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**ON THE JOINT DISTRIBUTION OF  $q$ -ADDITIVE  
FUNCTIONS ON POLYNOMIAL SEQUENCES**

WOLFGANG STEINER

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ABSTRACT. The joint distribution of sequences  $(f_\ell(P_\ell(n)))_{n \in \mathbb{N}}$ ,  $\ell = 1, 2, \dots, d$  and  $(f_\ell(P_\ell(p)))_{p \in \mathbb{P}}$  respectively, where  $f_\ell$  are  $q_\ell$ -additive functions and  $P_\ell$  polynomials with integer coefficients, is considered. A central limit theorem is proved for a larger class of  $q_\ell$  and  $P_\ell$  than by Drmota [3]. In particular, the joint limit distribution of the sum-of-digits functions  $s_{q_1}(n)$ ,  $s_{q_2}(n)$  is obtained for arbitrary integers  $q_1, q_2$ . For strongly  $q$ -additive functions with respect to the same  $q$ , a central limit theorem is proved for arbitrary polynomials  $P_\ell$  with the help of a joint representation of the digits of  $P_\ell(n)$  by a Markov chain.

1. INTRODUCTION

For a given integer  $q > 1$ , every non-negative integer  $n$  has a unique  $q$ -ary expansion

$$n = \sum_{k \geq 0} \epsilon_{q,k}(n) q^k$$

with  $\epsilon_{q,k}(n) \in \{0, 1, \dots, q-1\}$  (where the index  $q$  will often be omitted). Then the *sum-of-digits function* is given by

$$s_q(n) = \sum_{k \geq 0} \epsilon_{q,k}(n).$$

This is a special case of a  $q$ -additive function, i.e. a real-valued function  $f$  defined on the non-negative integers which satisfies  $f(0) = 0$  and

$$f(n) = \sum_{k \geq 0} f(\epsilon_{q,k}(n) q^k).$$

Such a function is said to be *strongly  $q$ -additive*, if

$$f(n) = \sum_{k \geq 0} f(\epsilon_{q,k}(n)).$$

Bassily and Kátai [1] proved the following central limit theorem.

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**Theorem 1 (Bassily and Kátai [1]).** *Let  $f$  be a  $q$ -additive function such that  $f(bq^k) = \mathcal{O}(1)$  as  $k \rightarrow \infty$  for all  $b \in \{0, 1, \dots, q-1\}$ . Assume  $\frac{D(N)}{(\log N)^\eta} \rightarrow \infty$  as  $N \rightarrow \infty$  for some  $\eta > 0$  and let  $P(n)$  be a polynomial with integer coefficients, degree  $r$  and positive leading term. Set*

$$\mu_k = \frac{1}{q} \sum_{b=0}^{q-1} f(bq^k), \quad \sigma_k^2 = \frac{1}{q} \sum_{b=0}^{q-1} f(bq^k)^2 - \mu_k^2$$

and

$$M(N) = \sum_{k=0}^{\lfloor \log_q N \rfloor} \mu_k, \quad D(N)^2 = \sum_{k=0}^{\lfloor \log_q N \rfloor} \sigma_k^2.$$

Then, as  $N \rightarrow \infty$ ,

$$\frac{1}{N} \# \left\{ n < N \mid \frac{f(P(n)) - M(N^r)}{D(N^r)} < x \right\} \rightarrow \Phi(x)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p \in \mathbb{P}, p < N \mid \frac{f(P(p)) - M(N^r)}{D(N^r)} < x \right\} \rightarrow \Phi(x),$$

where  $\Phi(x)$  denotes the distribution function of the normal law.

This theorem was only stated for  $\eta = \frac{1}{3}$ . However, a short inspection of the proof shows that  $\eta > 0$  is sufficient.

Drmota [3] generalised this theorem for certain joint distributions. From now on, denote by  $\mu_{\ell,k}, \sigma_{\ell,k}, M_\ell, D_\ell$  the  $\mu_k, \sigma_k, M, D$  of Theorem 1 with respect to  $f_\ell$ .

**Theorem 2 (Drmota [3]).** *Let  $f_\ell, 1 \leq \ell \leq d$ , be  $q_\ell$ -additive functions such that  $f_\ell(bq_\ell^k) = \mathcal{O}(1)$  as  $k \rightarrow \infty$  for all  $b \in \{0, 1, \dots, q_\ell - 1\}$ . Assume that  $\frac{D_\ell(N)}{(\log N)^\eta} \rightarrow \infty$ , as  $N \rightarrow \infty$ , for some  $\eta > 0$  and let  $P_\ell(x)$  be polynomials with integer coefficients of different degrees  $r_\ell$  and positive leading terms,  $1 \leq \ell \leq d$ . Then, as  $N \rightarrow \infty$ ,*

$$\frac{1}{N} \# \left\{ n < N \mid \frac{f_\ell(P_\ell(n)) - M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})} < x_\ell, 1 \leq \ell \leq d \right\} \rightarrow \Phi(x_1)\Phi(x_2)\cdots\Phi(x_d)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p < N \mid \frac{f_\ell(P_\ell(p)) - M_\ell(N^{r_\ell})}{D_{q_\ell}(N^{r_\ell})} < x_\ell, 1 \leq \ell \leq d \right\} \rightarrow \Phi(x_1)\Phi(x_2)\cdots\Phi(x_d).$$

Note that this theorem was stated only for coprime  $q_\ell$ , but this assumption is not used in the proof and therefore not necessary.

The problem is the case of polynomials of the same degree. For  $d = 2$ , we show the following theorem.

**Theorem 3.** *Let  $q_1, q_2 > 1$  be multiplicatively independent integers and let  $f_\ell$  be  $q_\ell$ -additive functions such that  $f_\ell(bq_\ell^k) = \mathcal{O}(1)$  as  $k \rightarrow \infty$  for all  $b \in \{0, 1, \dots, q_\ell - 1\}$ ,  $\ell = 1, 2$ . Assume that  $\frac{D_\ell(N)}{(\log N)^\eta} \rightarrow \infty$  as  $N \rightarrow \infty$ , for some  $\eta > 0$  and let  $P_\ell(n)$  be*

polynomials with integer coefficients of degree  $r$  and positive leading terms,  $\ell = 1, 2$ . Then, as  $N \rightarrow \infty$ ,

$$\frac{1}{N} \# \left\{ n < N \left| \frac{f_\ell(P_\ell(n)) - M_\ell(N^r)}{D_\ell(N^r)} < x_\ell, \ell = 1, 2 \right. \right\} \rightarrow \Phi(x_1)\Phi(x_2)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p < N \left| \frac{f_\ell(P_\ell(p)) - M_\ell(N^r)}{D_\ell(N^r)} < x_\ell, \ell = 1, 2 \right. \right\} \rightarrow \Phi(x_1)\Phi(x_2).$$

The first convergence was shown by Drmota [3] for linear polynomials and coprime integers  $q_1, q_2$ . In [4], Drmota and the author stated this theorem, but still only for coprime integers. We will prove the case of multiplicatively independent integers in Section 3.

Furthermore, we solve the problem of equal degrees of the polynomials for strongly  $q$ -additive functions with respect to the same  $q$  in the following section. Note that this covers the case of multiplicatively dependent  $q_1, q_2$  since  $q_1$ - and  $q_2$ -additive functions are  $q$ -additive, if  $q_1^{s_1} = q_2^{s_2} = q$ . Then the distributions clearly do not satisfy the independence relations of Theorems 2 and 3.

The main part of the proof of all theorems is a proposition similar to the following one (which proves Theorem 2).

**Proposition 1 (Drmota [3]).** *Let  $P_\ell(n)$ ,  $1 \leq \ell \leq d$ , be polynomials of different degrees  $r_\ell$  with integer coefficients and positive leading terms. Let  $\lambda > 0$  be an arbitrary constant and  $h_\ell$ ,  $1 \leq \ell \leq d$ , non-negative integers. Then, as  $N \rightarrow \infty$ ,*

$$\begin{aligned} \frac{1}{N} \# \left\{ n < N \left| \epsilon_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right. \right\} \\ = \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + \mathcal{O}((\log N)^{-\lambda}) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\pi(N)} \# \left\{ p < N \left| \epsilon_{q_\ell, k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right. \right\} \\ = \frac{1}{q_1^{h_1} q_2^{h_2} \cdots q_d^{h_d}} + \mathcal{O}((\log N)^{-\lambda}) \end{aligned}$$

uniformly for integers

$$(\log N^{r_\ell})^\eta \leq k_1^{(\ell)} < k_2^{(\ell)} < \cdots < k_{h_\ell}^{(\ell)} \leq \log_{q_\ell} N^{r_\ell} - (\log N^{r_\ell})^\eta \quad (1 \leq \ell \leq d)$$

(with some  $\eta > 0$ ) and  $b_j^{(\ell)} \in \{0, 1, \dots, q_\ell - 1\}$ .

For a list of references of other results for  $q$ -additive functions, we refer to Drmota [3].

2. STRONGLY  $q$ -ADDITIVE FUNCTIONS WITH RESPECT TO THE SAME  $q$ 

## 2.1. Results.

**Theorem 4.** Let  $f_\ell$ ,  $1 \leq \ell \leq d$ , be strongly  $q$ -additive functions with  $\sigma_\ell = \sigma_{\ell,k} > 0$  and  $P_\ell(n) = g_{r_\ell}^{(\ell)} n^{r_\ell} + \dots + g_1^{(\ell)} n + g_0^{(\ell)}$  polynomials with integer coefficients and positive leading terms. Then, as  $N \rightarrow \infty$ ,

$$\frac{1}{N} \# \left\{ n < N \mid \frac{f_\ell(P_\ell(n)) - M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})} < x_\ell, \ell = 1, \dots, d \right\} \rightarrow \Phi_V(x_1, \dots, x_d)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p < N \mid \frac{f_\ell(P_\ell(p)) - M_\ell(N^{r_\ell})}{D_\ell(N^{r_\ell})} < x_\ell, \ell = 1, 2, \dots, d \right\} \rightarrow \Phi_V(x_1, \dots, x_d)$$

where  $\Phi_V(x_1, \dots, x_d)$  denotes the distribution function of the  $d$ -dimensional normal law with covariance matrix  $V = (v_{i,j})_{1 \leq i, j \leq d}$  given by

$$v_{i,j} = \begin{cases} 1 & \text{if } i = j \\ C_{i,j} \left( \frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } g_{r_j}^{(j)} P_i(n) \equiv g_{r_i}^{(i)} P_j(n) \\ \frac{r_i - \max\{s \mid g_{r_i}^{(i)} g_s^{(j)} \neq g_{r_j}^{(j)} g_s^{(i)}\}}{r_i} C_{i,j} \left( \frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } r_i = r_j \\ 0 & \text{else,} \end{cases}$$

where

$$\begin{aligned} C_{i,j}(g_i, g_j) &= \frac{1}{\sigma_i \sigma_j} \sum_{l=0}^{R_j-1} \sum_{b_i=1}^{q-1} \sum_{b_j=1}^{q-1} \left( \pi_{b_i, b_j, g_i q^l, g_j} - \frac{1}{q^2} \right) f_i(b_i) f_j(b_j) \\ &\quad + \frac{1}{\sigma_i \sigma_j} \sum_{l=1}^{R_i-1} \sum_{b_i=1}^{q-1} \sum_{b_j=1}^{q-1} \left( \pi_{b_i, b_j, g_i, g_j q^l} - \frac{1}{q^2} \right) f_i(b_i) f_j(b_j) \end{aligned}$$

with  $R_\ell$  such that  $q \mid \frac{q^{R_\ell}}{(q^{R_\ell}, g_{r_\ell}^{(\ell)})}$  and

$$\begin{aligned} \pi_{b_i, b_j, g_i q^l, g_j} &= \pi_{b_i, b_j, g, g'} = \frac{1}{q^2} - \frac{\left( \overline{(b_i + 1)g'} - \overline{b_i g'} \right) \left( \overline{(b_j + 1)g} - \overline{b_j g} \right)}{g g' q^2} \\ &+ \frac{\min(\overline{b_i g'}, \overline{b_j g}) + \min(\overline{(b_i + 1)g'}, \overline{(b_j + 1)g}) - \min(\overline{(b_i + 1)g'}, \overline{b_j g}) - \min(\overline{b_i g'}, \overline{b_j g})}{g g' q} \end{aligned}$$

where  $g = \frac{g_i q^l}{(q^l, g_j)}$ ,  $g' = \frac{g_j}{(q^l, g_j)}$  and  $\bar{y}$  denotes the representative  $y'$  of  $y' \equiv y(q)$  with  $0 \leq y' < q$ . ( $\pi_{b_i, b_j, g_i, g_j q^l}$  is given symmetrically.)

*Remarks.* If  $V$  is positive definite, we have, with  $\mathbf{t} = (t_1, \dots, t_d)$ ,

$$\Phi_V(x_1, \dots, x_d) = \frac{1}{(2\pi)^{d/2} \sqrt{\det V}} \int_{-\infty}^{x_d} \dots \int_{-\infty}^{x_1} e^{-\frac{1}{2} \mathbf{t} V^{-1} \mathbf{t}} dt_1 \dots dt_d.$$

If  $g_{r_\ell}^{(\ell)}$  is coprime to  $q$ , then we have  $R_\ell = 1$ .

$l \geq R_j$  implies  $\pi_{b_i, b_j, g_i q^l, g_j} = \frac{1}{q^2}$  for all  $b_i, b_j$ .

The  $\pi_{b_i, b_j, g_i q^l, g_j}$  are the joint probabilities of digits  $k+l$  and  $k$  of  $g_i n$  and  $g_j n$  (which do not depend on  $k$ ):

$$\pi_{b_i, b_j, g_i q^l, g_j} = \mathbf{Pr}[\epsilon_k(g_i q^l n) = b_i, \epsilon_k(g_j) = b_j] = \mathbf{Pr}[\epsilon_{k+l}(g_i n) = b_i, \epsilon_k(g_j) = b_j].$$

Note that we need  $C_{i,j}(g_i, g_j)$  only for coprime  $g_i, g_j$ .

The constant term of the polynomials plays no role.

**Corollary 1.** *Let  $P_\ell(n) = g_{r_\ell}^{(\ell)} n^{r_\ell} + \dots + g_1^{(\ell)} n + g_0^{(\ell)}$  be polynomials with integer coefficients and positive leading terms. Then, as  $N \rightarrow \infty$ ,*

$$\begin{aligned} \frac{1}{N} \# \left\{ n < N \left| \frac{s_q(P_\ell(n)) - \frac{q-1}{2} \log_q N^{r_\ell}}{\sqrt{\frac{q^2-1}{12} \log_q N^{r_\ell}}} < x_\ell, \ell = 1, \dots, d \right. \right\} \\ \rightarrow \frac{1}{(2\pi)^{d/2} \sqrt{\det V}} \int_{-\infty}^{x_d} \dots \int_{-\infty}^{x_1} e^{-\frac{1}{2} \mathbf{t} V^{-1} \mathbf{t}} dt_1 \dots dt_d \end{aligned}$$

with the positive definite matrix  $V = (v_{i,j})_{1 \leq i, j \leq d}$  given by

$$v_{i,j} = \begin{cases} 1 & \text{if } i = j \\ C_{i,j} \left( \frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } g_{r_j}^{(j)} P_i(n) \equiv g_{r_i}^{(i)} P_j(n) \\ \frac{r_i - \max\{s | g_{r_i}^{(i)} g_s^{(j)} \neq g_{r_j}^{(j)} g_s^{(i)}\}}{r_i} C_{i,j} \left( \frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } r_i = r_j \\ 0 & \text{else,} \end{cases}$$

and

$$\begin{aligned} C_{i,j}(g_i, g_j) &= \frac{q^2 - (q, g_i)^2 - (q, g_j)^2 + 1}{g_i g_j (q^2 - 1)} \\ &+ \frac{1}{g_i g_j (q^2 - 1)} \left( \sum_{l=1}^{R_j-1} \frac{q^2 - \left(q, \frac{g_i q^l}{(q, g_j)}\right)}{q^l} + \sum_{l=1}^{R_i-1} \frac{q^2 - \left(q, \frac{g_j q^l}{(q, g_i)}\right)}{q^l} \right) \end{aligned}$$

*Remark.* For monomials  $P_\ell(n) = g_\ell n^r$  with  $(g_\ell, q) = 1$  we just have

$$v_{i,j} = \frac{(g_i, g_j)^2}{g_i g_j}.$$

For  $q = 2$  and  $r = 1$ , this was proved by W.M. Schmidt [6].

Furthermore, we can calculate the joint distribution of the sum-of-digits functions for multiplicatively dependent  $q_1, q_2$ .

**Corollary 2.** *For  $q_1 = \tilde{q}^{s_1}, q_2 = \tilde{q}^{s_2}$  with positive integers  $\tilde{q}, s_1, s_2$  and  $(s_1, s_2) = 1$ , we have, as  $N \rightarrow \infty$ ,*

$$\begin{aligned} \frac{1}{N} \# \left\{ n < N \left| \frac{s_{q_1}(n) - \frac{q_1-1}{2} \log_{q_1} N}{\sqrt{\frac{q_1^2-1}{12} \log_{q_1} N}} < x_1, \frac{s_{q_2}(n) - \frac{q_2-1}{2} \log_{q_2} N}{\sqrt{\frac{q_2^2-1}{12} \log_{q_2} N}} < x_2 \right. \right\} \\ \rightarrow \frac{1}{2\pi \sqrt{1-C^2}} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} e^{-\frac{1}{2(1-C^2)}(t_1^2 + t_2^2 - 2Ct_1 t_2)} dt_1 dt_2 \end{aligned}$$

with

$$C = \frac{\tilde{q} + 1}{\tilde{q} - 1} \sqrt{\frac{(q_1 - 1)(q_2 - 1)}{s_1 s_2 (q_1 + 1)(q_2 + 1)}}.$$

For general strongly  $q_\ell$ -additive functions, similar statements can be derived easily. The case of multiplicatively independent  $q_1, q_2$  is treated by Theorem 3.

## 2.2. A Markov chain and calculation of the covariance.

Define the polynomials

$$P_\ell^{(s)}(n) = g_{r_\ell}^{(\ell)} n^{r_\ell} + \cdots + g_s^{(\ell)} n^s \quad \text{for } 1 \leq s \leq r = \max_{1 \leq \ell \leq d} r_\ell.$$

and fix  $s$  in this subsection.

Furthermore, define vectors

$$\mathbf{w}_k^{(s)}(n) = (w_{k,s}, \dots, w_{k,r}) = \left( \left\{ \frac{n^s}{q^{k+1}} \right\}, \left\{ \frac{n^{s+1}}{q^{k+1}} \right\}, \dots, \left\{ \frac{n^r}{q^{k+1}} \right\} \right)$$

for  $0 \leq n < N$ , where  $\{x\}$  denotes the fractional part of  $x$  and see, by Proposition 1, that they asymptotically form a net to the base  $q$  if  $k \in [(\log N)^\eta, \log_q N^s - (\log N)^\eta]$  (but not for  $k > \log_q N^s$ ). Proposition 1 gives rather bad error terms if we want to calculate the number of  $\mathbf{w}_k^{(s)}(n)$  in an arbitrary set of  $\mathbb{T}^{r-s+1}$ . Nevertheless, this suggests that they are uniformly distributed and we use the Lebesgue measure as probability measure on  $\mathbb{T}^{r-s+1}$ .

We have  $\epsilon_k(P_\ell^{(s)}(n)) = b$  if and only if

$$\left\{ g_{r_\ell}^{(\ell)} w_{k,r_\ell} + \cdots + g_s^{(\ell)} w_{k,s} \right\} \in \left[ \frac{b}{q}, \frac{b+1}{q} \right).$$

This means that, for each digit  $b$ ,  $\{\mathbf{w}_k^{(s)}(n) \mid \epsilon_k(P_\ell^{(s)}(n)) = b\}$  (as a set of  $\mathbb{T}^{r-s+1}$ ) is contained in the stripe  $S_{b,\ell}^{(s)}$  between the hyperplanes  $g_{r_\ell}^{(\ell)} x_{r_\ell} + \cdots + g_s^{(\ell)} x_s = \frac{b}{q}$  (included) and  $g_{r_\ell}^{(\ell)} x_{r_\ell} + \cdots + g_s^{(\ell)} x_s = \frac{b+1}{q}$  (excluded). If  $P_\ell^{(s)}(n) \equiv 0$ , set  $S_{0,\ell}^{(s)} = \mathbb{T}^{r-s+1}$  and  $S_{b,\ell}^{(s)} = \emptyset$  for  $b \neq 0$ .

Thus, each set  $\{\mathbf{w}_k^{(s)}(n) \mid \epsilon_k(P_1^{(s)}(n)) = b_1, \dots, \epsilon_k(P_d^{(s)}(n)) = b_d\}$  is contained in  $S_{b_1,1}^{(s)} \cap \cdots \cap S_{b_d,d}^{(s)}$  and each of these intersections consists of a finite number of convex sets, the boundaries of which are the above hyperplanes. Let  $(W_j^{(s)})_{1 \leq j \leq \kappa_s}$  be the partition of  $\mathbb{T}^r$  induced by these sets (or equivalently by the hyperplanes). Then  $f_\ell|_{W_j^{(s)}}$  is constant for all  $\ell, j$ .

Furthermore, we have  $\epsilon_{k-j}(P_\ell^{(s)}(n)) = b$  if and only if  $T^j(\mathbf{w}_k^{(s)}(n)) \in S_{b,\ell}^{(s)}$  with the map  $T : \mathbb{T}^r \rightarrow \mathbb{T}^r$ ,  $T(w_{k,s}, \dots, w_{k,r}) = (qw_{k,s}, \dots, qw_{k,r})$ . Hence

$$\begin{aligned} & \left\{ n \mid \epsilon_0(P_\ell^{(s)}(n)) = b_0^{(\ell)}, \dots, \epsilon_k(P_\ell^{(s)}(n)) = b_k^{(\ell)} \right\} \\ &= \left\{ n \mid \mathbf{w}_k^{(s)}(n) \in T^{-k} S_{b_0^{(\ell)}, \ell}^{(s)}, \dots, \mathbf{w}_k^{(s)}(n) \in S_{b_k^{(\ell)}, \ell}^{(s)} \right\} \end{aligned}$$

and we define a sequence of random variables  $(Y_k^{(s)})_{k \geq 0}$  on  $\{W_1^{(s)}, W_2^{(s)}, \dots, W_{\kappa_s}^{(s)}\}$  by

$$\mathbf{Pr}[Y_0^{(s)} = W_{j_0}^{(s)}, \dots, Y_k^{(s)} = W_{j_k}^{(s)}] = \lambda_{r-s+1}(T^{-k} W_{j_0}^{(s)} \cap \cdots \cap T^{-1} W_{j_{k-1}}^{(s)} \cap W_{j_k}^{(s)})$$

for  $1 \leq j_i \leq \kappa_s$ ,  $0 \leq i \leq k$ . ( $\lambda_n$  denotes the  $n$ -dimensional Lebesgue measure.)

**Lemma 1.**  $(Y_k^{(s)})_{k \geq 0}$  is a Markov chain.

*Proof.* Let  $U$  be the subspace of  $\mathbb{R}^{r-s+1}$  spanned by the vectors  $(g_s^{(\ell)}, \dots, g_r^{(\ell)})$ ,  $1 \leq \ell \leq d$ . If  $U$  has (full) rank  $r-s+1$ , then  $T$  is injective on each  $W_j^{(s)}$ ,  $1 \leq j \leq \kappa_s$ . Otherwise,  $W_j^{(s)}$  contains with every point  $x$  all points  $x + U^\perp$  and  $T$  is  $q^\delta$ -to-one with  $\delta = r-s+1 - \text{rank}(U)$ . Furthermore,  $TW_j^{(s)}$  is the (disjoint) union of sets  $W_i^{(s)}$ , since the image of the hyperplane  $g_{r_\ell}^{(\ell)} x_{r_\ell} + \dots + g_s^{(\ell)} x_s = \frac{b}{q}$  is the hyperplane  $g_{r_\ell}^{(\ell)} x_{r_\ell} + \dots + g_s^{(\ell)} x_s = 0$ . Hence we have

$$\begin{aligned} \Pr[Y_0^{(s)} = W_{j_0}^{(s)}, \dots, Y_{k+1}^{(s)} = W_{j_{k+1}}^{(s)}] &= \lambda_{r-s+1}(T^{-(k+1)}W_{j_0}^{(s)} \cap \dots \cap W_{j_{k+1}}^{(s)}) \\ &= \frac{1}{q^\delta} \lambda_{r-s+1}(T^{-k}W_{j_0}^{(s)} \cap \dots \cap W_{j_k}^{(s)} \cap TW_{j_{k+1}}^{(s)}) \\ &= \begin{cases} \frac{1}{q^\delta} \lambda_{r-s+1}(T^{-k}W_{j_0}^{(s)} \cap \dots \cap W_{j_k}^{(s)}) & \text{if } W_{j_k}^{(s)} \subseteq TW_{j_{k+1}}^{(s)} \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} \Pr[Y_{k+1}^{(s)} = W_{j_{k+1}}^{(s)} | Y_0^{(s)} = W_{j_0}^{(s)}, \dots, Y_k^{(s)} = W_{j_k}^{(s)}] \\ = \begin{cases} \frac{1}{q^\delta} & \text{if } W_{j_k}^{(s)} \subseteq TW_{j_{k+1}}^{(s)} \\ 0 & \text{else} \end{cases} = \Pr[Y_{k+1}^{(s)} = W_{j_{k+1}}^{(s)} | Y_k^{(s)} = W_{j_k}^{(s)}], \end{aligned}$$

i.e. the Markov chain property is fulfilled.

As already noted, each  $f_\ell$  is constant on each  $W_j^{(s)}$  because of  $W_j^{(s)} \subseteq S_{b_{1,1}}^{(s)} \cap \dots \cap S_{b_{d,d}}^{(s)}$  for some  $b_i$ . Therefore we define the  $d$ -dimensional function  $f$  on  $(W_j^{(s)})_{1 \leq j \leq \kappa_s}$  by

$$f(W_j^{(s)}) = (f_1(W_j^{(s)}), \dots, f_d(W_j^{(s)})) = (f_1(b_1), \dots, f_d(b_d)).$$

Before stating a central limit theorem for  $f(Y_k^{(s)})$ , we study the covariance  $\text{Cov}(f_i(Y_{k_i}^{(s)}), f_j(Y_{k_j}^{(s)}))$ . To this effect, the following lemma, which will be proved together with Proposition 2, will be very useful. Note that  $Y_k^{(s)} \subseteq S_{b,\ell}^{(s)}$  is equivalent to  $f_\ell(Y_k^{(s)}) = b$ .

**Lemma 2.**

$$\Pr[Y_{k_i}^{(s)} \subseteq S_{b_i,i}^{(s)}, Y_{k_j}^{(s)} \subseteq S_{b_j,j}^{(s)}] = \sum_{m_i, m_j: \frac{m_i P_i^{(s)}(n)}{q^{k_i}} + \frac{m_j P_j^{(s)}(n)}{q^{k_j}} \equiv 0} c_{m_i, b_i, q} c_{m_j, b_j, q}, \quad (1)$$

where  $c_{m,b,q}$  are the Fourier coefficients of  $\mathbf{1}_{[b/q, (b+1)/q)}$

$$c_{0,b,q} = \frac{1}{q}, \quad c_{m,b,q} = \frac{e\left(-\frac{mb}{q}\right) - e\left(-\frac{m(b+1)}{q}\right)}{2\pi i m} \text{ for } m \neq 0.$$

By Lemma 2, we have

$$\Pr[Y_{k_i}^{(s)} \subseteq S_{b_i,i}^{(s)}, Y_{k_j}^{(s)} \subseteq S_{b_j,j}^{(s)}] = c_{0,b_i,q} c_{0,b_j,q} = \Pr[Y_{k_i}^{(s)} \subseteq S_{b_i,i}^{(s)}] \Pr[Y_{k_j}^{(s)} \subseteq S_{b_j,j}^{(s)}]$$



if the polynomials do not have the same degree or are not proportional. Then  $\mathbf{Cov}(f_i(Y_{k_i}^{(s)}), f_j(Y_{k_j}^{(s)})) = 0$ .

Now assume  $r_i = r_j$  and that the polynomials are proportional. Furthermore, let w.l.o.g.  $k_i \geq k_j$ . Then the  $m_i$  in (1) must satisfy  $m_i g_r^{(i)} \equiv 0 (q^{k_i - k_j})$ , i.e.  $m_i \equiv 0 \left( \frac{q^{k_i - k_j}}{(q^{k_i - k_j}, g_r^{(i)})} \right)$ . If  $k_i - k_j \geq R_i$ , this implies  $m_i \equiv 0 (q)$ . Hence we have  $c_{m_i, b_i, q} c_{m_j, b_j, q} = 0$  for  $(m_i, m_j) \neq (0, 0)$  and

$$\mathbf{Cov} \left( f_i(Y_{k_i}^{(s)}), f_j(Y_{k_j}^{(s)}) \right) = 0 \quad \text{if } k_i - k_j \geq R_i \text{ or } k_j - k_i \geq R_j.$$

(For  $k_j \geq k_i$ , we get the result by the symmetry of the covariance.)

Since the Markov chain  $(Y_k^{(s)})_{k \geq 0}$  is homogeneous, we obtain

$$\begin{aligned} \mathbf{Cov} \left( \sum_{k=A(N)}^{B(N)} f_i(Y_k^{(s)}), \sum_{k=A(N)}^{B(N)} f_j(Y_k^{(s)}) \right) \\ = \sum_{k=A(N)}^{B(N)} \sum_{l=\max(-R_i+1, A(N)-k)}^{\min(R_j-1, B(N)-k)} \mathbf{Cov} \left( f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)}) \right) \\ = (B(N) - A(N)) \sum_{l=-R_i+1}^{R_j-1} \mathbf{Cov} \left( f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)}) \right) + \mathcal{O}(1) \end{aligned}$$

for  $A(N) = \lceil (\log N)^\eta \rceil$ ,  $B(N) = \lfloor \log_q N \rfloor - \lceil (\log N)^\eta \rceil$ .

Now we can state the central limit theorem.

**Proposition 2.** *The sums of the random variables  $f(Y_k^{(s)})$  satisfy a multidimensional central limit theorem with convergence of moments. More precisely, we have, for all  $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$ , as  $N \rightarrow \infty$ ,*

$$\frac{\sum_{k=A(N)}^{B(N)} \sum_{\ell=1}^d \frac{a_\ell}{\sigma_\ell} f_\ell(Y_k^{(s)}) - \sum_{\ell=1}^d \frac{a_\ell}{\sigma_\ell} \overline{M}_\ell(N)}{\sqrt{B(N) - A(N)}} \rightarrow \mathcal{N} \left( 0, \mathbf{a} V^{(s)} \mathbf{a}^t \right), \quad (2)$$

where the covariance matrix  $V^{(s)} = \left( v_{i,j}^{(s)} \right)_{1 \leq i, j \leq d}$  is given by

$$v_{i,j}^{(s)} = \frac{1}{\sigma_i \sigma_j} \sum_{l=-R_i+1}^{R_j-1} \mathbf{Cov} \left( f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)}) \right)$$

and for all integers  $h_\ell \geq 0$  we have

$$\mathbf{E} \prod_{\ell=1}^d \left( \frac{\sum_{k=A(N)}^{B(N)} f_\ell(Y_k^{(s)}) - \overline{M}_\ell(N)}{\overline{D}_\ell(N)} \right)^{h_\ell} \rightarrow \int x_1^{h_1} \cdots x_d^{h_d} d\Phi_{V^{(s)}}(x_1, \dots, x_d). \quad (3)$$

*Proof.* We have

$$\begin{aligned} \mathbf{Var} \sum_{\ell=1}^d \sum_{k=A(N)}^{B(N)} \frac{a_\ell}{\sigma_\ell} f_\ell(Y_k^{(s)}) &= \sum_{i=1}^d \sum_{j=1}^d \mathbf{Cov} \left( \sum_{k=A(N)}^{B(N)} \frac{a_i}{\sigma_i} f_i(Y_k^{(s)}), \sum_{k=A(N)}^{B(N)} \frac{a_j}{\sigma_j} f_j(Y_k^{(s)}) \right) \\ &= (B(N) - A(N)) \sum_{i=1}^d \sum_{j=1}^d \frac{a_i a_j}{\sigma_i \sigma_j} \sum_{l=-R_i+1}^{R_j-1} \mathbf{Cov} \left( f_i(Y_k^{(s)}), f_j(Y_{k+l}^{(s)}) \right) + \mathcal{O}(1) \\ &= (B(N) - A(N)) \mathbf{a} V^{(s)} \mathbf{a}^t + \mathcal{O}(1). \end{aligned}$$

If  $\mathbf{a}V^{(s)}\mathbf{a}^t = 0$ , then  $\sum_{\ell=1}^d \sum_{k=A(N)}^{B(N)} a_{\ell} f_{\ell}(Y_k^{(s)}) = \mathcal{O}(1)$  and both sides in (2) are zero.

Otherwise, use the central limit theorem for stationary and homogeneous Markov chains or  $\varphi$ -mixing sequences (see e.g. Billingsley [2], p. 364) which holds if all states are recurrent and aperiodic. For  $Y_k^{(s)}$ , this condition is satisfied, since we clearly have an integer  $m$  such that  $T^m W_j^{(s)} = \mathbb{T}^{r-s+1}$  for all  $W_j^{(s)}$  and hence  $\Pr[Y_{k+l}^{(s)} = W_{j_{k+l}}^{(s)} | Y_k^{(s)} = W_{j_k}^{(s)}] > 0$  for all  $l \geq m$ . This implies the  $\varphi$ -mixing property for  $X_k = \sum_{\ell=1}^d a_{\ell} f_{\ell}(Y_k^{(s)})$  and the central limit theorem holds for  $X_k$ , too. (Note that  $X_k$  need not be a Markov chain, if  $\sum_{\ell=1}^d a_{\ell} f_{\ell}$  is not injective.)

For the convergence of moments, it suffices to show that they exist. The one-dimensional moments are

$$\mathbf{E} \left( \frac{\sum_{k=A(N)}^{B(N)} f_{\ell}(Y_k^{(s)}) - \overline{M}_{\ell}(N)}{\overline{D}_{\ell}(N)} \right)^{h_{\ell}} \sim \frac{1}{N} \sum_{n < N} \left( \frac{\sum_{k=A(N)}^{B(N)} f_{\ell}(\epsilon_k(n)) - \overline{M}_{\ell}(N)}{\overline{D}_{\ell}(N)} \right)^{h_{\ell}}$$

and converge therefore (cf. [1]). The multidimensional moments converge since  $\mathbf{E} \left| X_N^r \tilde{X}_N^s \right| \leq (\mathbf{E} X_N^{2r})^{\frac{1}{2}} (\mathbf{E} \tilde{X}_N^{2s})^{\frac{1}{2}}$  holds for all random variables  $X_N, \tilde{X}_N$ . Thus the proposition is proved.

For the calculation of  $\mathbf{Cov}(f_i(Y_k^{(s)}), f_j(Y_j^{(s)}))$ , it suffices to consider  $Y_k = Y_k^{(1)}$  and linear polynomials because of Lemma 2 and the succeeding remarks. For the sum-of-digits function, we even get explicit expressions.

**Lemma 3.** *Let  $P_1(n) = g_1 n$ ,  $P_2(n) = g_2 n$  and  $f_1(n) = f_2(n) = s_q(n)$ . Then the covariance of  $f_1(Y_k)$  and  $f_2(Y_k)$  is given by*

$$\mathbf{Cov}(f_1(Y_k), f_2(Y_k)) = \frac{(q^2 - d_1^2 - d_2^2 + 1)(g_1, g_2)^2}{12g_1g_2}, \quad (4)$$

where  $d_1 = \left(q, \frac{g_1}{(g_1, g_2)}\right)$  and  $d_2 = \left(q, \frac{g_2}{(g_1, g_2)}\right)$ .

*Proof.* The covariance is given by

$$\begin{aligned} \mathbf{Cov}(f_1(Y_k), f_2(Y_k)) &= \sum_{b_1=0}^{q-1} \sum_{b_2=0}^{q-1} \Pr[\epsilon_k(g_1 n) = b_1, \epsilon_k(g_2 n) = b_2] b_1 b_2 - \mathbf{E} f_1(Y_k) \mathbf{E} f_2(Y_k). \end{aligned} \quad (5)$$

Because of Lemma 2, the digit probability does not change if we replace  $g_1, g_2$  by  $\frac{g_1}{(g_1, g_2)}, \frac{g_2}{(g_1, g_2)}$ . Therefore assume  $(g_1, g_2) = 1$ . In order to get integers, set

$$\begin{aligned} a_{b_1, b_2} &= qg_1g_2 \Pr[\epsilon_k(g_1 n) = b_1, \epsilon_k(g_2 n) = b_2] \\ &= \# \left\{ x \in \{0, 1, \dots, qg_1g_2 - 1\} \left| \left[ \frac{x}{g_2} \right] \equiv b_1(q), \left[ \frac{x}{g_1} \right] \equiv b_2(q) \right. \right\}. \end{aligned}$$

We study  $A_{i,j} = \sum_{b_1=q-i}^{q-1} \sum_{b_2=q-j}^{q-1} a_{b_1, b_2}$  because of

$$\sum_{b_1=0}^{q-1} \sum_{b_2=0}^{q-1} a_{b_1, b_2} b_1 b_2 = \sum_{i=1}^{q-1} \sum_{b_1=q-i}^{q-1} \sum_{j=1}^{q-1} \sum_{b_2=q-j}^{q-1} a_{b_1, b_2}.$$

For every  $x$  in the set corresponding to  $a_{b_1, b_2}$ ,  $(qg_1g_2 - 1 - x)$  is in the set corresponding to  $a_{q-1-b_1, q-1-b_2}$ . Therefore we have  $a_{b_1, b_2} = a_{q-1-b_1, q-1-b_2}$  and

$$\begin{aligned} A_{i,j} &= \sum_{b_1=0}^{i-1} \sum_{b_2=0}^{j-1} a_{b_1, b_2} \\ &= \#\{x \in \{0, \dots, qg_1g_2 - 1\} \mid x \equiv 0, \dots, ig_2 - 1(qg_2), x \equiv 0, \dots, jg_1 - 1(qg_1)\} \end{aligned}$$

Since  $(qg_1, qg_2) = q$ , the system of congruences  $x \equiv x_1(qg_2)$  and  $x \equiv x_2(qg_1)$  has no solution  $x$  if  $x_1 \not\equiv x_2(q)$  and a unique solution modulo  $qg_1g_2$  for  $x_1 \equiv x_2(q)$ . If we denote the representative  $y'$  of  $y' \equiv y(q)$  with  $0 \leq y' < q$  by  $\overline{y}^{(q)}$ , then

$$\begin{aligned} A_{i,j} &= ig_2 \frac{jg_1 - \overline{jg_1}^{(q)}}{q} + \overline{jg_1}^{(q)} \frac{ig_2 - \overline{ig_2}^{(q)}}{q} + \min(\overline{ig_2}^{(q)}, \overline{jg_1}^{(q)}) \\ &= \frac{ig_2 jg_1}{q} - \frac{\overline{ig_2}^{(q)} \overline{jg_1}^{(q)}}{q} + \min(\overline{ig_2}^{(q)}, \overline{jg_1}^{(q)}). \end{aligned}$$

Hence

$$\sum_{i=1}^{q-1} \sum_{j=1}^{q-1} A_{i,j} = \frac{q(q-1)^2}{4} g_1 g_2 - \frac{q(q-d_1)(q-d_2)}{4} + d_1 d_2 \sum_{i=1}^{q'-1} \sum_{j=1}^{q''-1} \min(id_2, jd_1),$$

where  $q' = q/d_1$  and  $q'' = q/d_2$ . We have

$$\begin{aligned} &\sum_{i=1}^{q''-1} \sum_{j=1}^{q'-1} \min(id_2, jd_1) \\ &= \sum_{i=1}^{q''-1} id_2 \left( q' - 1 - \left\lfloor \frac{id_2}{d_1} \right\rfloor \right) + \sum_{j=1}^{q'-1} jd_1 \left( q'' - 1 - \left\lfloor \frac{j d_1}{d_2} \right\rfloor \right) + \sum_{i=1}^{\frac{q''}{d_1} - 1} id_1 d_2 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{q''-1} i \left( q' - 1 - \left\lfloor \frac{id_2}{d_1} \right\rfloor \right) &= (q' - 1) \sum_{i=1}^{q''-1} i - \frac{d_2}{d_1} \sum_{i=1}^{q''-1} i^2 - \frac{1}{d_1} \sum_{i=1}^{q''-1} \overline{id_2}^{(d_1)} i \\ &= \frac{(q' - 1)(q'' - 1)q''}{2} - \frac{q'(q'' - 1)(2q'' - 1)}{6} + \frac{1}{d_1} \sum_{j=0}^{\frac{q''}{d_1} - 1} \sum_{i=1}^{d_1 - 1} (jd_1 + i) \overline{id_2}^{(d_1)} \\ &= \frac{q'(q''^2 - 1)}{6} + \frac{q''}{4} \left( -q'' - \frac{q''}{d_1} - d_1 + 3 \right) + \frac{q''}{d_1^2} \sum_{i=1}^{d_1 - 1} \overline{id_2}^{(d_1)} i. \end{aligned}$$

With

$$\begin{aligned}
\frac{d_2}{d_1} \sum_{i=1}^{d_1-1} \overline{id_2}^{(d_1)} i &= \frac{d_2}{d_1} \left( \sum_{i=1}^{d_1-1} d_2 i^2 - \sum_{i=\lceil \frac{d_1}{d_2} \rceil + 1}^{\lceil \frac{2d_1}{d_2} \rceil} d_1 i - \dots - \sum_{i=\lceil \frac{(d_2-1)d_1}{d_2} \rceil + 1}^{d_1-1} (d_2-1)d_1 i \right) \\
&= d_2 \left( \sum_{i=1}^{d_1-1} \frac{d_2}{d_1} i^2 - (d_2-1) \sum_{i=1}^{d_1-1} i + \sum_{i=1}^{\lceil \frac{(d_2-1)d_1}{d_2} \rceil} i + \dots + \sum_{i=1}^{\lceil \frac{d_1}{d_2} \rceil} i \right) \\
&= \frac{d_2^2(d_1-1)(2d_1-1)}{6} - \frac{d_2(d_2-1)(d_1-1)d_1}{2} \\
&\quad + \sum_{j=1}^{d_2-1} \frac{(jd_1 - \overline{j d_1}^{(d_2)} + d_2)(jd_1 - \overline{j d_1}^{(d_2)})}{2d_2} \\
&= \frac{d_1^2 + d_2^2 + 1}{12} + \frac{d_1^2 d_2 + d_1 d_2^2 - 3d_1 d_2}{4} - \frac{d_1}{d_2} \sum_{j=1}^{d_2-1} \overline{j d_1}^{(d_2)} j
\end{aligned}$$

we obtain

$$\begin{aligned}
g_1 g_2 \mathbf{Cov}(f_1(Y_k), f_2(Y_k)) &= \frac{1}{q} \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} A_{i,j} - g_1 g_2 \frac{(q-1)^2}{4} \\
&= -\frac{(q-d_1)(q-d_2)}{4} + \frac{q^2 - d_2^2}{6} + \frac{-d_1 q - q - d_1^2 d_2 + 3d_1 d_2}{4} \\
&\quad + \frac{d_1^2 + d_2^2 + 1}{12} + \frac{d_1^2 d_2 + d_1 d_2^2 - 3d_1 d_2}{4} \\
&\quad + \frac{q^2 - d_1^2}{6} + \frac{-d_2 q - q - d_1 d_2^2 + 3d_1 d_2}{4} + \frac{q - d_1 d_2}{2} \\
&= \frac{q^2 - d_1^2 - d_2^2 + 1}{12}
\end{aligned}$$

and the lemma is proved.

Clearly we have

$$\mathbf{Pr}[\epsilon_k(g_1 n) = b_1, \epsilon_k(g_2 n) = b_2] = \frac{A_{b_i+1, b_j+1} - A_{b_i, b_j+1} - A_{b_i+1, b_j} + A_{b_i, b_j}}{q g_1 g_2}$$

for  $(g_1, g_2) = 1$ . Thus

$$\mathbf{Pr}[\epsilon_k(g_1 n) = b_1, \epsilon_k(g_2 n) = b_2] = \pi_{b_1, b_2, g_1, g_2}$$

first for  $(g_1, g_2) = 1$ , and, with Lemma 2, for general  $g_1, g_2$ . With the remarks succeeding Theorem 4, we get

$$v_{i,j}^{(s)} = \begin{cases} C_{i,j} \left( \frac{g_{r_i}^{(i)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})}, \frac{g_{r_j}^{(j)}}{(g_{r_i}^{(i)}, g_{r_j}^{(j)})} \right) & \text{if } g_{r_j}^{(j)} P_i^{(s)}(n) = g_{r_i}^{(i)} P_j^{(s)}(n) \\ 0 & \text{else.} \end{cases}$$

For  $q_1 = \tilde{q}^{s_1}$  and  $q_2 = \tilde{q}^{s_2}$ ,  $f_1(n) = s_{q_1}(n)$  and  $f_2(n) = s_{q_2}(n)$  are strongly  $q$ -additive functions with  $q = q_1^{s_2} = q_2^{s_1}$ . Then, for  $P_1(n) = P_2(n) = n$ ,  $(Y_k)_{k \geq 0}$  is clearly a sequence of independent random variables and

$$f_1(Y_k) = X_0 + \tilde{q}X_1 + \cdots + \tilde{q}^{s_1-1}X_{s_1-1} + X_{s_1} + \cdots + \tilde{q}^{s_1-1}X_{2s_1-1} + \cdots + \tilde{q}^{s_1-1}X_{s_1s_2-1},$$

$$f_2(Y_k) = X_0 + \tilde{q}X_1 + \cdots + \tilde{q}^{s_2-1}X_{s_2-1} + X_{s_2} + \cdots + \tilde{q}^{s_2-1}X_{2s_2-1} + \cdots + \tilde{q}^{s_2-1}X_{s_1s_2-1},$$

where  $(X_j)_{0 \leq j \leq s_1s_2-1}$  is a sequence of identically distributed independent random variables on  $\{0, 1, \dots, \tilde{q} - 1\}$ .

Hence we have

$$\mathbf{Cov}(f_1(Y_k), f_2(Y_k)) = \sum_{j=0}^{s_1s_2-1} c_j \mathbf{Var} X_j,$$

where  $c_j$  runs through  $\{\tilde{q}^{ab} : 0 \leq a \leq s_1 - 1, 0 \leq b \leq s_2 - 1\}$  because of  $(s_1, s_2) = 1$ . This implies

$$\begin{aligned} \mathbf{Cov}(f_1(Y_k), f_2(Y_k)) &= \frac{\tilde{q}^2 - 1}{12} (1 + \tilde{q} + \cdots + \tilde{q}^{s_1-1}) (1 + \tilde{q} + \cdots + \tilde{q}^{s_2-1}) \\ &= \frac{(\tilde{q} + 1)(\tilde{q}^{s_1} - 1)(\tilde{q}^{s_2} - 1)}{12(\tilde{q} - 1)}. \end{aligned}$$

With  $\sigma_1^2 = \mathbf{Var} f_1(Y_k) = s_2(q_1^2 - 1)/12$  and  $\sigma_2^2 = \mathbf{Var} f_2(Y_k) = s_1(q_2^2 - 1)/12$ , we get for the normalized covariance

$$\frac{\mathbf{Cov}(f_1(Y_k), f_2(Y_k))}{\sigma_1 \sigma_2} = \frac{\tilde{q} + 1}{\tilde{q} - 1} \frac{(q_1 - 1)(q_2 - 1)}{\sqrt{s_1 s_2 (q_1^2 - 1)(q_2^2 - 1)}}.$$

### 2.3. Comparison of moments.

It remains to compare the moments of  $f_\ell(P_\ell(n))$  to those in (3). We need the following proposition (cf. Proposition 1).

**Proposition 3.** *Let  $P_\ell(x)$ ,  $1 \leq \ell \leq d$ , be integer polynomials with positive leading terms,  $\lambda > 0$  an arbitrary constant and  $h_\ell$ ,  $1 \leq \ell \leq d$ , non-negative integers. Then for integers*

$$(\log N)^\eta \leq k_1^{(\ell)} < k_2^{(\ell)} < \cdots < k_{h_\ell}^{(\ell)} \leq \log_q N^{r_\ell} - (\log N)^\eta \quad (1 \leq \ell \leq d)$$

(with some  $\eta > 0$ ) which satisfy

$$k_j^{(\ell)} \notin \left( \log_q N^s - (\log N)^\eta, \log_q N^s + (\log N)^\eta \right)$$

for all  $1 \leq s \leq r_\ell - 1$ , we have uniformly, as  $N \rightarrow \infty$ ,

$$\begin{aligned} \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right\} \\ = \prod_{s=1}^r p_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)} + \mathcal{O}((\log N)^{-\lambda}) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\pi(N)} \# \left\{ p < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right\} \\ = \prod_{s=1}^r p_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)} + \mathcal{O}((\log N)^{-\lambda}) \end{aligned}$$

with

$$p_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)} = \begin{cases} \Pr \left[ Y_{k_j^{(\ell)}}^{(s)} \subseteq S_{b_j^{(\ell)}, \ell}^{(s)} \text{ for all } (j, \ell) \in K_s \right] & \text{if } K_s \neq \emptyset \\ 1 & \text{else,} \end{cases}$$

where

$$K_s = \left\{ (j, \ell) \mid k_j^{(\ell)} \in [\log_q N^{s-1} + (\log N)^\eta, \log_q N^s - (\log N)^\eta] \right\}.$$

*Proof.* We follow the proofs of Lemma 5 in [1] and Proposition 1 in [3]. Let  $\psi_{b,q,\Delta}(x)$  be defined by

$$\psi_{b,q,\Delta}(x) = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} \mathbf{1}_{[b/q, (b+1)/q)}(\{x+z\}) dz.$$

Its Fourier series  $\sum_{m \in \mathbb{Z}} d_{m,b,q,\Delta} e(mx)$  is given by  $d_{m,0,q,\Delta} = \frac{1}{q}$  and

$$d_{m,b,q,\Delta} = \frac{e\left(-\frac{mb}{q}\right) - e\left(-\frac{m(b+1)}{q}\right)}{2\pi i m} \frac{e\left(\frac{m\Delta}{2}\right) - e\left(-\frac{m\Delta}{2}\right)}{2\pi i m \Delta} \text{ for } m \neq 0.$$

Clearly we have

$$\psi_{b,q,\Delta}(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{b}{q} + \Delta, \frac{b+1}{q} - \Delta\right], \\ 0 & \text{if } x \in [0, 1] \setminus \left[\frac{b}{q} - \Delta, \frac{b+1}{q} + \Delta\right]. \end{cases}$$

If we set

$$t(y_1, \dots, y_d) = \prod_{\ell=1}^d \prod_{j=1}^{h_\ell} \psi_{b_j^{(\ell)}, q_\ell, \Delta} \left( \frac{y_\ell}{q_\ell^{k_j^{(\ell)}+1}} \right),$$

then we get for  $\Delta < 1/(2q)$

$$\begin{aligned} \left| \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right\} - \sum_{n < N} t(P_1(n), \dots, P_d(n)) \right| \\ \leq \sum_{\ell=1}^d \sum_{j=1}^{h_\ell} \# \left\{ n < N \mid \left\{ \frac{P_\ell(n)}{q_\ell^{k_j^{(\ell)}+1}} \right\} \in U_{b_j^{(\ell)}, q_\ell, \Delta} \right\} \ll \Delta N + N(\log N)^{-\lambda} \end{aligned}$$

with  $U_{b,q,\Delta} = [0, \Delta] \cup \bigcup_{b=1}^{q-1} \left[\frac{b}{q} - \Delta, \frac{b}{q} + \Delta\right] \cup [1 - \Delta, 1]$  and Lemma 4 of [1]. For primes, we get a similar statement.

Hence we have to consider the sums

$$\Sigma = \sum_{n < N} t(P_1(n), \dots, P_d(n)) = \sum_{\mathbf{M} \in \mathcal{M}} T_{\mathbf{M}} \sum_{n < N} e(\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(n)),$$

where  $\mathcal{M}$  is the set of all  $(\mathbf{m}_1, \dots, \mathbf{m}_d)$  with integer vectors  $\mathbf{m}_\ell = (m_1^{(\ell)}, \dots, m_{h_\ell}^{(\ell)})$ ,

$$T_{\mathbf{M}} = \prod_{\ell=1}^d \prod_{j=1}^{h_\ell} d_{m_j^{(\ell)}, b_j^{(\ell)}, q, \Delta}$$

$$\text{and } \mathbf{v}_\ell = \left( q_\ell^{-k_1^{(\ell)}-1}, \dots, q_\ell^{-k_{h_\ell}^{(\ell)}-1} \right).$$

First of all, set  $\Delta = (\log N)^{-\delta}$  with an arbitrary (but fixed) constant  $\delta > 0$ . Then we can restrict to those  $\mathbf{M}$  for which  $|m_j^{(\ell)}| < (\log N)^{2\delta}$  for all  $j, \ell$  because of

$$\begin{aligned} \sum_{\exists \ell, j: |m_j^{(\ell)}| \geq (\log N)^{2\delta}} |T_{\mathbf{M}}| &\ll \left( \sum_{m=[(\log N)^{2\delta}] }^{\infty} \frac{1}{\Delta m^2} \right) \left( \sum_{m=0}^{\infty} \min \left( 1, \frac{1}{m}, \frac{1}{\Delta m^2} \right) \right)^{h-1} \\ &\ll \frac{1}{\Delta} (\log N)^{-\delta} \left( \log \frac{1}{\Delta} \right)^{h-1} \ll (\log N)^{-\delta/2}, \end{aligned}$$

where  $h = h_1 + \dots + h_d$ . Furthermore, it is sufficient to consider just the case where  $m_j^{(\ell)} \neq 0$  for all  $j, \ell$ . (Otherwise, just reduce  $h_\ell$  to a smaller value.)

Set

$$Q_{\mathbf{M}}(n) = \mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \dots + \mathbf{m}_d \cdot \mathbf{v}_d P_d(n).$$

We have to check whether  $Q_{\mathbf{M}}(n)$  has degree  $r$  and satisfies the conditions of Lemmata 1 and 2 of [1] saying that

$$\begin{aligned} \frac{1}{N} \sum_{n < N} e(P(n)) &= \mathcal{O}((\log N)^{-\tau_0}), \\ \frac{1}{\pi(N)} \sum_{p < N} e(P(p)) &= \mathcal{O}((\log N)^{-\tau_0}), \end{aligned}$$

as  $N \rightarrow \infty$ , hold if the the leading coefficient of  $P(n)$  is  $\frac{A}{H}$  with  $(A, H) = 1$  and

$$(\log N)^\tau < H < N^r (\log N)^{-\tau} \quad (6)$$

for some  $\tau$  (depending on  $\tau_0$ ).

The coefficient of  $n^r$  is, if we set  $k_{\max} = \max_\ell k_{h_\ell}^{(\ell)}$ ,

$$\frac{A_{\mathbf{M}}}{H_{\mathbf{M}}} = \sum_{(j, \ell) \in K_r} \frac{g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max} - k_j^{(\ell)}}}{q^{k_{\max}}} + \sum_{(j, \ell) \notin K_r} \frac{g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max} - k_j^{(\ell)}}}{q^{k_{\max}}} \quad (7)$$

with  $(A_{\mathbf{M}}, H_{\mathbf{M}}) = 1$ . If  $A_{\mathbf{M}} \neq 0$ , then (6) is satisfied. If  $A_{\mathbf{M}} = 0$ , assume  $k_{\max} \in K_r$ . Then we obtain

$$\sum_{(j, \ell) \in K_r} g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max} - k_j^{(\ell)}} \equiv 0 \left( q^{k_{\max} - (\log_q N^{r-1} - (\log N)^\eta)} \right).$$

Because of  $|m_j^{(\ell)}| < (\log N)^{2\delta}$ , this implies  $\sum_{(j,\ell) \in K_r} g_r^{(\ell)} m_j^{(\ell)} q^{k_{\max} - k_j^{(\ell)}} = 0$ . Hence  $A_{\mathbf{M}} = 0$  if and only if both sums in (7) are zero and we have

$$\begin{aligned} & \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right\} \\ &= \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, (j, \ell) \in K_r \right\} \\ & \quad \times \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, (j, \ell) \notin K_r \right\} + \mathcal{O}((\log N)^{-\lambda}). \end{aligned}$$

Now we can repeat the arguments for  $(j, \ell) \in K_{r-1}$  and get inductively

$$\begin{aligned} & \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, 1 \leq j \leq h_\ell, 1 \leq \ell \leq d \right\} \\ &= \prod_{s=1}^r \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell(n)) = b_j^{(\ell)}, (j, \ell) \in K_s \right\} + \mathcal{O}((\log N)^{-\lambda}). \end{aligned}$$

Hence we may assume from now on that all  $k_j^{(\ell)}$  are contained in one set  $K_s$  for some  $s \leq r$ .

If the degree of  $Q_{\mathbf{M}}(n)$  is smaller than  $s$ , we have

$$|Q_{\mathbf{M}}(n)| \ll \frac{(\log N)^{2\delta} N^{s-1}}{q^{\log_q N^{s-1} + (\log N)^\eta}} = \frac{(\log N)^{2\delta}}{q^{(\log N)^\eta}}$$

for all  $n < N$  and, with  $e(y) = 1 + \mathcal{O}(y)$ ,

$$\sum_{|m_j^{(\ell)}| < (\log N)^{2\delta}, \deg(Q_{\mathbf{M}}(n)) < s} T_{\mathbf{M}} \left( \sum_{n < N} e(Q_{\mathbf{M}}(n)) - N \right) \ll \frac{N(\log N)^{2\delta(h+1)}}{q^{(\log N)^\eta}}.$$

Thus we can treat these  $Q_{\mathbf{M}}(n)$  as if they were the zero polynomial and it suffices to regard the polynomials  $P_\ell^{(s)}(n)$  and

$$Q_{\mathbf{M}}^{(s)}(n) = \mathbf{m}_1 \cdot \mathbf{v}_1 P_1^{(s)}(n) + \cdots + \mathbf{m}_d \cdot \mathbf{v}_d P_d^{(s)}(n).$$

(6) is satisfied if and only if  $Q_{\mathbf{M}}^{(s)}(n) \neq 0$  and we obtain

$$\begin{aligned} \Sigma = N \sum_{\mathbf{M} \in \mathcal{M}: Q_{\mathbf{M}}^{(s)}(n) \equiv 0} T_{\mathbf{M}} + \mathcal{O} \left( N(\log N)^{-\tau_0} \sum_{\mathbf{M} \in \mathcal{M}: |m_j^{(\ell)}| < (\log N)^{2\delta}, Q_{\mathbf{M}}^{(s)}(n) \neq 0} |T_{\mathbf{M}}| \right) \\ + \mathcal{O} \left( N(\log N)^{-\delta/2} \right) + \mathcal{O} \left( N(\log N)^{-\lambda} \right). \end{aligned}$$

Since the main term  $\sum_{\mathbf{M} \in \mathcal{M}: Q_{\mathbf{M}}^{(s)}(n) \equiv 0} T_{\mathbf{M}}$  depends on  $\Delta$ , we have to replace  $T_{\mathbf{M}}$  by

$$T'_{\mathbf{M}} = \prod_{\ell=1}^d \prod_{j=1}^{h_\ell} c_{m_j^{(\ell)}, b_j^{(\ell)}, q}.$$



Hence we have to estimate the difference  $\sum_{\mathbf{M} \in \mathcal{M}: Q_{\mathbf{M}}(n) \equiv 0} (T_{\mathbf{M}} - T'_{\mathbf{M}})$ .

We clearly have

$$d_{m_j^{(\ell)}, b_j^{(\ell)}, q, \Delta} = c_{m_j^{(\ell)}, b_j^{(\ell)}, q} \left( 1 + \mathcal{O} \left( m_j^{(\ell)} \Delta \right) \right)$$

as  $\Delta \rightarrow 0$  and therefore

$$T_{\mathbf{M}} = T'_{\mathbf{M}} \left( 1 + \mathcal{O} \left( \max_{j, \ell} m_j^{(\ell)} \Delta \right) \right). \quad (8)$$

First assume  $|m_j^{(\ell)}| < (\log N)^{\delta/2}$  for all  $j, \ell$ . From (8) and  $c_{m_j^{(\ell)}, b_j^{(\ell)}, q} \leq \min \left( 1, \frac{1}{m_j^{(\ell)}} \right)$ ,

we obtain

$$\begin{aligned} \sum_{\mathbf{M} \in \mathcal{M}: |m_j^{(\ell)}| < (\log N)^{\delta/2}} |T_{\mathbf{M}} - T'_{\mathbf{M}}| &\ll \sum_{\mathbf{M} \in \mathcal{M}: |m_j^{(\ell)}| < (\log N)^{\delta/2}} |T'_{\mathbf{M}}| (\log N)^{-\delta/2} \\ &\ll \left( \sum_{m=1}^{[(\log N)^{\delta/2}]^h} \frac{1}{m} \right)^h (\log N)^{-\delta/2} \leq \frac{(\log(\log N)^{\delta/2})^h}{(\log N)^{\delta/2}} \ll (\log N)^{-\delta/3} \end{aligned}$$

It remains to estimate the  $T_{\mathbf{M}}$  and  $T'_{\mathbf{M}}$  with  $|m_j^{(\ell)}| > (\log N)^{\delta/2}$  for some  $j, \ell$  which satisfy the equation  $Q_{\mathbf{M}}^{(s)}(n) \equiv 0$ , i.e.

$$\sum_{j, \ell} g_r^{(\ell)} q^{k_{\max} - k_j^{(\ell)}} m_j^{(\ell)} = 0.$$

By Lemma 14 of [4], we get

$$\sum_{\mathbf{M} \in \mathcal{M}: Q_{\mathbf{M}}^{(s)}(n) \equiv 0, |m_j^{(\ell)}| \geq (\log N)^{\delta/2} \text{ for some } j, \ell} T'_{\mathbf{M}} \ll (\log N)^{-\frac{\delta}{2(h-1)^2}}$$

and the same estimate for  $T_{\mathbf{M}}$ . Note that this lemma is stated for a linear equation where one of the coefficients is 1, but the proof can be easily adapted for general linear equations.

Hence

$$\sum_{\mathbf{M} \in \mathcal{M}: Q_{\mathbf{M}}^{(s)}(n) \equiv 0} T_{\mathbf{M}} = \tilde{p}_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)} + \mathcal{O} \left( (\log N)^{-\frac{\delta}{2(h-1)^2}} \right),$$

where

$$\tilde{p}_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)} = \sum_{\mathbf{M} \in \mathcal{M}: Q_{\mathbf{M}}^{(s)}(n) \equiv 0} T'_{\mathbf{M}}$$

and we get

$$\Sigma = N \tilde{p}_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)} + \mathcal{O} \left( (\log N)^{-\lambda} \right),$$

for  $\delta = 2(h-1)^2 \lambda$  and  $\tau_0 > \lambda$ .

It remains to prove that the  $\tilde{p}_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)}$  are the probabilities defined by the Markov chain.

We have

$$\begin{aligned} & \left\{ n < N \mid \epsilon_{k_j^{(\ell)}}(P_\ell^{(s)}(n)) = b_j^{(\ell)} \text{ for all } (j, \ell) \in K_s \right\} \\ &= \left\{ n < N \mid \mathbf{w}_{k_{\max}}^{(s)}(n) \in \bigcap_{(j, \ell) \in K_s} T_j^{k_j^{(\ell)} - k_{\max}} S_{b_j^{(\ell)}, \ell}^{(s)} \right\} \end{aligned}$$

and this intersection consists of a finite number of convex sets, which can be arbitrarily well approximated by elementary rectangles

$$\prod_{i=s}^r \left[ \sum_{j=1}^{J_i} \tilde{b}_j^{(i)} q^{-j}, \sum_{j=1}^{J_i} \tilde{b}_j^{(i)} q^{-j} + q^{-J_i} \right).$$

By Proposition 1, we get

$$\begin{aligned} & \frac{1}{N} \# \left\{ n < N \mid \mathbf{w}_{k_{\max}}^{(s)}(n) \in \prod_{i=s}^r \left[ \sum_{j=1}^{J_i} \tilde{b}_j^{(i)} q^{-j}, \sum_{j=1}^{J_i} \tilde{b}_j^{(i)} q^{-j} + q^{-J_i} \right] \right\} \\ &= \frac{1}{N} \# \left\{ n < N \mid \epsilon_{k_{\max-j+1}}(n^i) = \tilde{b}_j^{(i)}, 1 \leq j \leq J_i, s \leq i \leq r \right\} \rightarrow \frac{1}{q^{J_s} \dots q^{J_r}}, \end{aligned}$$

if  $k_{\max} \leq \log N - (\log N)^\eta$  and  $J_i \leq k_{\max} - (\log N)^\eta$ . This means that the density in each of this rectangles converges to its Lebesgue measure. Since we do not change  $\bigcap_{j, \ell} T_j^{k_j^{(\ell)} - k_{\max}} S_{b_j^{(\ell)}, \ell}^{(s)}$  if we shift all  $k_j^{(\ell)}$  and increase  $N$ , the  $J_i$  can be arbitrarily large. Therefore  $\tilde{p}_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)}$  must be its Lebesgue measure, which is just  $p_{k_1^{(1)}, \dots, k_{h_d}^{(d)}, b_1^{(1)}, \dots, b_{h_d}^{(d)}}^{(s)}$ .

This also implies Lemma 2 ( $d = 2, h_1 = h_2 = 1$ ).

Proposition 3 shows that we have to replace  $f_\ell$  by  $\bar{f}_\ell^{(N^{r_\ell})}$ ,

$$\bar{f}_\ell^{(N^{r_\ell})}(P_\ell(n)) = \sum_{s=1}^{r_\ell} \sum_{k=(s-1)\log_q N + A(N)}^{(s-1)\log_q N + B(N)} f_\ell(\epsilon_k(P_\ell(n))).$$

Note that  $\bar{f}_\ell^{(N^{r_\ell})}(P_\ell(n)) = f_\ell(P_\ell(n)) + \mathcal{O}((\log N)^\eta)$ . Similarly define  $\bar{M}_\ell(N^{r_\ell})$  and  $\bar{D}_\ell(N^{r_\ell})$  by taking the sum only over these  $k$ . Note that these definitions are slightly different from those in [3,4] (and [1], where  $\bar{f}$  is denoted by  $f_1$ ).

**Corollary 3.** *We have*

$$\begin{aligned} & \frac{1}{N} \sum_{n < N} \prod_{\ell=1}^d \left( \frac{\bar{f}_\ell^{(N^{r_\ell})}(P_\ell(n)) - \bar{M}_\ell(N^{r_\ell})}{\bar{D}_\ell(N^{r_\ell})} \right)^{h_\ell} \\ & - \mathbf{E} \prod_{\ell=1}^d \left( \frac{\sum_{s=1}^{r_\ell} \sum_{k=(s-1)\log_q N + A(N)}^{(s-1)\log_q N + B(N)} f_\ell(Y_k^{(s)}) - \bar{M}_\ell(N^{r_\ell})}{\bar{D}_\ell(N^{r_\ell})} \right)^{h_\ell} \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\pi(N)} \sum_{p < N} \prod_{\ell=1}^d \left( \frac{\overline{f}_\ell^{(Nr_\ell)}(P_\ell(p)) - \overline{M}_\ell(Nr_\ell)}{\overline{D}_\ell(Nr_\ell)} \right)^{h_\ell} \\ & - \mathbf{E} \prod_{\ell=1}^d \left( \frac{\sum_{s=1}^{r_\ell} \sum_{k=(s-1)\log_q N + A(N)}^{(s-1)\log_q N + B(N)} f_\ell(Y_k^{(s)}) - \overline{M}_\ell(Nr_\ell)}{\overline{D}_\ell(Nr_\ell)} \right)^{h_\ell} \rightarrow 0, \end{aligned}$$

where the  $Y_k^{(s)}$  and  $Y_{k'}^{(s')}$  are independent if  $s \neq s'$ .

*Proof.* The second terms are the sum over all integers

$$k_1^{(\ell)}, \dots, k_{h_\ell}^{(\ell)} \in [A(N), \log_q Nr_\ell - A(N)] \setminus \bigcup_{s=1}^{r_\ell-1} [\log_q N^s - A(N), \log_q N^s + A(N)],$$

$1 \leq \ell \leq d$ , of

$$\begin{aligned} & \mathbf{E} \prod_{\ell=1}^d \prod_{j=1}^{h_\ell} \frac{f_\ell\left(Y_{k_j^{(\ell)}}^{(s)}\right) - \mu_{\ell, k_j^{(\ell)}}}{D_\ell(Nr_\ell)} \\ & = \sum_{b_1^{(1)}=0}^{q-1} \dots \sum_{b_{h_d}^{(d)}=0}^{q-1} \prod_{\ell=1}^d \prod_{j=1}^{h_\ell} \frac{f_\ell(b_j^{(\ell)}) - \mu_{\ell, k_j^{(\ell)}}}{D_\ell(Nr_\ell)} \mathbf{Pr} \left[ Y_{k_j^{(\ell)}}^{(s)} \subseteq S_{b_j^{(\ell)}}^{(s)} \text{ for all } j, \ell \right], \end{aligned}$$

where the  $s$  are such that  $k_j^{(\ell)} \in K_s$ . Since the  $Y_{k_j^{(\ell)}}^{(s)}$  are independent for different  $s$ , we have

$$\mathbf{Pr} \left[ Y_{k_j^{(\ell)}}^{(s)} \subseteq S_{b_j^{(\ell)}}^{(s)} \text{ for all } (j, \ell) \right] = \prod_{s=1}^r \mathbf{Pr} \left[ Y_{k_j^{(\ell)}}^{(s)} \subseteq S_{b_j^{(\ell)}}^{(s)} \text{ for all } (j, \ell) \in K_s \right]$$

and, by Proposition 3, the corresponding first terms are the same up to an error term of  $\mathcal{O}((\log N)^{-\lambda})$ . Hence the convergences are valid with error terms  $\mathcal{O}((\log N)^{-\lambda+h-h\eta})$ .

Similarly to Corollary 2 of [3], we obtain

$$\begin{aligned} & \frac{1}{N} \sum_{n < N} \prod_{\ell=1}^d \left( \frac{f_\ell(P_\ell(n)) - M_\ell(Nr_\ell)}{D_\ell(Nr_\ell)} \right)^{h_\ell} \\ & - \frac{1}{N} \sum_{n < N} \prod_{\ell=1}^d \left( \frac{\overline{f}_\ell^{(Nr_\ell)}(P_\ell(n)) - \overline{M}_\ell(Nr_\ell)}{\overline{D}_\ell(Nr_\ell)} \right)^{h_\ell} \rightarrow 0 \end{aligned}$$

and therefore, by the method of moments (see e.g. Billingsley [2], p. 390),

$$\begin{aligned} & \frac{1}{N} \# \left\{ n < N \left| \frac{f_\ell(P_\ell(n)) - M_\ell(Nr_\ell)}{D_\ell(Nr_\ell)} < x_\ell, \ell = 1, 2, \dots, d \right. \right\} \\ & \rightarrow \mathbf{Pr} \left[ \frac{\sum_{s=1}^{r_\ell} \sum_{k=(s-1)\log_q N + A(N)}^{(s-1)\log_q N + B(N)} f_\ell(Y_k^{(s)}) - \overline{M}_\ell(Nr_\ell)}{\overline{D}_\ell(Nr_\ell)} < x_\ell, \ell = 1, \dots, d \right]. \end{aligned}$$

Clearly we have  $\overline{M}_\ell(N^{r_\ell}) = r_\ell \overline{M}_\ell(N)$ ,  $\overline{D}_\ell(N^{r_\ell})^2 = r_\ell \overline{D}_\ell(N)^2$  and

$$\begin{aligned} & \frac{\sum_{s=1}^{r_\ell} \sum_{k=(s-1) \log_q N + A(N)}^{(s-1) \log_q N + B(N)} f_\ell \left( Y_k^{(s)} \right) - \overline{M}_\ell(N^{r_\ell})}{\overline{D}_\ell(N^{r_\ell})} \\ &= \frac{1}{\sqrt{r_\ell}} \sum_{s=1}^{r_\ell} \frac{\sum_{k=A(N)}^{B(N)} f_\ell \left( Y_k^{(s)} \right) - \overline{M}_\ell(N)}{\sigma_\ell \sqrt{B(N) - A(N) + 1}} \rightarrow \frac{1}{\sqrt{r_\ell}} \left( Z_\ell^{(1)} + \dots + Z_\ell^{(r)} \right) \end{aligned}$$

by Proposition 2, where the  $Z^{(s)} = (Z_1^{(s)}, \dots, Z_d^{(s)})$  are independent normally distributed random vectors with covariance matrices  $V^{(s)}$ . (For  $s > r_\ell$ , we have  $f_\ell(Y_k^{(s)}) = 0 = Z_\ell^{(s)}$  because of  $P_\ell^{(s)}(n) \equiv 0$  and  $S_{0,\ell}^{(s)} = \mathbb{T}^{r-s+1}$ .)

Hence the sum is normally distributed and the elements of the covariance matrix  $V$  are given by

$$v_{i,j} = \frac{1}{\sqrt{r_i r_j}} \left( v_{i,j}^{(1)} + \dots + v_{i,j}^{(r)} \right).$$

For  $r_i \neq r_j$ , all  $v_{i,j}^{(s)}$  are zero, as well as for all  $s > r_i$ . If  $g_{r_j}^{(j)} P_i(n) \equiv g_{r_i}^{(i)} P_j(n)$ , then  $v_{i,j}^{(1)} = \dots = v_{i,j}^{(r_i)} = v_{i,j}$ . If we just have  $r_i = r_j$  and  $g_{r_j}^{(j)} g_s^{(i)} = g_{r_i}^{(i)} g_s^{(j)}$  for all  $s > s'$ , then  $v_{i,j}^{(s'+1)} = \dots = v_{i,j}^{(r_i)}$  and  $v_{i,j}^{(s)} = 0$  for  $s \leq s'$ . Therefore  $v_{i,j} = \frac{r_i - s'}{r_i} v_{i,j}^{(r_i)}$  and the covariance matrix has the stated form.

The corresponding statements for primes are obtained similarly and this concludes the proof of Theorem 4.

### 3. PROOF OF THEOREM 3

We have to prove the following proposition.

**Proposition 4.** *Let  $q_1, q_2$  be multiplicatively independent integers and  $P_1(n), P_2(n)$  integer polynomials with positive leading terms. Let  $\lambda > 0$  be an arbitrary constant and  $h_1, h_2$  non-negative integers. Then for integers*

$$(\log N^{r_\ell})^\eta \leq k_1^{(\ell)} < k_2^{(\ell)} < \dots < k_{h_\ell}^{(\ell)} \leq \log_{q_\ell} N^{r_\ell} - (\log N^{r_\ell})^\eta \quad (\ell = 1, 2)$$

(with some  $\eta > 0$ ), we have, as  $N \rightarrow \infty$ ,

$$\begin{aligned} & \frac{1}{N} \# \left\{ n < N \mid \epsilon_{q_1, k_j^{(1)}}(P_1(n)) = b_j^{(1)}, \epsilon_{q_2, k_j^{(2)}}(P_2(n)) = b_j^{(2)}, 1 \leq j \leq h_\ell \right\} \\ &= \frac{1}{q_1^{h_1} q_2^{h_2}} + \mathcal{O}((\log N)^{-\lambda}) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\pi(N)} \# \left\{ p < N \mid \epsilon_{q_1, k_j^{(1)}}(P_1(n)) = b_j^{(1)}, \epsilon_{q_2, k_j^{(2)}}(P_2(n)) = b_j^{(2)}, 1 \leq j \leq h_\ell \right\} \\ &= \frac{1}{q_1^{h_1} q_2^{h_2}} + \mathcal{O}((\log N)^{-\lambda}) \end{aligned}$$

uniformly for  $b_j^{(\ell)} \in \{0, \dots, q_\ell - 1\}$  and  $k_j^{(\ell)}$  in the given range, where the implicit constant of the error term may depend on  $q_\ell, P_\ell, h_\ell$  and  $\lambda$ .

For the proof we need the following three lemmata. The first one is a corollary to Baker's theorem on linear forms, in a version due to Waldschmidt [7].

**Lemma 4 (Corollary 3 in [3]).** *Let  $k_1, k_2$  be positive integers,  $q_1, q_2$  positive real numbers and  $m_1, m_2$  real numbers such that  $\frac{m_1}{q_1^{k_1}} + \frac{m_2}{q_2^{k_2}} \neq 0$ . Then there exists a constant  $C > 0$  such that*

$$\left| \frac{m_1}{q_1^{k_1}} + \frac{m_2}{q_2^{k_2}} \right| \geq \max \left( \frac{|m_1|}{q_1^{k_1}}, \frac{|m_2|}{q_2^{k_2}} \right) e^{-C \log q_1 \log q_2 \log(\max(k_1, k_2)) \log(\max(|m_1|, |m_2|))}.$$

The next lemma is an adapted version of Lemmata 1 and 2 of [1] which are due to Hua [5] and Vinogradov.

**Lemma 5 (Lemmata 10 and 11 in [4]).** *Let  $P(n)$  be a polynomial of degree  $r$  with leading coefficient  $\beta$ . For every  $\tau_0 > 0$ , we have a  $\tau > 0$  such that*

$$N^{-r}(\log N)^\tau < \beta < (\log N)^{-\tau}$$

implies

$$\frac{1}{N} \sum_{n < N} e(P(n)) = \mathcal{O}((\log N)^{-\tau_0})$$

and

$$\frac{1}{\pi(N)} \sum_{p < N} e(P(p)) = \mathcal{O}((\log N)^{-\tau_0})$$

as  $N \rightarrow \infty$ .

*Proof of Proposition 4.* As for Proposition 2, we have to estimate the sums

$$\Sigma = \sum_{(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{M}} T_{\mathbf{m}_1, \mathbf{m}_2} \sum_{n < N} e(\mathbf{m}_1 \cdot \mathbf{v}_1 P_1(n) + \mathbf{m}_2 \cdot \mathbf{v}_2 P_2(n)).$$

The case of different degrees of the polynomials is treated by Proposition 1. So we can assume that they have the same degree  $r_1 = r_2 = r$ .

As in the proof of Proposition 2, we fix  $\Delta = (\log N)^{-\delta}$  and restrict to those  $(\mathbf{m}_1, \mathbf{m}_2)$  for which  $|m_j^{(\ell)}| < (\log N)^{2\delta}$  and  $m_j^{(\ell)} \neq 0$  ( $q_\ell$ ) for all  $j, \ell$ .

Suppose now  $g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 \neq 0$  and set  $\varepsilon = \eta / (h_1 + h_2 - 1)$ . Then there exists an integer  $K$  with  $0 \leq K \leq h_1 + h_2 - 2$  such that for all  $j$  and  $\ell = 1, 2$

$$k_{j+1}^{(\ell)} - k_j^{(\ell)} \notin \left[ (\log N)^{K\varepsilon}, (\log N)^{(K+1)\varepsilon} \right).$$

So fix  $K$  with this property.

First suppose  $k_{j+1}^{(\ell)} - k_j^{(\ell)} < (\log N)^{K\varepsilon}$  for all  $j, \ell$ . Then we set

$$\overline{m}_\ell = g_r^{(\ell)} \sum_{j=1}^{h_\ell} m_j^{(\ell)} q_\ell^{k_{h_\ell}^{(\ell)} - k_j^{(\ell)}} \quad (\ell = 1, 2)$$

and have  $\log |\overline{m}_\ell| \ll (\log N)^{K\varepsilon}$ . We can apply Lemma 4 to

$$g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 = \frac{\overline{m}_1}{q_1^{k_{h_1}^{(1)} + 1}} + \frac{\overline{m}_2}{q_2^{k_{h_2}^{(2)} + 1}}$$

and obtain

$$\begin{aligned} \left| g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 \right| &\geq \max \left( q_1^{-k_{h_1}^{(1)}-1}, q_2^{-k_{h_2}^{(1)}-1} \right) e^{-c \log \log N (\log N)^{K\varepsilon}} \\ &\geq \frac{\max(q_1, q_2)^{(\log N)^\eta} e^{-c \log \log N (\log N)^{K\varepsilon}}}{N^r} \\ &\geq \frac{e^{\log(\max(q_1, q_2))(\log N)^\eta - c \log \log N (\log N)^{\frac{h_1+h_2-2}{h_1+h_2-1}}} }{N^r} \geq \frac{(\log N)^\tau}{N^r} \end{aligned}$$

for some constant  $c > 0$  and all  $\tau > 0$ . Because of

$$\left| g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 \right| \leq \frac{(h_1 + h_2)(\log N)^{2\delta}}{\min(q_1, q_2)^{-(\log N)^\eta}},$$

Lemma 5 can be applied.

Otherwise we have some  $s_\ell$ ,  $\ell = 1, 2$ , such that  $k_{j+1}^{(\ell)} - k_j^{(\ell)} < (\log N)^{K\varepsilon}$  for all  $j < s_\ell$  and  $k_{s_\ell+1}^{(\ell)} - k_{s_\ell}^{(\ell)} \geq (\log N)^{(K+1)\varepsilon}$ . Here we set

$$\bar{m}_\ell = g_r^{(\ell)} \sum_{j=1}^{s_\ell} m_j^{(\ell)} q_\ell^{k_{s_\ell}^{(\ell)} - k_j^{(\ell)}} \quad (\ell = 1, 2).$$

and have again  $\log |\bar{m}_\ell| \ll (\log N)^{K\varepsilon}$ . Furthermore, we can estimate the sums

$$\sum_{j=s_\ell+1}^{h_\ell} \frac{m_j^{(\ell)}}{q_\ell^{k_j^{(\ell)}+1}} = \mathcal{O} \left( (\log N)^{2\delta} q_\ell^{-k_{s_\ell} - (\log N)^{(K+1)\varepsilon}} \right).$$

Thus we get

$$\begin{aligned} \left| g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 \right| &\geq \left| \frac{\bar{m}_1}{q_1^{k_{s_1}^{(1)}+1}} + \frac{\bar{m}_2}{q_2^{k_{s_2}^{(2)}+1}} \right| - \left| \sum_{j_1=s_1+1}^{h_1} \frac{m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}+1}} \right| - \left| \sum_{j_2=s_2+1}^{h_2} \frac{m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}+1}} \right| \\ &\geq \max \left( q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(2)}-1} \right) e^{-c \log \log N (\log N)^{K\varepsilon}} \\ &\quad - \mathcal{O} \left( (\log N)^{2\delta} \max \left( q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(2)}-1} \right) e^{-(\log N)^{(K+1)\varepsilon}} \right) \\ &\gg \max \left( q_1^{-k_{s_1}^{(1)}-1}, q_2^{-k_{s_2}^{(2)}-1} \right) e^{-c \log \log N (\log N)^{K\varepsilon}} \end{aligned}$$

and Lemma 5 can again be applied.

If  $q_1$  and  $q_2$  are coprime, then we have  $g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2 = 0$  only for  $\mathbf{m}_1 = \mathbf{m}_2 = \mathbf{0}$ . Otherwise we may have other choices of  $(\mathbf{m}_1, \mathbf{m}_2)$ .

Set  $q = (q_1, q_2)$  and  $\tilde{q}_1 = q_1/q$ ,  $\tilde{q}_2 = q_2/q$ . Assume, w.l.o.g.,  $k_{h_1}^{(1)} \geq k_{h_2}^{(2)}$ . Then

we have

$$\begin{aligned}
& \sum_{j_1=1}^{h_1} \frac{g_r^{(1)} m_{j_1}^{(1)}}{q_1^{k_{j_1}^{(1)}}} + \sum_{j_2=1}^{h_2} \frac{g_r^{(2)} m_{j_2}^{(2)}}{q_2^{k_{j_2}^{(2)}}} \\
&= g_r^{(1)} \frac{m_1^{(1)} \tilde{q}_1^{k_{h_1}^{(1)} - k_1^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)} - k_1^{(2)}} q^{k_{h_1}^{(1)} - k_1^{(1)}} + \dots + m_{h_1-1}^{(1)} \tilde{q}_1^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)} - k_{h_1-1}^{(2)}} q^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}}}{\tilde{q}_1^{k_{h_1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}}} \\
& \quad + g_r^{(2)} \frac{m_1^{(2)} \tilde{q}_1^{k_{h_1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)} - k_1^{(2)}} q^{k_{h_1}^{(1)} - k_1^{(2)}} + \dots + m_{h_2}^{(2)} \tilde{q}_1^{k_{h_1}^{(1)}} q^{k_{h_1}^{(1)} - k_{h_2}^{(2)}}}{\tilde{q}_1^{k_{h_1}^{(1)}} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)}}},
\end{aligned}$$

where we have omit the “+1” in the denominator for simplicity. (Just consider  $k_j^{(\ell)} - 1$  instead of  $k_j^{(\ell)}$ .) Hence we must have

$$g_r^{(1)} \left( m_1^{(1)} \tilde{q}_1^{k_{h_1}^{(1)} - k_1^{(1)}} q^{k_{h_1}^{(1)} - k_1^{(1)}} + \dots + m_{h_1-1}^{(1)} \tilde{q}_1^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} q^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \right) \equiv 0 \left( \tilde{q}_1^{k_{h_1}^{(1)}} \right). \quad (9)$$

Of course this is useful only if  $\tilde{q}_1 > 1$ , which we assume first. We have to distinguish several cases. (9) implies

$$m_{j+1}^{(1)} q_1^{k_{h_1}^{(1)} - k_{j+1}^{(1)}} + \dots + m_{h_1-1}^{(1)} q_1^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} + \dots + m_{h_1}^{(1)} \equiv 0 \left( \tilde{q}_1^{k_{h_1}^{(1)} - k_j^{(1)}} \right) \quad (10)$$

for all  $j$ ,  $1 \leq j \leq h_1 - 1$ . If  $k_{j+1}^{(1)} - k_j^{(1)} \geq (\log N)^\varepsilon$  for some  $j$ , then  $|m_j^{(\ell)}| < (\log N)^{2\delta}$  implies that the left hand side of (10) must be zero. Hence  $m_{h_1}^{(1)} \equiv 0 (q_1)$  which implies  $T_{\mathbf{m}_1, \mathbf{m}_2} = 0$  since we have excluded  $m_{h_1}^{(1)} = 0$ . If  $k_{j+1}^{(1)} - k_j^{(1)} \leq (\log N)^\varepsilon$  for all  $j$ , then the left hand side of (9) must be zero and  $m_{h_1}^{(1)} \equiv 0 (q_1)$ .

Now consider the case  $\tilde{q}_1 = 1$ , i.e.  $q_1 | q_2$ . Then we have to check

$$\begin{aligned}
& g_r^{(1)} \left( m_1^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)} - k_1^{(1)}} + \dots + m_{h_1-1}^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}} q^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \tilde{q}_2^{k_{h_2}^{(2)}} \right) + \quad (11) \\
& g_r^{(2)} \left( m_1^{(2)} \tilde{q}_2^{k_{h_2}^{(2)} - k_1^{(2)}} q^{k_{h_1}^{(1)} - k_1^{(2)}} + \dots + m_{h_2-1}^{(2)} \tilde{q}_2^{k_{h_2}^{(2)} - k_{h_2-1}^{(2)}} q^{k_{h_1}^{(1)} - k_{h_2-1}^{(2)}} + m_{h_2}^{(2)} q^{k_{h_1}^{(1)} - k_{h_2}^{(2)}} \right) = 0.
\end{aligned}$$

This implies

$$g_r^{(2)} q^{k_{h_1}^{(1)} - k_{h_2}^{(2)}} \left( m_{j+1}^{(2)} q_2^{k_{h_2}^{(2)} - k_{j+1}^{(2)}} + \dots + m_{h_2-1}^{(2)} q_2^{k_{h_2}^{(2)} - k_{h_2-1}^{(2)}} + m_{h_2}^{(2)} \right) \equiv 0 \left( \tilde{q}_2^{k_{h_2}^{(2)} - k_j^{(2)}} \right) \quad (12)$$

for  $1 \leq j \leq h_2 - 1$  and for  $j = 0$ , if we set  $k_0^{(2)} = 0$ .

Assume first  $k_{h_1}^{(1)} - k_{h_2}^{(2)} \leq (\log N)^{\varepsilon/2}$ . Then we can do the same reasonings as above and obtain  $m_{h_2}^{(2)} \equiv 0 (q_2)$ .

The last (and most difficult) case is  $k_{h_1}^{(1)} - k_{h_2}^{(2)} \geq (\log N)^{\varepsilon/2}$ . First suppose that  $\tilde{q}_2$  has some prime divisor  $\tilde{p}_2 \nmid q$ . Then we get from (12)

$$g_r^{(2)} \left( m_{j+1}^{(2)} q_2^{k_{h_2}^{(2)} - k_{j+1}^{(2)}} + \dots + m_{j+1}^{(2)} q_2^{k_{h_2}^{(2)} - k_{h_2-1}^{(2)}} + m_{h_2}^{(2)} \right) \equiv 0 \left( \tilde{p}_2^{k_{h_2}^{(2)} - k_j^{(2)}} \right)$$

for  $0 \leq j \leq h_2 - 1$  and again  $m_{h_2}^{(2)} \equiv 0 (q_2)$ . Suppose next that  $q$  has some prime divisor  $p \nmid \tilde{q}_2$ . Then we have

$$g_r^{(1)} \left( m_1^{(1)} q^{k_{h_1}^{(1)} - k_1^{(1)}} + \cdots + m_{h_1-1}^{(1)} q^{k_{h_1}^{(1)} - k_{h_1-1}^{(1)}} + m_{h_1}^{(1)} \right) \equiv 0 \left( p^{k_{h_1}^{(1)} - k_{h_2}^{(2)}} \right)$$

and we can do the same reasonings with  $\varepsilon/h_1$  instead of  $\varepsilon$ .

It remains to consider  $q$  and  $\tilde{q}_2$  with prime factorisations  $q = p_1^{e_1} \cdots p_s^{e_s}$ ,  $\tilde{q}_2 = p_1^{\tilde{e}_1} \cdots p_s^{\tilde{e}_s}$ , where all  $e_i$  and  $\tilde{e}_i$  are positive integers. Let us rewrite (11):

$$\begin{aligned} & g_r^{(1)} \left( m_1^{(1)} \prod_{i=1}^s p_i^{k_{h_2}^{(2)} \tilde{e}_i + (k_{h_1}^{(1)} - k_1^{(1)}) e_i} + \cdots + m_{h_1}^{(1)} \prod_{i=1}^s p_i^{k_{h_2}^{(2)} \tilde{e}_i} \right) \\ & + g_r^{(2)} \left( m_1^{(2)} \prod_{i=1}^s p_i^{(k_{h_2}^{(2)} - k_1^{(2)}) \tilde{e}_i + (k_{h_1}^{(1)} - k_1^{(2)}) e_i} + \cdots + m_{h_2}^{(2)} \prod_{i=1}^s p_i^{(k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_i} \right) = 0. \end{aligned}$$

By assumption,  $q_1$  and  $q_2$  are multiplicatively independent. Thus we have  $s \geq 2$  and  $e_i/\tilde{e}_i \neq e_j/\tilde{e}_j$  for some  $i, j$ . Therefore  $k_{h_2}^{(2)} \tilde{e}_i - (k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_i$  cannot be zero for all  $i$  and the difference must be at least  $\frac{1}{2}(\log N)^{\varepsilon/2}$  for some  $i$ . Let

$$(k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_{i_0} - k_{h_2}^{(2)} \tilde{e}_{i_0} \geq \frac{1}{2}(\log N)^{\varepsilon/2}.$$

Then we have

$$g_r^{(1)} \left( m_1^{(1)} \prod_{i=1}^s p_i^{(k_{h_1}^{(1)} - k_1^{(1)}) e_i} + \cdots + m_{h_1}^{(1)} \right) \equiv 0 \left( p_{i_0}^{(k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_{i_0} - k_{h_2}^{(2)} \tilde{e}_{i_0}} \right)$$

and we can again do the same reasonings. Similarly

$$k_{h_2}^{(2)} \tilde{e}_{i_0} - (k_{h_1}^{(1)} - k_{h_2}^{(2)}) e_{i_0} \geq \frac{1}{2}(\log N)^{\varepsilon/2}$$

leads to

$$g_r^{(2)} \left( m_1^{(2)} \prod_{i=1}^s p_i^{(k_{h_2}^{(2)} - k_1^{(2)}) (\tilde{e}_i + e_i)} + \cdots + m_{h_2}^{(2)} \right) \equiv 0 \left( p_{i_0}^{\frac{1}{2}(\log N)^{\varepsilon/2}} \right)$$

and the same result.

Hence, we finally get

$$\begin{aligned} \sum_{(\mathbf{m}_1, \mathbf{m}_2) \neq (\mathbf{0}, \mathbf{0})} |T_{\mathbf{m}_1, \mathbf{m}_2}| \cdot \left| \frac{1}{N} \sum_{n < N} e \left( (g_r^{(1)} \mathbf{m}_1 \cdot \mathbf{v}_1 + g_r^{(2)} \mathbf{m}_2 \cdot \mathbf{v}_2) n \right) \right| \\ = \mathcal{O} \left( (\log N)^{-\delta/2} \right) + \mathcal{O} \left( (\log N)^{2(h_1+h_2)\delta-\lambda} \right), \end{aligned}$$

which completes the proof of Proposition 4.



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