



HAL
open science

Stability in discrete tomography: Linear programming, additivity and convexity

Sara Brunetti, Alain Daurat

► **To cite this version:**

Sara Brunetti, Alain Daurat. Stability in discrete tomography: Linear programming, additivity and convexity. 2003, pp.398-407, 10.1007/b94107 . hal-00023084

HAL Id: hal-00023084

<https://hal.science/hal-00023084>

Submitted on 19 Apr 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Stability in Discrete Tomography: Linear Programming, Additivity and Convexity

Sara Brunetti¹ and Alain Daurat²

¹ Dipartimento di Scienze Matematiche e Informatiche, Università di Siena,
Via del Capitano 15, 53100, Siena, Italy

sara.brunetti@unisi.it

² LSIT CNRS UMR 7005, Université Louis Pasteur (Strasbourg 1),
Pôle API, Boulevard Sébastien Brant, 67400 Illkirch-Graffenstaden, France
daurat@dpt-info.u-strasbg.fr

Abstract. The problem of reconstructing finite subsets of the integer lattice from X-rays has been studied in discrete mathematics and applied in several fields like image processing, data security, electron microscopy. In this paper we focus on the stability of the reconstruction problem for some lattice sets. First we show some theoretical bounds for additive sets, and a numerical experiment is made by using linear programming to deal with stability for convex sets.

keywords: Discrete Tomography, Linear Programming, Additivity.

1 Introduction

A *lattice set* is a non-empty finite subset of the integer lattice \mathbb{Z}^2 . A *lattice direction* is a direction directed by a vector in $\mathbb{Z}^2 \setminus \{0\}$, and it can also be given by an equation $p(x, y) = ax + by$ with $a, b \in \mathbb{Z}$. Further, the *X-ray* of a lattice set E in a lattice direction p is the function $X_p E$ giving the number of points in E on each line parallel to this direction, formally $X_p E(k) = |\{M \in E : p(M) = k\}|$. Discrete Tomography is the area of mathematics and computer science that deals with the inverse problem of reconstructing lattice sets from a finite set of X-rays. The reconstruction problem can be formulated as a linear program in terms of fuzzy sets instead of lattice sets and efficient algorithms based on the interior point can be provided for finding any solution or proving that no such solution exists [10,12]. This approach is also motivated by the computational complexity result stating that the reconstruction problem is NP-hard when the X-rays are taken in more than two directions ($m > 2$) so that (if $\mathbb{P} \neq \text{NP}$) any algorithm will take an exponential time.

In this paper we use linear programming to deal with the stability of the reconstruction problem. Stability is of main importance in practical applications where the X-rays are possibly affected by errors. For instance, in electron microscopy, techniques [7] that enable to count the number of atoms lying in a line up to an error of ± 1 are known. But in case of instability, the reconstructed set can be quite different from the original one even if the error on the data is small. In [1] the authors prove that when $m > 2$, the two sets can be even

disjoint, permitting an error of $2(m - 1)$ on the X-rays. In Section 3 we show that to obtain a stability result even with a very small error on the data the requirement of uniqueness for the sets is not enough (see Remark 1). If the sets are additive, then a stability result holds. Here we just recall to the reader that additivity implies uniqueness, whereas the converse is not true. Additionally, the notion of additivity should be regarded as a property of the solutions of the linear program.

In Section 4 we treat the stability of reconstructing convex sets. Experimental results suggest the conjecture that for the set of directions $\{x, y, 2x + y, -x + 2y\}$, convex sets are additive. This would imply that the results of Section 2 may hold for convex sets so giving a stability result that corresponds to the continuous case where the reconstruction problem for convex bodies is well-posed ([16]).

2 The Problem

The reconstruction problem is the task of determining any lattice set having the given X-rays. Stability concerns how sensitive is the problem to noisy data. Hence one can ask whether a perturbation of the data correspond solutions that are close. To study the problem we define a measure for the error on the X-rays and one for the distance of two solutions. Let \mathcal{D} be a set of m prescribed lattice directions with $m \geq 2$ and let E, F be lattice sets:

$$DX_{\mathcal{D}}(E, F) = \max_{p \in \mathcal{D}} \sum_{k \in \mathbb{Z}} |X_p E(k) - X_p F(k)|$$

and

$$\text{card}(E \Delta F) = \text{card}((E \setminus F) \cup (F \setminus E)).$$

The formulation of the problem that we consider is the following:

Problem 1. Let E be known. Determine F maximizing $\text{card}(E \Delta F)$, with the constraint that $DX_{\mathcal{D}}(E, F)$ is given.

Let us introduce some definitions that we need in the following.

Definition 1. A lattice set E is additive with respect to \mathcal{D} , or \mathcal{D} -additive, if there is a function e which gives a value $e_p(k)$ for each line $p = k$ parallel to a direction p of \mathcal{D} such that for all M in \mathbb{Z}^2 :

$$M \in E \text{ if and only if } \sum_{p \in \mathcal{D}} e_p(p(M)) > 0.$$

This definition introduced by Fishburn et al. can be better understood with linear programming: a lattice set E is additive if it is the unique solution of the linear programming problem which looks for a fuzzy set which has the same X-rays than E .

Definition 2. A lattice set E is unique with respect to \mathcal{D} , or \mathcal{D} -unique, if $F \subset \mathbb{Z}^2$ and $X_p E = X_p F$ for any $p \in \mathcal{D}$ imply $E = F$.

There is an intimate relationship between these two definitions: every \mathcal{D} -additive set is \mathcal{D} -unique and the converse is true if $m = 2$ (see[10]).

As a last remark we recall that if p and q are two directions, then a p -line does not always intersect a q -line. Indeed \mathbb{Z}^2 can be split in $\det(p, q)$ pq -lattices such that in each pq -lattice a p -line intersects with any q -line. Precisely a pq -lattice has the form:

$$L_i^{pq} = \{M : p(M) = i \pmod{\det(p, q)} \text{ and } q(M) = \kappa i \pmod{\det(p, q)}\}$$

where κ only depends on the directions p and q (see for example [6]). Moreover we denote by $\langle i, j \rangle_{pq}$ the point M such that $p(M) = i$ and $q(M) = j$. Notice that this point is in \mathbb{Z}^2 only if $p = i$ and $q = j$ are in the same pq -lattice.

3 Stability for Additive Sets

3.1 Error equal to 1

In this section we study the symmetric difference of any two \mathcal{D} -additive sets E and F verifying the condition $DX_{\mathcal{D}}(E, F) \leq 1$. In the first two lemmas additivity is not required.

The condition $DX_{\mathcal{D}}(E, F) \leq 1$ permits the X-rays of the two sets to differ by one in at most a line for each direction. Then, $p \in \mathcal{D}$ and an integer k_p exists such that $|X_p E(k_p) - X_p F(k_p)| = 1$ and $X_p E(k) = X_p F(k)$ for $k \neq k_p$.

Lemma 1. If $p \in \mathcal{D}$ and an integer k_p exist such that $|X_p E(k_p) - X_p F(k_p)| = 1$, then for every $q \in \mathcal{D}$ there is an integer k_q such that $|X_q F(k_q) - X_q E(k_q)| = 1$ and $\langle k_p, k_q \rangle_{pq} \in \mathbb{Z}^2$.

Proof. Let L_i^{pq} be the pq -lattice containing the line $p = k_p$, or equivalently $k_p \in p(L_i^{pq})$. Suppose that $X_p F(k_p) - X_p E(k_p) = +1$. Thus, we have that

$$\sum_{k \in p(L_i^{pq})} X_p F(k) = 1 + \sum_{k \in p(L_i^{pq})} X_p E(k).$$

Using the consistency of the X-rays for F and E , the previous identity leads to the following

$$\sum_{k \in q(L_i^{pq})} X_q F(k) = 1 + \sum_{k \in q(L_i^{pq})} X_q E(k),$$

for all $q \in \mathcal{D}$. From this, the thesis easily follows. \square

In the next lemma we show that all the lines with error 1 have a common point and this point is in \mathbb{Z}^2 . In the following, we assume that $\text{card}(F) > \text{card}(E)$ and for any $p \in \mathcal{D}$ the integer k_p is as in the previous lemma.

Lemma 2. *If $DX_{\mathcal{D}}(E, F) = 1$, then a point $W \in \mathbb{Z}^2$ exists such that*

$$\begin{aligned} X_p F(k) &= X_p E(k) + 1, \text{ if } k = p(W) \\ X_p F(k) &= X_p E(k), \text{ otherwise} \end{aligned}$$

for all the directions p in \mathcal{D} .

Proof. Let p, q and r be directions in \mathcal{D} and suppose that $A = \langle k_p, k_q \rangle_{pq}$, $B = \langle k_p, k_r \rangle_{pr}$, $C = \langle k_q, k_r \rangle_{qr}$ are three distinct points. Let a, b be such that $r = ap + bq$. Thus, summing up we can write:

$$\sum_{M \in F} r(M) = a \sum_{M \in F} p(M) + b \sum_{M \in F} q(M)$$

and by grouping line by line we obtain:

$$\sum_k kX_r F(k) = a \sum_k kX_p F(k) + b \sum_k kX_q F(k).$$

We can exhibit the corresponding identity for the set E . As a result of the difference of these two identities we obtain that $k_r = ak_p + bk_q$ and so $r(A) = r(B) = r(C)$. Thus, the three points A, B and C coincide and the claim is proved. \square

Suppose now that E and F are \mathcal{D} -additive, that is $E = \{M : e(M) > 0\}$ and $F = \{M : f(M) > 0\}$.

Proposition 1. *Let E and F be \mathcal{D} -additive lattice sets. If $DX_{\mathcal{D}}(E, F) = 1$, then $\text{card}(E \Delta F) = 1$.*

Proof. Let W be as in Lemma 2. At first suppose that $W \notin E$ and let $E' = E \cup \{W\}$. For each direction p in \mathcal{D} we have that $X_p E' = X_p F$. Finally, since additivity of F implies uniqueness of F , we conclude that $F = E \cup \{W\}$. On the contrary, if $W \in E$ we study the following:

$$\Phi_E = \sum_{M \in \mathbb{Z}^2} \sum_{p \in \mathcal{D}} e_p(p(M))(1_E(M) - 1_F(M)).$$

Rewriting it as

$$\sum_{M \in E} \sum_{p \in \mathcal{D}} e_p(p(M))(1_E(M) - 1_F(M)) + \sum_{M \notin E} \sum_{p \in \mathcal{D}} e_p(p(M))(1_E(M) - 1_F(M)),$$

we notice that $\Phi_E > 0$, because the additivity of E implies that if M is in E , then $e(M) > 0$ and $1_E(M) = 1$ holds, and otherwise $e(M) \leq 0$ and $1_E(M) = 0$. We can also explicit the terms $X_p E$ and $X_p F$ in Φ_E so obtaining that

$$\Phi_E = \sum_{k \notin p(W)} \sum_{p \in \mathcal{D}} e_p(k)(X_p E(k) - X_p F(k)) + \sum_{p \in \mathcal{D}} e_p(p(W))(X_p E(p(W)) - X_p F(p(W)))$$

that is strictly less than zero. \square

Remark 1. The comparison between uniqueness and additivity can be made following [10]. Given any three lattice directions we may construct two sets E, F in such a way that they are unique but non-additive. (We do not give the proof for reasons of space limit and we refer the reader to [10]). Figure 1 illustrates two such sets verifying the constraint $DX_{\mathcal{D}}(E, F) = 1$. Since they are disjoint, Proposition 1 does not hold for \mathcal{D} -unique sets.

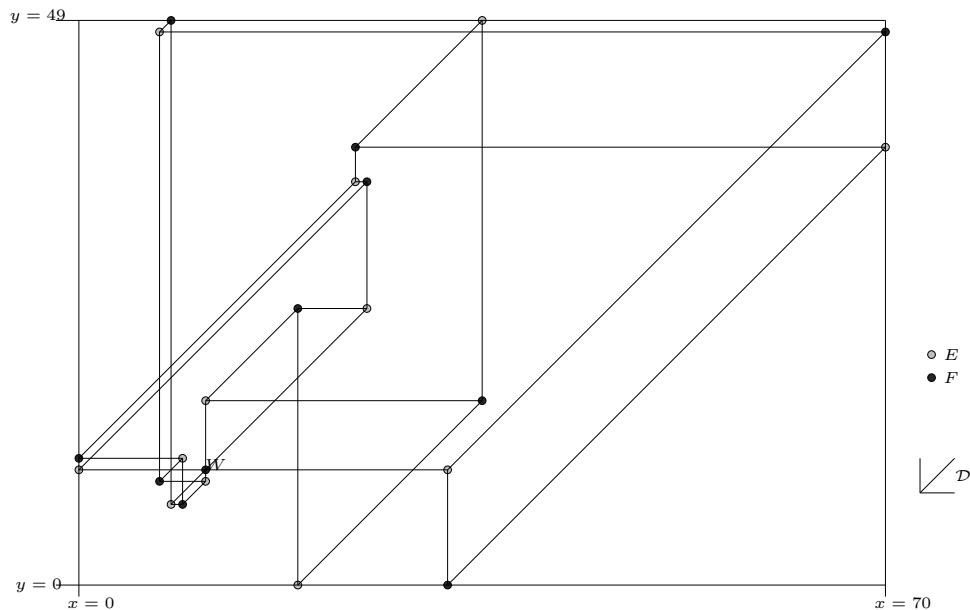


Fig. 1. E and F are non-additive sets of uniqueness such that $DX_{\mathcal{D}}(E, F) = 1$ and $E \cap F = \emptyset$.

3.2 Error larger than 1

In this section we consider the case where the error is larger than 1. Since even when the error is just equal to 2, we have instability if the number of lattice directions is larger than 2, we restrict our attention to the case of two directions.

More in detail, the instability follows from the result of [1, Theorem 1] because the sets constructed in the proof of [1] are actually \mathcal{D} -additive. Therefore we can restate it as follows:

Proposition 2 (see [1]). *For any n and a set \mathcal{D} of $m \geq 3$ directions there exist E and F \mathcal{D} -additive such that $|E| = |F| \geq n$, $DX_{\mathcal{D}}(E, F) = 2$ and $E \cap F = \emptyset$.*

As a result, our focus is on the case of two directions. In this case additivity is equivalent to uniqueness, and the construction used to prove Proposition 2 cannot be carried out.

Since pq -lattices are equivalent to \mathbb{Z}^2 we can see that it is sufficient to consider the case $\mathcal{D} = \{x, y\}$.

So, in the following, we suppose that E and F are unique with respect to $\mathcal{D} = \{x, y\}$. In [14, p17]) it is proved that $M = (x_M, y_M)$ is in E if and only if $e(M) = e_x(x_M) + e_y(y_M) \geq 0$ where $e_x(j) = X_x E(j)$ and $e_y(i) = -\text{card}(\{l : X_y E(l) \geq X_y E(i)\})$. Notice that this property implies directly the additivity of Definition 1 because we can add a small positive number to e in such a way that $e(M)$ remains negative if $M \notin E$. We define similarly $f(M) = f_x(x_M) + f_y(y_M)$. Then, as in the previous section, we can prove that:

$$\Phi_E = \sum_{(j,i) \in \mathbb{Z}^2} (e_x(j) + e_y(i))(1_E(j, i) - 1_F(j, i)) \geq 0$$

$$\Phi_F = \sum_{(j,i) \in \mathbb{Z}^2} (f_x(j) + f_y(i))(1_E(j, i) - 1_F(j, i)) \leq 0$$

Remark 2. By definition of e_x and f_x , if an error of $\pm a$ occurs in $x = j$ then $f_x(j) = e_x(j) \pm a$.

The relationship between $f_y(i)$ and $e_y(i)$ is more complex and will be studied in special cases. At first we begin with a short lemma:

Lemma 3. *Let P be a point of $F \setminus E$ such that $X_x F(x_P) = X_x E(x_P) + 1$ and $X_y F(y_P) \leq X_y E(y_P)$, then a point $Q \in E \setminus F$ exists satisfying $y_P = y_Q$, and for any such point we have $X_x E(x_Q) > X_x F(x_Q)$.*

Proof. Since $X_y F(y_P) \leq X_y E(y_P)$ there exists a point $Q \in E \setminus F$ such that $y_P = y_Q$. Let e_x, e_y be defined as above. We have:

$$e_x(x_Q) + e_y(y_Q) \geq 0 > f_x(x_Q) + f_y(y_Q) \quad (3.1)$$

$$e_x(x_P) + e_y(y_P) < 0 \leq f_x(x_P) + f_y(y_P) \quad (3.2)$$

Substituting $f_x(x_P) = e_x(x_P) + 1$ in (3.2) we get $e_y(y_P) < 0 \leq f_y(y_P) + 1$, that is, $e_y(y_P) \leq f_y(y_P)$ (because e_y and f_y are always integer). Since $e_y(y_P) = e_y(y_Q)$ and $f_y(y_P) = f_y(y_Q)$, equation (3.1) gives $e_x(x_Q) > f_x(x_Q)$. \square

Proposition 3. *Let $\mathcal{D} = \{x, y\}$; if E and F are any two \mathcal{D} -unique lattice sets satisfying $DX_{\mathcal{D}}(E, F) = 2$, then $\text{card}(E \triangle F) \leq 4$.*

The proof is omitted due to space constraints and is available in [4].

4 Stability for Convex Sets

In this section we experimentally study the stability of the reconstruction of convex sets via linear programming. Any convex set is the intersection of a convex polygon and the digital plane \mathbb{Z}^2 . The result of [11] states that convex sets are uniquely determined by their X-rays taken in a suitable set of directions.

In the “continuous” plane an analogous result holds and additionally the problem is well-posed ([16]). Moreover there is a connection between additive sets and convex sets, since an euclidean ball is additive with respect to two orthogonal directions ([9]). Our experiments support the suspect that these results have a correspondence in the “digital” plane.

Actually we consider in this section a class of lattice sets which is more general than the convex sets [3].

For each point $M = (x_M, y_M) \in \mathbb{Z}^2$ the four quadrants around M are defined by the following formulas:

$$\begin{aligned} R_0(M) &= \{(x, y) \in \mathbb{Z}^2 / x \leq x_M \text{ and } y \leq y_M\}, \\ R_1(M) &= \{(x, y) \in \mathbb{Z}^2 / x \geq x_M \text{ and } y \leq y_M\}, \\ R_2(M) &= \{(x, y) \in \mathbb{Z}^2 / x \geq x_M \text{ and } y \geq y_M\}, \\ R_3(M) &= \{(x, y) \in \mathbb{Z}^2 / x \leq x_M \text{ and } y \geq y_M\}. \end{aligned}$$

Definition 3. A lattice set E is Q-convex if and only if for each $M \notin E$ there exists $i \in \{0, 1, 2, 3\}$ such that $R_i(M) \cap E = \emptyset$.

An example of Q-convex set is given on the left part of Figure 2.

We have generated 184 Q-convex sets of semi-perimeter from 4 to 370 using a uniform generator ([5], inspired from [13]). Then their X-rays in the set of directions $\mathcal{D} = \{x, y, 2x + y, -x + 2y\}$ have been computed. (These directions have been chosen because the X-rays along them uniquely determine the convex sets ([11]) and they contain the horizontal and vertical directions). We then used these X-rays and any error $e \in \{0, 1, 2, 3\}$ as input data in the following linear-program:

Maximizing

$$\sum_{(i,j) \in E} (1 - v_{i,j}) + \sum_{(i,j) \notin E} v_{i,j} \quad (4.3)$$

such that

$$\sum_{p(i,j)=k} v_{i,j} = X_p E(k) + er_{p,k}^+ - er_{p,k}^- \quad (4.4)$$

$$\sum_k er_{p,k}^+ + er_{p,k}^- \leq e \quad (4.5)$$

$$0 \leq v_{i,j} \leq 1, er_{p,k}^+ \geq 0, er_{p,k}^- \geq 0 \quad (4.6)$$

We solved the linear program with the software soplex ([17]). Notice that solving this problem with $v_{i,j} \in \mathbb{Z}$ permits to exactly find the maximum of $\text{card}(E \Delta F)$ where F describes *all* the lattice sets such that $DX_{\mathcal{D}}(E, F) \leq e$. Unfortunately, integer-linear-program is an \mathbb{NP} -hard problem, and hence we solved the *relaxed* problem where the unknown variables can be fractional: this computation provides an upper bound to $\text{card}(E \Delta F)$. Figure 2 illustrates (on the right side-hand)

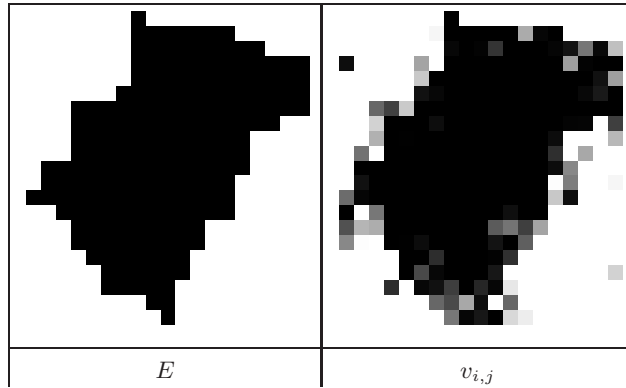


Fig. 2. A Q-convex set E and the corresponding extremal values of $v_{i,j}$ for $e = 3$. In this case we have $\text{card}(E) = 200$ and $\sum_{(i,j) \in E} (1 - v_{i,j}) + \sum_{(i,j) \in E^c} v_{i,j} = 33.7$.

a solution of the linear programming for $\text{card}(E) = 200$ and $e = 3$. The different grey-scale colors of the squares correspond to different values of $v_{i,j}$.

The complete results are summarized in Figures 3 and 4. In Figure 3 the upper bound to $\text{card}(E \Delta F)$ is divided by $\text{card}(E)$, so that each value gives an upper bound to the relative distance from a given set. Moreover the black squares show the values of the maximum of the quantity (4.3) when the constraints (4.4), (4.5) are replaced by $X_p E(k) - 1 \leq \sum_{p(i,j)=k} v_{i,j} \leq X_p E(k) + 1$: these values give an upper bound to $\text{card}(E \Delta F)$ when $DX'_D(E, F) = \max_{p \in \mathcal{D}} \max_{k \in \mathbb{Z}} |X_p E(k) - X_p F(k)| = 1$.

These experimental results bring out the following points:

- If $DX_D(E, F) = 0$, then we always found a null relative distance. In other words, according to our experiments every Q-convex set is \mathcal{D} -additive. In fact this property was first conjectured by L. Thorens ([15]) (with additivity replaced by uniqueness), and can be seen as a variant of Conjecture 4.6 of [2] and Theorem 5.7 of [11]. We may set out the conjecture as follows:

Conjecture 1. If \mathcal{D} is a set of directions which contains $\{x, y\}$, such that all the directions are not in the same quadrant and they uniquely determine the convex sets, then every Q-convex set is \mathcal{D} -additive.

Notice that the property about the quadrants is necessary because there is a counter-example with $\mathcal{D} = \{x, y, x + y, x + 5y\}$.

- For a fixed error e , the relative distance looks to converge to zero as $\text{card}(E)$ grows. If we divide by $\sqrt{\text{card}(E)}$ instead of $\text{card}(E)$, this ratio seems to be bounded so that in *average* $\text{card}(E \Delta F) = O(\sqrt{\text{card}(E)})$ according to our experiments (see Figure 4). It must be noticed that in the case $e = 1$, by the previous remark and Proposition 1 the real maximum error for lattice sets is always 1 for the generated cases; we obtain a stronger result in the experiment because the problem has been relaxed.

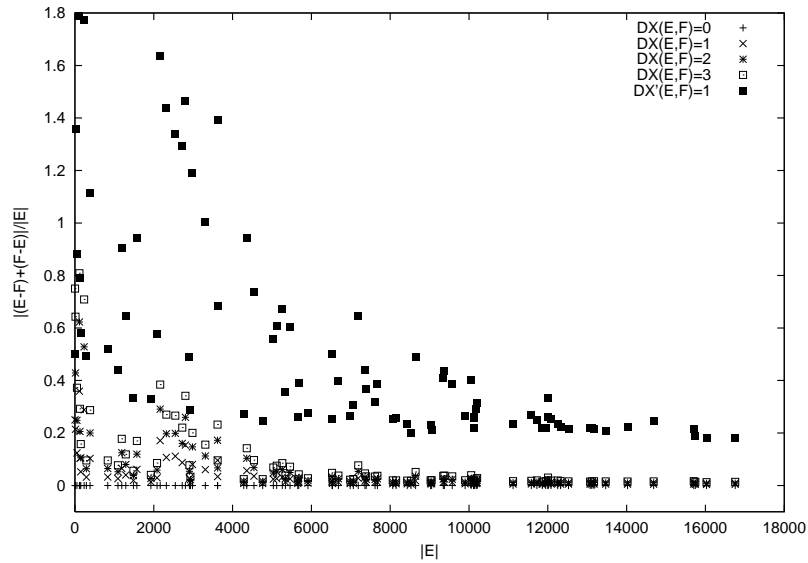


Fig. 3. An upper bound to $\frac{\text{card}(E \Delta F)}{\text{card}(E)}$ for the Q-convex generated sets. (Only 40 % of the 184 generated sets have been represented for readability)

- If $DX'_{\mathcal{D}}(E, F) = 1$, then the relative distance does not seem to converge to zero, but the computed values are only upper bounds, that is, we do not know if the fractional values mirror instability or they are just an artifact introduced by relaxing the integral constraints of the problem. In the former case, the reconstruction of convex sets would not be applied easily in the continuous world (as in medical imaging), because a rounding error of the measurements can always be of ± 1 .

References

1. A. Alpers, P. Grizmann, and L. Thorens. Stability and instability in discrete tomography. In *Dagstuhl Seminar: Digital and Image Geometry 2000*, volume 2243 of *Lecture Notes in Comp. Sci.*, pages 175–186. Springer, 2001.
2. E. Barucci, A. Del Lungo, M. Nivat, and R. Pinzani. X-rays characterizing some classes of discrete sets. *Linear Algebra Appl.*, 339:3–21, 2001.
3. S. Brunetti and A. Daurat. An algorithm reconstructing convex lattice sets. To be published in *Theoret. Comput. Sci.*
4. S. Brunetti and A. Daurat. Stability in discrete tomography: Linear programming, additivity and convexity. Complete version.
5. A. Daurat. Counting and generating Q-convex sets. in preparation.
6. A. Daurat, A. Del Lungo, and M. Nivat. Median points of discrete sets according to a linear distance. *Disc. and Comp. Geom.*, 23:465–483, 2000.

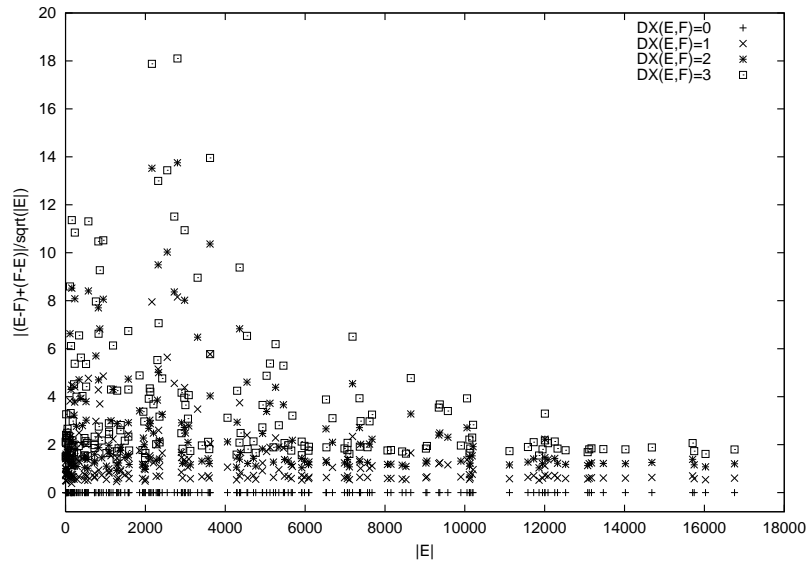


Fig. 4. An upper bound to $\frac{\text{card}(E \Delta F)}{\sqrt{\text{card}(E)}}$ for the 184 generated Q-convex sets

7. S. de Vries. In *Discrete Tomography, Packing and Covering, and Stable Set Problems. Polytopes and Algorithms*, Mathematik. Utz, Muenchen, DE, 1999.
8. A. Del Lungo and M. Nivat. Reconstruction of connected sets from two projections. In *Discrete tomography*, Appl. Numer. Harmon. Anal., pages 163–188. Birkhäuser Boston, 1999.
9. P. C. Fishburn, J. C. Lagarias, J. A. Reeds, and L. A. Shepp. Sets uniquely determined by projections on axes. I. Continuous case. *SIAM J. Appl. Math.*, 50(1):288–306, 1990.
10. P. C. Fishburn and L. A. Shepp. Sets of uniqueness and additivity in integer lattices. In *Discrete tomography*, Appl. Numer. Harmon. Anal., pages 35–58. Birkhäuser, 1999.
11. R. J. Gardner and P. Grizmann. Discrete tomography: determination of finite sets by X-rays. *Trans. Amer. Math. Soc.*, 349(6):2271–2295, 1997.
12. P. Grizmann, S. de Vries, and M. Wiegelmann. Approximating binary images from discrete X-rays. *SIAM J. Optim.*, 11(2):522–546, 2000.
13. W. Hochstättler, M. Loebl, and C. Moll. Generating convex polyominoes at random. *Discrete Math.*, 153(1-3):165–176, 1996.
14. A. Kuba and G. T. Herman. Discrete tomography: a historical overview. In *Discrete tomography*, Appl. Numer. Harmon. Anal., pages 3–34. Birkhäuser Boston, Boston, MA, 1999.
15. L. Thorens. Personal communication, 2000.
16. A. Volčič. Well-posedness of the Gardner-McMullen reconstruction problem. In *Measure theory, Oberwolfach 1983 (Oberwolfach, 1983)*, volume 1089 of *Lecture Notes in Math.*, pages 199–210. Springer, 1984.

17. R. Wunderling. *Paralleler und Objektorientierter Simplex-Algorithmus*. PhD thesis, ZIB technical report TR 96-09, 1996.