X)$.

Proof. It is sufficient to prove (29) when h and h' are simple adapted elements of the kind (25), and in this case the result follows from elementary computations. Since, according to Lemma 3, $\mathcal{E}_\pi(\mathfrak{H}, X)$ is dense in $L_\pi^2(\mathfrak{H}, X)$, the result is obtained from a standard density argument. ■

The following property, which is a consequence of the above discussion, follows immediately.

Corollary 5 For every $f \in L_\pi^2(\mathfrak{H}, X)$, the process

$$t \mapsto J_{\tilde{X}}^\pi(\pi_t f), \quad t \in [0, 1]$$

is a real valued \mathcal{F}_t^π - martingale initialized at zero.

Observe that the process $t \mapsto J_{\tilde{X}}^\pi(\pi_t f)$, $t \in [0, 1]$, need not have independent (nor conditionally independent) increments. On the other hand, due to the independence between X and \tilde{X} , and to (18), *conditionally* on the σ -field $\sigma(X)$, the increments of the process $t \mapsto J_{\tilde{X}}^\pi(\pi_t f)$ are independent (to see this, just consider the process $J_{\tilde{X}}^\pi(\pi_t f)$ for an elementary f as in (28), and observe that, in this case, conditioning on $\sigma(X)$ is equivalent to conditioning on the Φ_i 's; the general case is obtained once again by a density argument). It follows that the random process $J_{\tilde{X}}^\pi(\pi \cdot f)$ can be regarded as being *decoupled and tangent* to $J_X^\pi(\pi \cdot f)$, in a spirit similar to [14, Definition 4.1], [8] or [7]. We stress, however, that $J_{\tilde{X}}^\pi(\pi \cdot f)$ need not meet the definition of a tangent process given in such references, which is based on a notion of convergence in the Skorohod topology, rather than on the L^2 -convergence adopted in the present paper. The reader is referred to [8] for an exhaustive characterization of processes with conditionally independent increments.

Now, for $h \in \mathfrak{H}$ and $\lambda \in \mathbb{R}$, define the exponent $\psi_{\mathfrak{H}}(h; \lambda)$ according to (22), and observe that every $f \in L_\pi^2(\mathfrak{H}, X)$ is a random element with values in \mathfrak{H} . It follows that the quantity $\psi_{\mathfrak{H}}(f(\omega); \lambda)$ is well defined for every $\omega \in \Omega$ and every $\lambda \in \mathbb{R}$, and moreover, since $\psi_{\mathfrak{H}}(\cdot; \lambda)$ is $\mathcal{B}(\mathfrak{H})$ -measurable, for every $f \in L_\pi^2(\mathfrak{H}, X)$ and every $\lambda \in \mathbb{R}$, the complex-valued application $\omega \mapsto \psi_{\mathfrak{H}}(f(\omega); \lambda)$ is \mathcal{F} -measurable.

Proposition 6 For every $\lambda \in \mathbb{R}$ and every $f \in L_\pi^2(\mathfrak{H}, X)$,

$$\mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}}^\pi(f) \right) \mid \sigma(X) \right] = \exp [\psi_{\mathfrak{H}}(f; \lambda)], \quad a.s.-\mathbb{P}. \quad (31)$$

Proof. For $f \in \mathcal{E}_\pi(\mathfrak{H}, X)$, formula (31) follows immediately from the independence of X and \tilde{X} . Now fix $f \in L_\pi^2(\mathfrak{H}, X)$, and select a sequence $(f_n) \subset \mathcal{E}_\pi(\mathfrak{H}, X)$ such that

$$\mathbb{E} \left[\|f_n - f\|_{\mathfrak{H}}^2 \right] \rightarrow 0 \quad (32)$$

(such a sequence f_n always exists, due to Lemma 3). Since (32) implies that $\|f_n - f\|_{\mathfrak{H}} \xrightarrow{\mathbb{P}} 0$, for every subsequence n_k there exists a further subsequence $n_{k(r)}$ such that $\|f_{n_{k(r)}} - f\|_{\mathfrak{H}} \rightarrow 0$, a.s. - \mathbb{P} , thus implying $\psi_{\mathfrak{H}}(f_{n_{k(r)}}; \lambda) \rightarrow \psi_{\mathfrak{H}}(f; \lambda)$ for every $\lambda \in \mathbb{R}$, a.s. - \mathbb{P} . Then, for every $\lambda \in \mathbb{R}$, $\psi_{\mathfrak{H}}(f_n; \lambda) \xrightarrow{\mathbb{P}} \psi_{\mathfrak{H}}(f; \lambda)$, and therefore $\exp[\psi_{\mathfrak{H}}(f_n; \lambda)] \xrightarrow{\mathbb{P}} \exp[\psi_{\mathfrak{H}}(f; \lambda)]$. On the other hand,

$$\begin{aligned} \mathbb{E} \left| \mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}}^\pi(f_n) \right) - \exp \left(i\lambda J_{\tilde{X}}^\pi(f) \right) \mid \sigma(X) \right] \right| &\leq |\lambda| \mathbb{E} \left| J_{\tilde{X}}^\pi(f_n) - J_{\tilde{X}}^\pi(f) \right| \\ &\leq |\lambda| \mathbb{E} \left[\left(J_{\tilde{X}}^\pi(f_n) - J_{\tilde{X}}^\pi(f) \right)^2 \right]^{\frac{1}{2}} \\ &= |\lambda| \mathbb{E} \left[\|f_n - f\|_{\mathfrak{H}}^2 \right]^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

where the equality follows from (30), thus yielding

$$\exp[\psi_{\mathfrak{H}}(f_n; \lambda)] = \mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}}^\pi(f_n) \right) \mid \sigma(X) \right] \xrightarrow{\mathbb{P}} \mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}}^\pi(f) \right) \mid \sigma(X) \right],$$

and the desired conclusion is therefore obtained. ■

Examples – (a) Take $\mathfrak{H} = L^2([0, 1], dx)$ and suppose that $X(\mathfrak{H}) = \{X(h) : h \in \mathfrak{H}\}$ is a centered Gaussian family verifying (14). Define also $\pi = \{\pi_t : t \in [0, 1]\} \in \mathcal{R}(\mathfrak{H})$ according to (15), and write W to denote the Brownian motion introduced in (23). The subsequent discussion will make clear that $L^2_\pi(\mathfrak{H}, X)$ is, in this case, the space of square integrable processes that are adapted to the Brownian filtration $\sigma\{W_u : u \leq t\}$, $t \in [0, 1]$. Moreover, for every $t \in [0, 1]$ and $u \in L^2_\pi(\mathfrak{H}, X)$

$$J_X^\pi(\pi_t u) = \int_0^t u(s) dW_s \quad \text{and} \quad J_{\tilde{X}}^\pi(\pi_t u) = \int_0^t u(s) d\tilde{W}_s,$$

where the stochastic integration is in the Itô sense, and $\tilde{W}_t \triangleq \tilde{X}(\mathbf{1}_{[0,t]})$ is a standard Brownian motion independent of X .

(b) (*Orthogonalized Teugels martingales*, see [20]) Let $Z = \{Z_t : t \in [0, 1]\}$ be a real-valued and centered Lévy process, initialized at zero and endowed with a Lévy measure ν satisfying the condition: for some $\varepsilon, \lambda > 0$

$$\int_{(-\varepsilon, \varepsilon)^c} \exp(\lambda|x|) \nu(dx) < +\infty.$$

Then, for every $i \geq 2$, $\int_{\mathbb{R}} |x|^i \nu(dx) < +\infty$, and Z_t has moments of all orders. Starting from Z , for every $i \geq 1$ one can therefore define the *compensated power jump process* (or *Teugel martingale*) of order i , noted $Y^{(i)}$, as $Y_t^{(1)} = Z_t$ for $t \in [0, 1]$, and, for $i \geq 2$ and $t \in [0, 1]$,

$$Y_t^{(i)} = \sum_{0 < s \leq t} (\Delta Z_t)^i - \mathbb{E} \sum_{0 < s \leq t} (\Delta Z_t)^i = \sum_{0 < s \leq t} (\Delta Z_t)^i - t \int_{\mathbb{R}} x^i \nu(dx).$$

Plainly, each $Y^{(i)}$ is a centered Lévy process. Moreover, according to [20, pp. 111-112], for every $i \geq 1$ it is possible to find (unique) real coefficients $a_{i,1}, \dots, a_{i,i}$, such that $a_{i,i} = 1$ and the stochastic processes

$$H_t^{(i)} = Y_t^{(i)} + a_{i,i-1} Y_t^{(i-1)} + \dots + a_{i,1} Y_t^{(1)}, \quad t \in [0, 1], \quad i \geq 1,$$

are strongly orthogonal centered martingales (in the sense of [26, p.148]), also verifying $\mathbb{E} [H_t^{(i)} H_s^{(j)}] = \delta_{ij} (t \wedge s)$, where δ_{ij} is the Kronecker symbol. Observe that $H^{(i)}$ is again a Lévy process, and that, for every deterministic $g, f \in L^2([0, 1], ds)$, the integrals $\int_0^1 f(s) dH_s^{(i)}$ and $\int_0^1 g(s) dH_s^{(j)}$ are well defined and such that

$$\mathbb{E} \left[\int_0^1 f(s) dH_s^{(i)} \int_0^1 g(s) dH_s^{(j)} \right] = \delta_{ij} \int_0^1 g(s) f(s) ds. \quad (33)$$

Now define $\mathfrak{H} = L^2(\mathbb{N} \times [0, 1], \kappa(dm) \times ds)$, where $\kappa(dm)$ is the counting measure, and define, for $h(\cdot, \cdot) \in \mathfrak{H}$, $t \in [0, 1]$, and $(m, s) \in \mathbb{N} \times [0, 1]$,

$$\pi_t h(m, s) = h(m, s) \mathbf{1}_{[0,t]}(s).$$

It is clear that $\pi = \{\pi_t : t \in [0, 1]\} \in \mathcal{R}(\mathfrak{H})$. Moreover, for every $h(\cdot, \cdot) \in \mathfrak{H}$, we define

$$X(h) = \sum_{m=1}^{\infty} \int_0^1 h(m, s) dH_s^{(m)},$$

where the series is convergent in $L^2(\mathbb{P})$, since $\mathbb{E} X(h)^2 = \sum \int_0^1 h(m, s)^2 ds < +\infty$, due to (33) and the fact that $h \in \mathfrak{H}$. Since the $H^{(m)}$ are strongly orthogonal and (33) holds, one sees immediately that, for every $h, h' \in \mathfrak{H}$, $\mathbb{E}[X(h) X(h')] = (h, h')_{\mathfrak{H}}$, and moreover, since for every m and every h the process $t \mapsto \int_0^t \pi_t h(m, s) dH_s^{(m)} = \int_0^t h(m, s) dH_s^{(m)}$ has independent increments, $\pi \in \mathcal{R}_X(\mathfrak{H})$. We can also consider random h , and, by using [20], give the following characterization of random variables

$h \in L^2_\pi(\mathfrak{H}, X)$, and the corresponding integrals $J_X^\pi(h)$ and $J_{\tilde{X}}^\pi(h)$: (i) for every $h \in L^2_\pi(\mathfrak{H}, X)$ there exists a family $\{\phi_{m,t}^{(h)} : t \in [0, 1], m \geq 1\}$ of real-valued and \mathcal{F}_t^π -predictable processes such that for every fixed m , the process $t \mapsto \phi_{m,t}^{(h)}$ is a modification of $t \mapsto h(m, t)$; (ii) for every $h \in L^2_\pi(\mathfrak{H}, X)$,

$$J_X^\pi(h) = \sum_{m=1}^{\infty} \int_0^1 \phi_{m,t}^{(h)} dH_t^{(m)}, \quad (34)$$

where the series is convergent in $L^2(\mathbb{P})$; (iii) for every $h \in L^2_\pi(\mathfrak{H}, X)$,

$$J_{\tilde{X}}^\pi(h) = \sum_{m=1}^{\infty} \int_0^1 \phi_{m,t}^{(h)} d\tilde{H}_t^{(m)}, \quad (35)$$

where the series is convergent in $L^2(\mathbb{P})$, and the sequence $\{\tilde{H}^{(m)} : m \geq 1\}$ is an independent copy of $\{H^{(m)} : m \geq 1\}$. Note that by using [20, Theorem 1], one would obtain an analogous characterization in terms of iterated stochastic integrals of deterministic kernels.

4 Stable convergence

We shall now apply Theorem 1 to the setup outlined in the previous paragraph. Let \mathfrak{H}_n , $n \geq 1$, be a sequence of real separable Hilbert spaces, and, for each $n \geq 1$, let

$$X_n = X_n(\mathfrak{H}_n) = \{X_n(g) : g \in \mathfrak{H}_n\}, \quad (36)$$

be a centered, real-valued stochastic process, indexed by the elements of \mathfrak{H}_n and such that $\mathbb{E}[X_n(f)X_n(g)] = (f, g)_{\mathfrak{H}_n}$. The processes X_n are not necessarily Gaussian. As before, \tilde{X}_n indicates an independent copy of X_n , for every $n \geq 1$.

Theorem 7 *Let the previous notation prevail, and suppose that the processes X_n , $n \geq 1$, appearing in (36) (along with the independent copies \tilde{X}_n) are all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For every $n \geq 1$, let $\pi^{(n)} \in \mathcal{R}_{X_n}(\mathfrak{H}_n)$ and $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$. Suppose also that there exists a sequence $\{t_n : n \geq 1\} \subset [0, 1]$ and a collection of σ -fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left\| \pi_{t_n}^{(n)} u_n \right\|_{\mathfrak{H}_n}^2 \right] = 0$$

and

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n). \quad (37)$$

If

$$\exp[\psi_{\mathfrak{H}_n}(u_n; \lambda)] = \mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right) \mid \sigma(X_n) \right] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R}, \quad (38)$$

where $\psi_{\mathfrak{H}_n}(u_n; \lambda)$ is defined according to (22), $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall \lambda \in \mathbb{R}$,

$$\phi(\lambda) \in \vee_n \mathcal{U}_n \triangleq \mathcal{U}^*,$$

then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp \left(i\lambda J_{X_n}^{\pi^{(n)}}(u_n) \right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \quad (39)$$

and

$$J_{X_n}^{\pi^{(n)}}(u_n) \rightarrow_{(s, \mathcal{U}^*)} \mathbb{E}\mu(\cdot), \quad (40)$$

where $\mu \in \mathbf{M}$ verifies (2).

Remarks – (1) The proof of Theorem 7 uses Theorem 1, which assumes $\phi \in \widehat{\mathbf{M}}_0$, that is, ϕ is non-vanishing. If $\phi \in \widehat{\mathbf{M}}$ (instead of $\widehat{\mathbf{M}}_0$) and if, for example, there exists a subsequence n_k such that,

$$\mathbb{P} \left\{ \omega : \exp \left[\psi_{\mathfrak{H}_{n_k}} (u_{n_k}(\omega); \lambda) \right] \rightarrow \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R} \right\} = 1,$$

then, given the nature of $\psi_{\mathfrak{H}_{n_k}}$, $\phi(\lambda, \omega)$ is necessarily, for \mathbb{P} -a.e. ω , the Fourier transform of an infinitely divisible distribution (see e.g. [31, Lemma 7.5]), and therefore $\phi \in \widehat{\mathbf{M}}_0$. A similar remark applies to Theorem 12 below.

(2) For $n \geq 1$, the process $t \mapsto J_{X_n}^{\pi^{(n)}}(\pi_t^{(n)} u_n)$ is a martingale and hence admits a càdlàg modification. Then, an alternative approach to obtain results for stable convergence is to use the well-known criteria for the stable convergence of continuous-time càdlàg semimartingales, as stated e.g. in [5, Proposition 1 and Theorems 1 and 2] or [11, Chapter 4]. However, the formulation in terms of “principle of conditioning” yields, in our setting, more precise results, by using less stringent assumptions. For instance, (37) can be regarded as a weak version of the “nesting condition” used in [5, p. 126], whereas (39) is a refinement of the conclusions that can be obtained by means of [5, Proposition 1].

(3) Suppose that, under the assumptions of Theorem 7, there exists a càdlàg process $Y = \{Y_t : t \in [0, 1]\}$ such that, conditionally on \mathcal{U}^* , Y has independent increments and $\phi(\lambda) = \mathbb{E}[\exp(i\lambda Y_1) | \mathcal{U}^*]$. In this case, formula (40) is equivalent to saying that $J_{X_n}^{\pi^{(n)}}(u_n)$ converges \mathcal{U}^* -stably to Y_1 . See [8, Section 4] for several results concerning the stable convergence (for instance, in the sense of finite dimensional distributions) of semimartingales towards processes with conditionally independent increments.

Before proving Theorem 7, we consider the important case of a *nested* sequence of resolutions. More precisely, assume that $\mathfrak{H}_n = \mathfrak{H}$, $X_n = X$, for every $n \geq 1$, and that the sequence $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, $n \geq 1$, is nested in the following sense: for every $t \in [0, 1]$ and every $n \geq 1$,

$$\pi_t^{(n)} \mathfrak{H} \subseteq \pi_t^{(n+1)} \mathfrak{H} \quad (41)$$

(note that if $\pi^{(n)} = \pi$ for every n , then (41) is trivially satisfied); in this case, if t_n is non decreasing, the sequence $\mathcal{U}_n = \mathcal{F}_{t_n}^{\pi^{(n)}}(X)$, $n \geq 1$, automatically satisfies (37). We therefore have the following consequence of Theorem 7.

Corollary 8 *Under the above notation and assumptions, suppose that the sequence $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, $n \geq 1$, is nested in the sense of (41), and let $u_n \in L_{\pi^{(n)}}^2(\mathfrak{H}, X)$, $n \geq 1$. Suppose also that there exists a non-decreasing sequence $\{t_n : n \geq 1\} \subset [0, 1]$ s.t.*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left\| \pi_{t_n}^{(n)} u_n \right\|_{\mathfrak{H}}^2 \right] = 0. \quad (42)$$

If

$$\exp[\psi_{\mathfrak{H}}(u_n; \lambda)] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R},$$

where $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall \lambda \in \mathbb{R}$, $\phi(\lambda) \in \bigvee_n \mathcal{F}_{t_n}^{\pi^{(n)}}(X) \triangleq \mathcal{F}_*$, then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp(i\lambda J_X(u_n)) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X) \right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

and

$$J_X(u_n) \rightarrow_{(s, \mathcal{F}_*)} \mathbb{E} \mu(\cdot),$$

where $\mu \in \mathbf{M}$ verifies (2).

In the next result $\{u_n\}$ may still be random, but $\phi(\lambda)$ is non-random. It follows from Corollary 8 by taking $t_n = 0$ for every n , so that (42) is immaterial, and \mathcal{F}_* becomes the trivial σ -field.

Corollary 9 *Keep the notation of Corollary 8, and consider a (not necessarily nested) sequence $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, $n \geq 1$. If*

$$\exp[\psi_{\mathfrak{H}}(u_n; \lambda)] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

where ϕ is the Fourier transform of some non-random measure μ such that $\phi(\lambda) \neq 0$ for every $\lambda \in \mathbb{R}$, then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp \left(i \lambda J_{\tilde{X}}^{\pi^{(n)}}(u_n) \right) \right] \rightarrow \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

that is, the law of $J_{\tilde{X}}^{\pi^{(n)}}(u_n)$ converges weakly to μ .

Proof of Theorem 7 – Since $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$, there exists, thanks to Lemma 3 a sequence $u_n^e \in \mathcal{E}_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$, $n \geq 1$, such that (by using the isometry properties of $J_{\tilde{X}}^{\pi^{(n)}}$ and $J_{X_n}^{\pi^{(n)}}$, as stated in Proposition 4)

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[\|u_n - u_n^e\|_{\mathfrak{H}_n}^2 \right] = \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) - J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \right)^2 \right] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(J_{X_n}^{\pi^{(n)}}(u_n) - J_{X_n}^{\pi^{(n)}}(u_n^e) \right)^2 \right] \end{aligned} \quad (43)$$

and

$$\begin{aligned} 0 &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[\|\pi_{t_n}^{(n)} u_n^e\|_{\mathfrak{H}_n}^2 \right] = \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(J_{\tilde{X}_n}^{\pi^{(n)}}(\pi_{t_n}^{(n)} u_n^e) \right)^2 \right] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(J_{X_n}^{\pi^{(n)}}(\pi_{t_n}^{(n)} u_n^e) \right)^2 \right]. \end{aligned} \quad (44)$$

Without loss of generality, we can always suppose that u_n^e has the form

$$u_n^e = \sum_{i=1}^{N_n} \left[\sum_{j=1}^{M_n(i)} \Phi_j^{(n)}(t_{i-1}^{(n)}) \left(\pi_{t_i^{(n)}}^{(n)} - \pi_{t_{i-1}^{(n)}}^{(n)} \right) f_j^{(n)} \right]$$

where $0 = t_0^{(n)} < \dots < t_{N_n}^{(n)} = 1$, $f_j^{(n)} \in \mathfrak{H}_n$, $N_n, M_n(i) \geq 1$, $\Phi_j^{(n)}(t_{i-1}^{(n)})$ is square integrable and measurable with respect to $\mathcal{F}_{t_{i-1}^{(n)}}^{\pi^{(n)}}(X_n)$ where one of the $t_0^{(n)}, \dots, t_{N_n}^{(n)}$ equals t_n . Moreover, we have

$$\begin{aligned} J_{X_n}^{\pi^{(n)}}(u_n^e) &= \sum_{i=1}^{N_n} \left[\sum_{j=1}^{M_n(i)} \Phi_j^{(n)}(t_{i-1}^{(n)}) X_n \left((\pi_{t_i^{(n)}}^{(n)} - \pi_{t_{i-1}^{(n)}}^{(n)}) f_j^{(n)} \right) \right] \\ J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) &= \sum_{i=1}^{N_n} \left[\sum_{j=1}^{M_n(i)} \Phi_j^{(n)}(t_{i-1}^{(n)}) \tilde{X}_n \left((\pi_{t_i^{(n)}}^{(n)} - \pi_{t_{i-1}^{(n)}}^{(n)}) f_j^{(n)} \right) \right]. \end{aligned}$$

Now define for $n \geq 1$ and $i = 1, \dots, N_n$

$$\begin{aligned} X_{n,i}^{(1)} &= \sum_{j=1}^{M_n(i)} \Phi_j^{(n)}(t_{i-1}^{(n)}) X_n \left((\pi_{t_i^{(n)}}^{(n)} - \pi_{t_{i-1}^{(n)}}^{(n)}) f_j^{(n)} \right) \\ X_{n,i}^{(2)} &= \sum_{j=1}^{M_n(i)} \Phi_j^{(n)}(t_{i-1}^{(n)}) \tilde{X}_n \left((\pi_{t_i^{(n)}}^{(n)} - \pi_{t_{i-1}^{(n)}}^{(n)}) f_j^{(n)} \right) \end{aligned}$$

as well as $X_{n,0}^{(\ell)} = 0$, $\ell = 1, 2$; introduce moreover the filtration

$$\widehat{\mathcal{F}}_t^{(\pi^{(n)}, \mathfrak{H}_n)} = \mathcal{F}_t^{\pi^{(n)}}(X_n) \vee \sigma \left\{ \widetilde{X} \left(\pi_t^{(n)} f \right) : f \in \mathfrak{H}_n \right\}, \quad t \in [0, 1], \quad (45)$$

and let $\mathcal{G}_n = \sigma(X_n)$, $n \geq 1$. We shall verify that the array $X^{(2)} = \{X_{n,i}^{(2)} : 0 \leq i \leq N_n, n \geq 1\}$ is decoupled and tangent to $X^{(1)} = \{X_{n,i}^{(1)} : 0 \leq i \leq N_n, n \geq 1\}$, in the sense of Definition C of Section 2. Indeed, for $\ell = 1, 2$, the sequence $\{X_{n,i}^{(\ell)} : 0 \leq i \leq N_n\}$ is adapted to the discrete filtration

$$\mathcal{F}_{n,i} \triangleq \widehat{\mathcal{F}}_{t_i^{(n)}}^{(\pi^{(n)}, \mathfrak{H}_n)}, \quad i = 1, \dots, N_n; \quad (46)$$

also (5) is satisfied, since, for every j and every $i = 1, \dots, N_n$,

$$\Phi_j^{(n)} \left(t_{i-1}^{(n)} \right) \in \mathcal{F}_{t_{i-1}^{(n)}}^{\pi^{(n)}}(X_n) \subset \mathcal{F}_{n,i-1},$$

and

$$\begin{aligned} \mathbb{E} \left[\exp \left(i\lambda X_n \left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)} \right) \right) \mid \mathcal{F}_{n,i-1} \right] &= \mathbb{E} \left[\exp \left(i\lambda X_n \left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)} \right) \right) \right] \\ &= \mathbb{E} \left[\exp \left(i\lambda \widetilde{X}_n \left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_j^{(n)} \right) \right) \mid \mathcal{F}_{n,i-1} \right]. \end{aligned}$$

Since $\mathcal{G}_n = \sigma(X_n)$, we obtain immediately (6), because \widetilde{X}_n is an independent copy of X_n . We now want to apply Theorem 1 with

$$\begin{aligned} J_{X_n}^{\pi} \left(\pi_{t_n}^{(n)} u_n^e \right) &= \sum_{i=1}^{r_n} \left[\sum_{l=1}^{M_n(i)} \Phi_l^{(n)} \left(t_{i-1}^{(n)} \right) X_n \left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_l^{(n)} \right) \right] = \sum_{i=1}^{r_n} X_{n,i}^{(1)} = S_{n,r_n}^{(1)} \quad (47) \\ J_{\widetilde{X}_n}^{\pi} \left(\pi_{t_n}^{(n)} u_n^e \right) &= \sum_{i=1}^{r_n} \left[\sum_{l=1}^{M_n(i)} \Phi_l^{(n)} \left(t_{i-1}^{(n)} \right) \widetilde{X}_n \left((\pi_{t_i^{(n)}} - \pi_{t_{i-1}^{(n)}}) f_l^{(n)} \right) \right] = \sum_{i=1}^{r_n} X_{n,i}^{(1)} = S_{n,r_n}^{(2)}, \end{aligned}$$

where r_n is the element of $\{1, \dots, N_n\}$ such that $t_{r_n}^{(n)} = t_n$. To do so, we need to verify the remaining conditions of that theorem. To prove (7), use (45), (46) and (37), to obtain

$$\mathcal{F}_{n,r_n} = \widehat{\mathcal{F}}_{t_{r_n}^{(n)}}^{(\pi^{(n)}, \mathfrak{H}_n)} \supset \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \supseteq \mathcal{U}_n,$$

and hence (7) holds with $\mathcal{V}_n = \mathcal{U}_n$. To prove (8), observe that the asymptotic relation in (44) can be rewritten as

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(S_{n,r_n}^{(\ell)} \right)^2 \right] = 0, \quad \ell = 1, 2, \quad (48)$$

which immediately yields, as $n \rightarrow +\infty$,

$$S_{n,r_n}^{(1)} \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \mathbb{E} \left[\exp \left(i\lambda S_{n,r_n}^{(2)} \right) \mid \mathcal{G}_n \right] \xrightarrow{\mathbb{P}} 1$$

for every $\lambda \in \mathbb{R}$. To justify the last relation, just observe that (48) implies that $\mathbb{E} \left[\left(S_{n,r_n}^{(2)} \right)^2 \mid \mathcal{G}_n \right] \rightarrow 0$ in $L^1(\mathbb{P})$, and hence, for every diverging sequence n_k , there exists a subsequence n'_k such that, a.s.- \mathbb{P} ,

$$\mathbb{E} \left[\left(S_{n'_k, r_{n'_k}}^{(2)} \right)^2 \mid \mathcal{G}_{n'_k} \right] \xrightarrow{k \rightarrow +\infty} 0,$$

which in turn yields that, a.s.- \mathbb{P} ,

$$\mathbb{E} \left[\exp \left(i\lambda S_{n'_k, r_{n'_k}}^{(2)} \right) \mid \mathcal{G}_{n'_k} \right] \xrightarrow{k \rightarrow +\infty} 1.$$

To prove (9), observe that

$$\mathbb{E} \left| \exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \right) - \exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right) \right| \leq |\lambda| \mathbb{E} \left| J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) - J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right| \xrightarrow{n \rightarrow +\infty} 0,$$

by (43). Hence, since (38) holds for u_n , it also holds when u_n is replaced by the elementary sequence u_n^e . Since $J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) = J_{\tilde{X}_n}^{\pi^{(n)}}(\pi_1^{(n)} u_n^e) = S_{n, N_n}^{(2)}$ and $\mathcal{G}_n = \sigma(X_n)$, relation (9) holds. It follows that the assumptions of Theorem 1 are satisfied, and we deduce that necessarily, as $n \rightarrow +\infty$,

$$\begin{aligned} & \mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \right] \\ &= \mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \right) \mid \widehat{\mathcal{F}}_{t_n}^{(\pi^{(n)}, \mathfrak{S}_n)} \right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \end{aligned}$$

(the equality follows from the fact that X_n and \tilde{X}_n are independent). Theorem 1 also yields

$$J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \xrightarrow{(s, \mathcal{U}^*)} \mathbb{E}\mu(\cdot). \quad (49)$$

To go back from u_n^e to u_n , we use

$$\mathbb{E} \left| \exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \right) - \exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right) \right| \xrightarrow{n \rightarrow +\infty} 0, \quad (50)$$

which follows again from (43), and we deduce that

$$\mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n^e) \right) - \exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \right] \xrightarrow[n \rightarrow +\infty]{L^1} 0,$$

and therefore

$$\mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \right] \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Finally, by combining (49) and (50), we obtain

$$J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \xrightarrow{(s, \mathcal{U}^*)} \mathbb{E}\mu(\cdot).$$

■

By using the same approximation procedure as in the preceding proof, we may use Proposition 2 to prove the following refinement of Theorem 7.

Proposition 10 *With the notation of Theorem 7, suppose that the sequence $J_{\tilde{X}_n}^{\pi^{(n)}}(u_n)$ verifies (39), and that there exists a finite random variable $C(\omega) > 0$ such that, for some $\eta > 0$,*

$$\mathbb{E} \left[\left| J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right|^\eta \mid \mathcal{F}_{t_n}^{\pi^{(n)}} \right] < C(\omega), \quad \forall n \geq 1, \quad \text{a.s.-}\mathbb{P}.$$

Then, there is a subsequence $\{n(k) : k \geq 1\}$ such that, a.s. - \mathbb{P} ,

$$\mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}_n}^{\pi^{(n)}}(u_n) \right) \mid \mathcal{F}_{t_{n(k)}}^{\pi^{(n(k))}} \right] \xrightarrow{k \rightarrow +\infty} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

Theorem 7 can also be extended to a slightly more general framework. To this end, we introduce some further notation. Fix a closed subspace $\mathfrak{H}^* \subseteq \mathfrak{H}$. For every $t \in [0, 1]$, we denote by $\pi_{s \leq t} \mathfrak{H}^*$ the closed linear subspace of \mathfrak{H} , generated by the set $\{\pi_s f : f \in \mathfrak{H}^*, s \leq t\}$. Of course, $\pi_{s \leq t} \mathfrak{H}^* \subseteq \pi_t \mathfrak{H} = \pi_{s \leq t} \mathfrak{H}$. For a fixed $\pi \in \mathcal{R}_X(\mathfrak{H})$, we set $\mathcal{E}_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ to be the subset of $\mathcal{E}_\pi(\mathfrak{H}, X)$ composed of \mathfrak{H} -valued random variables of the kind

$$h = \Psi^*(t_1)(\pi_{t_2} - \pi_{t_1})g, \quad (51)$$

where $t_2 > t_1$, $g \in \mathfrak{H}^*$ and $\Psi^*(t_1)$ is a square integrable random variable verifying the measurability condition

$$\Psi^*(t_1) \in \sigma\{X(f) : f \in \pi_{s \leq t_1} \mathfrak{H}^*\},$$

whereas $L_\pi^2(\mathfrak{H}, \mathfrak{H}^*, X)$ is defined as the closure of $\mathcal{E}_\pi(\mathfrak{H}, \mathfrak{H}^*, X)$ in $L_\pi^2(\mathfrak{H}, X)$. Note that, plainly, $\mathcal{E}_\pi(\mathfrak{H}, X) = \mathcal{E}_\pi(\mathfrak{H}, \mathfrak{H}, X)$ and $L_\pi^2(\mathfrak{H}, X) = L_\pi^2(\mathfrak{H}, \mathfrak{H}, X)$. Moreover, for every $Y \in L_\pi^2(\mathfrak{H}, \mathfrak{H}^*, X)$ and every $t \in [0, 1]$, the following two properties are verified: (i) the random element $\pi_t Y$ takes values in $\pi_{s \leq t} \mathfrak{H}^*$, a.s.- \mathbb{P} , and (ii) the random variable $J_X^\pi(\pi_t h)$ is measurable with respect to the σ -field $\sigma\{X(f) : f \in \pi_{s \leq t} \mathfrak{H}^*\}$ (such claims are easily verified for h as in (51), and the general results follow once again by standard density arguments).

Remark – Note that, in general, even when $\text{rank}(\pi) = 1$ as in (15), and \mathfrak{H}^* is non-trivial, for $0 < t \leq 1$ the set $\pi_{s \leq t} \mathfrak{H}^*$ may be strictly contained in $\pi_t \mathfrak{H}$. It follows that the σ -field $\sigma\{X(f) : f \in \pi_{s \leq t} \mathfrak{H}^*\}$ can be strictly contained in $\mathcal{F}_t^\pi(X)$, as defined in (19). To see this, just consider the case $\mathfrak{H} = \bar{L}^2([0, 1], dx)$, $\mathfrak{H}^* = \{f \in L^2([0, 1], dx) : f = f \mathbf{1}_{[0, 1/2]}\}$, $\pi_s f = f \mathbf{1}_{[0, s]}$ ($s \in [0, 1]$), and take $t \in (1/2, 1]$. Indeed, in this case $X(\mathbf{1}_{[0, t]})$ is $\mathcal{F}_t^\pi(X)$ -measurable but is not $\sigma\{X(f) : f \in \pi_{s \leq t} \mathfrak{H}^*\}$ -measurable.

The following result can be proved along the lines of Lemma 3.

Lemma 11 *For every closed subspace \mathfrak{H}^* of \mathfrak{H} , a random element Y is in $L_\pi^2(\mathfrak{H}, \mathfrak{H}^*, X)$ if, and only if, $Y \in L^2(\mathfrak{H}, X)$ and, for every $t \in [0, 1]$,*

$$(Y, \pi_t h)_{\mathfrak{H}} \in \sigma\{X(f) : f \in \pi_{s \leq t} \mathfrak{H}^*\}.$$

The next theorem can be proved by using arguments analogous to the ones in the proof of Theorem 7. Here, $\mathfrak{H}_n = \mathfrak{H}$ and $X_n(\mathfrak{H}_n) = X(\mathfrak{H})$ for every n .

Theorem 12 *Under the above notation and assumptions, for every $n \geq 1$ let $\mathfrak{H}^{(n)}$ be a closed subspace of \mathfrak{H} , $\pi^{(n)} \in \mathcal{R}_X(\mathfrak{H})$, and $u_n \in L_{\pi^{(n)}}^2(\mathfrak{H}, \mathfrak{H}^{(n)}, X)$. Suppose also that there exists a sequence $\{t_n : n \geq 1\} \subset [0, 1]$ and a collection of closed subspaces of \mathfrak{H} , noted $\{\mathfrak{U}_n : n \geq 1\}$, such that*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left\| \pi_{t_n}^{(n)} u_n \right\|_{\mathfrak{H}}^2 \right] = 0$$

and

$$\mathfrak{U}_n \subseteq \mathfrak{U}_{n+1} \cap \pi_{s \leq t_n}^{(n)} \mathfrak{H}^{(n)}.$$

If

$$\exp[\psi_{\mathfrak{H}}(u_n; \lambda)] \xrightarrow{\mathbb{P}} \phi(\lambda) = \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R},$$

where $\phi \in \widehat{\mathbf{M}}_0$ and, $\forall \lambda \in \mathbb{R}$,

$$\phi(\lambda) \in \vee_n \sigma\{X(f) : f \in \mathfrak{U}_n\} \triangleq \mathcal{U}^*,$$

then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp \left(i \lambda J_X^{\pi^{(n)}}(u_n) \right) \mid X(f) : f \in \pi_{s \leq t_n}^{(n)} \mathfrak{H}^{(n)} \right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R},$$

and

$$J_X^{\pi^{(n)}}(u_n) \rightarrow_{(s, \mathcal{U}^*)} \mathbb{E} \mu(\cdot),$$

where $\mu \in \mathbf{M}$ verifies (2).

5 Stable limit theorems for multiple integrals with respect independently scattered measures

This section concerns multiple integrals with respect to independently scattered random measures (not necessarily Gaussian) and corresponding limit theorems. In particular, we will use Theorem 7 to obtain new central and non-central limit theorems for these multiple integrals, extending part of the results proved in [19] and [24] in the framework of multiple Wiener-Itô integrals with respect to Gaussian processes. A specific application is described in Section 3.3, where we deal with sequences of double integrals with respect to Poisson random measures. For further applications of the theory developed in Section 2 to the asymptotic analysis of Gaussian fields, the reader is referred to Section 6, as well as to the companion paper [23]. For a general discussion concerning multiple integrals with respect to random measures, see [4] and [29]. For limit theorems involving multiple stochastic integrals (and other related classes of random variables), see the two surveys by Surgailis [33] and [34], and the references therein.

5.1 Independently scattered random measures and multiple integrals

From now on (Z, \mathcal{Z}, μ) stands for a standard Borel space, with μ a positive, non-atomic and σ -finite measure on (Z, \mathcal{Z}) . We denote by \mathcal{Z}_μ the subset of \mathcal{Z} composed of sets of finite μ -measure. Observe that the σ -finiteness of μ implies that $\mathcal{Z} = \sigma(\mathcal{Z}_\mu)$.

Definition E – An *independently scattered random measure* M on (Z, \mathcal{Z}) , with *control measure* μ , is a collection of random variables

$$M = \{M(B) : B \in \mathcal{Z}_\mu\},$$

indexed by the elements of \mathcal{Z}_μ and such that: **(E-i)** for every $B \in \mathcal{Z}_\mu$ $M(B) \in L^2(\mathbb{P})$, **(E-ii)** for every finite collection of disjoint sets $B_1, \dots, B_m \in \mathcal{Z}_\mu$, the vector $(M(B_1), \dots, M(B_m))$ is composed of mutually independent random variables; **(E-iii)** for every $B, C \in \mathcal{Z}_\mu$,

$$\mathbb{E}[M(B)M(C)] = \mu(C \cap B). \quad (52)$$

Let $\mathfrak{H}_\mu = L^2(Z, \mathcal{Z}, \mu)$ be the Hilbert space of real-valued and square-integrable functions on (Z, \mathcal{Z}) (with respect to μ). Since relation (52) holds, it is easily seen that there exists a unique collection of centered and square-integrable random variables

$$X_M = X_M(\mathfrak{H}_\mu) = \{X_M(h) : h \in \mathfrak{H}_\mu\}, \quad (53)$$

such that the following two properties are verified: (a) for every elementary function $h \in \mathfrak{H}_\mu$ with the form $h(z) = \sum_{i=1, \dots, n} c_i \mathbf{1}_{B_i}(z)$, where $n = 1, 2, \dots$, $c_i \in \mathbb{R}$ and $B_i \in \mathcal{Z}_\mu$ are disjoint, $X_M(h) = \sum_{i=1, \dots, n} c_i M(B_i)$, and (b) for every $h, h' \in \mathfrak{H}_\mu$

$$\mathbb{E}[X_M(h)X_M(h')] = (h, h')_{\mathfrak{H}_\mu} \triangleq \int_Z h(z)h'(z)\mu(dz). \quad (54)$$

Property (a) implies in particular that, $\forall B \in \mathcal{Z}_\mu$, $M(B) = X_M(\mathbf{1}_B)$. Note that X_M is a collection of random variables of the kind defined in formula (17) of Section 3. Moreover, for every $h \in \mathfrak{H}_\mu$, the random variable $X_M(h)$ has an infinitely divisible law. It follows that, for every $h \in \mathfrak{H}_\mu$, there exists a unique pair $(c^2(h), \nu_h)$ such that $c^2(h) \in [0, +\infty)$ and ν_h is a (Lévy) measure on \mathbb{R} satisfying the three properties in (20), so that, for every $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(i\lambda X_M(h))] = \exp[\psi_{\mathfrak{H}_\mu}(h; \lambda)], \quad (55)$$

where the Lévy-Khinchine exponent $\psi_{\mathfrak{H}_\mu}(h; \lambda)$ is defined by (22).

We now give a characterization of $\psi_{\mathfrak{H}_\mu}(h; \lambda)$, based on the techniques developed in [27] (but see also [14, Section 5]).

Proposition 13 For every $B \in \mathcal{Z}_\mu$, let $(c^2(B), \nu_B)$ denote the pair such that $c^2(B) \in [0, +\infty)$, ν_B verifies (20) and

$$\psi_{\mathfrak{H}_\mu}(\mathbf{1}_B; \lambda) = -\frac{\lambda^2}{2}c^2(B) + \int_{\mathbb{R}} (\exp(i\lambda x) - 1 - i\lambda x) \nu_B(dx). \quad (56)$$

Then,

1. The application $B \mapsto c^2(B)$, from \mathcal{Z}_μ to $[0, +\infty)$, extends to a unique σ -finite measure $c^2(dz)$ on (Z, \mathcal{Z}) , such that $c^2(dz) \ll \mu(dz)$.
2. There exists a unique measure ν on $(Z \times \mathbb{R}, \mathcal{Z} \times \mathcal{B}(\mathbb{R}))$ such that $\nu(B \times C) = \nu_B(C)$, for every $B \in \mathcal{Z}_\mu$ and $C \in \mathcal{B}(\mathbb{R})$.
3. There exists a function $\rho_\mu : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty]$ such that (i) for every $z \in Z$, $\rho_\mu(z, \cdot)$ is a Lévy measure¹ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $\int_{\mathbb{R}} x^2 \rho_\mu(z, dx) < +\infty$, (ii) for every $C \in \mathcal{B}(\mathbb{R})$, $\rho_\mu(\cdot, C)$ is a Borel measurable function, (iii) for every positive function $g(z, x) \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$,

$$\int_Z \int_{\mathbb{R}} g(z, x) \rho_\mu(z, dx) \mu(dz) = \int_Z \int_{\mathbb{R}} g(z, x) \nu(dz, dx). \quad (57)$$

4. For every $(\lambda, z) \in \mathbb{R} \times Z$, define

$$K_\mu(\lambda, z) = -\frac{\lambda^2}{2}\sigma_\mu^2(z) + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x) \rho_\mu(z, dx), \quad (58)$$

where $\sigma_\mu^2(z) = \frac{dc^2}{d\mu}(z)$, then, for every $h \in \mathfrak{H}_\mu = L^2(Z, \mathcal{Z}, \mu)$, $\int_Z |K_\mu(\lambda h(z), z)| \mu(dz) < +\infty$ and the exponent $\psi_{\mathfrak{H}_\mu}$ in (55) is given by

$$\begin{aligned} \psi_{\mathfrak{H}_\mu}(h; \lambda) &= \int_Z K_\mu(\lambda h(z), z) \mu(dz) \\ &= -\frac{\lambda^2}{2} \int_Z \sigma_\mu^2(z) \mu(dz) + \int_Z \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x) \rho_\mu(z, dx) \mu(dz) \end{aligned} \quad (59)$$

Proof. The proof follows from results contained in [27, Section II]. Point 1 is indeed a direct consequence of [27, Proposition 2.1 (a)]. In particular, whenever $B \in \mathcal{Z}$ is such that $\mu(B) = 0$, then $M(B) = 0$, a.s.- \mathbb{P} (by applying (52) with $B = C$), and therefore $c^2(B) = 0$, thus implying $c^2 \ll \mu$. Point 2 follows from the first part of the statement of [27, Lemma 2.3]. To establish Point 3 define, as in [27, p. 456],

$$\gamma(A) = c^2(A) + \int_{\mathbb{R}} \min(1, x^2) \nu_A(dx),$$

whenever $A \in \mathcal{Z}_\mu$, and observe (see [27, Definition 2.2]) that $\gamma(\cdot)$ can be canonically extended to a σ -finite and positive measure on (Z, \mathcal{Z}) . Moreover, since $\mu(B) = 0$ implies $M(B) = 0$ a.s.- \mathbb{P} , the uniqueness of the Lévy-Khinchine characteristics implies as before $\gamma(A) = 0$, and therefore $\gamma(dz) \ll \mu(dz)$. Observe also that, by standard arguments, one can select a version of the density $(d\gamma/d\mu)(z)$ such that $(d\gamma/d\mu)(z) < +\infty$ for every $z \in Z$. According to [27, Lemma 2.3], there exists a function $\rho : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty]$, such that: (a) $\rho(z, \cdot)$ is a Lévy measure on $\mathcal{B}(\mathbb{R})$ for every $z \in Z$, (b) $\rho(\cdot, C)$ is a Borel measurable function for every $C \in \mathcal{B}(\mathbb{R})$, (c) for every positive function $g(z, x) \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R})$,

$$\int_Z \int_{\mathbb{R}} g(z, x) \rho(z, dx) \gamma(dz) = \int_Z \int_{\mathbb{R}} g(z, x) \nu(dz, dx). \quad (60)$$

In particular, by using (60) in the case $g(z, x) = \mathbf{1}_A(z) x^2$ for $A \in \mathcal{Z}_\mu$,

$$\int_A \int_{\mathbb{R}} x^2 \rho(z, dx) \gamma(dz) = \int_{\mathbb{R}} x^2 \nu_A(dx) < +\infty,$$

¹That is, $\rho_\mu(z, \{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, x^2) \rho_\mu(z, dx) < +\infty$

since $M(A) \in L^2(\mathbb{P})$, and we deduce that ρ can be chosen in such a way that, for every $z \in Z$, $\int_{\mathbb{R}} x^2 \rho(z, dx) < +\infty$. Now define, for every $z \in Z$ and $C \in \mathcal{B}(\mathbb{R})$,

$$\rho_\mu(z, C) = \frac{d\gamma}{d\mu}(z) \rho(z, C),$$

and observe that, due to the previous discussion, the application $\rho_\mu : Z \times \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty]$ trivially satisfies properties (i)-(iii) in the statement of Point 3, which is therefore proved. To Prove point 4, first define a function $h \in \mathfrak{H}_\mu$ to be *simple* if $h(z) = \sum_{i=1}^n a_i \mathbf{1}_{A_i}(z)$, where $a_i \in \mathbb{R}$, and (A_1, \dots, A_n) is a finite collection of disjoint elements of \mathcal{Z}_μ . Of course, the class of simple functions (which is a linear space) is dense in \mathfrak{H}_μ , and therefore for every $h \in \mathfrak{H}_\mu$ there exists a sequence h_n , $n \geq 1$, of simple functions such that $\int_Z (h_n(z) - h(z))^2 \mu(dz) \rightarrow 0$. As a consequence, since μ is σ -finite there exists a subsequence n_k such that $h_{n_k}(z) \rightarrow h(z)$ for μ -a.e. $z \in Z$ (and therefore for γ -a.e. $z \in Z$) and moreover, for every $A \in \mathcal{Z}$, the random sequence $X_M(\mathbf{1}_A h_n)$ (where we use the notation (53)) is a Cauchy sequence in $L^2(\mathbb{P})$, and hence it converges in probability. In the terminology of [27, p. 460], this implies that every $h \in \mathfrak{H}_\mu$ is M -integrable, and that, for every $A \in \mathcal{Z}$, the random variable $X_M(h \mathbf{1}_A)$, defined according to (53), coincides with $\int_A h(z) M(dz)$, i.e. the integral of h with respect to the restriction of $M(\cdot)$ to A , as defined in [27, p. 460]. As a consequence, by using a slight modification of [27, Proposition 2.6]², the function K_0 on $\mathbb{R} \times Z$ given by

$$K_0(\lambda, z) = -\frac{\lambda^2}{2} \sigma_0^2(z) + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x) \rho(z, dx),$$

where $\sigma_0^2(z) = (dc^2/d\gamma)(z)$, is such that $\int_Z |K_0(\lambda h(z), z)| \gamma(dz) < +\infty$ for every $h \in \mathfrak{H}_\mu$, and also

$$\mathbb{E}[\exp(i\lambda X_M(h))] = \int_Z K_0(\lambda h(z), z) \gamma(dz).$$

Relation (55) and the fact that, by definition,

$$K_\mu(\lambda h(z), z) = K_0(\lambda h(z), z) \frac{d\gamma}{d\mu}(z), \quad \forall z \in Z, \forall h \in \mathfrak{H}_\mu, \forall \lambda \in \mathbb{R},$$

yield (59). ■

Examples – (a) If M is a centered Gaussian measure with control μ , then $\nu = 0$ and, for $h \in \mathfrak{H}_\mu$,

$$\psi_{\mathfrak{H}_\mu}(h; \lambda) = -\frac{\lambda^2}{2} \int_Z h^2(z) \mu(dz).$$

(b) If M is a centered Poisson measure with control μ , then $c^2(\cdot) = 0$ and $\rho_\mu(z, dx) = \delta_1(dx)$ for all $z \in Z$, where δ_1 is the Dirac mass at x , and therefore, for $h \in \mathfrak{H}_\mu$,

$$\psi_{\mathfrak{H}_\mu}(h; \lambda) = \int_Z \left(e^{i\lambda h(z)} - 1 - i\lambda h(z) \right) \mu(dz).$$

For instance, one can take $Z = [0, +\infty) \times \mathbb{R} \times \mathbb{R}$, and $\mu(dx, du, dw) = dx du \nu(dw)$, where $\nu(dw) = \mathbf{1}_{|w| < 1} |w|^{-(1+\alpha)} dw$ and $\alpha \in (0, 2)$. In this case, the centered Poisson measure M generates the (standard) *Poissonized Telecom process* $\{Y_{P,\alpha}(t) : t \geq 0\}$, defined in [3, Section 4.1] as

$$Y_{P,\alpha}(t) = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} \left(((t+u) \wedge 0 + x)_+ - (u \wedge 0 + x)_+ \right) x^{-(1-\kappa)-1/\alpha} w M(dx, du, dw),$$

²The difference lies in the choice of the truncation.

with $\kappa \in (0, 1 - 1/\alpha)$.

We now want to define multiple integrals, of functions vanishing on diagonals, with respect to the random measure M . To this end, fix $d \geq 2$ and set μ^d to be the canonical product measure on (Z^d, \mathcal{Z}^d) induced by μ . We introduce the following standard notation: (i) $L^2(\mu^d) \triangleq L^2(Z^d, \mathcal{Z}^d, \mu^d)$ is the class of real-valued and square-integrable functions on (Z^d, \mathcal{Z}^d) ; (ii) $L_s^2(\mu^d)$ is the subset of $L^2(\mu^d)$ composed of square integrable and symmetric functions; (iii) $L_{s,0}^2(\mu^d)$ is the subset of $L_s^2(\mu^d)$ composed of square integrable and symmetric functions vanishing on diagonals.

Now define $\mathcal{S}_{s,0}(\mu^d)$ to be subset of $L_{s,0}^2(\mu^d)$ composed of functions with the form

$$f(z_1, \dots, z_d) = \sum_{\sigma \in \mathfrak{S}_d} \mathbf{1}_{B_1}(z_{\sigma(1)}) \cdots \mathbf{1}_{B_d}(z_{\sigma(d)}), \quad (61)$$

where $B_1, \dots, B_d \in \mathcal{Z}_\mu$ are pairwise disjoint sets, and \mathfrak{S}_d is the group of all permutations of $\{1, \dots, d\}$. Recall (see e.g. [29, Proposition 3]) that $\mathcal{S}_{s,0}(\mu^d)$ is total in $L_{s,0}^2(\mu^d)$. For $f \in L_{s,0}^2(\mu^d)$ as in (61), we set

$$I_d^M(f) = d! M(B_1) \times M(B_2) \times \cdots \times M(B_d) \quad (62)$$

to be the *multiple integral*, of order d , of f with respect to M . It is well known (see for instance [29, Theorem 5]) that there exists a unique linear extension of I_d^M , from $\mathcal{S}_{s,0}(\mu^d)$ to $L_{s,0}^2(\mu^d)$, satisfying the following: (a) for every $f \in L_{s,0}^2(\mu^d)$, $I_d^M(f)$ is a centered and square-integrable random variable, and (b) for every $f, g \in L_{s,0}^2(\mu^d)$

$$\mathbb{E}[I_d^M(f) I_d^M(g)] = d! (f, g)_{L^2(\mu^d)} \triangleq d! \int_{Z^d} f(\mathbf{z}_d) g(\mathbf{z}_d) \mu^d(d\mathbf{z}_d),$$

where $\mathbf{z}_d = (z_1, \dots, z_d)$ stands for a generic element of Z^d . Note that, by construction, if $d \neq d'$, $\mathbb{E}[I_d^M(f) I_{d'}^M(g)] = 0$ for every $f \in L_{s,0}^2(\mu^d)$ and every $g \in L_{s,0}^2(\mu^{d'})$. Again, for $f \in L_{s,0}^2(\mu^d)$, $I_d^M(f)$ is called the multiple integral, of order d , of f with respect to M . When $f \in L_s^2(\mu^d)$ (hence, f does not necessarily vanish on diagonals) we define

$$I_d^M(f) \triangleq I_d^M(f \mathbf{1}_{Z_0^d}), \quad (63)$$

where

$$Z_0^d \triangleq \{(z_1, \dots, z_d) \in Z^d : \text{the } z_j\text{'s are all different}\}, \quad (64)$$

so that (since μ is non atomic, and therefore the product measures do not charge diagonals), for every $f, g \in L^2(\mu^d)$, $\mathbb{E}[I_d^M(f) I_d^M(g)] = d! \int_{Z_0^d} f(\mathbf{z}_d) g(\mathbf{z}_d) \mu^d(d\mathbf{z}_d) = d! (f, g)_{L^2(\mu^d)}$. Note that, for $d = 1$, one usually sets $L_{s,0}^2(\mu^1) = L_s^2(\mu^1) = L^2(\mu^1) = \mathfrak{H}_\mu$, and $I_1^M(f) = X_M(f)$, $f \in \mathfrak{H}_\mu$.

In what follows, we shall show that, for some well chosen resolutions $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$, every multiple integral of the type $I_d^M(f)$, $f \in L_{s,0}^2(\mu^d)$, can be represented in the form of a generalized adapted integral of the kind introduced in Section 3. As a consequence, the asymptotic behavior of $I_d^M(f)$ can be studied by means of Theorem 7.

5.2 Representation of multiple integrals and limit theorems

Under the notation and assumptions of this section, consider a ‘‘continuous’’ increasing family $\{Z_t : t \in [0, 1]\}$ of elements of \mathcal{Z} , such that $Z_0 = \emptyset$, $Z_1 = Z$, $Z_s \subseteq Z_t$ for $s < t$, and, for every $g \in L^1(\mu)$ and every $t \in [0, 1]$,

$$\lim_{s \rightarrow t} \int_{Z_s} g(x) \mu(dx) = \int_{Z_t} g(x) \mu(dx). \quad (65)$$

For example, for $Z = [0, 1]^2$, one can take $Z_t = [0, t]^2$ or $Z_t = [(1-t)/2, (1+t)/2]^2$. To each $t \in [0, 1]$, we associate the following projection operator $\pi_t : \mathfrak{H}_\mu \mapsto \mathfrak{H}_\mu : \forall f \in \mathfrak{H}_\mu$,

$$\pi_t f(z) = \mathbf{1}_{Z_t}(z) f(z), \quad z \in Z, \quad (66)$$

so that, since M is independently scattered, the continuous resolution of the identity $\pi = \{\pi_t : t \in [0, 1]\}$ is such that, $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$. Note also that, thanks to (22) and by uniform continuity, for every $f \in \mathfrak{H}_\mu$, every $t \in (0, 1]$ and every sequence of partitions of $[0, t]$,

$$t^{(n)} = \left\{ 0 = t_0^{(n)} < t_1^{(n)} < \dots < t_{r_n}^{(n)} = t \right\}, \quad n \geq 1, \quad (67)$$

such that $\text{mesh}(t^{(n)}) \triangleq \max_{i=0, \dots, r_n-1} (t_{i+1}^{(n)} - t_i^{(n)}) \rightarrow 0$,

$$\max_{i=0, \dots, r_n-1} \left\| \left(\pi_{t_{i+1}^{(n)}} - \pi_{t_i^{(n)}} \right) f \right\|_{\mathfrak{H}_\mu}^2 \rightarrow 0, \quad (68)$$

and in particular, for every $B \in \mathcal{Z}_\mu$,

$$\max_{i=0, \dots, r_n-1} \mu \left(B \cap \left(Z_{t_{i+1}^{(n)}} \setminus Z_{t_i^{(n)}} \right) \right) \rightarrow 0. \quad (69)$$

The following result contains the key of the subsequent discussion.

Proposition 14 *For every $d \geq 2$, every random variable of the form $I_d^M(f \mathbf{1}_{Z_t^d})$, for some $f \in L_{s,0}^2(\mu^d)$ and $t \in (0, 1]$, can be approximated in $L^2(\mathbb{P})$ by linear combinations of random variables of the type*

$$M(B_1 \cap Z_{t_1}) \times M(B_2 \cap (Z_{t_2} \setminus Z_{t_1})) \times \dots \times M(B_d \cap (Z_{t_d} \setminus Z_{t_{d-1}})), \quad (70)$$

where the t_1, \dots, t_d are rational, $0 \leq t_1 < t_2 < \dots < t_d \leq t$ and $B_1, \dots, B_d \in \mathcal{Z}_\mu$ are disjoint. In particular, $I_d^M(f \mathbf{1}_{Z_t^d}) \in \mathcal{F}_t^\pi$, where the filtration \mathcal{F}_t^π , $t \in [0, 1]$, is defined as in (19).

Remark – Observe that, if $f \in \mathcal{S}_{s,0}(\mu^d)$ is such that

$$f(z_1, \dots, z_d) = \sum_{\sigma \in \mathfrak{S}_d} \mathbf{1}_{B_1 \cap Z_{t_1}}(z_{\sigma(1)}) \cdots \mathbf{1}_{B_d \cap (Z_{t_d} \setminus Z_{t_{d-1}})}(z_{\sigma(d)}), \quad (71)$$

then, by (62),

$$d! M(B_1 \cap Z_{t_1}) \times M(B_2 \cap (Z_{t_2} \setminus Z_{t_1})) \times \dots \times M(B_d \cap (Z_{t_d} \setminus Z_{t_{d-1}})) = I_d^M(f). \quad (72)$$

Proof. Observe first that, for every $f \in L_{s,0}^2(\mu^d)$, every $t \in (0, 1]$ and every sequence of rational numbers $t_n \rightarrow t$, $I_d^M(f \mathbf{1}_{Z_{t_n}^d}) \rightarrow I_d^M(f \mathbf{1}_{Z_t^d})$ in $L^2(\mathbb{P})$. By density, it is therefore sufficient to prove the statement for multiple integrals of the type $I_d^M(f \mathbf{1}_{Z_t^d})$, where $t \in \mathbb{Q} \cap (0, 1]$ and $f \in \mathcal{S}_{s,0}(\mu^d)$ is as in (61). Start with $d = 2$. In this case,

$$\frac{1}{2} I_2^M(f \mathbf{1}_{Z_t^2}) = M(B_1 \cap Z_t) M(B_2 \cap Z_t)$$

with B_1, B_2 disjoint, and also, for every partition $\{0 = t_0 < t_1 < \dots < t_r = t\}$ (with $r \geq 1$) of $[0, t]$,

$$\begin{aligned} \frac{1}{2} I_2^M(f) &= \sum_{i=1}^r M(B_1 \cap (Z_{t_i} \setminus Z_{t_{i-1}})) \sum_{j=1}^r M(B_2 \cap (Z_{t_j} \setminus Z_{t_{j-1}})) \\ &= \sum_{1 \leq i \neq j \leq r} M(B_1 \cap (Z_{t_i} \setminus Z_{t_{i-1}})) M(B_2 \cap (Z_{t_j} \setminus Z_{t_{j-1}})) + \\ &\quad + \sum_{i=1}^r M(B_1 \cap (Z_{t_i} \setminus Z_{t_{i-1}})) M(B_2 \cap (Z_{t_i} \setminus Z_{t_{i-1}})) \triangleq \Sigma_1 + \Sigma_2. \end{aligned}$$

The summands in the first sum Σ_1 have the desired form (70). It is therefore sufficient to prove that for every sequence of partitions $t^{(n)}$, $n \geq 1$, as in (67) and such that $\text{mesh}(t^{(n)}) \rightarrow 0$ and the $t_1^{(n)}, \dots, t_{r_n}^{(n)}$ are rational,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\sum_{i=1}^{r_n} M \left(B_1 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) M \left(B_2 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) \right)^2 \right] = 0. \quad (73)$$

Since B_1 and B_2 are disjoint, and thanks to the isometric properties of M ,

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{i=1}^{r_n} M \left(B_1 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) M \left(B_2 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) \right)^2 \right] \\ &= \sum_{i=1}^{r_n} \mathbb{E} \left[M \left(B_1 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right)^2 M \left(B_2 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right)^2 \right] \\ &= \sum_{i=1}^{r_n} \mu \left(B_1 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) \mu \left(B_2 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) \\ &\leq \mu(B_1) \max_{i=1, \dots, r_n} \mu \left(B_2 \cap \left(Z_{t_i^{(n)}} \setminus Z_{t_{i-1}^{(n)}} \right) \right) \rightarrow 0, \end{aligned}$$

thanks to (69). Now fix $d \geq 3$, and consider a random variable of the type

$$F = M(B_1 \cap Z_t) \times \cdots \times M(B_{d-1} \cap Z_t) \times M(B_d \cap Z_t), \quad (74)$$

where $B_1, \dots, B_d \in \mathcal{Z}_\mu$ are disjoint. The above discussion yields that F can be approximated by linear combinations of random variables of the type

$$\begin{aligned} & M(B_1 \cap Z_t) \times \cdots \times M(B_{d-3} \cap Z_t) \times \\ & \times [M(B_{d-2}) \times M(B_{d-1} \cap (Z_s \setminus Z_r)) \times M(B_d \cap (Z_v \setminus Z_u))], \end{aligned} \quad (75)$$

where $r < s < u < v \leq t$ are rational. We will proceed by induction focusing first on the terms in the brackets in (75). Express Z_t as the union of five disjoint sets $Z_t = (Z_t \setminus Z_v) \cup (Z_v \setminus Z_u) \cup (Z_u \setminus Z_s) \cup (Z_s \setminus Z_r) \cup Z_r$, and decompose $M(B_{d-2} \cap Z_t)$ accordingly. One gets

$$\begin{aligned} & M(B_{d-2} \cap Z_t) M(B_{d-1} \cap (Z_t \setminus Z_r)) M(B_d \cap (Z_v \setminus Z_u)) \\ &= M(B_{d-2} \cap (Z_s \setminus Z_r)) M(B_{d-1} \cap Z_s \setminus Z_r) M(B_d \cap (Z_v \setminus Z_u)) \\ & \quad + M(B_{d-2} \cap (Z_v \setminus Z_u)) M(B_{d-1} \cap Z_s \setminus Z_r) M(B_d \cap (Z_v \setminus Z_u)) \\ & \quad + M(B_{d-2} \cap (Z_u \setminus Z_s)) M(B_{d-1} \cap Z_s \setminus Z_r) M(B_d \cap (Z_v \setminus Z_u)) \\ & \quad + M(B_{d-2} \cap (Z_t \setminus Z_v)) M(B_{d-1} \cap Z_s \setminus Z_r) M(B_d \cap (Z_v \setminus Z_u)) \\ & \quad + M(B_{d-2} \cap (Z_r \setminus Z_0)) M(B_{d-1} \cap Z_s \setminus Z_r) M(B_d \cap (Z_v \setminus Z_u)). \end{aligned} \quad (76)$$

Observe that the last three summands involve disjoint subsets of Z and hence are of the form (70). Since each of the first two summands involve two identical subsets of Z (e.g. $(Z_s \setminus Z_r)$) and a disjoint subset (e.g. $(Z_s \setminus Z_r)$), they can be dealt with in the same way as (73) above. Thus, linear combinations of the five summands on the RHS of (76) can be approximated by linear combinations of random variables of the type

$$M(C_1 \cap (Z_{t_2} \setminus Z_{t_1})) M(C_2 \cap (Z_{t_3} \setminus Z_{t_2})) M(C_3 \cap (Z_{t_3} \setminus Z_{t_2})),$$

where $C_1, C_2, C_3 \in \mathcal{Z}_\mu$ are disjoint, and $t_1 < t_2 < t_3 \leq t$ are rational. The general result is obtained by recurrence. ■

Proposition 14 will be used to prove that, whenever there exists $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_\mu)$ defined as in formula (66), multiple integrals can be represented as generalized adapted integrals of the kind described in Section 3. To do this, we introduce a partial ordering on Z as follows: for every $z, z' \in Z$,

$$z \prec_\pi z' \quad (77)$$

if, and only if, there exists $t \in \mathbb{Q} \cap (0, 1)$ such that $z \in Z_t$ and $z' \in Z_t^c$, where Z_t^c stands for the complement of Z_t . For a fixed $d \geq 2$, we define the π -purely non-diagonal subset of Z^d as

$$Z_{\pi,0}^d = \{(z_1, \dots, z_d) \in Z^d : z_{\sigma(1)} \prec_{\pi} z_{\sigma(2)} \prec_{\pi} \dots \prec_{\pi} z_{\sigma(d)}, \text{ for some } \sigma \in \mathfrak{S}_d\}.$$

Note that $Z_{\pi,0}^d \in \mathcal{Z}^d$, and also that not every pair of distinct points of Z can be ordered, that is, in general, $Z_{\pi,0}^d \neq Z_0^d$, where $d \geq 2$ and Z_0^d is defined in (64) (for illustration, think of $Z = [0, 1]^2$, $Z_t = [0, t]^2$, $t \in [0, 1]$; indeed $((1/8, 1/4), (1/4, 1/4)) \in Z_0^2$, but $(1/4, 1/4)$ and $(1/8, 1/4)$ cannot be ordered). However, because of the continuity condition (65) and for every $d \geq 2$, the class of the elements of Z_0^d whose components cannot be ordered has measure μ^d equal to zero, as indicated by the following corollary.

Corollary 15 *For every $d \geq 2$ and every $f \in L_{s,0}^2(\mu^d)$,*

$$I_d^M(f) = I_d^M(f \mathbf{1}_{Z_{\pi,0}^d}).$$

As a consequence, $\mu^d(Z_0^d \setminus Z_{\pi,0}^d) = 0$, where Z_0 is defined in (64).

Proof. First observe that the class of r.v.'s of the type $I_d^M(f \mathbf{1}_{Z_{\pi,0}^d})$, $f \in L_{s,0}^2(\mu^d)$ is a closed vector space. Plainly, every $f \in \mathcal{S}_{s,0}(\mu^d)$ with the form (71) is such that $f(\mathbf{z}_d) = f(\mathbf{z}_d) \mathbf{1}_{Z_{\pi,0}^d}(\mathbf{z}_d)$ for every $\mathbf{z}_d \in Z^d$. Since, by Proposition 14 and relation (72), the class of functions of the type (71) are total in $L_{s,0}^2(\mu^d)$, the result is obtained by a density argument. The last assertion follows from the facts that $\forall f \in L_{s,0}^2(\mu^d)$ one has $f = f \mathbf{1}_{Z_{\pi,0}^d}$, a.e.- μ^d , and $I_d^M(f) = I_d^M(g)$ if and only if $f = g$, a.e.- μ^d . ■

For $\pi \in \mathcal{R}_{X_M}(\mathfrak{H}_{\mu})$ as in formula (66), the vector spaces $L_{\pi}^2(\mathfrak{H}_{\mu}, X_M)$ and $\mathcal{E}_{\pi}(\mathfrak{H}_{\mu}, X_M)$, composed respectively of adapted and elementary adapted elements of $L^2(\mathfrak{H}_{\mu}, X_M)$, are defined as in Section 3 (in particular, *via* formulae (24) and (25)). Recall that, according to Lemma 3, the closure of $\mathcal{E}_{\pi}(\mathfrak{H}_{\mu}, X_M)$ coincides with $L_{\pi}^2(\mathfrak{H}_{\mu}, X_M)$. For every $h \in L_{\pi}^2(\mathfrak{H}_{\mu}, X_M)$, the random variable $J_{X_M}^{\pi}(h)$ is defined by means of Proposition 4 and formula (27). The following result states that every multiple integral with respect to M is indeed a generalized adapted integral of the form $J_{X_M}^{\pi}(h)$, for some $h \in L_{\pi}^2(\mathfrak{H}_{\mu}, X_M)$. In what follows, for every $d \geq 1$, every $f \in L_{s,0}^2(\mu^d)$ and every fixed $z \in Z$, the symbol $f(z, \cdot) \mathbf{1}(\cdot \prec_{\pi} z)$ stands for the element of $L_{s,0}^2(\mu^{d-1})$, given by

$$(z_1, \dots, z_{d-1}) \mapsto f(z, z_1, \dots, z_{d-1}) \prod_{j=1}^{d-1} \mathbf{1}_{(z_j \prec_{\pi} z)}. \quad (78)$$

Proposition 16 *Fix $d \geq 2$, and let $f \in L_{s,0}^2(\mu^d)$. Then,*

1. *the random function*

$$z \mapsto h_{\pi}(f)(z) = d \times I_{d-1}^M(f(z, \cdot) \mathbf{1}(\cdot \prec_{\pi} z)), \quad z \in Z, \quad (79)$$

is an element of $L_{\pi}^2(\mathfrak{H}_{\mu}, X_M)$;

2. $I_M^d(f) = J_{X_M}^{\pi}(h_{\pi}(f))$, *where $h_{\pi}(f)$ is defined as in (79).*

Moreover, if a random variable $F \in L^2(\mathbb{P})$ has the form $F = \sum_{d=1}^{\infty} I_M^d(f^{(d)})$, where $f^{(d)} \in L_{s,0}^2(\mu^d)$ for $d \geq 1$ and the series is convergent in $L^2(\mathbb{P})$, then

$$F = J_{X_M}^{\pi}(h_{\pi}(F)), \quad (80)$$

where

$$h_\pi(F)(z) = \sum_{d=1}^{\infty} h_\pi(f^{(d)})(z), \quad z \in Z, \quad (81)$$

and the series in (81) is convergent in $L_\pi^2(\mathfrak{H}_\mu, X_M)$.

Proof. It is clear that $h_\pi(f) \in L^2(\mathfrak{H}_\mu, X_M)$ (the class of square integrable, but not necessarily adapted processes). Now observe that, thanks to Proposition 14, if $g \in L_{s,0}^2(\mu^d)$ has support in Z_t^d for some $t \in (0, 1]$, then $I_M^d(g) \in \mathcal{F}_t^\pi$. As a consequence, since for any fixed $z \in Z_t$, $t \in (0, 1]$, the symmetric function (on Z^{d-1}) $f(z, \cdot) \mathbf{1}(\cdot \prec_\pi z)$ has support in Z_t^d , for every $b \in \mathfrak{H}_\mu$ and $t \in (0, 1]$,

$$(h_\pi(f), \pi_t b)_{\mathfrak{H}_\mu} = \int_{Z_t} h_\pi(f)(z) b(z) \mu(dz) = d \int_{Z_t} b(z) I_{d-1}^M(f(z, \cdot) \mathbf{1}(\cdot \prec_\pi z)) \mu(dz) \in \mathcal{F}_t^\pi,$$

and therefore $h_\pi(f) \in L_\pi^2(\mathfrak{H}_\mu, X_M)$. This proves Point 1. By density, it is sufficient to prove Point 2 for random variables of the type $I_M^d(f)$, where $f \in \mathcal{S}_{s,0}(\mu^d)$ is as in (71). Indeed, for such an f and for every $(z, z_1, \dots, z_{d-1}) \in Z^d$

$$\begin{aligned} & f(z, z_1, \dots, z_{d-1}) \prod_{j=1}^{d-1} \mathbf{1}_{(z_j \prec_\pi z)} \\ &= \sum_{\sigma \in \mathfrak{S}_{d-1}} \mathbf{1}_{B_1 \cap Z_{t_1}}(z_{\sigma(1)}) \cdots \mathbf{1}_{B_{d-1} \cap (Z_{t_{d-1}} \setminus Z_{t_{d-2}})}(z_{\sigma(d-1)}) \mathbf{1}_{B_d \cap (Z_{t_d} \setminus Z_{t_{d-1}})}(z), \end{aligned}$$

so that

$$d \times h_\pi(f)(z) = d(d-1)! M(B_1 \cap Z_{t_1}) \times \cdots \times M(B_{d-1} \cap (Z_{t_{d-1}} \setminus Z_{t_{d-2}})) \mathbf{1}_{B_d \cap (Z_{t_d} \setminus Z_{t_{d-1}})}(z),$$

and finally, thanks to (27) and (72),

$$J_{X_M}^\pi(h_\pi(f)) = d! M(B_1 \cap Z_{t_1}) \times \cdots \times M(B_d \cap (Z_{t_d} \setminus Z_{t_{d-1}})) = I_M^d(f).$$

The last assertion in the statement is an immediate consequence of the orthogonality relations between multiple integrals of different orders. ■

Remarks – (1) Formula (79) implies that, for $t \in [0, 1]$ and $f \in L_{s,0}^2(\mu^d)$,

$$I_M^d(f \mathbf{1}_{Z_t^d}) = J_{X_M}^\pi(\pi_t h_\pi(f)),$$

and therefore, since $t \mapsto J_{X_M}^\pi(\pi_t h_\pi(f))$ is a \mathcal{F}_t^π -martingale (see Corollary 5),

$$\mathbb{E}[I_M^d(f) | \mathcal{F}_t^\pi] = I_M^d(f \mathbf{1}_{Z_t^d}), \quad t \in [0, 1]. \quad (82)$$

(2) The random process $z \mapsto dI_{d-1}^M(f(z, \cdot)) \triangleq D_z I_d^M(f)$ is a “formal” Malliavin-Shikegawa derivative of the random variable $I_d^M(f)$, whereas $z \mapsto dI_{d-1}^M(f(z, \cdot) \mathbf{1}(\cdot \prec_\pi z))$ is the projection of $D_z I_d^M(f)$ on the space of adapted integrands $L_\pi^2(\mathfrak{H}_\mu, X_M)$. In this sense, formula (80) can be interpreted as a “generalized Clark-Ocone formula”, in the same spirit of the results proved by L. Wu in [36]. See also the discussion contained in Section 6.

We now state the announced convergence result, which is a consequence of Proposition 16 and Theorem 7. In what follows, $(Z_n, \mathcal{Z}_n, \mu_n)$, $n \geq 1$, is a sequence of measurable spaces and, for each n , M_n is an independently scattered random measures on (Z_n, \mathcal{Z}_n) with control μ_n (the M_n 's are defined on the same probability space); also $\mathfrak{H}_{\mu_n} = L^2(Z_n, \mathcal{Z}_n, \mu_n)$. The collection of random variables $X_{M_n} = X_{M_n}(\mathfrak{H}_{\mu_n})$ is defined through formula (53), with Lévy-Khinchine exponent $\psi_{\mathfrak{H}_{\mu_n}}(h, \lambda)$, $h \in \mathfrak{H}_{\mu_n}$, $\lambda \in \mathbb{R}$, given by

(55). Moreover, for every $n \geq 1$, $\pi^{(n)} = \left\{ \pi_t^{(n)} : t \in [0, 1] \right\} \in \mathcal{R}_{X_{M_n}}(\mathfrak{H}_{\mu_n})$ is a continuous resolution of the identity defined as

$$\pi_t^{(n)} h(z) = \mathbf{1}_{Z_{n,t}}(z) h(z), \quad z \in Z, \quad h \in \mathfrak{H}_{\mu_n}, \quad (83)$$

where $Z_{n,t}$, $t \in [0, 1]$ is an increasing collection of measurable sets such that $Z_{n,0} = \emptyset$, $Z_{n,1} = Z_n$ and verifying the continuity condition (65).

Theorem 17 *Under the previous notation and assumptions, let d_n , $n \geq 1$, be a sequence of natural numbers such that $d_n \geq 1$, and let $\pi^{(n)} \in \mathcal{R}_{X_{M_n}}(\mathfrak{H}_{\mu_n})$ be as in (83). Let moreover $f_{d_n}^{(n)} \in L^2_{s,0}(\mu_n^{d_n})$, $n \geq 1$, and suppose there exists a sequence $\{t_n : n \geq 1\} \subset [0, 1]$ and σ -fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\lim_{n \rightarrow +\infty} d_n! \left\| f_{d_n}^{(n)} \mathbf{1}_{Z_{n,t_n}^{d_n}} \right\|_{L^2(\mu_n^{d_n})}^2 = 0 \quad (84)$$

and

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}_{t_n}^{\pi^{(n)}}(X_{M_n}). \quad (85)$$

Define also $h_{\pi^{(n)}}(f_{d_n}^{(n)}) \in L^2_{\pi^{(n)}}(\mathfrak{H}_{\mu_n}, X_{M_n})$ via formula (79) when $d_n \geq 2$, and set $h_{\pi^{(n)}}(f_{d_n}^{(n)}) = f_{d_n}^{(n)}$ when $d_n = 1$. If

$$\exp \left[\int_{Z_n} K_{\mu_n} \left(\lambda h_{\pi^{(n)}}(f_{d_n}^{(n)})(z), z \right) \mu_n(dz) \right] \xrightarrow{\mathbb{P}} \phi(\lambda, \omega), \quad \forall \lambda \in \mathbb{R}, \quad (86)$$

where $K_{\mu_n}(t, z)$, $(t, z) \in \mathbb{R} \times Z$, is given by (58), $\phi \in \widehat{\mathbf{M}}_0$ and $\phi(\lambda) \in \vee_n \mathcal{U}_n \triangleq \mathcal{U}^*$, then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp \left(i \lambda I_{d_n}^{M_n} \left(f_{d_n}^{(n)} \right) \right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_{M_n}) \right] \xrightarrow{\mathbb{P}} \phi(\lambda), \quad \forall \lambda \in \mathbb{R}, \quad (87)$$

and

$$I_{d_n}^{M_n} \left(f_{d_n}^{(n)} \right) \rightarrow_{(s, \mathcal{U}^*)} \mathbb{E} \mu(\cdot), \quad (88)$$

where $\mu \in \mathbf{M}$ is as in (2).

Proof. It is sufficient to observe that, thanks to Proposition 16, $I_{d_n}^{M_n} \left(f_{d_n}^{(n)} \right) = J_{X_{M_n}}^{\pi^{(n)}} \left(h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right) \right)$, $n \geq 1$. As a consequence, by using (82),

$$\begin{aligned} d_n! \left\| f_{d_n}^{(n)} \mathbf{1}_{Z_{n,t_n}^{d_n}} \right\|_{L^2(\mu_n^{d_n})}^2 &= \mathbb{E} \left[I_{d_n}^{M_n} \left(f_{d_n}^{(n)} \right)^2 \right] \\ &= \mathbb{E} \left[J_{X_{M_n}}^{\pi^{(n)}} \left(\pi_{t_n}^{(n)} h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right) \right)^2 \right] \\ &= \left\| \pi_{t_n}^{(n)} h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right) \right\|_{L^2_{\pi^{(n)}}(\mathfrak{H}_{\mu_n}, X_{M_n})}^2. \end{aligned}$$

Moreover, according to Proposition 13,

$$\int_{Z_n} K_{\mu_n} \left(\lambda h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right) (z), z \right) \mu_n(dz) = \psi_{\mathfrak{H}_{\mu_n}} \left(h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right), \lambda \right).$$

The conclusion is now a direct consequence of Theorem 7. ■

Remark – Starting from Theorem 17, one can prove an analogous of Corollary 8 (for nested resolutions) and Corollary 9 (for non random $\phi(\lambda)$). Moreover, Theorem 17 can be immediately extended

to sequences of random variables of the type $F_n = \sum_{d=1}^{\infty} I_{M_n}^d \left(f_n^{(d)} \right)$, $n \geq 1$, by using the last part of Proposition 16 (just replace $h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right)$ with $h_{\pi^{(n)}} (F_n)$).

Condition (86) can be difficult to verify, since it involves the sequence of random integrands $h_{\pi^{(n)}} \left(f_{d_n}^{(n)} \right)$, which may be complex functionals of the kernels $f_{d_n}^{(n)}$. In the next section, we will show that, in the specific framework of double Poisson integrals, one can establish neat sufficient conditions for (86), with a deterministic $\phi(\lambda)$, by using a version of the multiplication formula for multiple stochastic integrals. The techniques developed below can be extended to integrals of higher orders, to a random $\phi(\lambda)$, and even to non-Poissonian random measures, as long as a version of the multiplication formula is available (one might use, for instance, the general theory of “diagonal measures” developed in [4] and [29]). These extensions will be discussed in a separate paper.

5.3 Application: CLTs for double Poisson integrals

In this section (Z, \mathcal{Z}, μ) is a Borel measure space, with μ non-atomic, σ -finite and positive. Also, \widehat{N} stands for a compensated Poisson random measure on (Z, \mathcal{Z}) with control μ . This means that $\widehat{N} = \left\{ \widehat{N}(B) : B \in \mathcal{Z}_\mu \right\}$ is an independently scattered random measure as in Definition E, such that, for every $B \in \mathcal{Z}_\mu$

$$\widehat{N}(B) \stackrel{\text{law}}{=} N(B) - \mu(B),$$

where $N(B)$ is a Poisson random variable with parameter $\mu(B)$. Note that, for every $h \in L^2(Z, \mathcal{Z}, \mu) = \mathfrak{H}_\mu$,

$$X_{\widehat{N}}(h) = \int_Z h(z) \widehat{N}(dz),$$

where $X_{\widehat{N}}$ is defined by (53). Moreover, for every $h \in \mathfrak{H}_\mu$ and $\lambda \in \mathbb{R}$, the Lévy-Khinchine exponent $\psi_{\mathfrak{H}_\mu}(h, \lambda)$ appearing in (55), is such that (see again [31, Proposition 19.5])

$$\psi_{\mathfrak{H}_\mu}(h, \lambda) = \int_Z \exp(i\lambda h(z) - 1 - i\lambda h(z)) \mu(dz) \quad (89)$$

(recall that this corresponds to the case $\rho_\mu(z, dx) = \delta_1(dx)$ in Proposition 13).

As an application of the previous theory, we shall study the asymptotic behavior of a sequence of random variables of the type

$$F_n = I_2^{\widehat{N}}(f_n), \quad n \geq 1, \quad (90)$$

where $f_n \in L_{s,0}^2(\mu^2)$. In particular, we want to use Theorem 15 to establish sufficient conditions, ensuring that F_n converges in law to a standard Gaussian distribution. We will suppose the following:

Assumption N – (N₁) The sequence f_n , $n \geq 1$, in (90) verifies:

(N₁-i) (*Integrability condition*) $\forall n \geq 1$,

$$\int_Z f_n(z, \cdot)^2 \mu(dz) \in L^2(\mu); \quad (91)$$

(N₁-ii) (*Normalization condition*) As $n \rightarrow +\infty$,

$$2 \int_Z \int_Z f_n(z, z')^2 \mu(dz) \mu(dz') \rightarrow 1; \quad (92)$$

(N₁-iii) (*Fourth moment condition*) As $n \rightarrow +\infty$,

$$\int_Z \int_Z f_n(z, z')^4 \mu(dz) \mu(dz') \rightarrow 0 \quad (93)$$

(this implies, in particular, that $f_n \in L^4(\mu^2)$).

(N₂) For every $n \geq 1$, there exists a collection $\{Z_{n,t} : t \in [0, 1]\} \subset \mathcal{Z}$, such that $Z_{n,0} = \emptyset$, $Z_{n,1} = Z$, $Z_{n,s} \subseteq Z_{n,t}$ for $s < t$, and satisfying condition (65) (with $Z_{n,t}$ substituting Z_t). Note that one can take $Z_{n,t} = Z_{1,t}$ for every n and t .

Remarks – (1) Suppose there exists a set B , independent of n , such that $\mu(B) < +\infty$, and, for each n , $f_n = f_n \mathbf{1}_B$, a.e.- $d\mu^2$ (this is true, in particular, when μ is finite). Then, by the Cauchy-Schwarz inequality, if (93) is verified (f_n) must necessarily converge to zero in $L^2_{s,0}(\mu^2)$. To get more general sequences (f_n) we need to suppose $\mu(Z) = +\infty$.

(2) Assumption N is satisfied by a properly normalized sequence of uniformly bounded functions, with supports “slowly converging to Z ”. For instance, consider a sequence $g_n \in L^2_{s,0}(\mu^2)$ such that, for $n \geq 1$, $|g_n(\cdot, \cdot)| \leq c < +\infty$ (c independent of n) and the support of g_n is contained in a set of the type $B_n \times B_n$, where $0 < \mu(B_n) < +\infty$ and $\mu(B_n) \rightarrow +\infty$. Then, if

$$\mu(B_n)^{-2} \int_Z \int_Z g_n(z, z')^2 \mu(dz) \mu(dz') \rightarrow 1,$$

the sequence $f_n \triangleq \mu(B_n)^{-1} g_n$, $n \geq 1$, verifies Assumption N. Indeed, since $|f_n| \leq c\mu(B_n)^{-1}$,

$$\begin{aligned} \int_Z \int_Z f_n(z, z')^4 \mu(dz) \mu(dz') &\leq \frac{c^4}{\mu(B_n)^2} \rightarrow 0 \\ \int_Z \left(\int_Z f_n(z, z')^2 \mu(dz) \right)^2 \mu(dz') &\leq \frac{c^4}{\mu(B_n)} < +\infty. \end{aligned}$$

Before stating the main result of the section, we recall a useful version of the *multiplication formula* for multiple Poisson integrals. To this end, we define, for $q, p \geq 1$, $f \in L^2_{s,0}(\mu^p)$, $g \in L^2_{s,0}(\mu^q)$, $r = 0, \dots, q \wedge p$ and $l = 1, \dots, r$, the (contraction) kernel on $Z^{p+q-r-l}$, which reduces the number of variables in the product fg from $p+q$ to $p+q-r-l$ as follows: r variables are identified and, among these, l are integrated out. This contraction kernel is formally defined as follows:

$$\begin{aligned} &f \star_r^l g(\gamma_1, \dots, \gamma_{r-l}, t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) \\ &= \int_{Z^l} f(z_1, \dots, z_l, \gamma_1, \dots, \gamma_{r-l}, t_1, \dots, t_{p-r}) g(z_1, \dots, z_l, \gamma_1, \dots, \gamma_{r-l}, s_1, \dots, s_{q-r}) \mu^l(dz_1 \dots dz_l), \end{aligned}$$

and, for $l = 0$,

$$f \star_r^0 g(\gamma_1, \dots, \gamma_r, t_1, \dots, t_{p-r}, s_1, \dots, s_{q-r}) = f(\gamma_1, \dots, \gamma_r, t_1, \dots, t_{p-r}) g(\gamma_1, \dots, \gamma_r, s_1, \dots, s_{q-r}), \quad (94)$$

so that $f \star_0^0 g(t_1, \dots, t_p, s_1, \dots, s_q) = f(t_1, \dots, t_p) g(s_1, \dots, s_q)$. For example, if $p = q = 2$,

$$f \star_1^0 g(\gamma, t, s) = f(\gamma, t) g(\gamma, s), \quad f \star_1^1 g(t, s) = \int_Z f(z, t) g(z, s) \mu(dz) \quad (95)$$

$$f \star_2^1 g(\gamma) = \int_Z f(z, \gamma) g(z, \gamma) \mu(dz), \quad f \star_2^2 g = \int_Z \int_Z f(z_1, z_2) g(z_1, z_2) \mu(dz_1) \mu(dz_2). \quad (96)$$

The following product formula for two Poisson multiple integrals is proved e.g. in [13]: let $f \in L^2_{s,0}(\mu^p)$ and $g \in L^2_{s,0}(\mu^q)$, $p, q \geq 1$, and suppose moreover that $f \star_r^l g \in L^2(\mu^{p+q-r-l})$ for every $r = 0, \dots, p \wedge q$ and $l = 1, \dots, r$, then

$$I_p^{\widehat{N}}(f) I_q^{\widehat{N}}(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} \sum_{l=0}^r I_{q+p-r-l}^{\widehat{N}}(\widetilde{f \star_r^l g}), \quad (97)$$

where the tilde ($\widetilde{}$) stands for symmetrization (note that $f \star_r^l g$ need not vanish on diagonals, and that we use convention (63)).

The next result is the announced central limit theorem.

Theorem 18 *Define the sequence F_n and $f_n \in L_{s,0}^2(\mu^2)$, $n \geq 1$, as in (90), and suppose Assumption N holds. Then, $f_n \star_1^0 f_n \in L^2(\mu^3)$ and $f_n \star_1^1 f_n \in L_s^2(\mu^2)$ for every $n \geq 1$, and moreover:*

1. if

$$f_n \star_1^1 f_n \rightarrow 0 \text{ in } L_{s,0}^2(\mu^2) \quad \text{and} \quad f_n \star_2^1 f_n \rightarrow 0 \text{ in } L^2(\mu) \quad (98)$$

then

$$F_n \xrightarrow{\text{law}} N(0, 1), \quad (99)$$

where $N(0, 1)$ is a standard Gaussian random variable;

2. if $F_n \in L^4(\mathbb{P})$ for every n , then a sufficient condition to have (98) is that

$$\mathbb{E}(F_n^4) \rightarrow 3; \quad (100)$$

3. if the sequence $\{F_n^4 : n \geq 1\}$ is uniformly integrable, then conditions (98), (99) and (100) are equivalent.

Remarks – (a) Note that the statement of Theorem 18 does not involve any resolution of the identity. However, part (\mathbf{N}_2) of Assumption N will play a crucial role in the proof.

(b) Observe that

$$\|f_n \star_1^1 f_n\|_{L^2(\mu^2)}^2 = \int_Z \left(\int_Z f(a, z) f(b, z) \mu(dz) \right)^2 \mu(da) \mu(db) \quad (101)$$

$$\|f_n \star_2^1 f_n\|_{L^2(\mu)}^2 = \int_Z \left(\int_Z f(a, z)^2 \mu(da) \right)^2 \mu(dz). \quad (102)$$

(c) Let G be a Gaussian measure on (Z, \mathcal{Z}) , with control μ , and, for $n \geq 1$, let $H_n = I_2^G(h_n)$ be the double Wiener-Itô integral of a function $h_n \in L_{s,0}^2(\mu^2)$. In [19, Theorem 1] it is proved that, if $2 \|h_n\|^2 \rightarrow 1$ and *regardless of Assumption N*, the following three conditions are equivalent: (i) $H_n \xrightarrow{\text{law}} N(0, 1)$, (ii) $\mathbb{E}(H_n^4) \rightarrow 3$, (iii) $h_n \star_1^1 h_n \rightarrow 0$. Note also that Theorem 1 in [19] applies to multiple integrals of arbitrary order.

(d) A sufficient condition for the uniform integrability of (F_n^4) is clearly that $\sup_n \mathbb{E}(F_n^{4+\varepsilon}) < +\infty$ for some $\varepsilon > 0$. Note that in the Gaussian framework of [19, Theorem 1] the uniform integrability condition is always satisfied. Indeed, by noting $H_n = I_2^G(h_n)$ ($n \geq 1$) the sequence of double integrals introduced in the previous remark, for every $p > 2$ there exists a finite constant c_p such that $\sup_n \mathbb{E}(|H_n|^p) \leq c_p \sup_n \mathbb{E}(H_n^2) < +\infty$, where the last relation follows from the normalization condition $\mathbb{E}(H_n^2) = 2 \|h_n\|^2 \rightarrow 1$.

Proof of Theorem 18. Since

$$f_n \star_1^1 f_n(t, s) = \int_Z f_n(s, z) f_n(t, z) \mu(dz),$$

and $f \in L^2(\mu^2)$, the relation $f_n \star_1^1 f_n \in L_s^2(\mu^2)$ is a consequence of the Cauchy-Schwarz inequality. On the other hand, by (95),

$$\begin{aligned} \int_{Z^3} (f_n \star_1^0 f_n(\gamma, t, s))^2 \mu^3(d\gamma, dt, ds) &= \int_Z \left(\int_Z f_n(\gamma, t)^2 \mu(dt) \times \int_Z f_n(\gamma, s)^2 \mu(ds) \right) \mu(d\gamma) \\ &= \int_Z \left(\int_Z f_n(\gamma, s)^2 \mu(ds) \right)^2 \mu(d\gamma) < +\infty, \end{aligned}$$

due to part **(N₁-i)** in Assumption N, so that $f_n \star_1^0 f_n \in L^2(\mu^3)$.

(*Proof of point 1*) We shall apply Theorem 17 in the following case: $\phi(\lambda) = \exp(-\lambda^2/2)$, $\lambda \in \mathbb{R}$, and, for $n \geq 1$, $(Z_n, \mathcal{Z}_n) = (Z, \mathcal{Z})$, $d_n = 2$, $M_n = \widehat{N}$, $f_{d_n}^{(n)} = f_2^{(n)} = f_n$, $\mu_n = \mu$, $t_n = 0$ and $\pi^{(n)}h(z) = \mathbf{1}_{Z_{n,t}}(z)h(z)$, $\forall h \in \mathfrak{H}_\mu (= L^2(Z, \mathcal{Z}, \mu))$, where the sets $Z_{n,t}$ are defined as in part **(N₂)** of Assumption N. In this case, for $n \geq 1$, $Z_{n,t_n} = Z_{n,0} = \emptyset$ by definition, and therefore $\mathcal{F}_{t_n}^{\pi^{(n)}} = \{\emptyset, \Omega\}$, so that assumptions (84) and (85) are immaterial. Moreover, by using (89), for every $h \in \mathfrak{H}_\mu$

$$\begin{aligned} \int_{Z_n} K_{\mu_n}(\lambda h(z), z) \mu_n(dz) &= \int_Z K_\mu(\lambda h(z), z) \mu(dz) \\ &= \int_Z (\exp(i\lambda h(z)) - 1 - i\lambda h(z)) \mu(dz), \end{aligned} \quad (103)$$

where K_μ is defined by (58). Now define, for $n \geq 1$ and $z \in Z$,

$$h_{\pi^{(n)}}(f_n)(z) = 2I_1^{\widehat{N}}(f_n(z, \cdot) \mathbf{1}(\cdot \prec_{\pi^{(n)}} z)),$$

where the notation is the same as in (78). According to Theorem 17 and (103), to prove Point 1 it is sufficient to show that, when Assumption N is verified, condition (98) implies that, as $n \rightarrow +\infty$,

$$\int_Z (\exp(i\lambda h_{\pi^{(n)}}(f_n)(z)) - 1 - i\lambda h_{\pi^{(n)}}(f_n)(z)) \mu(dz) \xrightarrow{\mathbb{P}} -\frac{1}{2}.$$

To this end, we write

$$\begin{aligned} &\int_Z (\exp(i\lambda h_{\pi^{(n)}}(f_n)(z)) - 1 - i\lambda h_{\pi^{(n)}}(f_n)(z)) \mu(dz) \\ &= -\frac{1}{2} \int_Z (h_{\pi^{(n)}}(f_n)(z))^2 \mu(dz) \\ &\quad + \int_Z \left(\exp(i\lambda h_{\pi^{(n)}}(f_n)(z)) - 1 - i\lambda h_{\pi^{(n)}}(f_n)(z) + \frac{1}{2} (h_{\pi^{(n)}}(f_n)(z))^2 \right) \mu(dz) \\ &\triangleq U_n + V_n, \end{aligned} \quad (104)$$

and we shall show that, under the assumptions of Theorem 18, $U_n \xrightarrow{\mathbb{P}} -\frac{1}{2}$ and $V_n \xrightarrow{\mathbb{P}} 0$. We now apply (97) in the case $p = q = 1$, to have

$$\begin{aligned} (h_{\pi^{(n)}}(f_n)(z))^2 &= 4I_1^{\widehat{N}}(f_n(z, \cdot) \mathbf{1}(\cdot \prec_{\pi^{(n)}} z))^2 \\ &\triangleq 4 \int_Z f_n(z, x)^2 \mathbf{1}(x \prec_{\pi^{(n)}} z) \mu(dx) + 4I_1^{\widehat{N}}(f_n(z, \cdot)^2 \mathbf{1}(\cdot \prec_{\pi^{(n)}} z)) + 4I_2^{\widehat{N}}(g_n(z; \cdot, \cdot)) \end{aligned} \quad (105)$$

where $g_n(z; \cdot, \cdot) \in L_{s,0}^2(\mu^2)$ is given by

$$\begin{aligned} g_n(z; a, b) &= f_n \star_0^0 f_n(z, a; z, b) \mathbf{1}_{(a \prec_{\pi^{(n)}} z)} \mathbf{1}_{(b \prec_{\pi^{(n)}} z)} \\ &= f_n(z, a) f_n(z, b) \mathbf{1}_{(a \prec_{\pi^{(n)}} z)} \mathbf{1}_{(b \prec_{\pi^{(n)}} z)}. \end{aligned} \quad (106)$$

We deal with each term in (105) in succession. For the first term observe that, due to Corollary 15 and the symmetry of f_n ,

$$\begin{aligned} & \int_Z \int_Z f_n(z, x)^2 \mu(dx) \mu(dz) \\ &= \int_Z \int_Z f_n(z, x)^2 [\mathbf{1}(x \prec_{\pi(n)} z) + \mathbf{1}(z \prec_{\pi(n)} x)] \mu(dx) \mu(dz) \\ &= 2 \int_Z \int_Z f_n(z, x)^2 \mathbf{1}(x \prec_{\pi(n)} z) \mu(dx) \mu(dz) \end{aligned}$$

and therefore, thanks to Assumption N,

$$-\frac{1}{2} \int_Z 4 \int_Z f_n(z, x)^2 \mathbf{1}(x \prec_{\pi(n)} z) \mu(dx) \mu(dz) = -\frac{1}{2} \left[2 \int_Z \int_Z f_n(z, x)^2 \mu(dx) \mu(dz) \right] \rightarrow -\frac{1}{2}. \quad (107)$$

For the second term in (105) one has

$$\begin{aligned} \mathbb{E} \left[\int_Z I_1^{\widehat{N}} \left(f_n(z, \cdot)^2 \mathbf{1}(\cdot \prec_{\pi(n)} z) \right) \mu(dz) \right]^2 &= \mathbb{E} \left[I_1^{\widehat{N}} \left(\int_Z f_n(z, \cdot)^2 \mathbf{1}(\cdot \prec_{\pi(n)} z) \mu(dz) \right) \right]^2 \\ &= \int_Z \left[\int_Z f_n(z, x)^2 \mathbf{1}(x \prec_{\pi(n)} z) \mu(dx) \right]^2 \mu(dz) \\ &\leq \int_Z \left[\int_Z f_n(z, x)^2 \mu(dx) \right]^2 \mu(dz) \rightarrow 0, \end{aligned} \quad (108)$$

due to (101) and (98). Now consider the third term in (105), and observe that, by a Fubini argument,

$$\int_Z I_2^{\widehat{N}}(g_n(z; \cdot, \cdot)) \mu(dz) = I_2^{\widehat{N}}(h_n),$$

where, thanks to (106), $h_n \in L_{s,0}^2(\mu^2)$ is s.t.

$$h_n(a, b) = \int_Z f_n(z, a) f_n(z, b) \mathbf{1}(a \prec_{\pi(n)} z) \mathbf{1}(b \prec_{\pi(n)} z) \mu(dz). \quad (109)$$

We now want to show that $f_n \star_1^1 f_n(a, b) = \int_Z f_n(z, a) f_n(z, b) \mu(dz) \rightarrow 0$ implies that $h_n \rightarrow 0$. To do this, we start by observing that, a.e.- $\mu^2(da, db)$ and thanks to Corollary 15,

$$f_n(a, b) = f_n(a, b) \mathbf{1}_{(a \prec_{\pi(n)} b) \cup (b \prec_{\pi(n)} a)}.$$

As a consequence, by noting (for fixed z)

$$\begin{aligned} (z \prec_{\pi(n)} a \vee b) &= [(z \prec_{\pi(n)} a) \cap (z \prec_{\pi(n)} b)] \cup (a \prec_{\pi(n)} z \prec_{\pi(n)} b) \cup (b \prec_{\pi(n)} z \prec_{\pi(n)} a) \\ (a \vee b \prec_{\pi(n)} z) &= (a \prec_{\pi(n)} z) \cap (b \prec_{\pi(n)} z), \end{aligned}$$

we obtain that

$$\begin{aligned}
& \int_{Z^2} (f_n \star_1^1 f_n(a, b))^2 \mu^2(da, db) \\
&= \int_{Z^2} \left(\int_Z f_n(z, a) \mathbf{1}_{(a \prec_{\pi(n)} z) \cup (z \prec_{\pi(n)} a)} \mathbf{1}_{(z \prec_{\pi(n)} b) \cup (b \prec_{\pi(n)} z)} f_n(z, b) \mu(dz) \right)^2 \mu^2(da, db) \\
&= \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \left(\mathbf{1}_{(z \prec_{\pi(n)} a \vee b)} + \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \right) \mu(dz) \right)^2 \mu^2(da, db) \tag{110}
\end{aligned}$$

$$\begin{aligned}
&= \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(z \prec_{\pi(n)} a \vee b)} \mu(dz) \right)^2 \mu^2(da, db) \tag{111} \\
&+ \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \mu(dz) \right)^2 \mu^2(da, db) \\
&+ 2 \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \mu(dz) \right) \times \\
&\times \left(\int_Z f_n(z', a) f_n(z', b) \mathbf{1}_{(z \prec_{\pi(n)} a \vee b)} \mu(dz') \right) \mu^2(da, db).
\end{aligned}$$

Now we note $(a \prec_{\pi(n)} z \wedge z') = (a \prec_{\pi(n)} z') \cap (a \prec_{\pi(n)} z)$, so that, by a Fubini argument,

$$\begin{aligned}
& \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \mu(dz) \right)^2 \mu^2(da, db) \\
&= \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z', a) \mathbf{1}_{(a \prec_{\pi(n)} z \wedge z')} \mu(da) \right)^2 \mu^2(dz, dz') \tag{112}
\end{aligned}$$

and also, with obvious notation,

$$\begin{aligned}
& 2 \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \mu(dz) \right) \times \\
& \times \left(\int_Z f_n(z', a) f_n(z', b) \mathbf{1}_{(z' \prec_{\pi(n)} a \vee b)} \mu(dz') \right) \mu^2(da, db) \\
&= \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z \vee z')} \mu(dz) \right) \times \\
& \times \left(\int_Z f_n(z', a) f_n(z', b) \mathbf{1}_{(z' \wedge z \prec_{\pi(n)} a \vee b)} \mu(dz') \right) \mu^2(da, db),
\end{aligned}$$

so that the relation

$$\begin{aligned}
\mathbf{1}_{(a \vee b \prec_{\pi(n)} z \vee z')} \mathbf{1}_{(z' \wedge z \prec_{\pi(n)} a \vee b)} &= \mathbf{1}_{(z' \wedge z \prec_{\pi(n)} a, b \prec_{\pi(n)} z \vee z')} + \mathbf{1}_{(z' \wedge z \prec_{\pi(n)} a \prec_{\pi(n)} z \vee z')} \mathbf{1}_{(b \prec_{\pi(n)} z \vee z')} \\
&+ \mathbf{1}_{(z' \wedge z \prec_{\pi(n)} b \prec_{\pi(n)} z \vee z')} \mathbf{1}_{(a \prec_{\pi(n)} z \vee z')}
\end{aligned}$$

gives

$$\begin{aligned}
& 2 \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \mu(dz) \right) \times \\
& \times \left(\int_Z f_n(z', a) f_n(z', b) \mathbf{1}_{(z' \prec_{\pi(n)} a \vee b)} \mu(dz') \right) \mu^2(da, db) \\
& = \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z', a) \mathbf{1}_{(z \wedge z' \prec_{\pi(n)} a \prec_{\pi(n)} z \vee z')} \mu(da) \right)^2 \mu^2(dz, dz') \quad (113)
\end{aligned}$$

$$\begin{aligned}
& + 2 \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z', a) \mathbf{1}_{(z \wedge z' \prec_{\pi(n)} a \prec_{\pi(n)} z \vee z')} \mu(da) \right) \times \\
& \times \left(\int_Z f_n(z, a) f_n(z', a) \mathbf{1}_{(a \prec_{\pi(n)} z \wedge z')} \mu(da) \right) \mu^2(dz, dz'). \quad (114)
\end{aligned}$$

Observe that the terms (111), (112), (113) and (114) are integrals of terms respectively of the form $(A + B)^2$, A^2 , B^2 and $2AB$, whose sum therefore equals $2(A + B)^2$, yielding

$$\int_{Z^2} (f_n \star_1^1 f_n(a, b))^2 \mu^2(da, db) = 2 \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(z \prec_{\pi(n)} a \vee b)} \mu(dz) \right)^2 \mu^2(da, db). \quad (115)$$

Since h_n (as defined in (109)) is such that

$$\begin{aligned}
& \int_{Z^2} h_n(a, b)^2 \mu^2(da, db) \\
& = \int_{Z^2} \left(\int_Z f_n(z, a) f_n(z, b) \mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} \mu(dz) \right)^2 \mu^2(da, db)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{Z^2} (f_n \star_1^1 f_n(a, b))^2 \mu^2(da, db) \\
& = \int_{Z^2} \left[\int_Z f_n(z, a) f_n(z, b) \left(\mathbf{1}_{(a \vee b \prec_{\pi(n)} z)} + \mathbf{1}_{(z \prec_{\pi(n)} a \vee b)} \right) \mu(dz) \right]^2 \mu^2(da, db),
\end{aligned}$$

relation (115) gives the implication: if $f_n \star_1^1 f_n \rightarrow 0$ in $L^2(\mu^2)$, then $h_n \rightarrow 0$ in $L^2(\mu^2)$. This last result, combined with (107) and (108) implies that the sequence U_n , $n \geq 1$, as defined in (104), converges to $-\frac{1}{2}$ in probability.

To show that $V_n \xrightarrow{\mathbb{P}} 0$, observe that $|\exp(i\lambda x) - 1 - i\lambda x + \frac{1}{2}x^2| \leq |x|^3/6$, and consequently, by Cauchy-Schwarz

$$\begin{aligned}
|V_n| & \leq \frac{1}{6} \int_Z |h_{\pi(n)}(f_n)(z)|^3 \mu(dz) \\
& \leq \frac{1}{6} \left(\int_Z |h_{\pi(n)}(f_n)(z)|^4 \mu(dz) \right)^{\frac{1}{2}} \left(\int_Z |h_{\pi(n)}(f_n)(z)|^2 \mu(dz) \right)^{\frac{1}{2}}.
\end{aligned}$$

Since the first part of the proof implies that, under (98), $\left(\int_Z |h_{\pi(n)}(f_n)(z)|^2 \mu(dz) \right)^{\frac{1}{2}} \xrightarrow{\mathbb{P}} 1$, to conclude the proof of point 1 it is sufficient to show that, under Assumption N and (98), $\int_Z |h_{\pi(n)}(f_n)(z)|^4 \mu(dz) \rightarrow 0$ in $L^1(\mathbb{P})$. To do this, one can use (105) and the orthogonality of multiple integrals of different orders to

obtain that, for any fixed z ,

$$\begin{aligned}
\mathbb{E} \left[(h_{\pi^{(n)}}(f_n)(z))^4 \right] &= \mathbb{E} \left[\left((h_{\pi^{(n)}}(f_n)(z))^2 \right)^2 \right] \\
&= 16 \left(\int_Z f_n(z, x)^2 \mathbf{1}(x \prec_{\pi^{(n)}} z) \mu(dx) \right)^2 \\
&\quad + 16 \int_Z f_n(z, \cdot)^4 \mathbf{1}(\cdot \prec_{\pi^{(n)}} z) \mu(da) \\
&\quad + 32 \int_{Z^2} f_n(z, a)^2 f_n(z, b)^2 \mathbf{1}(a \prec_{\pi^{(n)}} z) \mathbf{1}(b \prec_{\pi^{(n)}} z) \mu^2(da, db),
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{E} \int_Z |h_{\pi^{(n)}}(f_n)(z)|^4 \mu(dz) &= \int_Z \mathbb{E} |h_{\pi^{(n)}}(f_n)(z)|^4 \mu(dz) \\
&\leq 16 \int_Z \left(\int_Z f_n(z, x)^2 \mu(dx) \right)^2 \mu(dz) \\
&\quad + 16 \int_Z \int_Z f_n(z, a)^4 \mu(da) \mu(dz) \\
&\quad + 32 \int_Z \left(\int_Z f_n(z, x)^2 \mu(dx) \right)^2 \mu(dz) \\
&\rightarrow 0,
\end{aligned}$$

since Assumption N and (98) are in order. This concludes the proof of Point 1.

(*Proof of Point 2*) To proof Point 2, use the product formula expansion (97) (from the term with $r = 0$ to the terms with $r = 2$) to write

$$\begin{aligned}
F_n^2 &= I_3^{\hat{N}}(f_n)^2 = I_4^{\hat{N}}(\widetilde{f_n \star_0^0 f_n}) + 4I_3^{\hat{N}}(\widetilde{f_n \star_1^0 f_n}) + 4I_2^{\hat{N}}(f_n \star_1^1 f_n) \\
&\quad + 2I_2^{\hat{N}}(f_n \star_2^0 f_n) + 2I_1^{\hat{N}}(f_n \star_2^1 f_n) + 2\|f_n\|_{L_{s,0}^2(\mu^2)}^2,
\end{aligned}$$

and observe that, since Assumption N holds and $f_n \star_2^0 f_n(a, b) = f_n(a, b)^2$ (by (94)), $I_2^{\hat{N}}(f_n \star_2^0 f_n) \rightarrow 0$ in $L^2(\mathbb{P})$ by (93), and therefore the assumption $\mathbb{E}(F_n^4) \rightarrow 3$ implies that

$$\begin{aligned}
&\mathbb{E} \left[\left(F_n^2 - 2I_2^{\hat{N}}(f_n \star_2^0 f_n) \right)^2 \right] \\
&= \mathbb{E} \left[\left(I_4^{\hat{N}}(\widetilde{f_n \star_0^0 f_n}) + 4I_3^{\hat{N}}(\widetilde{f_n \star_1^0 f_n}) + 4I_2^{\hat{N}}(f_n \star_1^1 f_n) \right. \right. \\
&\quad \left. \left. + 2I_1^{\hat{N}}(f_n \star_2^1 f_n) + 2\|f_n\|_{L_{s,0}^2(\mu^2)}^2 \right)^2 \right]
\end{aligned} \tag{116}$$

$$\rightarrow 3. \tag{117}$$

Now, due to (116),

$$\mathbb{E} \left[\left(F_n^2 - 2I_2^{\hat{N}}(f_n \star_2^0 f_n) \right)^2 \right] \tag{118}$$

$$\begin{aligned}
&= \left[\left(2\|f_n\|_{L_{s,0}^2(\mu^2)}^2 \right)^2 + \mathbb{E} \left(16I_2^{\hat{N}}(f_n \star_1^1 f_n)^2 \right) + \mathbb{E} \left(I_4^{\hat{N}}(\widetilde{f_n \star_0^0 f_n})^2 \right) \right] \\
&\quad + \mathbb{E} \left(16I_3^{\hat{N}}(\widetilde{f_n \star_1^0 f_n})^2 \right) + \mathbb{E} \left(4I_1^{\hat{N}}(f_n \star_2^1 f_n)^2 \right).
\end{aligned} \tag{119}$$

There are no cross terms because the multiple integrals have different orders, and hence are orthogonal. The most complicated term in the square bracket in (119) is $\mathbb{E} \left(I_4^{\widehat{N}} \left(\widetilde{f_n \star_0^0 f_n} \right)^2 \right)$. Since we are dealing with second order moments, the computations are as in the Gaussian case. We therefore obtain, using e.g. formula (2) in [24, p. 250], that

$$\begin{aligned} \mathbb{E} \left(I_4^{\widehat{N}} \left(\widetilde{f_n \star_0^0 f_n} \right)^2 \right) &= 4! \left\| \widetilde{f_n \star_0^0 f_n} \right\|_{L^2(\mu^2)}^2 \\ &= 2 \left(2 \|f_n\|_{L^2_{s,0}(\mu^2)}^2 \right)^2 + 16 \|f_n \star_1^1 f_n\|_{L^2(\mu^2)}^2. \end{aligned}$$

As a consequence, (118) equals

$$\begin{aligned} &\left(2 \|f_n\|_{L^2_{s,0}(\mu^2)}^2 \right)^2 + \left[2 \left(2 \|f_n\|_{L^2_{s,0}(\mu^2)}^2 \right)^2 + 16 \|f_n \star_1^1 f_n\|_{L^2(\mu^2)}^2 \right] \\ &+ 16 \times 2 \|f_n \star_1^1 f_n\|_{L^2(\mu^2)}^2 + 16 \times 3! \|f_n \star_1^0 f_n\|_{L^2(\mu^3)}^2 + 4 \|f_n \star_2^1 f_n\|_{L^2(\mu)}^2 \\ &= 3 \left(2 \|f_n\|_{L^2_{s,0}(\mu^2)}^2 \right)^2 + 48 \|f_n \star_1^1 f_n\|_{L^2(\mu^2)}^2 + 96 \|f_n \star_1^0 f_n\|_{L^2(\mu^3)}^2 + 4 \|f_n \star_2^1 f_n\|_{L^2(\mu)}^2. \quad (120) \end{aligned}$$

Since (120) converges to 3, by (117), and $2 \|f_n\|_{L^2_{s,0}(\mu^2)}^2 \rightarrow 1$ by Assumption **(N₁-ii)**, we conclude that $\|f_n \star_1^1 f_n\|_{L^2(\mu^2)}^2 \rightarrow 0$ and $\|f_n \star_2^1 f_n\|_{L^2(\mu)}^2 \rightarrow 0$, thus proving Point 2.

(*Proof of point 3*) If $F_n \xrightarrow{\text{law}} N(0, 1)$ and (F_n^4) is uniformly integrable, then necessarily $\mathbb{E}(F_n^4) \rightarrow \mathbb{E}(N(0, 1)^4) = 3$, so that the proof is obtained by combining Point 1 and Point 2 in the statement. ■

Example – We now exhibit an elementary example of a sequence $f_n \in L^2_{s,0}(\mu^2)$, $n \geq 1$, verifying conditions (91), (92), (93) and (98). To this end, let B_j , $j \geq 1$, be a sequence of disjoint subsets of Z such that $\mu(B_j) = 1$, $j \geq 1$, and set

$$B_{0,j}^2 = \{(x, y) \in B_j \times B_j : x \neq y\}, \quad j \geq 1;$$

note that, since μ is non-atomic, $\mu^2(B_{0,j}^2) = \mu^2(B_j \times B_j) = 1$. For $n \geq 1$ and $(x, y) \in Z^2$, we define $f_n(x, y) = (2n)^{-1/2} \sum_{j=1}^n \mathbf{1}_{B_{0,j}^2}(x, y)$. Of course, $f_n \in L^2_{s,0}(\mu^2)$ by definition, and we shall prove that (f_n) also satisfies (91), (92), (93) and (98). Indeed, $\int_Z f_n(z, \cdot)^2 \mu(dz) = 2n^{-1} \sum_{j=1}^n \mathbf{1}_{B_j}(\cdot) \in L^2(\mu)$ and $2 \|f_n\|^2 = 1$ by definition, so that (f_n) verifies (91) and (92). On the other hand,

$$\begin{aligned} \int_Z \int_Z f_n(x, y)^4 \mu^2(dx, dy) &= \frac{1}{4n^2} \int_Z \int_Z \left(\sum_{j=1}^n \mathbf{1}_{B_{0,j}^2}(x, y) \right)^4 \mu^2(dx, dy) \\ &= \frac{1}{4n} \rightarrow 0, \end{aligned}$$

and therefore (93) is verified. Finally,

$$\int_Z \left(\int_Z f(z, x)^2 \mu(dz) \right)^2 \mu(dx) = \frac{1}{4n} \rightarrow 0$$

and

$$\begin{aligned} \int_Z \left(\int_Z f(x, z) f(y, z) \mu(dz) \right)^2 \mu^2(dx, dy) &= \frac{1}{4n^2} \int_Z \int_Z \left(\sum_{j=1}^n \mathbf{1}_{B_{0,j}^2}(x, y) \right)^2 \mu^2(dx, dy) \\ &= \frac{1}{4n} \rightarrow 0, \end{aligned}$$

thus yielding that (f_n) satisfies (98), by (101) and (102). Of course, since (due e.g. to (97))

$$I_2^{\widehat{N}}(f_n) = n^{-1/2} \sum_{j=1}^n 2^{-1/2} \left(\widehat{N}(B_j)^2 - \widehat{N}(B_j) - 1 \right),$$

the central limit result $I_2^{\widehat{N}}(f_n) \xrightarrow{law} N(0, 1)$ can be verified directly, by using a standard version of the Central Limit Theorem, as well as the fact that \widehat{N} is independently scattered and the B_j 's are disjoint.

6 Stable convergence of functionals of Gaussian processes

We shall now use Theorem 7 to prove general sufficient conditions, ensuring the stable convergence of functionals of Gaussian processes towards mixtures of normal distributions. This extends part of the results contained in [19] and [24], and leads to quite general criteria for the stable convergence of Skorohod integrals and multiple Wiener-Itô integrals. However, to keep the length of this paper within bounds, we have deferred the discussion about multiple Wiener-Itô integrals, as well as some relations with Brownian martingales to a separate paper, see [23].

6.1 Preliminaries

Consider a real separable Hilbert space \mathfrak{H} , as well as a continuous resolution of the identity $\pi = \{\pi_t : t \in [0, 1]\} \in \mathcal{R}(\mathfrak{H})$ (see Definition D). Throughout this paragraph, $X = X(\mathfrak{H}) = \{X(f) : f \in \mathfrak{H}\}$ stands for a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, indexed by the elements of \mathfrak{H} and satisfying the isomorphic condition (14). Note, that due to the Gaussian nature of X , every vector as in (18) is composed of independent random variables, and therefore, in this case, $\mathcal{R}(\mathfrak{H}) = \mathcal{R}_X(\mathfrak{H})$. When (14) is satisfied and $X(\mathfrak{H})$ is a Gaussian family, one usually says that $X(\mathfrak{H})$ is an *isonormal Gaussian process*, or a *Gaussian measure*, over \mathfrak{H} (see e.g. [17, Section 1] or [18]). As before, we write $L^2(\mathfrak{H}, X)$ to indicate the (Hilbert) space of \mathfrak{H} -valued and $\sigma(X)$ -measurable random variables. The filtration $\mathcal{F}^\pi(X) = \{\mathcal{F}_t^\pi(X) : t \in [0, 1]\}$ (which is complete by definition) is given by formula (19).

In what follows, we shall apply to the Gaussian measure X some standard notions and results from Malliavin calculus (the reader is again referred to [17] and [18] for any unexplained notation or definition). For instance, $D = D_X$ and $\delta = \delta_X$ stand, respectively, for the usual Malliavin derivative and Skorohod integral with respect to the Gaussian measure X (the dependence on X will be dropped, when there is no risk of confusion); for $k \geq 1$, $\mathbb{D}_X^{1,2}$ is the space of differentiable functionals of X , endowed with the norm $\|\cdot\|_{1,2}$ (see [17, Chapter 1] for a definition of this norm); $dom(\delta_X)$ is the domain of the operator δ_X . Note that D_X is an operator from $\mathbb{D}_X^{1,2}$ to $L^2(\mathfrak{H}, X)$, and also that $dom(\delta_X) \subset L^2(\mathfrak{H}, X)$. For every $d \geq 1$, we define $\mathfrak{H}^{\otimes d}$ and $\mathfrak{H}^{\odot d}$ to be, respectively, the d th tensor product and the d th *symmetric* tensor product of \mathfrak{H} . For $d \geq 1$ we will denote by I_d^X the isometry between $\mathfrak{H}^{\odot d}$ equipped with the norm $\sqrt{d!} \|\cdot\|_{\mathfrak{H}^{\otimes d}}$ and the d th Wiener chaos of X .

The vector spaces $L_\pi^2(\mathfrak{H}, X)$ and $\mathcal{E}_\pi(\mathfrak{H}, X)$, composed respectively of adapted and elementary adapted elements of $L^2(\mathfrak{H}, X)$, are once again defined as in Paragraph 3. We now want to link the above defined operators δ_X and D_X to the theory developed in the previous sections. In particular, we shall use the facts that (i) for any $\pi \in \mathcal{R}_X(\mathfrak{H})$, $L_\pi^2(\mathfrak{H}, X) \subseteq dom(\delta_X)$, and (ii) for any $u \in L_\pi^2(\mathfrak{H}, X)$ the random variable $J_X^\pi(u)$ can be regarded as a Skorohod integral. They are based on the following (simple) result, proved for instance in [36, Lemme 1].

Proposition 19 *Let the assumptions of this section prevail. Then, $L_\pi^2(\mathfrak{H}, X) \subseteq dom(\delta_X)$, and for every $h_1, h_2 \in L_\pi^2(\mathfrak{H}, X)$*

$$\mathbb{E}(\delta_X(h_1) \delta_X(h_2)) = (h_1, h_2)_{L_\pi^2(\mathfrak{H}, X)}. \quad (121)$$

Moreover, if $h \in \mathcal{E}_\pi(\mathfrak{H}, X)$ has the form $h = \sum_{i=1}^n h_i$, where $n \geq 1$, and $h_i \in \mathcal{E}_\pi(\mathfrak{H}, X)$ is such that

$$h_i = \Phi_i \times \left(\pi_{t_2^{(i)}} - \pi_{t_1^{(i)}} \right) f_i, \quad f_i \in \mathfrak{H}, \quad i = 1, \dots, n,$$

with $t_2^{(i)} > t_1^{(i)}$ and Φ_i square integrable and $\mathcal{F}_{t_1^{(i)}}^{\pi}(X)$ -measurable, then

$$\delta_X(h) = \sum_{i=1}^n \Phi_i \times \left[X \left(\pi_{t_2^{(i)}} f_i \right) - X \left(\pi_{t_1^{(i)}} f_i \right) \right]. \quad (122)$$

Relation (121) implies, in the terminology of [36], that $L_\pi^2(\mathfrak{H}, X)$ is a closed subspace of the *isometric subset of dom* (δ_X) , defined as the class of those $h \in \text{dom}(\delta_X)$ s.t. $\mathbb{E} \left(\delta_X(h)^2 \right) = \|h\|_{L^2(\mathfrak{H}, X)}^2$ (note that, in general, such an isometric subset is not even a vector space; see [36, p. 170]). Relation (122) applies to simple integrands h , but by combining (121), (122) and Proposition 4, we deduce immediately that, for every $h \in L_\pi^2(\mathfrak{H}, X)$,

$$\delta_X(h) = J_X^\pi(h), \quad \text{a.s.-}\mathbb{P}. \quad (123)$$

where the random variable $J_X^\pi(h)$ is defined according to Proposition 4 and formula (27). Observe that the definition of J_X^π involves the resolution of the identity π , whereas the definition of δ does not involve any notion of resolution.

The next crucial result, which is partly a consequence of the continuity of π , is an abstract version of the *Clark-Ocone formula* (see [17]): it is a direct corollary of [36, Théorème 1, formula (2.4) and Théorème 3], to which the reader is referred for a detailed proof.

Proposition 20 (Abstract Clark-Ocone formula; Wu, 1990) *Under the above notation and assumptions (in particular, π is a continuous resolution of the identity as in Definition D), for every $F \in \mathbb{D}_X^{1,2}$,*

$$F = \mathbb{E}(F) + \delta \left(\text{proj} \{ D_X F \mid L_\pi^2(\mathfrak{H}, X) \} \right), \quad (124)$$

where $D_X F$ is the Malliavin derivative of F , and $\text{proj} \{ \cdot \mid L_\pi^2(\mathfrak{H}, X) \}$ is the orthogonal projection operator on $L_\pi^2(\mathfrak{H}, X)$.

Remarks – (a) Note that the right-hand side of (124) is well defined since $D_X F \in L^2(\mathfrak{H}, X)$ by definition, and therefore

$$\text{proj} \{ D_X F \mid L_\pi^2(\mathfrak{H}, X) \} \in L_\pi^2(\mathfrak{H}, X) \subseteq \text{dom}(\delta_X),$$

where the last inclusion is stated in Proposition 19.

(b) Formula (124) has been proved in [36] in the context of abstract Wiener spaces, but in the proof of (124) the role of the underlying probability space is immaterial. The extension to the framework of isonormal Gaussian processes is therefore standard (see e.g. [18, Section 1.1]).

(c) Since $\mathbb{D}_X^{1,2}$ is dense in $L^2(\mathbb{P})$ and $\delta_X(L_\pi^2(\mathfrak{H}, X))$ is an isometry (due to relation (121)), the Clark-Ocone formula (124) implies that every $F \in L^2(\mathbb{P}, \sigma(X))$ admits a unique “predictable” representation of the form

$$F = \mathbb{E}(F) + \delta_X(u), \quad u \in L_\pi^2(\mathfrak{H}, X); \quad (125)$$

see also [36, Remarque 2, p. 172].

(d) Since (123) holds, formula (124) can be rewritten as

$$F = \mathbb{E}(F) + J_X^\pi \left(\text{proj} \{ D_X F \mid L_\pi^2(\mathfrak{H}, X) \} \right). \quad (126)$$

Now consider, as before, an independent copy of X , noted $\tilde{X} = \{\tilde{X}(f) : f \in \mathfrak{H}\}$, and, for $h \in L^2_\pi(\mathfrak{H}, X)$, define the random variable $J_{\tilde{X}}^\pi(h)$ according to Proposition 4 and (28). The following result is an immediate consequence of Proposition 13, and characterizes $J_{\tilde{X}}^\pi(h)$, $h \in L^2_\pi(\mathfrak{H}, X)$, as a conditionally Gaussian random variable.

Proposition 21 *For every $h \in L^2_\pi(\mathfrak{H}, X)$ and for every $\lambda \in \mathbb{R}$,*

$$\mathbb{E} \left[\exp \left(i\lambda J_{\tilde{X}}^\pi(h) \right) \mid \sigma(X) \right] = \exp \left(-\frac{\lambda^2}{2} \|h\|_{\mathfrak{H}}^2 \right).$$

6.2 Stable convergence to a mixture of Gaussian distributions

The following result, based on Theorem 7, gives general sufficient conditions for the stable convergence of Skorohod integrals to a mixture of Gaussian distributions. In what follows, \mathfrak{H}_n , $n \geq 1$, is a sequence of real separable Hilbert spaces, and, for each $n \geq 1$, $X_n = X_n(\mathfrak{H}_n) = \{X_n(g) : g \in \mathfrak{H}_n\}$, is an isonormal Gaussian process over \mathfrak{H}_n ; for $n \geq 1$, \tilde{X}_n is an independent copy of X_n (note that \tilde{X}_n appears in the proof of the next result, but not in the statement). Recall that $\mathcal{R}(\mathfrak{H}_n)$ is a class of resolutions of the identity π (see Definition D), and that the Hilbert space $L^2_\pi(\mathfrak{H}_n, X_n)$ is defined after Relation (24).

Theorem 22 *Suppose that the isonormal Gaussian processes $X_n(\mathfrak{H}_n)$, $n \geq 1$, are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let, for $n \geq 1$, $\pi^{(n)} \in \mathcal{R}(\mathfrak{H}_n)$ and $u_n \in L^2_{\pi^{(n)}}(\mathfrak{H}_n, X_n)$. Suppose also that there exists a sequence $\{t_n : n \geq 1\} \subset [0, 1]$ and σ -fields $\{\mathcal{U}_n : n \geq 1\}$, such that*

$$\left\| \pi_{t_n}^{(n)} u_n \right\|_{\mathfrak{H}_n}^2 \xrightarrow{\mathbb{P}} 0 \quad (127)$$

and

$$\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n). \quad (128)$$

If

$$\|u_n\|_{\mathfrak{H}}^2 \xrightarrow{\mathbb{P}} Y, \quad (129)$$

for some $Y \in L^2(\mathbb{P})$ such that $Y \neq 0$, $Y \geq 0$ and $Y \in \mathcal{U}^* \triangleq \bigvee_n \mathcal{U}_n$, then, as $n \rightarrow +\infty$,

$$\mathbb{E} \left[\exp \left(i\lambda \delta_{X_n}(u_n) \right) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \right] \xrightarrow{\mathbb{P}} \exp \left(-\frac{\lambda^2}{2} Y \right), \quad \forall \lambda \in \mathbb{R},$$

and

$$\delta_{X_n}(u_n) \rightarrow_{(s, \mathcal{U}^*)} \mathbb{E} \mu(\cdot),$$

where $\mu \in \mathbf{M}$ verifies $\hat{\mu}(\lambda) = \exp \left(-\frac{\lambda^2}{2} Y \right)$ (see (3) for the definition of $\hat{\mu}$).

Proof. Since $\delta_{X_n}(u_n) = J_{X_n}^{\pi^{(n)}}(u_n)$ for every n , the result follows immediately from Theorem 7 by observing that, due to Proposition 21,

$$\mathbb{E} \left[\exp \left(i\lambda J_{X_n}^{\pi^{(n)}}(u_n) \right) \mid \sigma(X_n) \right] = \exp \left(-\frac{\lambda^2}{2} \|u_n\|_{\mathfrak{H}}^2 \right),$$

and therefore (129) that $\mathbb{E} \left[\exp \left(i\lambda J_{X_n}^{\pi^{(n)}}(u_n) \right) \mid \sigma(X_n) \right] \rightarrow \exp(-\lambda^2 Y/2)$ if, and only if, (129) is verified. \blacksquare

By using the Clark-Ocone formula stated in Proposition 20, we deduce immediately, from Theorem 22, a useful criterion for the stable convergence of (Malliavin) differentiable functionals.

Corollary 23 Let $\mathfrak{H}_n, X_n(\mathfrak{H}_n), \pi^{(n)}, t_n$ and $\mathcal{U}_n, n \geq 1$, satisfy the assumptions of Theorem 22 (in particular, (37) holds), and consider a sequence of random variables $\{F_n : n \geq 1\}$, such that $\mathbb{E}(F_n) = 0$ and $F_n \in \mathbb{D}_{X_n}^{1,2}$ for every n . Then, a sufficient condition to have that

$$F_n \xrightarrow{(s, \mathcal{U}^*)} \mathbb{E}\mu(\cdot)$$

and

$$\mathbb{E} \left[\exp(i\lambda F_n) \mid \mathcal{F}_{t_n}^{\pi^{(n)}}(X_n) \right] \xrightarrow{\mathbb{P}} \exp\left(-\frac{\lambda^2}{2} Y\right), \quad \forall \lambda \in \mathbb{R},$$

where $\mathcal{U}^* \triangleq \vee_n \mathcal{U}_n, Y \geq 0$ is s.t. $Y \in \mathcal{U}^*$ and $\hat{\mu}(\lambda) = \exp\left(-\frac{\lambda^2}{2} Y\right), \forall \lambda \in \mathbb{R}$, is

$$\left\| \pi_{t_n}^{(n)} \text{proj} \{D_{X_n} F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\} \right\|_{\mathfrak{H}_n}^2 \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \left\| \text{proj} \{D_{X_n} F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\} \right\|_{\mathfrak{H}_n}^2 \xrightarrow{\mathbb{P}} Y. \quad (130)$$

Proof. Since, for every n, F_n is a centered random variable in $\mathbb{D}_{X_n}^{1,2}$, the abstract Clark-Ocone formula ensures that $F_n = \delta_{X_n}(\text{proj} \{D_{X_n} F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\})$, the result follows from Theorem 22, by putting

$$u_n = \text{proj} \{D_{X_n} F_n \mid L_{\pi^{(n)}}^2(\mathfrak{H}_n, X_n)\}.$$

■

6.3 Example: a “switching” sequence of quadratic Brownian functionals

We conclude the paper by providing a generalization, as well as a new proof, of a result contained in [25, Proposition 2.1]. Let $W_t, t \in [0, 1]$, be a standard Brownian motion initialized at zero, and define the time-reversed Brownian motion W^* by

$$W_t^* = W_1 - W_{1-t}, \quad t \in [0, 1].$$

Observe that

$$W_0^* = 0 \quad \text{and} \quad W_1^* = W_1. \quad (131)$$

We are interested in the asymptotic behavior, for $n \rightarrow +\infty$, of the “switching” sequence

$$A_n = \int_0^1 t^{2n} \left[\left(W_1^{(n)}\right)^2 - \left(W_t^{(n)}\right)^2 \right] dt, \quad n \geq 1,$$

where

$$W^{(n)} = \begin{cases} W & n \text{ odd} \\ W^* & n \text{ even.} \end{cases}$$

In particular, we would like to determine the speed at which A_n converges to zero as $n \rightarrow +\infty$, by establishing a stable convergence result. We start by observing that the asymptotic study of A_n can be reduced to that of a sequence of double stochastic integrals. As a matter of fact, from the relation $\left(W_t^{(n)}\right)^2 = t + 2 \int_0^t W_s^{(n)} dW_s^{(n)}$ one gets

$$A_n = \int_0^1 t^{2n} \left[2 \int_0^1 W_s^{(n)} \mathbf{1}_{(t \leq s)} dW_s^{(n)} + 1 - t \right] dt,$$

and it is easily deduced that

$$\begin{aligned} \sqrt{n}(2n+1) A_n &= 2\sqrt{n} \int_0^1 dW_s^{(n)} W_s^{(n)} s^{2n+1} + \sqrt{n}(2n+1) \int_0^1 (1-t) t^{2n} dt \\ &= 2\sqrt{n} \int_0^1 dW_s^{(n)} W_s^{(n)} s^{2n+1} + o(1). \end{aligned}$$

Now define $\sigma(W)$ to be the σ -field generated by W (or, equivalently, by W^*): we have the following

Proposition 24 As $n \rightarrow +\infty$,

$$\sqrt{n}(2n+1)A_n \xrightarrow{(s, \sigma(W))} \mathbb{E}\mu_1(\cdot), \quad (132)$$

where $\mu_1(\cdot)$ verifies, for $\lambda \in \mathbb{R}$,

$$\widehat{\mu}_1(\lambda) = \exp\left(-\frac{\lambda^2}{2}W_1^2\right),$$

or, equivalently, for every $Z \in \sigma(W)$

$$(Z, \sqrt{n}(2n+1)A_n) \xrightarrow{law} (Z, W_1 \times N'), \quad (133)$$

where N' is a standard Gaussian random variable independent of W .

Remark – In particular, if $W^{(n)} = W$ for every n (no switching between W and W^*), one gets the same convergence in law (133). This last result was proved in [25, Proposition 2.1] by completely different methods.

Proof of Proposition 24. The proof of (132) is based on Theorem 22. First observe that the Gaussian family

$$X_W(h) = \int_0^1 h(s) dW_s, \quad h \in L^2([0, 1], ds), \quad (134)$$

defines an isonormal Gaussian process over the Hilbert space $\mathfrak{H}_1 \triangleq L^2([0, 1], ds)$; we shall write X_W to indicate the isonormal Gaussian process given by (134). Now define the following sequence of continuous resolutions of the identity on \mathfrak{H}_1 : for every $n \geq 1$, every $t \in [0, 1]$ and every $h \in \mathfrak{H}_1$

$$\pi_t^{(n)}h = \begin{cases} \pi_t^o h \triangleq h\mathbf{1}_{[0,t]}, & n \text{ odd} \\ \pi_t^e h \triangleq h\mathbf{1}_{[1-t,1]}, & n \text{ even.} \end{cases},$$

so that, for $t \in [0, 1]$

$$\mathcal{F}_t^{\pi^{(n)}}(X_W) = \begin{cases} \mathcal{F}_t^{\pi^o}(X_W) \triangleq \sigma\{W_u : u \leq t\}, & n \text{ odd} \\ \mathcal{F}_t^{\pi^e}(X_W) \triangleq \sigma\{W_1 - W_s : s \geq 1-t\}, & n \text{ even.} \end{cases}. \quad (135)$$

In this case, the class of adapted processes $L_{\pi^o}^2(\mathfrak{H}_1, X_W)$ (resp. $L_{\pi^e}^2(\mathfrak{H}_1, X_W)$) is given by those elements of $L^2(\mathfrak{H}_1, X_W)$ that are adapted to the filtration $\mathcal{F}_t^{\pi^o}(X_W)$ (resp. $\mathcal{F}_t^{\pi^e}(X_W)$), as defined in (135). Moreover, by defining, for $n \geq 1$,

$$u_n(t) = 2\sqrt{n}W_t^{(n)}t^{2n+1}, \quad t \in [0, 1],$$

we see immediately that $u_n \in L_{\pi^{(n)}}^2(\mathfrak{H}_1, X_W)$ for every n , and hence

$$2\sqrt{n} \int_0^1 W_s^{(n)} s^{2n+1} dW_s^{(n)} = \int_0^1 u_n(s) dW_s^{(n)} = \delta_W(u_n),$$

where δ_W stands for a Skorohod integral with respect to X_W . Indeed, if n is even, $\int_0^1 u_n(s) dW_s^{(n)}$ is an Itô integral with respect to W , and, when n is odd, $\int_0^1 u_n(s) dW_s^{(n)}$ is again an Itô integral with respect to a time reversed Brownian motion (see e.g. [22, Section 4] for more details). Now fix $\varepsilon \in (1/2, 1)$, and set $t_n = \varepsilon^{1/\sqrt{n}}$, so that, $t_n > 1/2$ and $t_n \uparrow 1$. Then,

$$\mathbb{E} \left[\left\| \pi_{t_n}^{(n)} u_n \right\|_{\mathfrak{H}_1}^2 \right] = 4n \int_0^{t_n} s^{4n+3} ds = \frac{4n}{4n+4} \varepsilon^{\frac{4n+4}{\sqrt{n}}} \xrightarrow{n \rightarrow +\infty} 0,$$

and

$$\begin{aligned}
\|u_n\|_{\mathfrak{H}_1}^2 &= 4n \int_0^1 ds \left(W_s^{(n)}\right)^2 s^{4n+2} \\
&= \frac{4n}{4n+4} + \frac{8n}{4n+3} \int_0^1 dW_u^{(n)} W_u^{(n)} (1-u^{4n+3}) \\
&= o_{\mathbb{P}}(1) + 1 + \frac{8n}{4n+3} \int_0^1 dW_u^{(n)} W_u^{(n)} \xrightarrow{\mathbb{P}} W_1^2 = (W_1^*)^2,
\end{aligned} \tag{136}$$

by (131), where $o_{\mathbb{P}}(1)$ stands for a sequence converging to zero in probability (as $n \rightarrow +\infty$). We thus have shown that relations (127) and (129) of Theorem 22 are satisfied. It remains to verify relation (37), namely to show that there exists a sequence of σ -fields $\{\mathcal{U}_n : n \geq 1\}$ verifying $\mathcal{U}_n \subseteq \mathcal{U}_{n+1} \cap \mathcal{F}_{t_n}^{\pi^{(n)}}(X_W)$ and $\bigvee_n \mathcal{U}_n = \sigma(W)$. The sequence

$$\mathcal{U}_n \triangleq \sigma\{W_u - W_s : 1 - t_n \leq s < u \leq t_n\},$$

which is increasing and such that $\mathcal{U}_n \subseteq \mathcal{F}_{t_n}^{\pi^{(n)}}(X_W)$ (see (135)), verifies these properties. Therefore, Theorem 22 applies, and we obtain the stable convergence result (132). ■

Remarks – (a) The sequence $A'_n \triangleq \sqrt{n}(2n+1)A_n$, although stably convergent and such that (136) is verified, *does not* admit a limit in probability. Indeed, simple computations show that A'_n is not a Cauchy sequence in $L^2(\mathbb{P})$ and therefore, since the L^2 and L^0 topologies coincide on any finite sum of Wiener chaoses (see e.g. [32]), A'_n cannot converge in probability.

(b) Observe that, by using the notation introduced above (see e.g. 15), $\text{rank}(\pi^o) = \text{rank}(\pi^e) = 1$.

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References

- [1] Billingsley P. (1969). *Convergence of probability measures*. Birkhäuser.
- [2] Brodskii M.S. (1971), *Triangular and Jordan Representations of Linear Operators*. Transl. Math. Monographs **32**, AMS, Providence.
- [3] Cohen S. and Taqqu M. (2004). Small and large scale behavior of the Poissonized Telecom process. *Methodology and Computing in Applied Probability* **6**, 363-379
- [4] Engel D.D. (1982). The multiple stochastic integral. *Mem. Am. Math. Society* **38**, 1-82.
- [5] Feigin P. D. (1985). Stable convergence of semimartingales. *Stochastic Processes and their Applications* **19**, 125-134.
- [6] Giné E. and de la Peña V.H. (1999). *Decoupling*. Springer Verlag.
- [7] Jacod, J. (1984). Une généralisation des semimartingales : les processus admettant un processus à accroissements indépendants tangent. In: *Séminaire de probabilités XX??*, 91-118. LNM **1059**, Springer Verlag.
- [8] Jacod, J. (2002). On processes with conditional independent increments and stable convergence in law. In: *Séminaire de probabilités XXXVI*, 383-401. LNM **1801**, Springer Verlag.

- [9] Jacod J., Kłopotowski A. and Mémin J. (1982). Théorème de la limite centrale et convergence fonctionnelle vers un processus à accroissements indépendants : la méthode des martingales. *Annales de l'I.H.P. section B*, **1**, 1-45
- [10] Jacod J. and Sadi H. (1997). Processus admettant un processus à accroissements indépendants tangent : cas général. In: *Séminaire de Probabilités XXI*, 479-514. LNM **1247**, Springer Verlag.
- [11] Jacod J. and Shiryaev A.N. (1987). *Limit Theorems for Stochastic Processes*. Springer, Berlin.
- [12] Jakubowski A. (1986). Principle of conditioning in limit theorems for sums of random variables. *The Annals of Probability* **11**(3), 902-915
- [13] Y. Kabanov (1975). On extended stochastic integrals. *Theory of Probability and its applications* **20**, 710-722.
- [14] Kwapien S. and Woyczyński W.A. (1991). Semimartingale integrals via decoupling inequalities and tangent processes. *Probability and Mathematical Statistics* **12**(2), 165-200
- [15] Kwapien S. and Woyczyński W.A. (1992). *Random Series and Stochastic Integrals: Single and Multiple*. Birkhäuser.
- [16] Lipster R.Sh. and Shiryaev A.N. (1980). A functional central limit theorem for semimartingales. *Theory of Probability and Applications* **XXV**, 667-688.
- [17] Nualart D. (1995). *The Malliavin Calculus and related topics*. Springer Verlag.
- [18] Nualart D. (1998). Analysis on Wiener space and anticipating stochastic calculus. In: *Lectures on Probability Theory and Statistics. École de probabilités de St. Flour XXV (1995)*, LNM **1690**, Springer, 123-227.
- [19] Nualart D. and Peccati G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *The Annals of Probability* **33**(1), 177-193.
- [20] Nualart D. and Schoutens W. (2000). Chaotic and predictable representation for Lévy processes. *Stochastic Processes and their Applications* **90**, 109-122.
- [21] Nualart D. and J. Vives J. (1990). Anticipative calculus for the Poisson space based on the Fock space. *Séminaire de Probabilités XXIV*, LNM **1426**, Springer, 154-165.
- [22] Peccati G., Thieullen M. and Tudor C.A. (2006). Martingale structure of Skorohod integral processes. *The Annals of Probability*, to appear
- [23] Peccati G. and Taqqu M. (2006). Stable convergence of multiple Wiener-Itô integrals. Preprint.
- [24] Peccati G. and Tudor C.A. (2004). Gaussian limits for vector-valued multiple stochastic integrals. In: *Séminaire de Probabilités XXXVIII*, 247-262, Springer Verlag
- [25] Peccati G. and Yor M. (2004). Four limit theorems for quadratic functionals of Brownian motion and Brownian bridge. In: *Asymptotic Methods in Stochastics*, AMS, Fields Institute Communication Series, 75-87
- [26] Protter P. (1992). *Stochastic Integration and Differential Equation*. Springer-Verlag, Berlin-New York
- [27] Rajput B.S. and Rosinski J. (1989). Spectral representation of infinitely divisible processes. *Probability Theory and Related Fields* **82**, 451-487
- [28] Revuz D. and Yor M. (1999). *Continuous martingales and Brownian motion*. Springer-Verlag
- [29] Rota G.-C. and Wallstrom C. (1997). Stochastic integrals: a combinatorial approach. *The Annals of Probability*, **25**(3), 1257-1283.

- [30] Samorodnitsky G. and Taqqu M. (1994). *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*, Chapman and Hall. New York, London.
- [31] Sato K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics **68**. Cambridge University Press.
- [32] Schreiber M. (1969). Fermeture en probabilité de certains sous-espaces d'un espace L^2 . *Zeitschrift W. v. G.* **14**, 36-48
- [33] Surgailis D. (2000). CLTs for Polynomials of Linear Sequences: Diagram Formulae with Applications. In: *Long Range Dependence*, 111-128, Birkhäuser.
- [34] Surgailis D. (2000). Non-CLT's: U-Statistics, Multinomial Formula and Approximations of Multiple Wiener-Itô integrals. In: *Long Range Dependence*, 129-142, Birkhäuser.
- [35] Üstünel A.S. and Zakai M. (1997). The Construction of Filtrations on Abstract Wiener Space. *Journal of Functional Analysis*, **143**, 10-32.
- [36] Wu L.M. (1990). Un traitement unifié de la représentation des fonctionnelles de Wiener. In: *Séminaire de Probabilités XXIV*, 166-187. LNM **1426**, Springer Verlag.
- [37] Xue, X.-H. (1991). On the principle of conditioning and convergence to mixtures of distributions for sums of dependent random variables. *Stochastic Processes and their Applications* **37**(2), 175-186.
- [38] Yosida K. (1980). *Functional analysis*. Springer-Verlag, Berlin-New York.