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GENERALIZED DE BRUIJN DIGRAPHS AND THE DISTRIBUTION OF PATTERNS IN α-EXPANSIONS*

WOLFGANG STEINER**

Abstract. A generalization of de Bruijn digraphs is defined for Parry's α-expansions and it is shown that the characteristic polynomial of these graphs is in principle that of α. With the help of this result we prove that certain functionals of α-expansions, e.g. the number of specific digital patterns, satisfy a central limit theorem, which is an extension of a result due to Drmota [3].

1. Introduction

Our starting points are the α-expansion of real numbers due to Parry [14] and the induced linear recurrence (cf. Loraud [12]). Let $\alpha > 1$ be a real number. Then the α-expansion of an arbitrary real number x is given by

$$x = \zeta_1 + \frac{\zeta_2}{\alpha} + \frac{\zeta_3}{\alpha^2} + \cdots ,$$

where $\zeta_1 = [x]$, the integer part of x, and the other digits are computed with the transformation $T(x) = \{\alpha x\}$ (where $\{x\}$ denotes the fractional part of x): $\zeta_n = [\alpha T^{n-2}(x)]$. Then the digits ζ_j satisfy

$$(\zeta_n, \zeta_{n+1}, \dots) < (a_1, a_2, \dots) \text{ for } n \geq 2,$$

where the a_j are the digits of the α-expansion of α, i.e.

$$\alpha = a_1 + \frac{a_2}{\alpha} + \frac{a_3}{\alpha^2} + \cdots ,$$

and "$<$" denotes the lexicographic order. In particular we have

$$(a_n, a_{n+1}, \dots) < (a_1, a_2, \dots) \text{ for } n \geq 2. \tag{1.1}$$

Conversely, if a sequence $(a_1, a_2, \dots)$ satisfies (1.1), we have a real number α with α-expansion $(a_1, a_2, \dots)$.

Those α which have recurrent "tails" in their α-expansions, i.e. $a_{j+m} = a_j$ for all $j > n$ for some integers n and m, are called α-numbers. The α-numbers which have a finite α-expansion are called simple α-numbers.

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A simple α-number ($a_r \neq 0$, $a_j = 0$ for all $j > r$) is a root of the polynomial

$$x^r - a_1 x^{r-1} - \cdots - a_{r-1}x - a_r$$

which is called characteristic polynomial of α. If α is a non-simple α-number, it is a root of the polynomial

$$(x^{n+m} - a_1 x^{n+m-1} - \cdots - a_{n+m-1}x - a_{n+m}) - (x^n - a_1 x^{n-1} - \cdots - a_{n-1}x - a_n).$$

This polynomial is called characteristic polynomial, if n and m are minimal with this property.

For the induced linear recurrence we distinguish between simple α-numbers and other real numbers. For simple α-numbers we define

$$G_0 = 1,\ G_j = \begin{cases} \sum\limits_{i=1}^{j} a_i G_{j-i} + 1 & \text{for } j < r \\ \sum\limits_{i=1}^{r} a_i G_{j-i} & \text{for } j \geq r \end{cases} \tag{1.2}$$

and for the others

$$G_0 = 1,\ G_j = \sum_{i=1}^{j} a_i G_{j-i} + 1 \text{ for } j > 0. \tag{1.3}$$

With $G = (G_n)$, every non-negative integer n has a (unique) proper G-ary digital expansion

$$n = \sum_{j \geq 0} \epsilon_j(n) G_j$$

with integer digits $\epsilon_j(n) \geq 0$ such that

$$\sum_{j=0}^{k} \epsilon_j(n) G_j < G_{k+1} \text{ for } k \geq 0.$$

The digits $\epsilon_j = \epsilon_j(n)$ satisfy

$$(\epsilon_k, \epsilon_{k-1}, \dots) < (a_1, a_2, \dots) \text{ for } k \geq 0 \tag{1.4}$$

(cf. [12]), where we have set $\epsilon_j = 0$ for all $j < 0$. Conversely, a sequence $(\epsilon_0, \epsilon_1, \dots)$ is the digital expansion of an integer n if (1.4) holds and $\epsilon_j > 0$ only for a finite number of $j \geq 0$.

We will study functions depending on subblocks of these digital expansions. Let

$$\mathcal{B}_L = \{(\epsilon_{L-1}(n), \epsilon_{L-2}(n), \dots, \epsilon_0(n)) : n < G_L\}$$

be the set of blocks $B \in \{0, 1, \dots, a_1\}^L$ of length L which actually occur in G-ary digital expansions. Let $F : \mathcal{B}_{L+1} \to \mathbb{R}$ be any given function (for some $L \geq 0$) with $F(0, 0, \dots, 0) = 0$. Furthermore, set

$$s_F(n) = \sum_{j \geq 0} F\left(\epsilon_{j+L}(n), \epsilon_{j+L-1}(n), \dots, \epsilon_j(n)\right).$$

This means that we consider a weighted sum over all subsequent digital patterns of length $L+1$ of the digital expansion of n. For example, for $L=0$ and $F(\epsilon)=\epsilon$ we just obtain the sum-of-digits function, or if $L=1$ and $F(\epsilon,\eta)=1-\delta_{\epsilon,\eta}$ ($\delta_{x,y}$ denoting the Kronecker delta) then $s_F(n)$ is just counting the number of times that a digit is different from the preceding one etc.

In order to get an insight into the distribution of $s_F(n)$, it is convenient to consider a related sequence of random variables X_N, $N \geq 1$, defined by

$$\mathbf{Pr}[X_N \leq x] = \frac{1}{N}|\{n < N : s_F(n) \leq x\}|$$

In Section 3 we will show that Drmota's methods in [3] can be applied to prove asymptotic normality of the distribution of X_N. Drmota showed this for the special case of finite recurrences of the type (1.2) with $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$. It should be noted that the case $L=0$ for general α is treated by Drmota and Gajdosik [4].

In the case of simple α-numbers we will use a generalization of de Bruijn digraphs (Definition 2.1) and, in particular, the property that the characteristic polynomial of these graphs is the characteristic polynomial of α multiplied with a factor x^n. This property has been conjectured by Drmota [3] and is shown in Section 2.

Remark 1.1. Sometimes different generalizations of de Bruijn digraphs can be found in the literature. Imase and Itoh [9] and Reddy, Pradhan and Kuhl [16] introduced independently the graph $G_B(n,d)$, where $n > d$, the set of vertices is $\{0,1,\ldots,n-1\}$ and the set of edges consists of

$$i \to di + r \pmod{n} \text{ for } 0 \leq i < n,\ 0 \leq r < d.$$

Later, Imase and Itoh [10] introduced also the graph $G_l(n,d)$ whose set of vertices is the same as that of $G_B(n,d)$ and the set of edges consists of

$$i \to d(n-1-i) + r \pmod{n} \text{ for } 0 \leq i < n,\ 0 \leq r < d.$$

Du and Hwang [5] showed that these graphs have some of the properties of de Bruijn digraphs. Their characteristic polynomial has been given by Xueliang and Fuji [18].

2. Generalized de Bruijn digraphs

For a q-ary expansion which is a special case of Parry's expansion with $\alpha = q$ an integer, we have $\mathcal{B}_L = \{0,1,\ldots,q-1\}^L$. Then $\mathcal{B}_L$ is the set of vertices of a directed de Bruijn digraph with edges $B = (\eta_1,\ldots,\eta_L) \to C = (\theta_1,\ldots,\theta_L)$ where

$$(\eta_2,\ldots,\eta_L) = (\theta_1,\ldots,\theta_{L-1}). \tag{2.1}$$

With this characterization of de Bruijn digraphs we can generalize them on Parry's α-expansions with simple α-numbers:

Definition 2.1. Let $(a_1, a_2, \dots, a_r)$ be a r-tuple which satisfies (1.1) (if we set $a_n := 0$ for $n > r$) and $\mathcal{B}_L$ be the set of blocks $(\eta_1, \dots, \eta_L)$ which satisfy

$$(\eta_k, \eta_{k+1}, \dots) < (a_1, a_2, \dots) \text{ for } k \geq 1$$

(if we set $\eta_n := 0$ for $n > L$). Then the *generalized de Bruijn digraph* of $\mathcal{B}_L$ is defined by the set of vertices $\mathcal{B}_L$ and the edges $B \to C$ where $B = (\eta_1, \dots, \eta_L)$ and $C = (\theta_1, \dots, \theta_L)$ satisfy (2.1) and

$$(\eta_1, \dots, \eta_L, \theta_L) \in \mathcal{B}_{L+1}. \tag{2.2}$$

Remark 2.1. Because of (1.4), this definition of $\mathcal{B}_L$ is equivalent to the one given in the Introduction.

Remark 2.2. If (2.1) holds and $L \geq r$, then (2.2) is automatically satisfied. For $L = r - 1$, the only exceptions are

$$(\eta_1, \dots, \eta_{r-1}) = (a_1, \dots, a_{r-1}), \qquad (\theta_1, \dots, \theta_{r-1}) = (a_2, \dots, a_{r-1}, x)$$

with $a_r \leq x \leq a_1$.

Remark 2.3. An important property of de Bruijn digraphs is that the line graph of the graph of blocks of length L is the graph of blocks of length $L + 1$. For our generalization, this property holds if we have $L \geq r - 1$.

Denote by $\mathbf{A}_L$ the adjacency matrix of the generalized de Bruijn digraph of $\mathcal{B}_L$.

Example 2.1. For the q-ary expansion with $q = 2$ (or equivalently the G-ary expansion with $r = 1$, $a_1 = 2$), we have $\mathcal{B}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$ and

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Example 2.2. In the case of the Zeckendorf expansion ($r = 2$, $a_1 = a_2 = 1$, the G_j are the Fibonacci numbers) we have $\mathcal{B}_2 = \{(0,0), (0,1), (1,0)\}$ and

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of these graphs is strongly connected to the characteristic polynomial of the corresponding simple α-number:

Theorem 2.1. *For $L \geq r-1$, the characteristic polynomial of the adjacency matrix $\mathbf{A}_L$ of the generalized de Bruijn digraph of $\mathcal{B}_L$ is*

$$\chi(\mathbf{A}_L)(x) = x^{G_L - r} p(x),$$

where $G = (G_j)_{j \geq 0}$ is defined by the finite linear recurrence (1.2) and

$$p(x) = x^r - a_1 x^{r-1} - a_2 x^{r-2} - \dots - a_{r-1} x - a_r$$

is the characteristic polynomial of the linear recurrence and of the corresponding simple α-number.

Proof. First we remark that $\#(\mathcal{B}_L) = G_L$ and for each $B \in \mathcal{B}_L$

$$B = (\epsilon_{L-1}(i-1), \epsilon_{L-2}(i-1), \dots, \epsilon_0(i-1))$$

for some $i \in \{1, 2, \dots, G_L\}$ where the ϵ_j are the digits in the G-ary expansion.

The conditions (2.1) and (2.2) can thus be written for $i, j \in \{1, 2, \dots, G_L\}$ as

$$(\epsilon_{L-2}(i-1), \dots, \epsilon_0(i-1)) = (\epsilon_{L-1}(j-1), \dots, \epsilon_1(j-1)) \tag{2.3}$$

and

$$(\epsilon_{L-1}(i-1), \epsilon_{L-1}(j-1), \dots, \epsilon_0(j-1)) \in \mathcal{B}_{L+1} \tag{2.4}$$

respectively.

Therefore the coefficients of $\mathbf{A}_L = (a_{ij}^{(L)})_{1 \le i,j \le G_L}$ are

$$a_{ij}^{(L)} = \begin{cases} 1 & \text{if (2.3) and (2.4) hold} \\ 0 & \text{otherwise.} \end{cases}$$

First assume $L \ge r$. Then we can omit (2.4) (cf. Remark 2.2) and have

$$a_{i+kG_{L-1},j}^{(L)} = a_{i,j}^{(L)}.$$

We define the matrix $\mathbf{P}_L := (p_{ij}^{(L)})_{1 \le i,j \le G_L}$ for $L \ge r$ by

$$p_{ij}^{(L)} := \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } j \le G_{L-1},\ i = j + kG_{L-1},\ k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathbf{P}_L^{-1} = (p_{ij}^{(-L)})_{1 \le i,j \le G_L}$ has the coefficients

$$p_{ij}^{(-L)} = \begin{cases} 1 & \text{if } i = j \\ 1 & \text{if } j \le G_{L-1},\ i = j + kG_{L-1},\ k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

With $\mathbf{P}_L$ we define a matrix similar to $\mathbf{A}_L$ by

$$\mathbf{A}'_L = (a_{ij}^{\prime(L)})_{1 \le i,j \le G_L} := \mathbf{P}_L \mathbf{A}_L \mathbf{P}_L^{-1}.$$

In the construction of $\mathbf{A}'_L$ the rows i of $\mathbf{A}_L$ are subtracted from the rows $i + kG_{L-1}$ and the columns $i + kG_{L-1}$ are added to the columns i. Therefore

$$a_{ij}^{\prime(L)} = 0 \text{ for all } i \in \{G_{L-1}+1, G_{L-1}+2, \dots, G_L\}, j \in \{1, 2, \dots, G_L\}$$

and it suffices to continue with the matrix

$$\mathbf{A}_{L-1} = (a_{ij}^{(L-1)})_{1 \le i,j \le G_{L-1}} := \mathbf{A}'_L \begin{pmatrix} 1 & 2 & \dots & G_{L-1} \\ 1 & 2 & \dots & G_{L-1} \end{pmatrix}$$

(this notation means that we take only the rows and columns $1, 2, \dots, G_{L-1}$ of $\mathbf{A}'_L$) because of

$$\chi(\mathbf{A}_L)(x) = \chi(\mathbf{A}'_L)(x) = x^{G_L - G_{L-1}} \chi(\mathbf{A}_{L-1})(x).$$

The so defined $\mathbf{A}_{L-1}$ is the adjacency matrix of the generalized de Bruijn digraph of $\mathcal{B}_{L-1}$, because we have $a_{ij}^{(L-1)} = 1$ not only if (2.3) holds, but also for

$$\begin{aligned}(\epsilon_{L-2}(i-1),\dots,\epsilon_0(i-1)) &= (\epsilon_{L-1}(j+kG_{L-1}-1),\dots,\epsilon_1(j+kG_{L-1}-1))\\ &= (k,\epsilon_{L-2}(j-1),\dots,\epsilon_1(j-1))\end{aligned}$$

with a $k \in \{0,1,\dots,a_1\}$ (and $j+kG_{L-1} \le G_L$), and therefore

$$a_{ij}^{(L-1)} = [(\epsilon_{L-3}(i-1),\dots,\epsilon_0(i-1)) = (\epsilon_{L-2}(j-1),\dots,\epsilon_1(j-1))],$$

where we use the Iversonian notation $[\mathit{expression}] = 1$ if *expresssion* is true and 0 otherwise. For $L = r$ we have to check additionally $a_{ij}^{(r-1)} = 0$ for

$$(\epsilon_{r-2}(i-1),\dots,\epsilon_0(i-1)) = (a_1,a_2,\dots,a_{r-1}),$$
$$(\epsilon_{r-2}(j-1),\dots,\epsilon_0(j-1)) = (a_2,\dots,a_{r-1},x)$$

with $a_r \le x \le a_1$ (cf. Remark 2.2). This is true, because for these i and j, $a_{ij}^{(r-1)} = 1$ would imply that we have a $j' = j + kG_{r-1}$ with $a_{ij'}^{(r)} = 1$, i.e.

$$(\epsilon_{r-1}(j'-1),\dots,\epsilon_0(j'-1)) = (a_1,a_2,\dots,a_{r-1},x),$$

but this violates (1.4) and is therefore impossible.

Hence we iterate the construction until we get $\mathbf{A}_{r-1}$ and obtain

$$\chi(\mathbf{A}_L)(x) = x^{G_L-G_{r-1}}\chi(\mathbf{A}_{r-1})(x). \tag{2.5}$$

We continue with $L = r-1$.

For $i < G_{r-1},\ j \le G_{r-1}$ (but not for $i = G_{r-1}$) we have

$$a_{ij}^{(r-1)} = [(\epsilon_{r-3}(i-1),\dots,\epsilon_0(i-1)) = (\epsilon_{r-2}(j-1),\dots,\epsilon_1(j-1))]$$

and

$$a_{i+kG_{r-2},j}^{(r-1)} = a_{i,j}^{(r-1)}.$$

For $1 \le s \le r-1$, we define the matrix $\mathbf{P}_s := (p_{ij}^{(s)})_{1\le i,j\le G_{r-1}}$ by

$$p_{ij}^{(s)} := \begin{cases} 1 & \text{if } i = j\\ -1 & \text{if } j \le G_{s-1},\ i = j+kG_{s-1},\ k>0 \text{ and } i < G_s\\ 0 & \text{otherwise.}\end{cases}$$

With $\mathbf{A}'_{r-1} := \mathbf{P}_{r-1}\mathbf{A}_{r-1}\mathbf{P}_{r-1}^{-1}$, we get

$a_{ij}'^{(r-1)} = 0$ for all $i \in \{G_{r-2}+1, G_{r-2}+2,\dots,G_{r-1}-1\}, j \in \{1,2,\dots,G_{r-1}\}$.

and build the matrix $\mathbf{A}_{r-2} := \mathbf{A}'_{r-1}\binom{1\ 2\ \dots\ G_{r-2}\ G_{r-1}}{1\ 2\ \dots\ G_{r-2}\ G_{r-1}}$, where the numeration of the rows and columns is kept, i.e. $\mathbf{A}_{r-2} = (a_{ij}^{(r-2)})_{i,j\in\{1,2,\dots,G_{r-2},G_{r-1}\}}$. This matrix satisfies

$$\chi(\mathbf{A}_{r-1})(x) = \chi(\mathbf{A}'_{r-1})(x) = x^{G_{r-1}-G_{r-2}-1}\chi(\mathbf{A}_{r-2})(x)$$

and for $i < G_{r-2},\ j \le G_{r-1}$

$$a_{ij}^{(r-2)} = [(\epsilon_{r-4}(i-1),\dots,\epsilon_0(i-1)) = (\epsilon_{r-3}(j-1),\dots,\epsilon_1(j-1))]$$

(but $\mathbf{A}_{r-2}$ is not the adjacency matrix of a generalized de Bruijn digraph).

Hence we iterate this procedure by defining

$$\mathbf{A}_{s-1} := (\mathbf{P}_s\mathbf{A}_s\mathbf{P}_s^{-1})\begin{pmatrix}1 & 2 & \dots & G_{s-1} & G_s & \dots & G_{r-1}\\ 1 & 2 & \dots & G_{s-1} & G_s & \dots & G_{r-1}\end{pmatrix}$$

for $1 \le s \le r-2$ and get, for $i < G_s$, $j \le G_s$.

$$a_{ij}^{(s)} = [(\epsilon_{s-2}(i-1), \dots, \epsilon_0(i-1)) = (\epsilon_{s-1}(j-1), \dots, \epsilon_1(j-1))].$$

Therefore we have

$$\chi(\mathbf{A}_{r-1})(x) = x^{G_{r-1}-r}\chi(\mathbf{A}_0)(x). \tag{2.6}$$

The definitions of $\mathbf{P}_s$ and $\mathbf{A}_{s-1}$ imply for $1 \le s \le r-1$, $j \le G_{s-1}$ and $i \in \{1, 2, \dots, G_{s-1}, G_s, \dots, G_{r-1}\}$,

$$a_{ij}^{(s-1)} = \sum_{j'=1}^{G_s-1}[\epsilon_s(j') \ge \epsilon_s(j)][(\epsilon_{s-2}(j'), \dots, \epsilon_0(j')) = (\epsilon_{s-2}(j), \dots, \epsilon_0(j))]a_{ij'}^{(s)}$$
$$= \dots = \sum_{j'=1}^{G_{r-1}-1}[\epsilon_s(j') \ge \epsilon_s(j)][(\epsilon_{s-2}(j'), \dots, \epsilon_0(j')) = (\epsilon_{s-2}(j), \dots, \epsilon_0(j))]a_{ij'}^{(r-1)}$$

and thus

$$a_{iG_s}^{(0)} = a_{iG_s}^{(s)} = \sum_{\substack{1\le j<G_{r-1}:\epsilon_s(j)\ge 1,\\ (\epsilon_{s-1}(j),\dots,\epsilon_0(j))=(0,0,\dots,0)}} a_{ij}^{(r-1)}. \tag{2.7}$$

Now we can easily calculate $\mathbf{A}_0 = (a_{ij}^{(0)})_{i,j\in\{G_0,G_1,\dots,G_{r-1}\}}$. For $i = G_t$ with $0 \le t \le r-1$ we have

$$(\epsilon_{r-2}(G_t-1), \dots, \epsilon_0(G_t-1)) = (0, \dots, 0, 0, a_1, a_2, \dots, a_t).$$

Therefore we have $a_{G_t,j}^{(r-1)} = 1$ if and only if

$$(\epsilon_{r-2}(j-1), \dots, \epsilon_0(j-1)) = (0, \dots, 0, a_1, a_2, \dots, a_t, x) \tag{2.8}$$

with $0 \le x \le a_{t+1}$ for $t < r-1$ and if and only if

$$(\epsilon_{r-2}(j-1), \dots, \epsilon_0(j-1)) = (a_2, a_3, \dots, a_{r-1}, x) \tag{2.9}$$

with $0 \le x < a_r$ for $t = r-1$ respectively.

For the first column, i.e. $s = 0$, the sum in (2.7) runs over all j with $\epsilon_0(j) \ge 1$. $x = a_{t+1}$ in (2.8) implies $j = G_{t+1}$ and $\epsilon_0(j) = 0$, otherwise $\epsilon_0(j) = x+1 \ge 1$. For $i = G_t$ we have therefore a_{t+1} terms $a_{ij}^{(r-1)} = 1$ and $a_{G_tG_0}^{(0)} = a_{t+1}$.

For $1 \le s \le r-1$ and $i = G_t$, the conditions on j in (2.7) imply

$$(\epsilon_{s-1}(j-1), \dots, \epsilon_0(j-1)) = (a_1, \dots, a_s)$$

and, with (2.8) and (2.9), we have $a^{(r-1)}_{G_t j} = 1$ for at most one j. Together with (1.1) we get $x = a_{t+1}$. We have $a^{(r-1)}_{G_{r-1} j} = 0$ for this j and hence $a^{(0)}_{G_{r-1} G_s} = 0$. For $0 \le t < r-1$, the G-ary expansion of this j is

$$(\epsilon_{t+1}(j), \dots, \epsilon_0(j)) = (1, 0, 0, \dots, 0).$$

Thus we obtain $a^{(0)}_{G_t G_s} = [t+1 = s]$ for $s > 0$.

Recapitulating, $\mathbf{A}_0$ has the form

$$\mathbf{A}_0 = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_r & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

and its characteristic polynomial is clearly $p(x)$. With (2.5) and (2.6), the theorem is proved. □

Remark 2.4. For de Bruijn digraphs we have $\chi(\mathbf{A}_L)(x) = x^{q^L - 1}(x - q)$.

Remark 2.5. For general digraphs D, the characteristic polynomial of the line graph $\mathcal{L}(D)$ is a multiple of that of D:

$$\chi(\mathcal{L}(D))(x) = \chi(D)(x) x^{e(D) - v(D)},$$

where e(D) denotes the number of edges and v(D) the number of vertices. This property could be used in the first part of the proof of Theorem 2.1 but anyway our construction is needed for the second part of the proof.

3. Asymptotic Properties of Functions Depending on Subblocks of α-Expansions

Now we study the random variables X_N defined in the introduction. Expected value and variance of X_N are given by

$$\mathbf{E}X_N = \frac{1}{N} \sum_{n<N} s_F(n) \quad \text{and by} \quad \mathbf{V}X_N = \frac{1}{N} \sum_{n<N} (s_F(n) - \mathbf{E}X_N)^2 \tag{3.1}$$

We introduce the function

$$c_N(z) = \sum_{n<N} z^{s_F(n)}$$

and consider for any block $B = (\eta_1, \dots, \eta_L) \in \mathcal{B}_L$ the functions

$$a_j^B(z) := \sum_{n<G_j, (\epsilon_{j-1}(n), \dots, \epsilon_{j-L}(n)) = B} z^{s_F(n)}.$$

Then

$$\sum_{B \in \mathcal{B}_L} a_j^B(z) = c_N(z).$$

In order to obtain recurrent relations for the functions a_j^B we need the following notation:

For $B = (\eta_1, \dots, \eta_L) \in \mathcal{B}_L$ let $B' = (\eta_2, \dots, \eta_L)$ denote the block consisting of the last $L-1$ elements of B and η_B the first element η_1, i.e. $B = (\eta_B, B')$. (Similarly $'B = (\eta_1, \dots, \eta_{L-1})$.) Furthermore, for $(\epsilon, B) = (\epsilon, \eta_1, \dots, \eta_L) \in \mathcal{B}_{L+1}$ set

$$\begin{aligned}\kappa(\epsilon, B) &= \sum_{i=0}^{L-1} (F(0, \dots, 0, \epsilon, \eta_1, \dots, \eta_{L-i}) - F(0, \dots, 0, 0, \eta_1, \dots, \eta_{L-i})) + \\ &\quad F(0, \dots, 0, 0, \epsilon).\end{aligned}$$

Note that $\kappa(0, B) = 0$.

3.1. **Simple α-numbers.** In the case of simple α-numbers, we may assume, without loss of generality, that $L \geq r-1$. (If we are only interested in $L+1$ subsequent digits with $L < r-1$, then we consider a new function $\tilde{F} : \mathcal{B}_r \to \mathbf{R}$ that does not depend on the first $(r-L-1)$ digits.)

Lemma 3.1. *The functions $a_j^B(z)$, $j > 0$, are recursively given by*

$$a_j^B(z) = \sum_{C \in \mathcal{B}_L:\ 'C = B',\ (\eta_B, C) \in \mathcal{B}_{L+1}} a_{j-1}^C(z) z^{\kappa(\eta_B, C)}.$$

Proof. The set

$$\{n < G_j : (\epsilon_{j-1}(n), \dots, \epsilon_{j-L}(n)) = B\}$$

is divided into subsets of the form

$$\{n < G_j : \epsilon_{j-1}(n) = \eta_B,\ (\epsilon_{j-2}(n), \epsilon_{j-3}(n), \dots, \epsilon_{j-L-1}(n)) = (B', \epsilon) = C\} =$$

$$\{n < G_{j-1} : (\epsilon_{j-2}(n), \epsilon_{j-3}(n), \dots, \epsilon_{j-L-1}(n)) = C,\ (\eta_B, C) \in \mathcal{B}_{L+1}\} + \eta_B G_{j-1}.$$

Because of $\{(\eta_B, C) : B, C \in \mathcal{B}_L, 'C = B'\} \supseteq \mathcal{B}_{L+1}$ we cover all possible cases.

Furthermore, for $n < G_{j-1}$ with $(\epsilon_{j-2}(n), \epsilon_{j-3}(n), \dots, \epsilon_{j-L-1}(n)) = C$ and $(\eta_B, C) \in \mathcal{B}_{L+1}$, we have

$$s_F(n + \eta_B G_{j-1}) = s_F(n) + \kappa(\eta_B, C).$$

□

Corollary 3.1. *The vector $\mathbf{a}_j(z) = (a_j^B(z))_{B \in \mathcal{B}_L}$ satisfies the matrix recursion*

$$\mathbf{a}_j(z) = \mathbf{A}_L(z) \mathbf{a}_{j-1}(z) \qquad (j > 0),$$

where the $G_L \times G_L$-matrix $\mathbf{A}_L(z) = (a_{B,C}(z))_{B,C \in \mathcal{B}_L}$ is given by

$$a_{B,C}(z) = \begin{cases} z^{\kappa(\eta_B, C)} & \text{if } 'C = B' \text{ and } (\eta_B, C) \in \mathcal{B}_{L+1} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbf{A}_L(1)$ is the adjacency matrix of the generalized de Bruijn digraph of $\mathcal{B}_L$, its characteristic polynomial is therefore (Theorem 2.1)

$$\chi(\mathbf{A}_L(1))(x) = x^{G_L - r}(x^r - a_1 x^{r-1} - a_2 x^{r-2} - \cdots - a_{r-1}x - a_r)$$

and α is an eigenvalue of $\mathbf{A}_L(1)$. Lemma 3.2 shows that the other eigenvalues of $\mathbf{A}_L(1)$ have absolute value less than α.

Lemma 3.2. *The conjugates of a simple α-number α with respect to the characteristic polynomial have absolute value less than α.*

Proof. Set

$$g(x) := 1 - x^r p(x^{-1}) = \sum_{j=1}^{r} a_j x^j.$$

If $|x| > \alpha$, then $|g(x^{-1})| \le g(|x^{-1}|) < g(\alpha^{-1}) = 1$ and $p(x) \neq 0$.

If $|x| = \alpha$, $x \neq \alpha$, then we either have $|g(x^{-1})| < g(|x^{-1}|) = 1$ or all powers x^j with $a_j > 0$ have the same argument, which must be different from 0 because of $a_1 > 0$. In both cases we have $g(x^{-1}) \neq 1$ and $p(x) \neq 0$.

Because of $g'(\alpha^{-1}) > 0$, α is a simple root of $p(x)$ and the lemma is proved. □

Now we have all prerequesites for Drmota's proofs of the following lemma and theorems:

Lemma 3.3 (cf. [3], Lemma 3.3)**.** *Let $G(t,z) = \det(t\mathbf{I} - \mathbf{A}_L(z))$ be the characteristic polynomial of the matrix $\mathbf{A}_L(z)$. Then there exists a (complex) neighbourhood of $z = 1$ such that $G(t,z) = 0$ has a unique solution $t = \alpha(z)$ of maximal modulus. Furthermore, the function $\alpha(z)$ is analytic in this neighbourhood.*

Theorem 3.1 (cf. [3], Theorem 2.1)**.**

$$\mathbf{E}X_N = \frac{1}{N}\sum_{n<N} s_F(n) = \mu\frac{\log N}{\log \alpha} + \mathcal{O}(1)$$

and

$$\mathbf{V}X_N = \frac{1}{N}\sum_{n<N}(s_F(n) - \mathbf{E}X_N)^2 = \sigma^2\frac{\log N}{\log \alpha} + \mathcal{O}(1),$$

where

$$\mu = \frac{\alpha'(1)}{\alpha} \text{ and } \sigma^2 = \frac{\alpha''(1)}{\alpha} + \mu - \mu^2.$$

Theorem 3.2 (cf. [3], Theorem 2.2)**.** *If $\sigma^2 \neq 0$, then for every $\varepsilon > 0$*

$$\frac{1}{N}|\{n < N : s_F(n) < \mathbf{E}X_N + x\mathbf{V}X_N\}| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt + \mathcal{O}((\log N)^{-1/2+\varepsilon})$$

uniformly for all real x as $N \to \infty$.

Theorem 3.3 (cf. [3], Theorem 2.2)**.** *If $\sigma^2 \neq 0$, F just attains integer values and*

$$d = \gcd\{\kappa(\epsilon, B) : (\epsilon, B) \in \mathcal{B}_{L+1}\} = 1,$$

then for every $\varepsilon > 0$

$$|\{n < N : s_F(n) = k\}| = \frac{N}{\sqrt{2\pi \mathbf{V} X_N}} \left(\exp\left(-\frac{(k - \mathbf{E}X_N)^2}{2\mathbf{V}X_N} \right) + \mathcal{O}((\log N)^{-1/2+\varepsilon}) \right)$$

uniformly for all non-negative integers k as $N \to \infty$.

3.2. **Non-simple α-numbers.** Now we treat the case of non-simple α-numbers, i.e.

$$(a_{n+m+1}, a_{n+m+2}, \dots) = (a_{n+1}, a_{n+2}, \dots)$$

for some integers $m,\ n$ and $m,\ n$ minimal with this property.

Let F be a function $F : \mathcal{B}_{l+1} \to \mathbf{R}$ as above. We may assume that $l = km \geq n + m - 1$ for some $k \in \mathbb{N}$. (If we are interested in $l+1$ subsequent digits with $l < km$ then we consider a new function $\tilde{F} : \mathcal{B}_{km+1} \to \mathbf{R}$ that does not depend on the first $(km - l)$ digits.)

Lemma 3.4. *For all $l \in \mathbb{N}$ we can find an integer $L \geq l$ such that*

$$(a_j, a_{j+1}, \dots, a_{L+1}) < (a_1, a_2, \dots, a_{L-j+2}) \text{ for all } j \in \{2, \dots, L+1\} \tag{3.2}$$

Proof. If (3.2) holds for $L := l$, we are finished. Otherwise we have a $j \leq l+1$ such that

$$(a_j, a_{j+1}, \dots, a_{l+1}) = (a_1, a_2, \dots, a_{l-j+2})$$

and an integer $g > l + 1$ such that

$$(a_j, a_{j+1}, \dots, a_{g-1}) = (a_1, a_2, \dots, a_{g-j}) \text{ and } a_g < a_{g-j+1}.$$

If (3.2) holds for $L := g - 1$, we are finished. Otherwise we have a $j' \leq g$ such that

$$(a_{j'}, a_{j'+1}, \dots, a_g) = (a_1, a_2, \dots, a_{g-j'+1}).$$

For $j' \geq j$, we had

$$(a_{j'}, a_{j'+1}, \dots, a_g) < (a_{j'-j+1}, a_{j'-j+2}, \dots, a_{g-j+1}) \leq (a_1, a_2, \dots, a_{g-j'+1}).$$

Therefore $j' < j$. We can find $g' > g$ such that

$$(a_{j'}, a_{j'+1}, \dots, a_{g'-1}) = (a_1, a_2, \dots, a_{g'-j'}),\ a_{g'} < a_{g'-j'+1}$$

and repeat this procedure.

Since $j > j' > j'' > \dots > 1$, we find a L that satisfies (3.2) after a finite number of steps. □

Remark 3.1. If (3.2) holds for L, it holds for $L + m$ since

$$(a_j, \dots, a_{L+m+1}) = (a_{j-m}, \dots, a_{L+1}) \text{ for all } j \in \{L+2, \dots, L+m+1\}$$

and, by induction, for $L + km$.

Now we consider the functions

$$a_j^B(z) := \sum_{n<G_j,(\epsilon_{j-1}(n),\dots,\epsilon_{j-L}(n))=B} z^{s_F(n)} \qquad (B \in \mathcal{B}_L).$$

Lemma 3.5. *Let L satisfy (3.2) and $L \ge km \ge n+m-1$. Then the functions a_j^B, $j > L$, $B \in \mathcal{B}_L$ are recursively given by*

$$a_j^B(z) = \sum_{\substack{C\in\mathcal{B}_L:\ 'C=B',\\ (\eta_B,C)\in\mathcal{B}_{L+1}}} a_{j-1}^C(z)\ z^{\kappa(\eta_B,C)}$$

if $B \ne (a_1, a_2, \dots, a_L)$ and

(3.3)

$$a_j^{(a_1,\dots,a_L)}(z) = \sum_{\substack{C=(a_2,\dots,a_L,\eta_L),\\ \eta_L\le a_{L+1}}} a_{j-1}^C(z)\ z^{\kappa(a_1,C)} - \sum_{D\in\mathcal{C}_{km}} b_{j-L-1}^D(z)\ z^{\lambda(a_1,\dots,a_{L+1},D)},$$

where

$$\mathcal{C}_{km} := \{D \in \mathcal{B}_{km} : D \ge (a_{L+2}, a_{L+3}, \dots, a_{L+km+1})\},$$

$$b_j^D(z) := \sum_{\substack{i<G_j:\ (\epsilon_{j-1}(i),\dots,\epsilon_{j-km}(i))=D,\\ (\epsilon_{j-1}(i),\dots,\epsilon_0(i))>(a_{L+2},\dots,a_{L+j+1})}} z^{s_F(i)}$$

and

$$\lambda(\theta_1, \theta_2, \dots, \theta_{L+1}, D) := s_F(n_1) - s_F(n_2)$$

with

$$D = (\zeta_1, \zeta_2, \dots, \zeta_{km}), \qquad \epsilon_i(n_1) = \epsilon_i(n_2) = 0 \text{ for all } i \ge km + L + 1$$

$$(\epsilon_{km+L}(n_1), \epsilon_{km+L-1}(n_1), \dots, \epsilon_0(n_1)) = (\theta_1, \theta_2, \dots, \theta_{L+1}, \zeta_1, \zeta_2, \dots, \zeta_{km}),$$

$$(\epsilon_{km+L}(n_2), \epsilon_{km+L-1}(n_2), \dots, \epsilon_0(n_2)) = (0, \dots, 0, \zeta_1, \zeta_2, \dots, \zeta_{km}).$$

The functions $b_j^D(z)$, $j \ge km$, $D \in \mathcal{C}_{km}$ are recursively given by

$$b_j^D(z) = \begin{cases} \displaystyle\sum_{B\in\mathcal{B}_L:\ (\eta_1,\dots,\eta_{km})=D} a_j^B(z) & \text{if } D > (a_{L+2}, \dots, a_{L+km+1}) \\ \displaystyle\sum_{E\in\mathcal{C}_{km}} b_{j-km}^E(z) z^{\lambda(0,\dots,0,a_{L+2},\dots,a_{L+km+1},E)} & \text{if } D = (a_{L+2}, \dots, a_{L+km+1}) \end{cases}$$

Proof. The proof of the first equation is the same as that of Lemma 3.1. We just have to check

$$(\eta_B, \epsilon_{j-2}(n), \dots, \epsilon_0(n), 0, \dots) < (a_1, a_2, \dots) \tag{3.4}$$

for $C = (\epsilon_{j-2}(n), \dots, \epsilon_{j-L-1}(n))$, $'C = B'$, $(\eta_B, C) \in \mathcal{B}_{L+1}$.

(3.4) can only be violated, if $(\eta_B, C) = (a_1, \dots, a_{L+1})$ which implies $B = (a_1, \dots, a_L)$. We have

$$a_j^{(a_1,\dots,a_L)}(z) = \sum_{\substack{(\varphi_{j-1},\dots,\varphi_0):\\ (\varphi_{j-1},\dots,\varphi_{j-L})=(a_1,\dots,a_L),\ \varphi_{j-L-1}\le a_{L+1},\\ (\varphi_{j-2},\dots,\varphi_0)\in\mathcal{B}_{j-1}}} z^{s_F(\varphi_{j-1},\dots,\varphi_0)} - \tag{3.5}$$
$$\sum_{\substack{(\varphi_{j-1},\dots,\varphi_0):\\ (\varphi_{j-1},\dots,\varphi_{j-L-1})=(a_1,\dots,a_{L+1}),\\ (\varphi_{j-2},\dots,\varphi_0)\in\mathcal{B}_{j-1},\\ (\varphi_{j-L-2},\dots,\varphi_0)>(a_{L+2},\dots,a_j)}} z^{s_F(\varphi_{j-1},\dots,\varphi_0)}$$

where $s_F(\varphi_{j-1}, \dots, \varphi_0) := s_F(n)$ for the (unique) integer n with G-ary expansion $(\varphi_{j-1}, \dots, \varphi_0)$ (and $\varphi_i = 0$ for all $i \ge j$).

The first sum of (3.5) is equal to the first sum of (3.3) (cf. Lemma 3.1 and consider (3.2)). The patterns of the second sum of (3.5) are exactly those of the first sum which do not satisfy $(\varphi_{j-1}, \dots, \varphi_0) \in \mathcal{B}_j$. Because of (3.2) the choice of patterns $(\varphi_{j-L-2}, \dots, \varphi_0)$ in the second sum of (3.5) is not influenced by the patterns $(\varphi_{j-1}, \dots, \varphi_{j-L-1})$. Therefore this sum is equal to the second sum of (3.3).

The equation for $b_j^D(z)$, $D > (a_{L+2}, \dots, a_{L+km+1})$ is clear. For $D = (a_{L+2}, \dots, a_{L+km+1})$ we have to consider Remark 3.1 and that $(a_{L+km+2}, \dots, a_{L+2km+1}) = (a_{L+2}, \dots, a_{L+km+1})$. □

Corollary 3.2. *The vector*

$$\mathbf{a}_j(z) = (a_j^{B_1}(z), \dots, a_j^{B_{G_L}}(z), b_{j-1}^{D_1}(z), \dots, b_{j-1}^{D_M}(z), \dots, b_{j-L}^{D_1}(z), \dots, b_{j-L}^{D_M}(z))^T$$

with

$$B_i := (\epsilon_{L-1}(i-1), \dots, \epsilon_0(i-1)), \qquad M := \#(\mathcal{C}_{km}),$$

$$D_1 := (a_{L+2}, \dots, a_{L+km+1}),\ \dots,\ D_M := (a_1, \dots, a_{km})$$

satisfies the matrix recursion

$$\mathbf{a}_j(z) = \mathbf{A}_L(z)\mathbf{a}_{j-1}(z) \qquad (j > L)$$

where $\mathbf{A}_L(z) = (a_{i,j}(z))_{1\le i,j\le G_L+LM}$ *is given by*

$$a_{i,j}(z) = \begin{cases} z^{\kappa(\eta_{B_i}, B_j)} & \text{if } i,\ j \le G_L,\ {}'B_j = B_i', \\ & (\eta_{B_i}, B_j) \in \mathcal{B}_{L+1} \\ -z^{\lambda(a_1,\dots,a_{L+1}, D_{j-G_L-(L-1)M})} & \text{if } i = G_L,\ j > G_L + (L-1)M \\ z^{\lambda(0,\dots,0,a_{L+2},\dots,a_{L+km+1}, D_{j-G_L-(km-1)M})} & \text{if } i = G_L+1,\ j \le G_L + kmM, \\ & j > G_L + (km-1)M \\ 1 & \text{if } G_L + 2 \le i \le G_L + M, \\ & D_{i-G_L} = (\eta_1, \dots, \eta_{km})(B_j) \\ & \text{or } i > G_L + M,\ j = i - M \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if $L > km$, $\mathbf{A}_L(1)$ has the form

$$\left(\begin{array}{c|c|c|c|c|c|c}
\tilde{\mathbf{A}}_L & 0 & \cdots & \cdots & \cdots & 0 & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & -1 \end{matrix} \\
\hline
\begin{matrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{matrix} & 0 & \cdots & \begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & \cdots & 0 & 0 \\
\hline
0 & E_M & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
\vdots & \ddots & \ddots & \ddots & & & \vdots \\
\hline
\vdots & & \ddots & \ddots & \ddots & & \vdots \\
\hline
\vdots & & & \ddots & \ddots & \ddots & \vdots \\
\hline
0 & \cdots & \cdots & \cdots & 0 & E_M & 0
\end{array}\right)$$

where $\tilde{\mathbf{A}}_L$ is the matrix $\mathbf{A}_L$ of the generalized de Bruijn digraph of the $(L+1)$-tuple $(a_1, a_2, \ldots, a_L, a_{L+1}+1)$ and E_M is the identity matrix of size M.

$\mathbf{A}_{km}(1)$ has the form

$$\left(\begin{array}{c|c|c|c|c|c}
\tilde{\mathbf{A}}_{km} & 0 & \cdots & \cdots & 0 & \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -1 & \cdots & -1 \end{matrix} \\
\hline
\begin{matrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 1 \end{matrix} & 0 & \cdots & \cdots & 0 & \begin{matrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} \\
\hline
0 & E_M & 0 & \cdots & \cdots & 0 \\
\hline
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\hline
\vdots & & \ddots & \ddots & \ddots & \vdots \\
\hline
0 & \cdots & \cdots & 0 & E_M & 0
\end{array}\right)$$

Theorem 3.4. *The characteristic polynomial of* $\mathbf{A}_L(1)$ *is*

$$\chi(\mathbf{A}_L(1))(x) = p(x)\ (x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)\ x^{G_L + LM - km - n} \tag{3.6}$$

where

$$p(x) = (x^{n+m} - a_1 x^{n+m-1} - \cdots - a_{n+m}) - (x^n - a_1 x^{n-1} - \cdots - a_n)$$

is the characteristic polynomial of α.

Proof. First we construct a matrix $\mathbf{A}' = (a'_{ij})_{1\le i,j\le G_L+L}$ with

$$\chi(\mathbf{A}_L(1))(x) = \chi(\mathbf{A}')(x)\ x^{LM-L}.$$

To get $\mathbf{A}'$, we define $\mathbf{P}_h = (p_{ij}^{(h)})_{1\le i,j\le G_L+LM},\ 0\le h<L$, with

$$p_{i,i}^{(h)} := 1 \text{ for all } i \le G_L + LM,$$

$$p_{G_L+hM+1,j}^{(h)} := 1 \text{ for all } j \text{ with } G_L + hM < j \le G_L + (h+1)M.$$

Then

$$\mathbf{A}' := \bar{\mathbf{A}}\begin{pmatrix} 1 & 2 & \dots & G_L+1 & G_L+M+1 & \dots & G_L+(L-1)M+1 \\ 1 & 2 & \dots & G_L+1 & G_L+M+1 & \dots & G_L+(L-1)M+1 \end{pmatrix}$$

where

$$\bar{\mathbf{A}} := \mathbf{P}_0\mathbf{P}_1\dots\mathbf{P}_{L-1}\mathbf{A}_L\mathbf{P}_{L-1}^{-1}\mathbf{P}_L^{-1}\dots\mathbf{P}_0^{-1}.$$

$\mathbf{A}'$ has the form

$$\left(\begin{array}{c|c|c}
\multicolumn{1}{c|}{\tilde{\mathbf{A}}_L} & 0 & \begin{matrix}0\\ \vdots\\ 0\\ -1\end{matrix} \\ \hline
0\ \cdots\ 0\ 1\ \cdots\ 1 & 0\ \cdots\ 0\ 1\ 0\ \cdots\ 0 & 0 \\ \hline
0 & E_{L-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}
\end{array}\right)$$

if $L > km$ and

$$\left(\begin{array}{c|c|c}
\multicolumn{1}{c|}{\tilde{\mathbf{A}}_{km}} & 0 & \begin{matrix}0\\ \vdots\\ 0\\ -1\end{matrix} \\ \hline
0\ \cdots\ 0\ 1\ \cdots\ 1 & 0\ \cdots\ 0 & 1 \\ \hline
0 & E_{L-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix}
\end{array}\right)$$

if $L = km$ respectively.

Since $\tilde{\mathbf{A}}_L$ is the matrix $\mathbf{A}_L$ of the generalized de Bruijn digraph of the $(L+1)$-tuple $(a_1, a_2, \dots, a_L, a_{L+1}+1)$, it can be transformed to

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_L & \vdots & & \ddots & 1 \\ a_{L+1}+1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

(see proof of Theorem 2.1). In the transformation, the last row is never added to or subtracted from another row. Because of this and the fact that $a'_{ij} = 0$ for all $i < G_L,\ j > G_L$, we can apply this transformation to the whole matrix $\mathbf{A}'$ and get the $(2L+1) \times (2L+1)$-matrix $\mathbf{A}''$ which has the form

$$\begin{pmatrix}
a_1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
a_2 & 0 & 1 & \ddots & & & & & & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & & & & & \vdots \\
a_L & \vdots & & \ddots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
a_{L+1}+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\
y_0 & y_1 & \cdots & \cdots & y_{l-1} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & \ddots & \ddots & \ddots & & & & \vdots \\
\vdots & & & & & & \ddots & \ddots & \ddots & & & \vdots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0
\end{pmatrix}$$

and

$$\begin{pmatrix}
a_1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
a_2 & 0 & 1 & \ddots & & & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & & & \vdots \\
a_L & \vdots & & \ddots & 1 & 0 & \cdots & \cdots & 0 \\
a_{L+1}+1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & -1 \\
y_0 & y_1 & \cdots & \cdots & y_{l-1} & 0 & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0
\end{pmatrix}$$

respectively.

$$\chi(\mathbf{A}')(x) = \chi(\mathbf{A}'')(x)\ x^{G_L - L - 1}$$

and the $y_j,\ 0 \le j \le L$, are given by $y_j := \#(\mathcal{D}_j)$ with

$$\mathcal{D}_j = \{B = (\eta_1, \dots, \eta_L) \in \mathcal{B}_L :\ (\eta_1, \dots, \eta_{km}) > (a_{L+2}, \dots, a_{L+km+1}),$$
$$(\eta_{L-j+1}, \dots, \eta_L) = (a_1, \dots, a_j), (\eta_{L+1-h}, \dots, \eta_L) \ne (a_1, \dots, a_h) \text{ for all } h > j\}$$

Lemma 3.6. *The $y_{L-j},\ 0 \le j \le L$, are recursively given by*

$$y_L = 1,\ y_{L-j} = \sum_{h=1}^{j} y_{L-j+h} a_h - \begin{cases} a_{L+j+1} & \text{if } 1 \le j < km \\ a_{L+1+km} + 1 & \text{if } j = km \\ 0 & \text{if } j > km \end{cases}$$

Proof. $y_L = 1$ is obvious.

If $B = (\eta_1, \dots, \eta_L) \in \mathcal{D}_{L-j}$, let $h \geq 1$ be maximal with the property

$$B = (\eta_1, \dots, \eta_{j-h}, a_1, \dots, a_{h-1}, \eta_j, a_1, \dots, a_{L-j})$$

Then

$$(\eta_1, \dots, \eta_{j-h}, a_1, \dots, a_{L-j+h}) \in \mathcal{D}_{L-j+h}.$$

If we take a L-tuple $(\theta_1, \dots, \theta_{j-h}, a_1, \dots, a_{L-j+h}) \in \mathcal{D}_{L-j+h}$, $1 \leq h < j$, then

$$(\eta_1, \dots, \eta_L) := (\theta_1, \dots, \theta_{j-h}, a_1, \dots, a_{h-1}, \eta_j, a_1, \dots, a_{L-j}) \in \mathcal{D}_{l-j}$$

if and only if $\eta_j < a_h$ and $(\eta_1, \dots, \eta_{km}) > (a_{L+2}, \dots, a_{L+km+1})$.

If $j > km$, the definition of $(\eta_1, \dots, \eta_L)$ guarantees that the last condition holds. If $j \leq km$, it provides

$$(\eta_1, \dots, \eta_{j-1}) \geq (a_{L+2}, \dots, a_{L+j}).$$

Therefore $(\eta_1, \dots, \eta_{km}) > (a_{L+2}, \dots, a_{L+km+1})$ is violated if and only if

$$j \leq km,\ (\eta_1, \dots, \eta_{j-1}) = (a_{L+2}, \dots, a_{L+j}),\ \eta_j < a_{L+j+1}$$

or

$$j = km,\ (\eta_1, \dots, \eta_{j-1}) = (a_{L+2}, \dots, a_{L+j}),\ \eta_{km} \leq a_{L+km+1}.$$

□

We calculate $\chi(\mathbf{A}'')$ by expanding $\det(x\mathbf{I} - \mathbf{A}'')$ at the columns $(2L+1)$ and $(L+km+1)$:

$$\begin{aligned}
\chi(\mathbf{A}'')(x) &= x^{L-km}\Big(x^{km}(x^{L+1} - a_1x^L - \cdots - a_Lx - a_{L+1} - 1) + \\
&\quad (-1)^{km}(-1)^{L+km+1+L+2}(x^{L+1} - a_1x^L - \cdots - a_Lx - a_{L+1} - 1)\Big) + \\
&\quad (-1)^{L-1}(-1)^{2L+1+L+1}\det(\hat{\mathbf{A}}_L) \\
&= x^{L-km}(x^{km} - 1)(x^{L+1} - a_1x^L - \cdots - a_Lx - a_{L+1} - 1) - \det(\hat{\mathbf{A}}_L) \\
&= x^{L-km}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)(x^m - 1) \\
&\quad \times(x^{L+1} - a_1x^L - \cdots - a_Lx - a_{L+1} - 1) - \det(\hat{\mathbf{A}}_L)
\end{aligned}$$

where $\hat{\mathbf{A}}_j,\ 1 \leq j \leq L$, is the matrix

$$\hat{\mathbf{A}}_j = \begin{pmatrix}
x - a_1 & -1 & 0 & \cdots & \cdots & 0 \\
-a_2 & x & -1 & \ddots & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
-a_j & 0 & \cdots & 0 & x & -1 \\
-y_0 & -y_1 & \cdots & \cdots & \cdots & -y_j
\end{pmatrix}$$

$$\begin{aligned}
\det(\hat{\mathbf{A}}_L) &= -y_L(x^L - a_1x^{L-1} - \cdots - a_L) + \det(\hat{\mathbf{A}}_{L-1}) = \cdots \\
&= -y_Lx^L + (a_1y_L - y_{L-1})x^{L-1} + (a_2y_L + a_1y_{L-1} - y_{L-2})x^{L-2} \\
&\quad + \cdots + (a_Ly_L + \cdots + a_1y_1 - y_0) \\
&= -x^L + a_{L+2}x^{L-1} + \cdots + a_{L+km}x^{L-km+1} + (a_{L+km+1} + 1)x^{L-km} \\
&= -x^{L-km}(x^{km} - a_{L+2}x^{km-1} - \cdots - a_{L+km}x - a_{L+km+1} - 1) \\
&= -x^{L-km}(x^{(k-1)m} + \cdots + 1)(x^m - a_{L+2}x^{m-1} - \cdots - a_{L+m+1} - 1)
\end{aligned}$$

Hence

$$\begin{aligned}
\chi(\mathbf{A}'')(x) &= x^{L-km}(x^{(k-1)m} + \cdots + 1)\Big((x^{L+m+1} - a_1x^{L+m} - \cdots - a_{L+m+1} - 1) \\
&\quad -(x^{L+1} - a_1x^L - \cdots - a_{L+1} - 1)\Big) \\
&= x^{2L+1-km-n}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)p(x)
\end{aligned}$$

Therefore

$$\begin{aligned}
\chi(\mathbf{A}_L(1))(x) &= x^{G_L-L-1+LM-L}\chi(\mathbf{A}'')(x) \\
&= x^{G_L+LM-km-n}(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)p(x)
\end{aligned}$$

and the theorem is proved. □

Hence α is an eigenvalue of $\mathbf{A}_L(1)$ and the other eigenvalues are 0, the roots of $(x^{(k-1)m} + x^{(k-2)m} + \cdots + 1)$ which are km-th roots of unity and the roots of the characteristic polynomial of α. As for simple α-numbers, α is the eigenvalue with the largest absolute value, which is shown in Lemma 3.7.

Lemma 3.7. *The conjugates of a non-simple α-number α with respect to the characteristic polynomial have absolute value less than α.*

Proof. Set, for $|x| > 1$,

$$f(x) := 1 - \frac{a_1}{x} - \frac{a_2}{x^2} - \frac{a_3}{x^3} - \cdots$$

and, for $|x| < 1$,

$$g(x) := 1 - f(x^{-1}) = \sum_{j=1}^{\infty} a_jx^j.$$

Then, for the same reasons as in the proof of Lemma 3.2, the roots of $f(x)$ have absolute value less than α for $x \neq \alpha$.

If we set

$$p_k(x) := p(x)(1 + x^m + x^{2m} + \cdots + x^{(k-1)m}),$$

then

$$p_k(x) = (x^{n+km} - a_1x^{n+km-1} - \cdots - a_{n+km}) - (x^n - a_1x^{n-1} - \cdots - a_n).$$

With $q_k(x) := x^{-n-km}p_k(x)$ we have

$$q_k(x) = (1 - \frac{a_1}{x} - \cdots - \frac{a_{n+km}}{x^{n+km}}) - (\frac{1}{x^{km}} - \frac{a_1}{x^{km+1}} - \cdots - \frac{a_n}{x^{n+km}})$$

and, for $|x| > 1$, $q_k(x) \to f(x)$ as $k \to \infty$.

Therefore $f(x) = 0$ is a necessary condition for $p(x) = 0$, $|x| > 1$, and the roots of $p(x)$ have absolute value less than α for $x \neq \alpha$.

Since $q_k'(\alpha) > 0$ for some sufficiently big k, α is a simple root of $p(x)$ and the lemma is proved. □

Hence Lemma 3.3, Theorem 3.1 and Theorem 3.2 are also valid for non-simple α-numbers, whereas the proof of Theorem 3.3 cannot be directly applied since it uses properties of non-negative matrices and $\mathbf{A}_L(1)$ contains negative elements in this case.

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