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An algorithm reconstructing convex lattice sets

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Abstract

In this paper, we study the problem of reconstructing special lattice sets from X-rays in a finite set of prescribed directions. We present the class of “Q-convex” sets which is a new class of subsets of \mathbb{Z}^2 having a certain kind of weak connect-
edness. The main result of this paper is a polynomial-time algorithm solving the reconstruction problem for the “Q-convex” sets. These sets are uniquely determined by certain finite sets of directions. As a result, this algorithm can be used for re-
constructing convex subsets of \mathbb{Z}^2 from their X-rays in some suitable sets of four lattice directions or in any set of seven mutually non parallel lattice directions.

Key words: Algorithms; Combinatorial problems; Convexity; Discrete tomography; Lattice sets.

1 Introduction

The present paper studies the problem of reconstructing special “lattice sets” from a set of X-rays in certain directions. A *lattice set* is a non-empty finite subset of the integer lattice \mathbb{Z}^2 . A directing vector $p \in \mathbb{Z}^2 \setminus \{0\}$ is called a *lattice direction*. Further, the *X-ray* of a lattice set F in a lattice direction p is the function $X_p F$ giving the number of points in F on each line parallel to this direction.

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The computational complexity of various inverse problems in discrete tomography is studied in [10] and the general problem of reconstructing two-dimensional lattice sets from their X-rays in a set of $m \geq 3$ pairwise nonparallel directions is shown to be NP-hard. In most practical applications there is some a priori information concerning the sets to be reconstructed. The algorithms can take advantage of this information to reconstruct the set. Mathematically, it can be described in terms of properties of the subsets of \mathbb{Z}^2 , namely, of classes of lattice sets the solution must belong to. Many authors have studied the case of determining a lattice set from its X-rays in the horizontal and vertical directions and, in particular, there are polynomial-time algorithms to reconstruct special sets having some convexity and connectivity properties like, for example, horizontally and vertically convex polyominoes [3,4,6]. In [2] the authors reconstruct connected lattice sets which are convex in the directions of the X-rays including $(1, 0)$, $(0, 1)$ and $(1, 1)$. In this paper we present a new class of lattice sets whose definition involves a certain kind of weak connectedness and convexity. These sets are called “Q-convex” sets. Then, the basic question is whether it is possible to reconstruct a “Q-convex” set from its X-rays in a finite set \mathcal{D} of lattice directions. Let us point out that we allow arbitrary lattice directions. We provide a polynomial-time algorithm for solving this reconstruction problem. Moreover, the problems studied in [3,4,6] are solvable as special cases of our problem.

The class of convex lattice sets (i.e., finite subsets F with $F = \mathbb{Z}^2 \cap \text{conv}F$) is another well-known and studied class in discrete tomography. Gardner and Gritzmann [11] proved that the X-rays in four suitable or any seven prescribed mutually nonparallel lattice directions uniquely determine all the convex lattice sets. The complexity of the reconstruction problem on this class is an open problem raised by Gritzmann during the workshop: *Discrete Tomography: Algorithms and Complexity* (1997). Since the class of “Q-convex” sets contains that of convex lattice sets and “Q-convex” sets are uniquely determined by certain finite sets of directions ([8],[7]), for such directions the proposed algorithm solves the reconstruction problem for the class of convex sets too.

2 Definitions and notations

2.1 Classical definitions

Lattice direction. A direction is an equivalence class for the relation of parallelism on the straight lines of the plane. It can be given by an equation $\lambda x + \mu y = \text{const}$ or by a directing vector $(-\mu, \lambda)$. If λ and μ are integer then, the direction is a *lattice direction*, and we can suppose that λ and μ are coprime. The horizontal direction is directed by $(1, 0)$, the vertical one by $(0, 1)$, the diagonal one by $(1, 1)$.

Convexity. A lattice set F is *line-convex* with respect to a direction p if the intersections of all lines of p with F are the sets of the points with integer coordinates of straight line segments. In particular, F is *hv-convex* (resp. *hvd-convex*) if it is line-convex with respect to the horizontal and vertical directions (resp. the horizontal, vertical and diagonal directions). Finally, a lattice set is convex if it is the intersection between \mathbb{Z}^2 and its convex hull.

Connectivity. A 4-path (resp. an 8-path, a 6-path) is a finite sequence (M_0, M_1, \dots, M_n) of points of \mathbb{Z}^2 such that $M_{i+1} - M_i$ is in the set $\{(\pm 1, 0), (0, \pm 1)\}$ (resp. $\{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$, $\{(\pm 1, 0), (0, \pm 1), (1, 1), (-1, -1)\}$). A lattice set F is *4-connected* (resp. *8-connected*, *6-connected*) if for any A, B in F there is a 4-path (resp. an 8-path, a 6-path) from A to B . A 4-connected lattice set is also called a *polyomino*.

2.2 New definitions and first properties

Let \mathcal{D} be a set of two prescribed lattice directions $p = \lambda_p x + \mu_p y$ and $q = \lambda_q x + \mu_q y$. Furthermore we call a *p-line* and a *q-line* any line having equation $p(M) = \text{const}$ and $q(M) = \text{const}$ for each $M \in \mathbb{Z}^2$, respectively. We point out that if $\delta = |\det(p, q)| = |\lambda_p \mu_q - \lambda_q \mu_p| \neq 1$, the intersection of a *p-line* and a *q-line* is not always in \mathbb{Z}^2 as the reader may note in subsection 3.1. In [9] the authors give a condition to determine whether the intersection of these lines is a point of \mathbb{Z}^2 : a point M belongs to \mathbb{Z}^2 if and only if $j \equiv \kappa i \pmod{\delta}$, where $p(M) = i$, $q(M) = j$ and $\kappa = (\lambda_q u + \mu_q v) \text{sign}(\lambda_p \mu_q - \lambda_q \mu_p) \pmod{\delta}$, $\lambda_p u + \mu_p v = 1$.

We denote by $\langle i, j \rangle_{p,q}$ (or $\langle i, j \rangle$ if there is no ambiguity) the point M which satisfies $p(M) = i$ and $q(M) = j$.

We consider two directions p and q and a point $M = \langle i, j \rangle$; it defines the following four zones (called quadrants, see Fig. 1a)):

$$\begin{aligned} Z_0(\langle i, j \rangle) &= \{\langle i', j' \rangle \in \mathbb{Z}^2 : i' \leq i \text{ and } j' \leq j\}, \\ Z_1(\langle i, j \rangle) &= \{\langle i', j' \rangle \in \mathbb{Z}^2 : i' \geq i \text{ and } j' \leq j\}, \\ Z_2(\langle i, j \rangle) &= \{\langle i', j' \rangle \in \mathbb{Z}^2 : i' \geq i \text{ and } j' \geq j\}, \\ Z_3(\langle i, j \rangle) &= \{\langle i', j' \rangle \in \mathbb{Z}^2 : i' \leq i \text{ and } j' \geq j\}. \end{aligned}$$

Definition 2.1. A lattice set F is *Q-convex* (quadrant-convex) around $\mathcal{D} = \{p, q\}$ if $Z_t(M) \cap F \neq \emptyset$ for all $t \in \{0, 1, 2, 3\}$ implies $M \in F$.

We denote the class of lattice sets which are *Q-convex* around the directions of \mathcal{D} by $\mathcal{Q}(\mathcal{D})$. When a lattice set is *Q-convex* around the specified set of directions, we shortly say that the set is *Q-convex*. Fig. 1 shows some examples of lattice sets having different kinds of convexity, when the considered directions are $p = x$ and $q = y$.

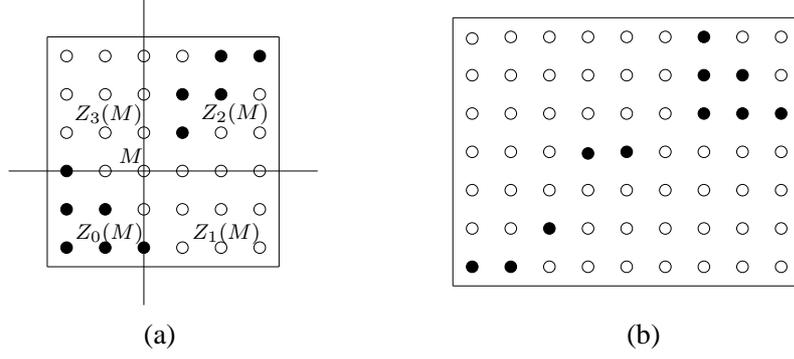


Fig. 1. a) A lattice set which is line-convex with respect to $(1, 0)$ and $(0, 1)$, but not Q -convex. b) A lattice set Q -convex around $(1, 0)$ and $(0, 1)$.

Definition 2.2. A lattice set F is *indivisible* for the direction p , or p -*indivisible*, if $\{i \in \mathbb{Z} : |\{N \in F \mid p(N) = i\}| > 0\}$ is made up of consecutive integers.

By definition, if F is p -indivisible lattice set, then there are $i_1, i_2 \in \mathbb{Z}$ such that the line $p = i$ contains a point of F if and only if $i_1 \leq i \leq i_2$. If F is p - and q -indivisible with $\mathcal{D} = \{p, q\}$, we say that F is \mathcal{D} -*indivisible* or shortly, *indivisible*. The lattice set shown in Fig. 1a) is indivisible, whereas that in Fig. 1b) is not. An example of an indivisible lattice set which is line-convex with respect to the directions $p = x - y$ and $q = x + y$, but not Q -convex, is given in Fig. 2.

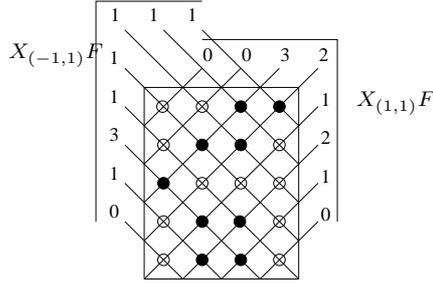


Fig. 2. An indivisible lattice set which is line-convex with respect to $(1, 1)$ and $(-1, 1)$, but not Q -convex.

In case $p = x$ and $q = y$, we can establish the following interesting relationship between indivisible Q -convex sets and hv -convex 8-connected sets.

Proposition 2.3. *Let $p = x$ and $q = y$. An indivisible lattice set F belongs to $\mathcal{Q}(\{p, q\})$ if and only if F is 8-connected and line-convex with respect to directions p and q .*

Proof. Let $p = x$ and $q = y$ and let F be an 8-connected and hv -convex set. The set F is 8-connected, so it is indivisible. Suppose that for any $i \in \{0, \dots, 3\}$ we have a point $M_i \in Z_i(M) \cap F$. We are going to prove that $M \in F$. Consider first M_0 and M_1 and let $M_0 = A_0, \dots, A_i, \dots, A_k = M_1$ the shortest 8-path in F (it is the path which minimizes k). Since this path is the shortest one and F is hv -convex, the path is monotone, namely, the two sequences $(p(A_i))$ and $(q(A_i))$ are monotone. So, there is a point $N_1 \in F \cap (Z_0(M) \cap Z_1(M))$ which

is in the path (see Fig. 3a)). By considering a path from M_2 to M_3 we can prove in a similar way that there exists a point $N_2 \in F \cap (Z_2(M) \cap Z_3(M))$. Since the point M is in the vertical segment $[N_1, N_2]$, M belongs to F .

Conversely, suppose F is an indivisible Q-convex set. F is hv -convex because of the Q-convexity. Let M and N be two points of F and $x_M < x_N$ and $y_M < y_N$ (see Fig. 3b)). We construct an 8-path from M to N . Let $M_1 =$

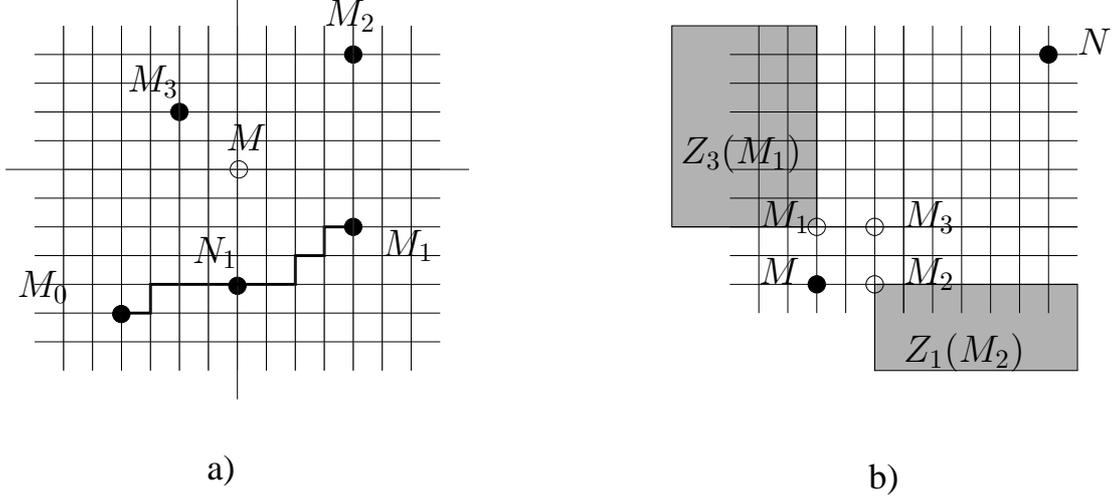


Fig. 3. a) An 8-path from M_0 to M_1 intersecting $Z_0(M) \cap Z_1(M)$. b) Constructing an 8-path from M to N .

$(x_M, y_M + 1)$, $M_2 = (x_M + 1, y_M)$ and $M_3 = (x_M + 1, y_M + 1)$. Suppose that none of them belongs to F . Since F is Q-convex, if $M_1 \notin F$ there is at least one zone $Z_i(M_1)$ such that $Z_i(M_1) \cap F = \emptyset$. We deduce $i = 3$, because $N \in Z_2(M_1)$ and $M \in Z_0(M_1) \cap Z_1(M_1)$. By proceeding analogously for M_2 , we deduce $Z_1(M_2) \cap F = \emptyset$ and for M_3 , we have $Z_1(M_3) \cap F = \emptyset$ or $Z_3(M_3) \cap F = \emptyset$. If $Z_1(M_3) \cap F = \emptyset$, then the line $x = x_M + 1$ does not contain points of F , contradicting the hypothesis of indivisibility. If $Z_3(M_3) \cap F = \emptyset$, then the line $y = y_M + 1$ does not contain points of F , also contradicting the hypothesis of indivisibility. Therefore one of the three points M_1, M_2, M_3 belongs to F , say M_1 , and (M, M_1) constitutes the first step in the construction of any 8-path from M to N . Continuing in this way, we obtain the searched path. \square

Let us now introduce the reconstruction problem. Consider any finite subset F of \mathbb{Z}^2 : the X -ray of F in a lattice direction p is the function $X_p F : \mathbb{Z} \rightarrow \mathbb{N}$ defined by: $X_p F(i) = |\{N \in F \mid p(N) = i\}|$, where $i \in \mathbb{Z}$. By definition, $X_p F$ gives the number of points in F on each line parallel to p . Let us define

$$\begin{aligned} pmin &= \min\{i : X_p F(i) > 0\}, & pmax &= \max\{i : X_p F(i) > 0\}, \\ qmin &= \min\{j : X_q F(j) > 0\}, & qmax &= \max\{j : X_q F(j) > 0\}, \\ m &= pmax - pmin + 1, & n &= qmax - qmin + 1. \end{aligned}$$

The set F is finite and so the set of lines intersecting F is also finite. Thus, a vector of nonnegative integers gives a suitable representation for any X-ray of F . The inverse reconstruction problem can be formulated as follows:

Reconstruction2Qconv (Reconstruction of Q-convex sets from X-rays in two directions)

Instance: Two directions p and q and two vectors $\mathbf{p} = (p_{pmin}, \dots, p_{pmax})$, $\mathbf{q} = (q_{qmin}, \dots, q_{qmax})$ of nonnegative integers.

Task: Reconstruct a set $F \in \mathcal{Q}(\{p, q\})$ such that $X_p F(i) = p_i$, $X_q F(j) = q_j$ for all $i \in [pmin, pmax]$ and $j \in [qmin, qmax]$, if one exists.

3 Reconstruction algorithm for two directions

In this section we suppose that one instance of **Reconstruction2Qconv** is given. Without loss of generality we can assume $p_{pmin} > 0$, $p_{pmax} > 0$, $q_{qmin} > 0$, $q_{qmax} > 0$. Let Δ denote the parallelogram:

$$\Delta = \{M = \langle i, j \rangle_{p,q} \in \mathbb{Z}^2 : pmin \leq i \leq pmax, qmin \leq j \leq qmax\}.$$

If α and β are two subsets of \mathbb{Z}^2 we denote the set of all the solutions F of **Reconstruction2Qconv** which verify $\alpha \subseteq F \subseteq \beta$ by $\mathcal{E}(\alpha, \beta)$. The reconstruction problem just consists in determining if $\mathcal{E}(\emptyset, \mathbb{Z}^2)$ is empty or not and in reconstructing a member of it in the latter case. We have trivially $\mathcal{E}(\emptyset, \mathbb{Z}^2) = \mathcal{E}(\emptyset, \Delta)$. We cannot determine $\mathcal{E}(\emptyset, \Delta)$ directly, but if $\mathcal{E}(\emptyset, \Delta) \neq \emptyset$ then there exist U_1 and $U_2 \in \Delta$ with $p(U_1) = pmin$ and $p(U_2) = pmax$ such that $\mathcal{E}(\{U_1, U_2\}, \Delta)$ is not empty.

In the next part, we fix $U_1, U_2 \in \Delta$ such that $p(U_1) = pmin$, $p(U_2) = pmax$. (These points are called the p -base points). Our aim is to check if $\mathcal{E}(\{U_1, U_2\}, \Delta)$ is empty or not. Moreover we suppose that $q(U_1) \leq q(U_2)$. (The case $q(U_1) \geq q(U_2)$ is similar.)

3.1 The set H

The first step consists in finding a set H such that $\mathcal{E}(\{U_1, U_2\}, \Delta) = \mathcal{E}(\{U_1, U_2\}, H)$. For this we define the four partial sums:

$$\begin{aligned} S_0(\langle i, j \rangle) = S_0(i) &= \sum_{i' \leq i} p_{i'} & S_2(\langle i, j \rangle) = S_2(i) &= \sum_{i' \geq i} p_{i'} \\ S_1(\langle i, j \rangle) = S_1(j) &= \sum_{j' \leq j} q_{j'} & S_3(\langle i, j \rangle) = S_3(j) &= \sum_{j' \geq j} q_{j'}. \end{aligned} \quad (3.1)$$

If $S_0(pmax) = S_2(pmin) = \sum_{i=pmin}^{pmax} p_i$ is different from $S_1(qmax) = S_3(qmin) = \sum_{j=qmin}^{qmax} q_j$, then we know that there cannot be any solution. So we suppose that these two numbers are equal. Let us define S by:

$$S = \sum_{pmin \leq i \leq pmax} p_i = \sum_{qmin \leq j \leq qmax} q_j. \quad (3.2)$$

These sums satisfy the following easy but fundamental lemma:

Lemma 3.1. *Let $M = \langle i, j \rangle$ with $i, j \in \mathbb{Z}$. If $S_k(M) + S_{k+1}(M) > S$, then $F \cap Z_k(M) \neq \emptyset$ for any $F \in \mathcal{E}(\emptyset, \Delta)$, where $k + 1 = 0$ for $k = 3$.*

Proof. At first we take $k = 0$ into consideration. If $F \cap Z_0(M) = \emptyset$, then $S_0(M) + S_1(M) = |F \cap (Z_3(M) \cup Z_1(M))| \leq S$. Analogously, cases $k = 1, 2, 3$ can be proven. \square

For each line $p = i$ such that $p_i > 0$ we can define two q -indices, as follows:

$$a_i = \min\{j : S_1(j) + S_2(i) > S\} \quad (3.3)$$

$$b_i = \max\{j : S_3(j) + S_0(i) > S\}. \quad (3.4)$$

Lemma 3.2. *If $p_i > 0$, then $a_i \leq b_i$, for $i \in [pmin, pmax]$.*

Proof. By (3.3) we have that $S_1(a_i - 1) + S_2(i) \leq S$. Since $S_1(a_i - 1) = S - S_3(a_i)$ and $S_2(i) = S - S_0(i - 1)$, the inequality can be rewritten as $S_3(a_i) + S_0(i - 1) \geq S$. If $p_i > 0$, then $S_0(i - 1) < S_0(i)$ and therefore, $S_3(a_i) + S_0(i) > S$. In view of (3.4), this implies $a_i \leq b_i$. \square

Now we define the sequence c_i as follows:

$$\begin{aligned} c_i &= q(U_1), & \text{if } a_i < q(U_1) \\ c_i &= a_i, & \text{if } q(U_1) \leq a_i \leq b_i \leq q(U_2) \\ c_i &= q(U_2), & \text{if } b_i > q(U_2) \end{aligned}$$

Lemma 3.3. *Let $F \in \mathcal{E}(\{U_1, U_2\}, \Delta)$ and $C = \langle i, c_i \rangle \in \mathbb{Q}^2$. If $p_i > 0$, then*

$$Z_k(C) \cap F \neq \emptyset, \quad \forall k \in \{0, \dots, 3\}.$$

Proof. • If $a_i < q(U_1)$, we have $C = \langle i, q(U_1) \rangle$ and so $U_1 \in Z_0(C) \cap Z_3(C)$ and $U_2 \in Z_2(C)$ because of $q(U_1) < q(U_2)$. Moreover, by the definition of a_i it follows that $S_1(C) + S_2(C) > S$ and then, by Lemma 3.1, we conclude $Z_1(C) \cap F \neq \emptyset$.

• If $q(U_1) \leq a_i \leq b_i \leq q(U_2)$, then $C = \langle i, a_i \rangle$. So, $U_1 \in Z_0(C)$ and $U_2 \in Z_2(C)$. By the definition of a_i , $S_1(C) + S_2(C) > S$ and therefore $Z_1(C) \cap F \neq \emptyset$. Finally, we use the fact that $q(C) = a_i \leq b_i$ and $S_3(C) + S_0(C) > S$ to conclude that $Z_3(C) \cap F \neq \emptyset$.

• If $b_i > q(U_2)$, then we have $C = \langle i, q(U_2) \rangle$. It follows that $U_2 \in Z_1(C) \cap Z_2(C)$ and $U_1 \in Z_0(C)$. By the definition of b_i , $S_3(C) + S_0(C) > S$ and so $Z_3(C) \cap F \neq \emptyset$. \square

Thus, if the point C is in \mathbb{Z}^2 , then it is also in F for any $F \in \mathcal{E}(\{U_1, U_2\}, \Delta)$. But the point $C = \langle i, c_i \rangle$ can be in $\mathbb{Q}^2 \setminus \mathbb{Z}^2$. Let us define $c'_i = \max\{j : j \leq$

c_i and $\langle i, j \rangle \in \mathbb{Z}^2$ and $\delta = |\det(p, q)|$. We have

$$c'_i \leq c_i \leq c'_i + \delta \text{ and } \langle i, c'_i \rangle, \langle i, c'_i + \delta \rangle \in \mathbb{Z}^2.$$

Lemma 3.4. *Let $F \in \mathcal{E}(\{U_1, U_2\}, \Delta)$. If $p_i > 0$, then $F \cap \{\langle i, c'_i \rangle, \langle i, c'_i + \delta \rangle\} \neq \emptyset$.*

Proof. Let $A = \langle i, c'_i \rangle$, $B = \langle i, c'_i + \delta \rangle$, and $C = \langle i, c_i \rangle$. Lemma 3.4 states that $Z_k(C) \cap F \neq \emptyset$, for any k . Since $p_i > 0$, there exists a point $N \in F$ such that $p(N) = i$.

- If $q(N) \leq q(C)$, then $N \in Z_0(A) \cap Z_1(A)$. We have $Z_2(C) \subseteq Z_2(A)$, $Z_3(C) \subseteq Z_3(A)$ and therefore $Z_2(A) \cap F \neq \emptyset$, $Z_3(A) \cap F \neq \emptyset$. By the Q-convexity of F we deduce $A \in F$.
- If $q(N) \geq q(C)$, then $N \in Z_2(B) \cap Z_3(B)$. Since $Z_0(C) \subseteq Z_0(B)$, $Z_1(C) \subseteq Z_1(B)$, by the same arguments as above we can conclude that $B \in F$.

□

Now let us introduce the following set H :

$$H = \{\langle i, j \rangle \in \mathbb{Z}^2 : p_i > 0, q_j > 0, c_i - \delta p_i < j \leq c_i + \delta p_i\}.$$

Using this definition we can reformulate the previous lemma as follows:

Lemma 3.5. $\mathcal{E}(\{U_1, U_2\}, \Delta) = \mathcal{E}(\{U_1, U_2\}, H)$.

By the definition of H we also have:

Lemma 3.6. *In each line $p = i$ there are at most $2p_i$ points of H for all $i \in \{pmin, \dots, pmax\}$.*

3.2 The filling operations

The previous section shows that $\mathcal{E}(\{U_1, U_2\}, \mathbb{Z}^2) = \mathcal{E}(\{U_1, U_2\}, H)$. Now we look for more precise pairs $\alpha, \beta \subset \mathbb{Z}^2$ such that $\mathcal{E}(\{U_1, U_2\}, \mathbb{Z}^2) = \mathcal{E}(\alpha, \beta)$, where α is a subset of any $F \in \mathcal{E}(\{U_1, U_2\}, \mathbb{Z}^2)$, whereas $\beta \setminus \alpha$ contains *indeterminate points* in the sense that we do not know whether they are in F or not.

So, at the beginning we instantiate $\alpha = \{U_1, U_2\}$ and $\beta = H$, and then we expand α and reduce β by means of some operations. All the operations are performed separately on the lines $p = i$ and $q = j$.

Let us denote the set of points of the intersection between $p = i$ ($q = j$) and β by β_p^i (β_q^j) and the set of points of the intersection between $p = i$ ($q = j$) and α by α_p^i (α_q^j). We also define:

$$g(\alpha_p^i) = \min_{M \in \alpha_p^i} q(M), \quad d(\alpha_p^i) = \max_{M \in \alpha_p^i} q(M), \quad g(\beta_p^i) = \min_{M \in \beta_p^i} q(M), \quad d(\beta_p^i) = \max_{M \in \beta_p^i} q(M).$$

Here are the four operations $\oplus, \otimes, \ominus, \odot$ already described in [3] adapted to any direction p .

- $\beta_p^i = \{\langle i, j \rangle : g(\alpha_p^i) - \delta(p_i - |\alpha_p^i|) \leq j \leq d(\alpha_p^i) + \delta(p_i - |\alpha_p^i|)\};$
- t_2 , if $\alpha_p^i = \emptyset$ and β_p^i is made up of $2p_i$ consecutive points. So we have

$$\beta_p^i = \{\langle i, j \rangle : g(\beta_p^i) \leq j < g(\beta_p^i) + 2\delta p_i\};$$

- t_3 , if $\alpha_p^i = \emptyset$ and β_p^i consists of two separated sequences of p_i points. So:

$$\beta_p^i = \{\langle i, j \rangle : g(\beta_p^i) \leq j \leq g(\beta_p^i) + \delta(p_i - 1) \text{ or } d(\beta_p^i) - \delta(p_i - 1) \leq j \leq d(\beta_p^i)\} \text{ with } d(\beta_p^i) - g(\beta_p^i) \geq 2p_i$$

Since we know that $\beta \subseteq H$, thanks to Lemma 3.6 we can claim that:

Proposition 3.7. *After performing the filling operations, each line having equation $p = i$ is of type t_0 , t_1 or t_2 , $i \in [pmin, pmax]$.*

Let $p = i$ be any p -line. From Proposition 3.7, we deduce that:

$$|\beta_p^i| = 2p_i - |\alpha_p^i| \text{ for all } i \in [pmin, pmax].$$

By summing over i we have

$$|\beta| = 2S - |\alpha|. \quad (3.5)$$

Consider now the q -lines and let $q = j$ be the equation of any line containing indeterminate points. Thanks to the operations \otimes and \odot' , we have:

$$|\beta_q^j| \geq 2q_j - |\alpha_q^j|$$

and therefore,

$$|\beta| = \sum_j |\beta_q^j| \geq \sum_j (2q_j - |\alpha_q^j|) = 2S - |\alpha|.$$

By (3.5) we deduce:

$$|\beta_q^j| = 2q_j - |\alpha_q^j| \text{ for all } j \in [qmin, qmax];$$

otherwise we get a contradiction. We note that this result allows us to determine the type of the q -lines. In fact, when $|\alpha_q^j| > 0$ we know that $q = j$ is a line of type t_1 . If $|\alpha_q^j| = 0$ then we have $|\beta_q^j| = 2q_j$; thanks to the operation \odot' this means that the set β_q^j is made up of two sequences having the same length, being either consecutive (in this case $q = j$ of type t_2) or separate (in this case $q = j$ of type t_3).

Proposition 3.8. *After performing the filling operations, each line having equation $q = j$ is of type t_0 , t_1 , t_2 or t_3 , $j \in [qmin, qmax]$.*

3.4 Reduction to a 2-SAT formula

For each point $M \in \Delta \setminus \alpha$ we associate a boolean variable V_M expressing the presence (resp. absence) of M in the final solution if V_M (resp. $\overline{V_M}$) is true.

Each instantiation of the boolean variables V_M gives a set $\alpha \subseteq F(V) \subseteq \beta$ where

$$F(V) = \alpha \cup \{M \in \beta \setminus \alpha : V_M = TRUE\}.$$

Now we construct a boolean formula whose variables are $(V_M)_{M \in \beta \setminus \alpha}$ in such a way that $F(V)$ is a Q-convex set having the given X-rays. Therefore the reconstruction problem will be reduced to the search of a truth assignment of the variables for the formula. Since this formula is a 2-SAT formula, its satisfiability can be easily checked (see [1]).

3.4.1 Expression of $X_q F(j) = q_j$

We fix a line $q = j$. This line is of type t_i with $i \in \{0, \dots, 3\}$ and so, there are exactly $2(q_j - |\alpha_p^i|)$ unknown points on each line $q = j$. If the line is of type t_1 or t_2 , and $A = \langle i, j \rangle, B = \langle i + \delta p_i, j \rangle \in \beta \setminus \alpha$, then for any set $F \in \mathcal{E}(\alpha, \beta)$ we have

$$A \in F \text{ if and only if } B \notin F$$

so we can express $X_q F(j) = q_j$ by the formula:

$$FQ_j = \bigwedge_{\substack{\langle i, j \rangle, \langle i', j \rangle \in \beta \setminus \alpha \\ i' - i = \delta q_j}} V_{\langle i, j \rangle} \iff \overline{V_{\langle i', j \rangle}}.$$

If the line is of type t_3 , then this line is made up of two sequences of consecutive indeterminate points. Since we know that each set $F \in \mathcal{E}(\alpha, \beta)$ contains exactly one of these sequences, in this case we can express $X_q F(j) = q_j$ by the formula:

$$FQ_j = \bigwedge_{\substack{\langle i, j \rangle, \langle i', j \rangle \in \beta \setminus \alpha \\ i' - i > \delta q_j}} V_{\langle i, j \rangle} \iff \overline{V_{\langle i', j \rangle}}.$$

In the same way we can express that $X_p F(i) = p_i$ by a similar formula FP_i .

3.4.2 Expression of the Q-convexity

Now we impose that the set $F(V)$ is Q-convex. We can find a direct boolean formula which expresses that for any $M \notin F(V)$ there is a quadrant $Z_i(M)$ containing no points of $F(V)$ but this formula is a disjunction of 5 variables or negations of variables, that is not a 2-SAT formula.

Remark 3.9. Let $M = \langle i, j \rangle_{p,q}$ be a point of $\mathbb{Z}^2 \setminus \alpha$ which verifies one of the following properties:

- $q(M) = j$ (resp., $p(M) = i$) is a t_1 q -line (resp., p -line) or a t_2 q -line (resp., p -line).
 - $q(M) = j$ is a t_3 q -line such that $d(\beta_q^j) - \delta(q_j - 1) \leq i$ or $i \leq g(\beta_q^j) + \delta(q_j - 1)$.
- Then, one of the two semi-lines $\Lambda_q^-(M) = \{\langle i', j \rangle : i' \leq i\}$ and $\Lambda_q^+(M) = \{\langle i', j \rangle : i' \geq i\}$ (resp., $\Lambda_p^-(M) = \{\langle i, j' \rangle : j' \leq j\}$ and $\Lambda_p^+(M) = \{\langle i, j' \rangle : j' \geq j\}$) contains a point of F for any $F \in \mathcal{E}(\alpha, \beta)$. We denote this semi-line by $\Lambda_q(M)$ (resp., $\Lambda_p(M)$). In fact:

- if the line is of type t_1 , then $\Lambda_q(M)$ is the semi-line containing a point of α_q^j ;
- if the line is of type t_2 or t_3 and $M \notin \beta$, then $\Lambda_q(M)$ is the semi-line containing all the points of β_q^j ;
- if the line is of type t_2 or t_3 and $M \in \beta$, then we have $\Lambda_q^-(M) \cap \Lambda_q^+(M) = \{M\} \subseteq \beta_q^j$ and $|\beta_q^j| = 2q_j$. So, one of the semi-lines verifies $|\Lambda_q^{\pm}(M) \cap \beta_q^j| > q_j$. This semi-line contains at least one point of any $F \in \mathcal{E}(\alpha, \beta)$.

Let $g'(\beta_q^j) = g(\beta_q^j) + \delta(q_j - 1)$ and $d'(\beta_q^j) = d(\beta_q^j) - \delta(q_j - 1)$; as a summary, if $g'(\beta_q^j) \geq i$, then $\Lambda_q(M) = \Lambda_q^+(M)$, whereas if $d'(\beta_q^j) \leq i$, then $\Lambda_q(M) = \Lambda_q^-(M)$.

Now we will express the Q-convexity of $F(V)$ around $M \in \Delta \setminus \alpha$ by means of a 2-SAT boolean formula.

- At first, suppose that $p(M) = i$ and $q(M) = j$ verify Remark 3.9. Thanks to the semi-lines $\Lambda_p(M)$ and $\Lambda_q(M)$ we can find an integer $k \in \{0, \dots, 3\}$ such that for any $l \neq k$ and any $F \in \mathcal{E}(\alpha, \beta)$ we have $Z_l(M) \cap F \neq \emptyset$.
 - If $M \notin \beta$ and $Z_k(M) \cap \alpha \neq \emptyset$, then we have $\mathcal{E}(\alpha, \beta) = \emptyset$, so we can express the Q-convexity by the formula

$$FALSE. \tag{3.6}$$

- If $M \notin \beta$ and $Z_k(M) \cap \alpha = \emptyset$, the formula is:

$$\overline{V_N} \tag{3.7}$$

for any $N \in Z_k(M) \cap \beta$.

- If $M \in \beta$ and $Z_k(M) \cap \alpha \neq \emptyset$, the formula is:

$$V_M. \tag{3.8}$$

- If $M \in \beta$ and $Z_k(M) \cap \alpha = \emptyset$, the formula is:

$$V_N \Rightarrow V_M \tag{3.9}$$

for any $N \in Z_k(M) \cap \beta$.

- Suppose now that $M \notin \beta$, and at least one of the lines $p(M) = i$ and $q(M) = j$ does not verify the conditions in Remark 3.9. Since we know that $U_1, U_2 \in \alpha$, there are at most two quadrants which do not contain any point of α . If there is no or only one such quadrant, then we can express the Q-convexity by formulas (3.6) and (3.7). Otherwise there are exactly two quadrants $Z_{l_1}(M)$ and $Z_{l_2}(M)$ which do not contain any point of α .
 - If $p = i$ is a t_0 -line, or $q = j$ is a t_0 -line or a t_3 -line such that $g(\beta_q^j) + \delta q_j \leq i \leq d(\beta_q^j) - \delta q_j$, then we can express the Q-convexity around M by the formula:

$$\overline{V_{N_1}} \vee \overline{V_{N_2}} \tag{3.10}$$

for any $N_1 \in \beta \cap Z_{l_1}(M)$, $N_2 \in \beta \cap Z_{l_2}(M)$.

Now we briefly summarize the reconstruction procedure, describing its main steps and their complexities. The analysis of the computational cost of every step is given in the appendix.

The algorithm checks whether the given X-rays satisfy the necessary condition on the cumulated sums to get a solution, and then it fixes two p -base points or two q -base points depending on the sizes of the X-ray vector. (If $m < n$, the p -base points are chosen). The cost of this choice is $\min\{m^2, n^2\}$, the number of possible positions of the base points. Furthermore let us assume that the p -base points U_1 and U_2 are chosen. At this point, since $\mathcal{E}(\emptyset, \Delta) \supseteq \mathcal{E}(\{U_1, U_2\}, \Delta)$, the problem is reduced to checking if $\mathcal{E}(\{U_1, U_2\}, \Delta)$ is empty or not. To this goal, the algorithm works to reduce the set containing the solution by computing the set H (only depending on the X-rays and p -base points). This is made in $O(mn)$ time. Then, sets α and β are initialized and the filling operations are performed to expand α and reduce β in such a way that the following property is preserved: $\mathcal{E}(\{U_1, U_2\}, H) = \mathcal{E}(\alpha, \beta)$. The computational cost of this step is $O(mn(m+n))$. Finally, the algorithm builds a boolean formula such that each assignment of values of the variables satisfying the formula gives rise to a solution $F(V)$ of our reconstruction problem. Since both the construction and the satisfiability of the formula take $O(mn(m+n))$ time, one knows if $\mathcal{E}(\alpha, \beta)$ is empty or not in $O(mn(m+n))$ time.

Proposition 3.10. *Reconstruction2Qconv is solvable in $O(\min\{m^2, n^2\}(mn(m+n)))$ time.*

Remark 3.11. From the previous remark and Lemma 2.3 it follows that our algorithm solves the problem of reconstructing 8-connected hv -convex sets from its X-rays in directions $p = x, q = y$ in polynomial time.

4 More than two X-rays

In this section, we study the general problem with more than two X-rays. Now the question is the following: is it possible to reconstruct a Q -convex set from its X-rays taken in a prescribed set of d directions? Let us concentrate on the case $d = 3$. Let \mathcal{D} be a set of three lattice directions $p = \lambda_p x + \mu_p y, q = \lambda_q x + \mu_q y, r = \lambda_r x + \mu_r y$. Moreover we assume $\det(p, r) = \lambda_p \mu_r - \mu_p \lambda_r \neq 0$ and $\det(q, r) = \lambda_q \mu_r - \mu_q \lambda_r \neq 0$. Now a point M of \mathbb{Z}^2 is the intersection of three lines having equations $p(M) = i, q(M) = j$ and $r(M) = k$ and it determines three kinds of quadrants $Z_t^{pq}(M), Z_t^{qr}(M)$ and $Z_t^{rp}(M)$, for $t = 0, 1, 2, 3$ related to the pairs of directions $\{p, q\}, \{q, r\}$ and $\{r, p\}$, respectively.

Definition 4.1. A lattice set F is Q -convex around $\{p, q, r\}$ if it is Q -convex around $\{p, q\}, \{q, r\}$ and $\{r, p\}$. More generally, a lattice set is Q -convex around a set \mathcal{D} of directions if it is Q -convex around any pair of direction included in \mathcal{D} .

Proposition 4.2. *Let $p = x, q = y$ and $r = x - y$ be the horizontal, vertical and diagonal directions. An indivisible lattice set F belongs to $\mathcal{Q}(\mathcal{D})$ if and*

only if F is 6-connected and line-convex with respect to the directions q , p and r .

Proof. If F is 6-connected, it is also 8-connected and therefore F is indivisible and Q-convex around $\{p, q\}$. Since there exists an isomorphism of \mathbb{Z}^2 which transforms r -lines into q -lines but leaves p -lines invariant, we can conclude that F is indivisible and Q-convex around $\{p, r\}$. Analogously, the isomorphism of \mathbb{Z}^2 which transforms r -lines into p -lines, leaving q -lines invariant, allows us to say that F is indivisible and Q-convex around $\{q, r\}$.

Conversely, suppose that F is an indivisible Q-convex set; we show that for each pair (M, N) of F points there is 6-path from $M = (x_M, y_M)$ to $N = (x_N, y_N)$ in F . Let N be such that $x_M < x_N$, $y_M < y_N$ and $y_N \leq x_N$ (the other cases can be proven by symmetry). Moreover, let $M_1 = (x_M + 1, y_M + 1)$ and $M_2 = (x_M + 1, y_M)$. By the indivisibility of F , there is a point M' of F on the line $x = x_M + 1$. If $y_{M'} \leq y_{M_2}$ (see Fig. 4a)), by the Q-convexity of F around $\{p, q\}$, M_2 belongs to F so that the first step of the path is determined. So, let us suppose $y_{M'} > y_{M_2}$ (see Fig. 4b)); by the Q-convexity of F around $\{p, r\}$ M_1 belongs to F so that the first step of the path is determined. \square

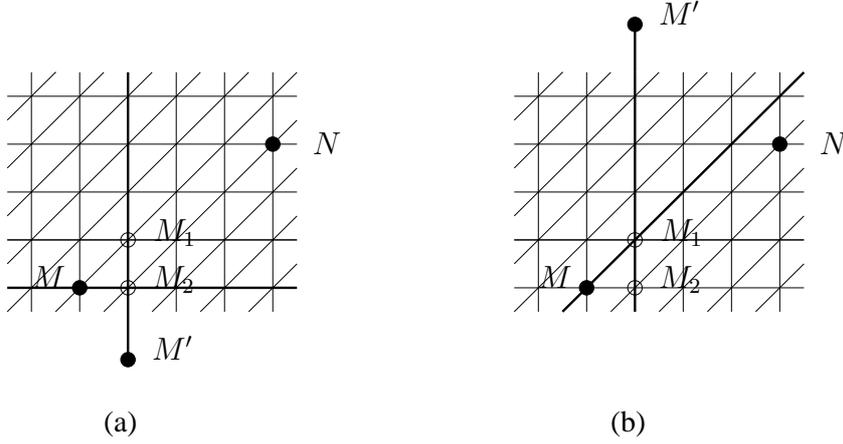


Fig. 4. a) $y_{M'} \leq y_{M_2}$. b) $y_{M'} > y_{M_2}$.

Our algorithm can be easily extended in order to work for a set $\mathcal{D} = \{p, q, r\}$ of three directions for reconstructing lattice sets which are Q-convex around \mathcal{D} . First the algorithm chooses the p -base-points U_1, U_2 ; then it constructs the set H just considering the pairs p, q of directions and after that it performs the filling operations in all the given directions. In this way, the set of all the solutions is more accurately specified at each step:

$$\mathcal{E}(\{U_1, U_2\}, \Delta) = \mathcal{E}(\{U_1, U_2\}, H) = \mathcal{E}(\alpha, \beta).$$

It is easy to see that in this case too, all the p -lines, q -lines, r -lines are of type t_i , $i \in \{0, \dots, 3\}$. All the formulas expressing that $F(V)$ is a solution are easily generalizable to the three-directions case except the formulas (3.10)

because in expressing the Q-convexity in M around q, r the two points U_1 and U_2 can be in the same quadrant.

To study this case, we generalize Remark 3.9 to points of \mathbb{Q}^2 .

Remark 4.3. Let $M = \langle j, k \rangle_{q,r}$ be a point of $\mathbb{Q}^2 \setminus \alpha$ which verifies one of the following properties: (M can be in $\mathbb{Q}^2 \setminus \mathbb{Z}^2$.)

- $q(M) = j$ is a t_1 q -line.
- $q(M) = j$ is a t_2 or t_3 q -line and $d'(\beta_q^j) \leq k$ or $k \leq g'(\beta_q^j)$.

Then, one the two semi-lines $\Lambda_q^-(M) = \{\langle j, k' \rangle_{q,r} : k' \leq k\}$ and $\Lambda_q^+(M) = \{\langle j, k' \rangle_{q,r} : k' \geq k\}$ contains a point of F for any $F \in \mathcal{E}(\alpha, \beta)$. More precisely, if $g'(\beta_q^j) \geq k$, then $\Lambda_q(M) = \Lambda_q^+(M)$, whereas if $d'(\beta_q^j) \leq k$, then $\Lambda_q(M) = \Lambda_q^-(M)$.

Suppose that $M = \langle j, k \rangle_{q,r}$ is a point of $\Delta \setminus \beta$ not verifying the conditions of Remark 3.9 on the lines $q(M) = j$ and $r(M) = k$. Moreover, U_1 and U_2 are in the same quadrant, for example $Z_0^{qr}(M)$. If there exists a point N such that $r(N) = r(M)$ and $q(N) \geq q(M)$ verifying the conditions of Remark 4.3, then we know that $Z_h^{qr}(M) \cap F \neq \emptyset$ with $h \in \{1, 2\}$ for any $F \in \mathcal{E}(\alpha, \beta)$ and this case is analogous to one of the two-directions cases. Therefore, we can suppose that $q(M) = j$ and $r(M) = k$ are t_0 or t_3 lines and all the lines $q = j'$ with $j' \geq j$ are of type t_0 and, t_2 or t_3 such that $g'(\beta_q^{j'}) < k < d'(\beta_q^{j'})$. The same assumption is made for all the lines $r = k'$ with $k' \geq k$. As a consequence of formulas FQ and FR , each indeterminate point $N \in Z_2^{qr}(M)$ is associated to a point $N' \in Z_1^{qr}(M)$ and to a point $N'' \in Z_3^{qr}(M)$, by the formulas

$$V_N \equiv \overline{V_{N'}} \equiv \overline{V_{N''}}. \quad (4.11)$$

Therefore, we can express the Q-convexity in M as:

$$V_{N_1} \Leftrightarrow V_{N_2} \quad (4.12)$$

for any $N_1, N_2 \in (\beta \setminus \alpha) \cap Z_2^{qr}(M)$. The algorithm constructs the boolean formula and checks its satisfiability in $O(n^3)$ time where $n = \max(p_{max} - p_{min}, q_{max} - q_{min}, r_{max} - r_{min})$. Since performing the filling operations takes $O(n^3)$ and the number of attempts is bound by n^2 (=number of possible choices for U_1 and U_2), the complexity of this algorithm is $O(n^5)$.

Remark 4.4. From Lemma 4.2 it follows that our algorithm solves the problem of reconstructing 6-connected hvd -convex sets from their X-rays in directions $p = x, q = y, r = x - y$ in polynomial time.

We can generalize the algorithm in order to work with any number d of directions. The precise problem we solve is the following:

ReconstructionQconv

Instance: A set of directions $\mathcal{D} = \{v_1, \dots, v_d\}$ and d vectors $(p_{v_1}(\min_{v_1}), \dots, p_{v_i}(\max_{v_i}))_{1 \leq i \leq d}$.

Task: Reconstruct a set $F \in \mathcal{Q}(\mathcal{D})$ such that $X_{v_i} F(j) = p_{v_i}(j)$ for all $i \in [1, d]$ and $j \in [\min_{v_i}, \max_{v_i}]$.

In this case the Q-convexity is expressed for all the pairs (v_i, v_j) such that $1 \leq i < j \leq d$, so we can construct a solution in $O(d^2n^5)$, where n is the maximal length of the X-rays ($n = \max(\max_{v_i} - \min_{v_i})$).

Proposition 4.5. ReconstructionQconv *is solvable in $O(d^2n^5)$ time.*

This generalization is particularly interesting in view of a new result of Daurat [8,7] establishing when subsets of $\mathcal{Q}(\mathcal{D})$ are uniquely determined by the data:

Theorem 4.6. *If $|\mathcal{D}| \geq 7$, or if one of the cross-ratios of the slopes of any four directions, arranged in increasing order, is not in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$ then for any sets $E, F \in \mathcal{Q}(\mathcal{D})$ we have:*

$$\forall p \in \mathcal{D} \ X_p E = X_p F \implies E = F.$$

In this paper we establish the complexity of the related algorithmic problem, showing that the reconstruction problem in $\mathcal{Q}(\mathcal{D})$ is solvable in polynomial time. Let us stress that the class of Q-convex sets contains the class of sets that are equal to the intersection of their convex hull and \mathbb{Z}^2 , namely, the class of convex sets. Thus, the most important application of the proposed algorithm is that it allows to reconstruct a convex lattice set from its X-rays taken in any certain set of directions, so answering the question proposed by Gritzmann during the workshop held in Dagstuhl in 1997. Precisely let us define the following problem:

ReconstructionConv

Instance: A set of directions $\mathcal{D} = \{v_1, \dots, v_d\}$ such that $d \geq 7$ or one of the cross-ratios of the slopes of four directions, arranged in increasing order is not in $\{\frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$ and d vectors $(p_{v_i}(\min_{v_i}), \dots, p_{v_i}(\max_{v_i}))_{1 \leq i \leq d}$.

Task: Reconstruct a convex set such that $X_{v_i} F(j) = p_{v_i}(j)$ for all $i \in [1, d]$ and $j \in [\min_{v_i}, \max_{v_i}]$.

Theorem 4.7. ReconstructionConv *can be solved in $O(d^2n^5)$ time.*

Proof. Let $(\mathcal{D}, (p_{v_i})_{1 \leq i \leq |\mathcal{D}|})$ be an instance of **ReconstructionConv**. By the algorithm of Proposition 4.5, we can check if there is a Q-convex set around \mathcal{D} which has X-rays (p_{v_i}) . If there is no Q-convex solution, then *a fortiori* there is no convex solution. Otherwise we have found the Q-convex set F whose X-rays are (p_{v_i}) . By Theorem 4.6 the set F is in fact the unique Q-convex set having (p_{v_i}) as X-rays. So, we only have to check if F is convex. This check can be done by computing the convex hull of F in \mathbb{R}^2 (see for example [12]) and then filling the convex polygon to check that $F = \text{conv}(F) \cap \mathbb{Z}^2$. \square

Remark 4.8. After the submission of this paper, the authors and A. Del Lungo have found an algorithm designed for approximate reconstruction problems. It has already been published in [5]. The class of lattice sets studied in [5] is obtained by extending the definition 2.1 in a different way than 4.1. The resulting class, so-called strongly Q-convex, is a subclass of that of Q-convex sets. The algorithm proposed in [5] also permits to reconstruct the convex sets

as a special case, but it is much slower since d (number of directions) appears at the exponent of n .

5 Some considerations and conclusions

The most significant result of this paper is a polynomial-time algorithm for the reconstruction of uniquely determined convex lattice sets. But this result leads to two questions:

Can we decrease the complexity of our algorithm ? This question is important because the complexity of our algorithm can look still high for real applications. At a closer look, the number of possible choices for U_1 and U_2 causes the growth of the exponent of n from 3 to 5. Thus, in a smarter implementation of the algorithm, at first no base-points are chosen but the filling operations are performed; then, one base-point is fixed, and finally, only if necessary to reach the solution after failing the previous attempts, two base-points are selected. This variant probably improves the average case. Preliminary experiments ([8]) indicate an estimated average case complexity of $O(n^{2.8})$ because in most cases the bases need not to be chosen a priori, but much work should be done, and in particular we need an algorithm which generates uniformly convex lattice sets of a given size at random.

Secondly what can we say about the reconstruction problem for any set of lattice directions not uniquely determining convex lattice sets ? Does there exist a polynomial algorithm in this case ? We could apply our algorithm until the reduction to a 2-SAT formula, but then we do not see any way to express the convexity by a formula whose satisfiability could be checked in polynomial time.

6 Appendix

In this section we analyze the computational cost of computing the main steps of the reconstruction algorithm.

6.1 Performing the filling operations

In implementing the filling operations we use 5 supplementary variables for each p -line and each q -line. Consider the line $p = i$; we denote these variables by $\text{toput}\oplus_p^i$, $\text{toput}\otimes_p^i$, $\text{toremove}\ominus_p^i$, $\text{toremove}\odot_p^i$, $\text{toremove}\odot_p^i$ containing the points to be modified if we would apply the corresponding filling operations. Algorithm 1 describes the procedure performing the filling operations. Let us now compute the time-complexity of this algorithm. The procedure `compute_points_to_change` needs $O(m + n)$ operations. It follows that `put_in_alpha` and `remove_from_beta` have also a complexity of $O(m + n)$. In the main procedure executing the **repeat** loop (lines 8-18) takes $O(m +$

Algorithm 1 Implementation of the filling operations

compute_points_to_change(p, i)

- 1: **for all** $\circlearrowleft \in \{\oplus, \otimes\}$ **do**
 - 2: toput $\circlearrowleft_p^i \leftarrow \circlearrowleft(\alpha_p^i) \setminus \alpha_p^i$
 - 3: **end for**
 - 4: **for all** $\circlearrowright \in \{\ominus, \odot, \odot'\}$ **do**
 - 5: toremove $\circlearrowright_p^i \leftarrow \beta_p^i \setminus \circlearrowright(\beta_p^i)$
 - 6: **end for**
-

put_in_alpha(M)

- 1: **if** $M \notin \beta$ **then**
 - 2: EXIT(no solution)
 - 3: **end if**
 - 4: $\alpha \leftarrow \alpha \cup \{M\}$
 - 5: compute_points_to_change($p, p(M)$)
 - 6: compute_points_to_change($q, q(M)$)
-

remove_from_beta(M)

- 1: **if** $M \in \alpha$ **then**
 - 2: EXIT(no solution)
 - 3: **end if**
 - 4: $\beta \leftarrow \beta \setminus \{M\}$
 - 5: compute_points_to_change($p, p(M)$)
 - 6: compute_points_to_change($q, q(M)$)
-

main()

- 1: **for all** $i \in \{pmin, \dots, pmax\}$ **do**
 - 2: compute_points_to_change(p, i)
 - 3: **end for**
 - 4: **for all** $j \in \{qmin, \dots, qmax\}$ **do**
 - 5: compute_points_to_change(q, j)
 - 6: **end for**
 - 7: **repeat**
 - 8: **for all** $i \in \{pmin, \dots, pmax\}$ **do**
 - 9: **for all** $\circlearrowleft \in \{\oplus, \otimes\}$ and $M \in \text{toput}\circlearrowleft_p^i$ **do**
 - 10: put_in_alpha(M)
 - 11: **end for**
 - 12: **for all** $\circlearrowright \in \{\ominus, \odot, \odot'\}$ and $M \in \text{toremove}\circlearrowright_p^i$ **do**
 - 13: remove_from_beta(M)
 - 14: **end for**
 - 15: **end for**
 - 16: **for all** $j \in \{qmin, \dots, qmax\}$ **do**
 - 17: Idem by replacing p by q
 - 18: **end for**
 - 19: **until** all the sets toput \circlearrowleft and toremove \circlearrowright are empty
-

$n + k_i(m + n)$) operations, where k_i is the number of the calls of the procedures `put_in_alpha` and `remove_from_beta` at the i th iteration. Each time that `put_in_alpha` or `remove_from_beta` are called, $|\beta \setminus \alpha|$ decreases except when an EXIT call is made. Therefore, $k_1 + k_2 + \dots + k_r \leq mn + 1$. Moreover we have that the number of repeat loop iterations is bounded by mn . Therefore the final complexity of the algorithm 1 is $O(mn(m + n))$.

6.2 Constructing and satisfying the boolean formula

6.2.1 Two directions

In this part we prove that we can build a 2-SAT formula expressing the existence of a solution $F \in \mathcal{E}(\alpha, \beta)$ in $O((m + n)mn)$ time.

The formulas FP and FQ . The formulas FP_i, FQ_j associated to t_1 and t_2 lines can be trivially found in $O(mn(m + n))$ time. If $q = j$ is of type t_3 , constructing the formula FQ_j takes $O(mn)$ time, since FQ_j is equivalent to the formula:

$$\begin{aligned} V_{\langle g(\beta_q^j), j \rangle} &\iff V_{\langle g(\beta_q^j) + \delta, j \rangle} \iff \dots \iff V_{\langle g(\beta_q^j) + (q_j - 1)\delta, j \rangle} \\ &\iff \overline{V_{\langle d(\beta_q^j) - (q_j - 1)\delta, j \rangle}} \iff \dots \iff \overline{V_{\langle d(\beta_q^j), j \rangle}} \end{aligned}$$

Therefore, all the formulas FP_i, FQ_j can be found in $O(mn(m + n))$ time.

Let us now search the formulas which express the Q-convexity. At first we suppose that we have precomputed the function g, d, g', d' for any p -line or q -line. This computation takes $O(mn)$ operations. So now, we can suppose that the time-complexity is constant.

The formulas associated to the points M which verify the remark 3.9. We build formulas (3.6), (3.7), (3.8), (3.9) line by line. Let us consider any line $q = j$; we show that the formulas associated to the points M of this line can be found in $O(mn)$ time.

We look for the minimum and maximum p -indices such that $\Lambda_p(M) = \Lambda_p^-(M)$ and for the minimum and maximum p -indices such that $\Lambda_p(M) = \Lambda_p^+(M)$. Formally, we define

$$i_1 = \min\{i : d'(\beta_p^i) \leq j\}, i_2 = \max\{i : d'(\beta_p^i) \leq j\},$$

$$i'_1 = \min\{i : g'(\beta_p^i) \geq j\}, i'_2 = \max\{i : g'(\beta_p^i) \geq j\}.$$

These numbers can be computed in $O(m)$ -time.

Thus, the points $M \in \Delta \setminus \alpha$ of $q = j$ under the hypothesis of the remark 3.9 verify:

$$p(M) \in ([i_1, i_2] \cup [i'_1, i'_2]) \cap ([pmin, g'(\beta_q^j)] \cup [d'(\beta_q^j), pmax]).$$

For instance, we look for all the clauses corresponding to the points M such that $p(M) \in [i_1, i_2] \cap [pmin, g'(\beta_q^j)]$. (The other cases are similar). Since $p(M) \in [i_1, i_2]$, $Z_l(M) \cap F \neq \emptyset$, $l = 0, 1$, while by $p(M) \in [pmin, g'(\beta_q^j)]$ it follows that $Z_2(M) \cap F \neq \emptyset$, for any $F \in \mathcal{E}(\alpha, \beta)$. If $Z_3(M) \cap \alpha \neq \emptyset$, clauses (3.6),(3.8) can be easily found in $O(mn)$ time. Clauses (3.7) can be found in $O(mn)$ time, since we construct $\overline{V_N}$ for any $N \in Z_3(M)$, where $M \notin \beta$ is the point of $q = j$ maximizing p such that $p(M) \in [i_1, i_2] \cap [pmin, g'(\beta_q^j)]$.

Now we build formulas (3.9). At first we express the line-convexity along the line $q = j$ by the formula:

$$V_{\langle g(\beta_q^j), j \rangle} \Rightarrow V_{\langle g(\beta_q^j) + \delta, j \rangle} \Rightarrow \dots \Rightarrow V_{\langle g(\beta_q^j) + \delta(q_j - 1 - |\alpha_q^j|), j \rangle}. \quad (6.13)$$

Let l_{min}, l_{max} defined by:

$$\{l : \langle g(\beta_q^j) + \delta l, j \rangle \in \beta \setminus \alpha \text{ and } g(\beta_q^j) + \delta l \in [i_1, i_2] \cap [pmin, g'(\beta_q^j)]\} = \{l_{min}, \dots, l_{max}\}.$$

Then we construct the additional clauses

$$V_{N_l} \Rightarrow V_{\langle g(\beta_q^j) + l\delta, j \rangle} \quad (6.14)$$

for all $N_l \in Z_3(\langle g(\beta_q^j) + l\delta, j \rangle)$ if $l = l_{min}$ and for all $N_l \in Z_3(\langle g(\beta_q^j) + l\delta, j \rangle) \setminus Z_3(\langle g(\beta_q^j) + (l-1)\delta, j \rangle)$ if $l \in \{l_{min} + 1, \dots, l_{max}\}$. It is easy to see that formulas (6.13) and (6.14) are equivalent to formulas (3.9) associated to all the points M on the line $q = j$. When the q -line is fixed, all these formulas can be found in $O(mn)$ time. Since there are n q -lines, the global construction takes $O(mn^2)$ time.

The formulas associated to the other points Now we show that also clauses (3.10) can be found in $O((mn(m+n)))$ time. These formulas are built for the points not in β lying on a t_0 p -line or on a t_0, t_3 q -line. Since there are n q -lines (resp., m p -lines), if for each q -line (resp., p -line) we find these formulas in $O(mn + m^2)$ -time (resp., $O(mn + n^2)$), then the construction will take $O(mn(m+n))$ time.

The points M that are on a t_0 - q -line $q = j$. Three cases should be considered:

$$j < q(U_1), q(U_1) < j < q(U_2), q(U_2) < j.$$

Let us examine the case: $q(U_1) < j < q(U_2)$. Let us define

$$i_1 = \max\{i : d'(\beta_p^i) \leq j\}, i_2 = \min\{i : g'(\beta_p^i) \geq j\}$$

the maximum and minimum p -index such that $\Lambda_p(\langle i_1, j \rangle) = \Lambda_p^-(\langle i_1, j \rangle)$ and $\Lambda_p(\langle i_2, j \rangle) = \Lambda_p^+(\langle i_2, j \rangle)$. Thus, $Z_1(\langle i_1, j \rangle_{p,q}) \cap F \neq \emptyset$ and $Z_3(\langle i_2, j \rangle_{p,q}) \cap F \neq \emptyset$ for any $F \in \mathcal{E}(\alpha, \beta)$. Since $\langle i_1, j \rangle_{p,q}$ may not be in \mathbb{Z}^2 , let $i'_1 = \max\{i : i \leq i_1, \langle i, j \rangle_{p,q} \in \mathbb{Z}^2\}$. So, we impose the formula $\overline{V_N}$ for any $N \in Z_3(\langle i'_1, j \rangle_{p,q})$. Analogously, let $i'_2 = \min\{i : i \geq i_2, \langle i, j \rangle_{p,q} \in \mathbb{Z}^2\}$: we impose $\overline{V_N}$ for any $N \in Z_1(\langle i'_2, j \rangle_{p,q})$. Moreover, in each line $p = i$ such that $i_1 < i < i_2$ and $p_i > 0$ we select two special points $A_i, B_i \in \beta \setminus \alpha$ defined by:

$$q(A_i) = \max_{h < j} \{h : \langle i, h \rangle_{p,q} \in \beta \setminus \alpha\}, \quad q(B_i) = \min_{j < h} \{h : \langle i, h \rangle_{p,q} \in \beta \setminus \alpha\}.$$

To express the Q-convexity around the points of $q = j$ we impose the following clauses:

$$\overline{V_{A_i}} \Rightarrow \overline{V_{\langle i, q(A_i) - \delta \rangle}} \Rightarrow \overline{V_{\langle i, q(A_i) - 2\delta \rangle}} \Rightarrow \dots \Rightarrow \overline{V_{\langle i, g(\beta_p^i) \rangle}}, \quad (6.15)$$

$$\overline{V_{B_i}} \Rightarrow \overline{V_{\langle i, q(B_i) + \delta \rangle}} \Rightarrow \overline{V_{\langle i, q(B_i) + 2\delta \rangle}} \Rightarrow \dots \Rightarrow \overline{V_{\langle i, d(\beta_p^i) \rangle}}, \quad (6.16)$$

So, less than mn clauses are constructed. To these ones we should also add the clauses

$$\overline{V_{A_{h_1}}} \vee \overline{V_{B_{h_2}}}, \quad (6.17)$$

for all pairs (h_1, h_2) with $h_2 \leq i \leq h_1$ such that $\langle i, j \rangle_{p,q} \in \mathbb{Z}^2$ and $i'_1 + \delta \leq i \leq i'_2 - \delta$. The pairs (h_1, h_2) can be found in $O(m^2)$ -time by the algorithm 2.

Algorithm 2 Enumeration of the pairs (h_1, h_2) with $a \leq h_2 \leq h_1 \leq b$ when there exists $i = a + \delta i'$ with $i' \in \mathbb{Z}$, $h_2 \leq i \leq h_1$. The complexity of this algorithm is $O((b - a)^2)$

```

 $h_1 \leftarrow a$ 
 $i \leftarrow a$ 
while  $h_1 \leq b$  do
  while  $h_1 < i + \delta$  do
     $h_2 \leftarrow a$ 
    while  $h_2 \leq i$  do
      PRINT  $(h_1, h_2)$ 
       $h_2 \leftarrow h_2 + 1$ 
    end while
     $h_1 \leftarrow h_1 + 1$ 
  end while
   $i \leftarrow i + \delta$ 
end while

```

It is easy to check that if $N_1 \in Z_1(M)$ and $N_2 \in Z_3(M)$ the constructed clauses are equivalent to $\overline{V_{N_1}} \vee \overline{V_{N_2}}$. Since we can build clauses (6.15) and (6.16) in $O(n)$ time and clauses (6.17) in $O(m^2)$ time, constructing clauses (3.10) takes $O(mn(m + n))$ time in case $q(U_1) < j < q(U_2)$.

Now let us take case $j < q(U_1)$ into consideration. If $\{i : d'(\beta_p^i) \leq j\}$ is not empty, let i_1 and i_2 be the minimum and maximum element of the set, respectively and again $i'_1 = \max\{i : i \leq i_2, \langle i, j \rangle_{p,q} \in \mathbb{Z}^2\}$ and $i'_2 =$

$\min\{i : i \geq i_1, \langle i, j \rangle_{p,q} \in \mathbb{Z}^2\}$. If $i'_1 \geq i'_2$, then there is no solution in $\mathcal{E}(\alpha, \beta)$; otherwise $i'_2 = i'_1 + \delta$ and so the formulas $\overline{V_N}$, for any $N \in Z_0(\langle i'_1, j \rangle_{p,q})$ and $N \in Z_1(\langle i'_2, j \rangle_{p,q})$ are imposed. If $\{i : d'(\beta_p^i) \leq j\}$ is empty, then on each line $p = i$ such that $p_i > 0$ we select the point A_i such that $q(A_i) = \max\{h : \langle i, h \rangle_{p,q} \in \beta \setminus \alpha \text{ and } 1 \leq h < j\}$.

Therefore, the Q-convexity around the points of $q = j$ is expressed by

$$\overline{V_{A_i}} \Rightarrow \overline{V_{\langle i, q(A_i) - \delta \rangle}} \Rightarrow \overline{V_{\langle i, q(A_i) - 2\delta \rangle}} \Rightarrow \dots \Rightarrow \overline{V_{\langle i, g(\beta_p^i) \rangle}},$$

and

$$\overline{V_{A_{h_1}}} \vee \overline{V_{A_{h_2}}},$$

for all $h_2 \leq i \leq h_1$ such that $\langle i, j \rangle_{p,q} \in \mathbb{Z}^2$ with $p_{min} \leq i \leq p_{max}$. Since less than $mn + m^2$ clauses are built, constructing clauses (3.10) takes $O(mn(m+n))$ time in case $j < q(U_1)$.

Case $q(U_2) > j$ can be similarly treated being the symmetric case.

The points M that are on a t_0 - p -line $p = i$. This case is very similar to the previous one. As in the previous case we define:

$$j_1 = \min\{j : d'(\beta_q^j) \leq i\}, \quad j_2 = \max\{j : d'(\beta_q^j) \leq i\}$$

$$j'_1 = \min\{j : g'(\beta_q^j) \geq i\}, \quad j'_2 = \max\{j : g'(\beta_q^j) \geq i\}.$$

It follows that $j_1 \leq q(U_1) \leq j_2$ and $j'_1 \leq q(U_2) \leq j'_2$, and moreover $j_1 \leq j'_2$ by $q(U_1) \leq q(U_2)$.

The Q-convexity around the points $\langle i, j \rangle$ which verify $j_1 \leq j \leq j_2$ or $j'_1 \leq j \leq j'_2$ are expressed by the clauses (3.7).

The Q-convexity around the points $\langle i, j \rangle$ such that $j \leq \min(j_1, j'_1)$ or $j_2 \leq j \leq j'_1$ or $j \geq \max(j_2, j'_2)$ can be expressed by similar formulas to (6.15), (6.16), (6.17) in $O(mn + n^2)$ -time.

The points M that are on a t_3 - q -line $q = j$. We should express the Q-convexity in all the points $\langle i, j \rangle$ such that $g(\beta_q^j) + \delta q_j \leq i \leq d(\beta_q^j) - \delta q_j$. Let N be an arbitrary indeterminate point such that $q(N) = j$ and $q(N) \leq g'(\beta_q^j)$. Since for each $N', N'' \in \beta_q^j \setminus \alpha$ with $p(N') \leq g'(\beta_q^j) < d'(\beta_q^j) \leq p(N'')$ we have the formula $V_N \equiv V_{N'} \equiv \overline{V_{N''}}$ as a consequence of FQ_j , we can express the Q-convexity by the clauses

$$(V_N \Rightarrow \overline{V_{N_2}}, (\overline{V_N} \Rightarrow \overline{V_{N_1}}) \tag{6.18}$$

for each $N_1 \in Z_{l_1}(\langle d(\beta_q^j) - \delta q_j, j \rangle) \cap \beta$ and $N_2 \in Z_{l_2}(\langle g(\beta_q^j) + \delta q_j, j \rangle) \cap \beta$ where

$$\begin{aligned} (l_1, l_2) &= (0, 1) && \text{if } j \leq q(U_1) \\ &= (3, 1) && \text{if } q(U_1) \leq j \leq q(U_2) \\ &= (3, 2) && \text{if } j \geq q(U_2). \end{aligned}$$

These clauses can be found in $O(mn)$ time for each line $q = j$.

6.2.2 More than two directions

The expression of the Q-convexity in the points M which verify the Remark 3.9 can be computed exactly like in the case of two directions. Here we studied in details the additional cases that need to be taken into account. So, let us consider a line $q = j$ of type t_0 or t_3 . We should impose the Q-convexity around the two directions (q, r) in any point $M = \langle j, k \rangle_{q,r}$ of $\Delta \setminus \beta$ not verifying the conditions of Remark 3.9. Moreover we suppose $q(U_1) \leq j$ and that U_1 and U_2 are in the same quadrant Z_l^{qr} with $l \in \{0, \dots, 3\}$. From a brutal computation, it may seem that the cost of the construction of the formulas (4.12) is $O(n^4)$, so changing the performance of the reconstruction algorithm considerably. Actually, we show that it takes $O(n^3)$. We determine the gaps where the r -indices of the points in $q = j$ need to be considered. Let

$$k_1 = \min_{h \geq j} (d'(\beta_q^h)), \quad k_2 = \max_{h \geq j} (g'(\beta_q^h))$$

and

$$k'_1 = \min\{k : g'(\beta_r^k) \geq j \text{ or } d'(\beta_r^k) \leq j\}, \quad k'_2 = \max\{k : g'(\beta_r^k) \geq j \text{ or } d'(\beta_r^k) \leq j\}.$$

The minima and maxima are taken only over lines intersecting β . The case in which all lines are t_0 -lines is trivial. Since the line $r = r(U_1)$ is a t_1 -line, by definitions of k'_1, k'_2 the inequality $k'_1 < r(U_1) < k'_2$ follows.

We express the Q-convexity in all the points $M = \langle j, k \rangle_{qr}$ of the q -line $q = j$.

- If $k \geq k_1$, then $\exists h > j \ \Lambda_q(\langle h, k \rangle_{q,r}) = \Lambda_q^-(\langle h, k \rangle_{q,r})$. We know that $F \cap Z_1^{qr}(\langle j, k_1 \rangle_{q,r}) \neq \emptyset$ for any $F \in \mathcal{E}(\alpha, \beta)$. So we can express the Q-convexity by the formulas (3.10), which can be built by the method which is used for two directions.
- The case $k \leq k_2$ is similar to the previous one ($\exists h > j \ \Lambda_q(\langle h, k \rangle_{q,r}) = \Lambda_q^+(\langle h, k \rangle_{q,r})$).
- If $k \in [k'_1, k'_2]$, then we know that there exist $l_1 \in \{0, 1\}, l_2 \in \{2, 3\}$ such that $Z_{l_1}^{qr}(\langle j, k'_1 \rangle) \cap F \neq \emptyset$ and $Z_{l_2}^{qr}(\langle j, k'_2 \rangle) \cap F \neq \emptyset$ for any $F \in \mathcal{E}(\alpha, \beta)$. So, we can also express the Q-convexity by the formulas (3.10).
- It remains the case $k \in]k_2, k_1[\setminus [k'_1, k'_2]$. If $]k_2, k_1[\cap] - \infty, k'_1[\neq \emptyset$, define: $k_3 = \max\{k < \min(k'_1, k_2) : \langle j, k \rangle_{q,r} \in \mathbb{Z}^2\}$. The Q-convexity in all the points $\langle j, k \rangle$ with $k \in]k_2, k_1[\cap] - \infty, k'_1[$ can be expressed by the formula:

$$V_{M_1} \Leftrightarrow \dots \Leftrightarrow V_{M_l}$$

where M_1, \dots, M_l are the indeterminate points of $Z_1^{qr}(\langle j, k_3 \rangle_{q,r})$.

In the same way if $]k_2, k_1[\cap]k'_2, +\infty[\neq \emptyset$ we define: $k_4 = \min\{k > \max(k_1, k'_2) : \langle j, k \rangle_{q,r} \in \mathbb{Z}^2\}$. The Q-convexity in all the points $\langle j, k \rangle$ with $k \in]k_2, k_1[\cap]k'_2, +\infty[$ can be expressed by the formula:

$$V_{M_1} \Leftrightarrow \dots \Leftrightarrow V_{M_l}$$

where M_1, \dots, M_l are the indeterminate points of $Z_2^{qr}(\langle j, k_4 \rangle_{q,r})$. Therefore, the construction of the formulas can be done in $O(n^2)$ time, for each fixed line. In conclusion, since there are at most $2n$ q -lines and r -lines the clauses to express the Q -convexity can be found in $O(n^3)$ time.

References

- [1] B. Aspvall, M. Plass, R. Tarjan, A linear-time algorithm for testing the truth of certain quantified boolean formulas, *Inform. Process. Lett.* 8 (3) (1979) 121–123.
- [2] E. Barcucci, S. Brunetti, A. Del Lungo, M. Nivat, Reconstruction of discrete sets from three or more X-rays, *Proc. of CIAC 2000, Lecture Notes in Comput. Sci.*, 1767 (2000) 199–210.
- [3] E. Barcucci, A. D. Lungo, M. Nivat, R. Pinzani, Reconstructing convex polyominoes from horizontal and vertical projections, *Theoret. Comput. Sci.* 155 (2) (1996) 321–347.
- [4] E. Barcucci, A. Del Lungo, M. Nivat, R. Pinzani, Medians of polyominoes: a property for the reconstruction, *Int. J. Imaging Systems and Technol.* 8 (1998) 69–77.
- [5] S. Brunetti, A. Daurat, A. Del Lungo, Approximate X-rays reconstruction of special lattice sets, *Pure Math. Appl.* 11 (3) (2000) 409–425.
- [6] M. Chrobak, C. Dürr, Reconstructing HV-convex polyominoes from orthogonal projections, *Inform. Process. Lett.* 69 (6) (1999) 283–289.
- [7] A. Daurat, Uniqueness of the reconstruction of Q -convexes from their projections., preprint LLAIC1 (2000).
- [8] A. Daurat, Convexité dans le plan discret, application à la tomographie discrète, Ph.D. thesis, LLAIC1 and LIAFA, Université Paris 7 (2000).
- [9] A. Daurat, A. Del Lungo, M. Nivat, Median points of discrete sets according to a linear distance, *Discrete Comput. Geom.* 23 (2000) 465–483.
- [10] R. J. Gardner, P. Gritzmann, D. Prangenberg, On the computational complexity of reconstructing lattice sets from their X-rays, *Discrete Math.* 202 (1999) 45–71.
- [11] R. J. Gardner, P. Gritzmann, Discrete tomography: Determination of finite sets by X-rays., *Trans. Amer. Math. Soc.* 349 (6) (1997) 2271–2295.
- [12] R. L. Graham, An efficient algorithm for determining the convex hull of a finite planar set, *Inform. Process. Lett.* 1 (4) (1972) 132–133.
- [13] G. H. Woeginger, The reconstruction of polyominoes from horizontal and vertical projections, *Inform. Process. Lett.* 77 (5-6) (2001) 225–229.