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Use of Augmented Lagrangian Methods for the Optimal Control of Obstacle Problems

M. Bergounioux [1]

Abstract. We investigate optimal control problems governed by variational inequalities involving constraints on the control, and more precisely the example of the obstacle problem. In this paper we discuss some augmented Lagrangian algorithms to compute the solution.

KeyWords : Optimal control, Lagrange multipliers, augmented Lagrangian, variational inequalities

1 Introduction

We investigate optimal control problems governed by variational inequalities and involving constraints on both the control and the state. These problems have been widely studied, by V. Barbu (Ref.1), F. Mignot and J.P. Puel (Ref.2) or Zheng-Xu He (Ref.3) for instance.

We may consider these problems from many points of view. One of the most important is the approximation of the variational inequality by an equation where the maximal monotone operator (which is in this case the subdifferential of a lipschitz function) is approached by a differentiable single-value mapping, with Moreau-Yosida approximations techniques. This method (mainly due to Barbu (Ref.1)) leads to several existence results and to first-order optimality systems. As the passage to the limit in the optimality system is difficult and impossible without specific assumptions, one usually compute the solution of the approximate problem which is a non-linear optimal control problem.

In Refs.4-6 first-order necessary optimality conditions have been obtained : these results are based on a relaxation of the original problem via a splitting operator method. This leads to the reformulation of the problem as an optimal control problem with constraints coupling the state and the control ; these constraints are not convex. The method has been developed for the example of the obstacle problem in Refs.4-5 (and extended later to general variational inequalities in Ref.6) and we briefly recall the main results in the first part of this paper. The next section is devoted to a saddle-point formulation of the optimality system, and we discuss many viewpoints since the problem is not convex. Then, we present some Lagrangian and augmented Lagrangian algorithms ; in particular we point out relaxed (in the sense of Fortin and Glowinski (Ref.7)) augmented Lagrangian algorithms. The last section is devoted to numerical experiments.

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2 Problem Setting

Let Ω be an open, bounded subset of $\mathbb{R}^n$ ($n \leq 3$) with a smooth boundary, and consider the following obstacle problem :

$$\min \left\{ J(y,v) = \frac{1}{2}\int_\Omega (y-z_d)^2\,dx + (\nu/2)\int_\Omega v^2\,dx \right\} \tag{$\mathcal{P}_0$}$$

such that

$$a(y,\varphi - y) \geq \langle v+f, \varphi - y\rangle \quad \forall \varphi \in K_o, \tag{2.1}$$

$$y \in K_o, \tag{2.2}$$

$$v \in U_{ad}, \tag{2.3}$$

where

- (i) $z_d \in L^2(\Omega)$, $v \in L^2(\Omega)$, $f \in L^2(\Omega)$ and $\nu > 0$;
- (ii) a is a bilinear form defined on $H^1_o(\Omega) \times H^1_o(\Omega)$, with the classical properties (ellipticity, coercivity, and continuity ; see Refs.1,8 for example),
- (iii) $\langle\ ,\ \rangle$ denotes the duality between $H^1_o(\Omega)$ and $H^{-1}(\Omega)$, $(\ ,\ )$ the scalar-product of $L^2(\Omega)$ and $\|\ \|$ the $L^2(\Omega)$-norm ;
- (iv) U_{ad} is a nonempty, closed, convex subset of $L^2(\Omega)$, and

$$K_o = \{y \in H^1_o(\Omega) \mid y \geq 0 \text{ a.e. in } \Omega\}. \tag{2.4}$$

The equations (2.1) and (2.2) may be interpreted as follows [see Barbu (Ref.1) or Mignot and Puel (Ref.2)] :

$$Ay = v + \xi, \qquad y \geq 0, \qquad \xi \geq 0, \qquad \langle y, \xi\rangle = 0, \tag{2.5}$$

where $A \in \mathcal{L}(H^1_o(\Omega), H^{-1}(\Omega))$ such that $\langle A\phi, \psi\rangle = a(\phi,\psi)$. The solution y of (2.5) is known to be unique. Then, the optimal control problem appears as a problem governed by a state **equation** (instead of **inequation**) with mixed state and control constraints :

$$\min \left\{ J(y,v) = \frac{1}{2}\int_\Omega (y-z_d)^2\,dx + (\nu/2)\int_\Omega v^2\,dx \right\}, \tag{$\mathcal{P}$}$$

$$Ay = v + f + \xi \text{ in } \Omega, \quad y = 0 \text{ on } \Gamma, \tag{2.6}$$

$$(y,v,\xi) \in \mathcal{D}, \tag{2.7}$$

where

$$\mathcal{D} = \{(y,v,\xi) \in H^1_o(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mid v \in U_{ad},\ y \geq 0,\ \xi \geq 0,\ \langle y,\xi\rangle = 0\}. \tag{2.8}$$

Remark 2.1 *In fact ξ should be taken from $H^{-1}(\Omega)$, but a regularity result of Friedman (Ref.8) allows us to choose it from $L^2(\Omega)$. So y given by (2.6) belongs to $H^2(\Omega) \cap H^1_o(\Omega)$ and we set*

$$K = K_o \cap (H^2(\Omega) \cap H^1_o(\Omega))\ . \tag{2.9}$$

Problem $(\mathcal{P})$ has at least one solution, denoted by $(\bar{y}, \bar{v}, \bar{\xi})$. Similar problems have been studied also in Bergounioux and Tiba (Ref.9) in the case where the constraint set $\mathcal{D}$ is convex. This does not hold here : $\mathcal{D}$ is nonconvex and its interior is, in some sense, "very" empty. Therefore, we relax the bilinear constraint "$\langle y, \xi \rangle = 0$" using "$0 \leq \langle y, \xi \rangle \leq \alpha$" instead, where $\alpha > 0$. This approach is motivated by the numerical point of view : during the computation, equality conditions like "$\cdots = 0$" are usually expressed as "$|\cdots| \leq \alpha$" where α can be arbitrarily small, but strictly positive.

In order to ensure the existence of a solution for the relaxed problem and to avoid the use of an adapted penalization, we also assume that the L^2- norm of ξ is bounded by some positive constant R (greater that $\|\bar{\xi}\|$). Again, this is not very restrictive from the numerical point of view, since R can be arbitrarily large. Let us denote

$$V_{ad} = \{ \ \xi \in L^2(\Omega) \mid \xi \geq 0 \ \text{ a.e in } \Omega, \quad \|\xi\| \leq R \ \} \ .$$

The associated relaxed problem $(\mathcal{P}_\alpha)$ becomes :

$$\begin{array}{l} \min J(y,v) = \dfrac{1}{2}\displaystyle\int_\Omega (y - z_d)^2 \, dx + (\nu/2) \int_\Omega v^2 \, dx \\ Ay = v + f + \xi \text{ in } \Omega, y = 0 \text{ on } \Gamma, \\ (y, v, \xi) \in \mathcal{D}_\alpha \ , \end{array} \tag{$\mathcal{P}_\alpha$}$$

where

$$\mathcal{D}_\alpha = \{(y, v, \xi) \in H_o^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mid v \in U_{ad}, \ y \geq 0, \ \xi \in V_{ad}, \ (y, \xi) \leq \alpha \ \}. \tag{2.10}$$

Theorem 2.1 *Problem $(\mathcal{P}_\alpha)$ has (at least) has a solution $(y_\alpha, v_\alpha, \xi_\alpha)$. Moreover when $\alpha \to 0$, y_α converges to $\bar{y}$ weakly in $H_o^1(\Omega)$, v_α converges to $\bar{v}$ strongly in $L^2(\Omega)$ and ξ_α converges to $\bar{\xi}$ weakly in $L^2(\Omega)$.*

Moreover we may derive optimality conditions :

Theorem 2.2 *Let $(y_\alpha, v_\alpha, \xi_\alpha)$ be a solution of $(\mathcal{P}_\alpha)$ and assume the following qualification conditions :*

$$\begin{array}{c} \forall \alpha \text{ such that } (y_\alpha, \xi_\alpha) = \alpha \ , \exists (\tilde{y}, \tilde{v}, \tilde{\xi}) \in K \times U_{ad} \times V_{ad} \quad \text{ such that} \\ A\tilde{y} = \tilde{v} + f + \tilde{\xi} \quad \text{ and } \quad (\tilde{y}, \xi_\alpha) + \left(y_\alpha, \tilde{\xi}\right) < 2\alpha \ , \end{array} \tag{$\mathcal{H}_1$}$$

$$\begin{array}{c} \exists p \in [1, +\infty], \ \exists \ \rho > 0 \ , \quad \forall \ \chi \in L^p(\Omega), \ \|\chi\|_{L^p(\Omega)} \leq 1 \ , \\ \exists (y_\chi, v_\chi, \xi_\chi) \text{ bounded in } K \times U_{ad} \times V_{ad} \text{ (by a constant independent of } \chi) \ , \\ \text{such that } Ay_\chi = v_\chi + f + \xi_\chi + \rho\chi \quad \text{in } \Omega. \end{array} \tag{$\mathcal{H}_2$}$$

Then there exist $q_\alpha \in L^2(\Omega)$ and $r_\alpha \in \mathbb{R}_+$ such that

$$(y_\alpha - z_d, y - y_\alpha) + (q_\alpha, A(y - y_\alpha)) + r_\alpha \, (\xi_\alpha, y - y_\alpha) \geq 0 \quad \text{ for all } y \in K, \tag{2.11}$$

$$(\nu v_\alpha - q_\alpha, v - v_\alpha) \geq 0 \quad \text{ for all } v \in U_{ad}, \tag{2.12}$$

$$(r_\alpha y_\alpha - q_\alpha, \xi - \xi_\alpha) \geq 0 \quad \text{ for all } \xi \in V_{ad}, \tag{2.13}$$

$$r_\alpha \, [(y_\alpha, \xi_\alpha) - \alpha] = 0 \ . \tag{2.14}$$

Proof.- See Ref.4 □

Remark 2.2 *In the applications, the conditions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *are quite often satisfied; it is true, for example, if* $U_{ad} = L^2(\Omega)$ *or* $U_{ad} = \{ v \in L^2(\Omega) \mid v \geq \Psi \geq 0$ *a.e. in* $\Omega\}$, *where* $\Psi \in L^2(\Omega)$.

Now we are interested in the exploitation of the above optimality system. As we decide to use a lagrangian point of view, we are going to formulate it as a saddle-point existence result.

3 Saddle-Point Formulation

In the sequel, conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are supposed to be satisfied so that we always get the existence of the optimality system of Theorem 2.2. Let L^α be the lagrangian function associated to problem $(\mathcal{P}_\alpha)$, defined on $H^2(\Omega) \cap H^1_o(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}$ by

$$L^\alpha(y, v, \xi, q, r) = J(y, v) + (q, Ay - v - f - \xi) + r[(y, \xi) - \alpha] \ . \tag{3.15}$$

A direct consequence of Theorem 2.2 is the following result :

Theorem 3.1 *Assume conditions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *are ensured and let* $(y_\alpha, v_\alpha, \xi_\alpha)$ *be a solution of* $(\mathcal{P}_\alpha)$, *then* $(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha)$ *satisfies*

$$L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha) \geq L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q, r) \quad \textit{for all } (q, r) \in L^2(\Omega) \times \mathbb{R}^+ \ , \tag{3.16}$$

and

$$\nabla_{y,v,\xi} L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha)(y - y_\alpha, v - v_\alpha, \xi - \xi_\alpha) \geq 0 \tag{3.17}$$

for all $(y, v, \xi) \in K \times U_{ad} \times V_{ad}$.

Proof - Relation (3.16) comes from the fact that for all $(q, r) \in L^2(\Omega) \times \mathbb{R}^+$,

$$L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q, r) = J(y_\alpha, v_\alpha) + r[(y_\alpha, \xi_\alpha) - \alpha] \leq J(y_\alpha, v_\alpha) \ ,$$

and

$$J(y_\alpha, v_\alpha) = L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha) \ ,$$

because of relation (2.14) .
Moreover, adding (2.11), (2.12) and (2.13) we obtain exactly relation (3.17). □
Let us call

$$C = K \times U_{ad} \times V_{ad} \times L^2(\Omega) \times \mathbb{R}^+ \ ; \tag{3.18}$$

Theorem 3.1 is not sufficient to ensure the existence of a saddle point of L^α on C, because of the non-convexity of L^α. Nevertheless as the non-convex part is bilinear, it is easy to see that this theorem is still valid if we replace L^α by the linearized Lagrangian function $\mathcal{L}^\alpha$:

$$\mathcal{L}^\alpha(y, v, \xi, q, r) = J(y, v) + (q, Ay - v - f - \xi) + r[(y, \xi_\alpha) + (y_\alpha, \xi) - 2\alpha] \ . \tag{3.19}$$

More precisely we have

Theorem 3.2 *With the assumptions and notations of Theorem 3.1, $(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha)$ is a saddle point of the linearized Lagrangian function $\mathcal{L}^\alpha$ on C :*

$$\mathcal{L}^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q, r) \leq \mathcal{L}^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha) \leq \mathcal{L}^\alpha(y, v, \xi, q_\alpha, r_\alpha) \qquad \textit{for all } (y, v, \xi, q, r) \in C \ .$$

Proof - We get first the left hand part of the above inequality since for any $(q, r) \in L^2(\Omega) \times \mathbb{R}^+$

$$\mathcal{L}^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q, r) = J(y_\alpha, v_\alpha) + 2r\ [(y_\alpha, \xi_\alpha) - \alpha] \leq J(y_\alpha, v_\alpha) = \mathcal{L}^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha) \ .$$

The right hand part comes from

$$\nabla_{y,v,\xi}\mathcal{L}^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha) = \nabla_{y,v,\xi}L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha)$$

with relation (3.17) and the convexity of $\mathcal{L}^\alpha$. □

In the case where the bilinear constraint is not active : $(y_\alpha, \xi_\alpha) < \alpha$, we get $r_\alpha = 0$. It is easy to see that $\mathcal{L}^\alpha(y, v, \xi, q_\alpha, r_\alpha)$ is then equal to $L^\alpha(y, v, \xi, q_\alpha, r_\alpha)$ and Theorem 3.2 yields

$$L^\alpha(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha) \leq L^\alpha(y, v, \xi, q_\alpha, r_\alpha) \qquad \text{for all } (y, v, \xi, q, r) \in C \ .$$

So with relation (3.16) we get

Corollary 3.1 *If $(y_\alpha, v_\alpha, \xi_\alpha)$ is a solution of $(\mathcal{P}_\alpha)$ where the inequality constraint is inactive ($(y_\alpha, \xi_\alpha) < \alpha$) and if condition $(\mathcal{H}_2)$ is satisfied, then $(y_\alpha, v_\alpha, \xi_\alpha, q_\alpha, r_\alpha)$ is a saddle point of L^α on C.*

Remark 3.1 *We may also define the augmented Lagrangian function*

$$\mathcal{L}_c^\alpha(y, v, \xi, q, r) = \mathcal{L}^\alpha(y, v, \xi, q, r) + (c/2)\left[\|Ay - v - f - \xi\|^2 + [(y, \xi_\alpha) + (y_\alpha, \xi) - 2\alpha]_+^2\right] \ ,$$

where $c > 0$ and $g_+ = \max(0, g)$.
Then we may note that the previous theorem is still valid with $\mathcal{L}_c^\alpha$ instead of $\mathcal{L}^\alpha$.

Remark 3.2 *The converse result of Theorem 3.2 is false since we have used the linearized Lagrangian and not the genuine one.*

4 Lagrangian Algorithms

The previous section suggests to test Lagrangian algorithms usually used to compute saddle-points in a convex frame. We are going to present some of them in the very case where the inequality constraint is not active that is : $(y_\alpha, \xi_\alpha) < \alpha$. Indeed, we have seen with Corollary 3.1 that the problem turns to be (locally) convex in this case and we get an existence result for saddle point of L^α. Of course, this is restrictive but we have to remember that the original problem was formulated with the constraint $(y, \xi) = 0$; so we may hope that this assumption is not too unrealistic.

Then we shall see how these methods work in the case where the constraint may be active. Though we are not able to prove convergence in this case we may interpret the solution(s) of the optimality system as the fixed point(s) of a functional Φ and Lagrangian algorithms may be viewed as successive approximations methods to compute the fixed point. Unfortunately, though we are able to show that Φ is locally Lipschitz-continuous (with sensitivity analysis techniques), we cannot estimate precisely the Lipschitz constant (and prove that Φ is contractive).

4.1 Inactive constraint $(y_\alpha, \xi_\alpha) \leq \alpha$

The basic method to compute a saddle point is the Uzawa algorithm (see Ref.7 for example). We have already used this kind of method coupled with a Gauss-Seidel splitting to solve optimal control problems in Refs. 10-11. Concerning our problem this gives

Algorithm A0

- Step 1. Initialization : Set $n = 0$, choose $q_o \in L^2(\Omega),\ r_o \in \mathbb{R}^+,\ (v_{-1}, \xi_{-1}) \in U_{ad} \times V_{ad}$.
- Step 2. Compute

$$\begin{aligned} y_n &= \underset{y \in K}{\arg\min}\ L^\alpha(y, v_{n-1}, \xi_{n-1}, q_n, r_n) \\ (v_n, \xi_n) &= \underset{(v,\xi) \in U_{ad} \times V_{ad}}{\arg\min}\ L^\alpha(y_n, v, \xi, q_n, r_n) \end{aligned}$$

- Step 3. Compute

$$q_{n+1} = q_n + \rho_1\ (Ay_n - v_n - f - \xi_n) \quad \text{where} \quad \rho_1 \geq \rho_o > 0\ ,$$
$$r_{n+1} = r_n + \rho_2\ \left[(y_n, \xi_n) - \alpha\right]_+ \quad \text{where} \quad \rho_2 \geq \rho_o > 0\ .$$

We notice that the second minimization problem of Step 2 can be decoupled ; it is equivalent to

$$\begin{aligned} v_n &= \underset{v \in U_{ad}}{\arg\min}\ (\nu/2)\ \|v\|^2 - (q_n, v) \\ \xi_n &= \underset{\xi \in V_{ad}}{\arg\min}\ (r_n y_n - q_n, \xi)\ . \end{aligned}$$

So $v_n = P_{U_{ad}}(q_n/\nu)$ where $P_{U_{ad}}$ denotes the $L^2(\Omega)$-projection on U_{ad} and ξ_n is given by solving a linear programming problem on a bounded set. To avoid this and to get a better convergence rate one usually consider the augmented Lagrangian function

$$L_c^\alpha(y, v, \xi, q, r) = L^\alpha(y, v, \xi, q, r) + (c/2)\left[\ \|Ay - v - f - \xi\|^2 + [(y, \xi) - \alpha]_+^2\right]\ , \quad (4.20)$$

where $c > 0$. Then the above algorithm used with L_c^α instead of L^α leads to

Algorithm A1

- Step 1. Initialization : Set $n = 0$, choose $c > 0,\ q_o \in L^2(\Omega),\ r_o \in \mathbb{R}^+,\ (v_{-1}, \xi_{-1}) \in U_{ad} \times V_{ad}$.
- Step 2. Compute

$$\begin{aligned} y_n &= \underset{y \in K}{\arg\min} \quad \tfrac{1}{2}\|y - z_d\|^2 + (q_n, Ay) + r_n\,(y, \xi_{n-1}) \\ &\qquad\qquad +(c/2)\ \left[\|Ay - v_{n-1} - f - \xi_{n-1}\|^2 + [(y, \xi_{n-1}) - \alpha]_+^2\right] \\ v_n &= P_{U_{ad}}([q_n + c(Ay_n - f - \xi_{n-1})]/[\nu + c]) \\ \xi_n &= \underset{\xi \in V_{ad}}{\arg\min}\ (r_n y_n - q_n, \xi) + (c/2)\ \left(\|Ay_n - v_n - f - \xi\|^2 + [(y_n, \xi) - \alpha]_+^2\right) \end{aligned}$$

– Step 3. Compute

$$q_{n+1} = q_n + \rho_1 \ (Ay_n - v_n - f - \xi_n) \quad \text{where} \ \ \rho_1 \geq \rho_o > 0 \ ,$$

$$r_{n+1} = r_n + \rho_2 \ \ [(y_n, \xi_n) - \alpha]_+ \quad \text{where} \ \ \rho_2 \geq \rho_o > 0 \ .$$

Remark 4.1 *The Gauss-Seidel splitting (see Refs.7,11,12 for example) leads to the following minimization problem to get the control in Step 2.*

$$\begin{array}{rcl} v_n & = & \textit{Arg min} \ (\nu/2) \ \|v\|^2 - (q_n, v) + (c/2) \ \|Ay_n - v - f - \xi_{n-1}\|^2 \\ & & v \in U_{ad} \ , \end{array}$$

that is $v_n = P_{U_{ad}}([q_n + c \ (Ay_n - f - \xi_{n-1})]/[\nu + c])$.

The above algorithm $\mathcal{A}_1$ is based on the most "natural" penalization of the inequality constraint. We investigate now a variant of this algorithm, where the augmentation term for the inequality constraint has been modified. This algorithm, due to Ito and Kunisch (Ref.13), has been developed for some more general problems with equality and inequality constraints in Hilbert spaces. We present it now.
We consider the general problem

$$\begin{array}{ll} \text{minimize} & \varphi(x) \\ \text{s.t.} & e(x) = 0, \ g(x) \leq 0, \ l(x) \leq \hat{z}, \end{array} \tag{4.21}$$

where
(i) $\varphi \ : X \to \mathbb{R}, \ e \ : X \to Y, \ g \ : X \to \mathbb{R}^m, \ l \ : X \to Z$ with X, Y, Z Hilbert spaces.
(ii) l is a bounded linear operator on X and $\hat{z} \in Z$.
(iii) There exists $x^* \in X$ and $(\lambda^*, \mu^*, \eta^*) \in Y \times \mathbb{R}^m_+ \times Z^*_+$ such that

$$\text{the functions } \varphi, e, g \text{ are Fréchet - } \mathcal{C}^2 \text{ in a neighborhood of } x^* \ ; \tag{4.22}$$

(iv)

$$\begin{array}{c} \varphi, g_i : X \to \mathbb{R}, \ \ \text{for } i = 1, \cdots, m, \ \text{are weakly lower semi continuous and} \\ e \text{ maps weakly convergent sequences to weakly convergent sequences.} \end{array} \tag{4.23}$$

(v) x^* is stationary for (4.21) with Lagrange multiplier $(\lambda^*, \mu^*, \eta^*)$, i.e

$$\varphi'(x^*)h + \langle \lambda^*, e'(x^*)h \rangle + \langle \mu^*, g'(x^*)h \rangle_m + \langle \eta^*, l(h) \rangle = 0 \qquad \text{for all } h \in X \ ,$$

and

$$e(x^*) = 0, \ \ \langle \mu^*, g(x^*) \rangle_{R^m} = 0, \quad \langle \eta^*, l(x^*) - \hat{z} \rangle_Z = 0 \ ; \tag{4.24}$$

(vi)

$$e'(x^*) : X \to Y \text{ is surjective} \ \ ; \tag{4.25}$$

(vii)

$$\begin{array}{c} \text{There exists a constant } \sigma > 0 \text{ such that} \\ L''(x^*)(h, h) \geq \sigma |h|^2 \text{ for all } h \neq 0 \text{ in } \Xi \end{array} \tag{4.26}$$

where

$$\Xi = \{ \ h \in X \mid e'(x^*)h = 0, \ g_i'(x^*)h \leq 0 \text{ for } i \in I_1, g_i'(x^*)h = 0 \text{ for } i \in I_2 \ \} \ ,$$

$$I_1 = \{ \ i \mid g_i(x^*) = 0, \ \mu_i^* = 0 \ \}, \ I_2 = \{ \ i \mid g_i(x^*) = 0, \ \mu_i^* \neq 0 \ \} \ ,$$

$$L(x) = L(x, (\lambda^*, \mu^*, \eta^*)) = \varphi(x) + \langle \lambda^*, e(x) \rangle + \langle \mu^*, g(x) \rangle + \langle \eta^*, l(x) \rangle \ ;$$

Finally we define the augmented Lagrangian function

$$\mathcal{L}_n^c(x) = \varphi(x) + \langle \lambda_n, e(x) \rangle + \langle \mu_n, g(x) \rangle + (c/2) \left[|e(x)|^2 + |\hat{g}(x, \mu_n, c)|^2 \right] \ ,$$

where $\hat{g}(x, \mu, c) = \max(g(x), -\mu/c)$ (componentwise). We have then the following result

Theorem 4.1 *Under the above assumptions, for an initial choice of (λ_o, μ_o), let x_n satisfy*

$$\mathcal{L}_n^c(x_n) \leq \mathcal{L}_n^c(x^*) \ \textit{and} \ l(x_n) \leq \hat{z} \ ,$$

and let $(\lambda_{n+1}, \mu_{n+1})$ be defined with

$$\lambda_{n+1} = \lambda_n + \rho_n e(x_n), \ \mu_{n+1} = \mu_n + \rho_n \hat{g}(x_n, \mu_n, c), \quad 0 < \rho_n < c \ .$$

Then we have

$$\sum_{n=1}^{+\infty} \rho_n |x_n - x^*|^2 \leq C_o(|\lambda_o - \lambda^*|^2 + |\mu_o - \mu^*|^2) \ .$$

Proof - See Ref.13 □

We are going to apply this result to our problem. More precisely :

- (i) $X = H^2(\Omega) \cap H_o^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, $Y = L^2(\Omega)$;
  $x = (y, v, \xi), \ \varphi(x) = J(y, v), \ e(x) = Ay - v - f - \xi$ and $m = 1, \ g(x) = (y, \xi) - \alpha$.
  φ is quadratic, e is affine and g is bilinear so they are $\mathcal{C}^2$ and assumption (4.22) is ensured. It is easy to see that (4.23) is also satisfied for φ and e. Moreover, g is weakly sequentially continuous because of the compactness of the injection of $H_o^1(\Omega)$ into $L^2(\Omega)$. So it is weakly lower semi-continuous.
- (ii) In the case where U_{ad} is described as following

  $$U_{ad} = \{ \ v \in L^2(\Omega) \mid \Lambda_i(v) \leq b_i, \ i = 1 \cdots p \ \} \ ,$$

  where $\Lambda_i : L^2(\Omega) \to L^2(\Omega)$ is linear, bounded, we set $Z = H_o^1(\Omega) \times L^2(\Omega)^p \times L^2(\Omega)$, $\hat{z} = (0, b_1, \cdots, b_p, 0)$ and $l = (-Id_{H_o^1(\Omega)}, \Lambda_1, \cdots, \Lambda_p, -Id_{L^2(\Omega)})$ where Id_E is the identity mapping of the space E.
  This is not really restrictive since it involves for example the cases where $U_{ad} = \{ \ v \in L^2(\Omega) \mid a(x) \leq v(x) \leq b(x)$ a.e. on $\Omega \ \}$ or $U_{ad} = L^2(\Omega)$.
- (iii) Assumption (4.24) is ensured by Theorem (2.2) with $x^* = (y_\alpha, v_\alpha, \xi_\alpha), \ \lambda^* = q_\alpha$ and $\mu^* = r_\alpha$ if we suppose that $(\mathcal{H}_2)$ is satisfied.
- (iv) Moreover as e is affine then $e'(x^*) = e + f$, which is obviously surjective.

– (v) At last I_1 and I_2 are empty so that $\Xi = X$.

$$L(x) = L^\alpha(y, v, \xi, q_\alpha, r_\alpha) + \langle \eta^*, l(y, v, \xi)\rangle \ ,$$

$$L''(x^*)(x, x) = J''(y_\alpha, v_\alpha)((y, v), (y, v)) + r_\alpha \left[(y_\alpha, \xi) + (y, \xi_\alpha)\right] \ ,$$

because all the linear terms "disappear".
Furthermore $r_\alpha = 0$ so that $L''(x^*)(x, x) = J''(y_\alpha, v_\alpha)((y, v), (y, v))$ and (4.26) is fulfilled.

Henceforth, we get the convergence of the algorithm below

Algorithm A2

Step 1. Initialization : Set $n = 0$, choose $c > 0$, $q_o \in L^2(\Omega)$, $r_o \in \mathbb{R}^+$.
Step 2. Compute

$$(y_n, v_n, \xi_n) = \underset{(y,v,\xi) \in K \times U_{ad} \times V_{ad}}{\arg\min} \hat{L}_c^\alpha(y, v, \xi, q_n, r_n)$$

where

$$\begin{aligned} \hat{L}_c^\alpha(y, v, \xi, q, r) &= J(y, v) + (q, Ay - v - f - \xi) + r \max((y, \xi) - \alpha, -r/c) \\ &\quad +(c/2) \left[\|Ay - v - f - \xi\|^2 + [\max((y, \xi) - \alpha, -r/c)]^2\right] \ , \end{aligned} \tag{4.27}$$

Step 3.

$$q_{n+1} = q_n + \rho_1 \ (Ay_n - v_n - f - \xi_n) \quad \text{where} \ \ \rho_1 \in]0, c[\ ,$$

$$r_{n+1} = r_n + \rho_2 \ \max((y_n, \xi_n) - \alpha, -r_n/c) \quad \text{where} \ \ \rho_2 \in]0, c[\ .$$

4.2 Active Constraint

When the inequality constraint may be active, we have no longer any information on r_α nor local convexity of the Lagrangian function. Anyway we shall test the Algorithms of the previous section a priori, since we do not know the solution, so we do not know if the constraint is active or not. A justification of this point of view is that these Algorithms may be interpretated as successive approximations method to compute the fixed-points of a function Φ that we are going to define. We are able to prove that Φ is locally lipschitz continuous but we cannot estimate precisely the Lipschitz constant. Our feeling is that an appropriate choice for the augmentation parameters allows to make this constant strictly less that 1, so that Φ is contractive. To reinterpretate Algorithm A1, we define some functions φ_i as following :
(i) $\varphi_1 \ : L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}_+ \to H^2(\Omega) \cap H_o^1(\Omega)$:

$$\begin{aligned} \varphi_1(v, \xi, q, r) = y^* = \underset{y \in K}{\text{Arg min}} \quad & \tfrac{1}{2}\|y - z_d\|^2 + (q, Ay) + r\,(y, \xi) \\ & +(c/2) \ \left[\|Ay - v - f - \xi\|^2 + [(y, \xi) - \alpha]_+^2\right] \ . \end{aligned} \tag{4.28}$$

(ii) $\varphi_2 \ : \ H^2(\Omega) \cap H_o^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega)$:

$$\varphi_2(y, q, \xi) = v^* = P_{U_{ad}}([q + c(Ay - f - \xi)]/[\nu + c]).$$

(iii) $\varphi_3 \ : H^2(\Omega) \cap H^1_o(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}_+ \to L^2(\Omega)$:

$$\varphi_3(y, vq, r) = \xi^* = \underset{\xi \in V_{ad}}{\text{Arg min}} \quad (ry - q, \xi) + (c/2) \ \left[\|Ay - v - f - \xi\|^2 + [(y, \xi) - \alpha]^2_+\right] \ .$$

(iv) $\varphi_4 \ : H^2(\Omega) \cap H^1_o(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}_+ \to L^2(\Omega) \times \mathbb{R}_+$:

$$\varphi_4(y, v, \xi, q, r) = (q^*, r^*) = (q + \rho_1(Ay - v - f - \xi), r + \rho_2 \ [(y, \xi) - \alpha]_+) \ .$$

At last, let us define $\Phi : L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R} \to L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}_+$:

$$\Phi(v, \xi, q, r) = (\bar{v}, \bar{\xi}, \bar{q}, \bar{r}) \ ,$$

with

$$\bar{y} = \varphi_1(v, \xi, q, r) \ ,$$
$$\bar{v} = \varphi_2(\bar{y}, q, \xi) = \varphi_2(\varphi_1(v, \xi, q, r), q, \xi) \ ,$$
$$\bar{\xi} = \varphi_3(\bar{y}, \bar{v}, q, r) = \varphi_3(\varphi_1(v, \xi, q, r), \varphi_2(\varphi_1(v, \xi, q, r), q, \xi), q, r) \ ,$$
$$(\bar{q}, \bar{r}) = \varphi_4[\bar{y}, \bar{v}, \bar{\xi}, q, r]$$
$$= \varphi_4[\varphi_1(v, \xi, q, r), \varphi_2(\varphi_1(v, \xi, q, r), q, \xi), \varphi_3(\varphi_1(v, \xi, q, r), \varphi_2(\varphi_1(v, \xi, q, r), q, \xi), q, r), q, r] \ .$$

All product spaces are endowed with the product norm. So Algorithm A1 turns to be exactly the successive approximation method applied to Φ, to solve

$$\Phi(v, \xi, q, r) = (v, \xi, q, r) \ . \tag{4.29}$$

To prove the convergence we should prove first that Φ is contractive. Then, we have to show that the solution $(\tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})$ of (4.29) satisfies the optimality system of Theorem 2.2 with $\tilde{y} = \varphi_1(\tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})$.

Theorem 4.2 *The function Φ defined above is locally Lipschitz continuous.*

Proof - It is sufficient to prove that every $\varphi_i, \ i = 1, \cdots, 4$, is Lipschitz-continuous.
(i) It is clear that φ_2 is Lipschitz continuous because of the properties of the projection operator and the Lipschitz constant k_2 satisfies

$$k_2 \leq c \ \max(\|A\|, 1, 1/c)/(\nu + c) \ ,$$

where $\|A\|$ denotes the norm of the operator A.
(ii) The mapping $\psi \ : H^2(\Omega) \cap H^1_o(\Omega) \times L^2(\Omega) \to \mathbb{R}$ such that $\psi(y, \xi) = (y, \xi)$ is bilinear continuous, so it is $\mathcal{C}^1$ and therefore locally Lipschitz continuous. As the projection $x \mapsto x_+$ from $\mathbb{R}$ to $\mathbb{R}_+$ is globally Lipschitz continuous as well we get the property for

$$\begin{array}{rccc} \Psi : & H^2(\Omega) \cap H^1_o(\Omega) \times L^2(\Omega) & \to & \mathbb{R}_+ \\ & (y, \xi) & \mapsto & [(y, \xi) - \alpha]_+ \ , \end{array}$$

and φ_4 is locally Lipschitz continuous as well.
(iii) To study φ_1 and φ_3 we use some sensitivity and stability techniques (see Malanowski Ref.14 for example). We prove that φ_1 is locally Lipschitz continuous (the method is the same for φ_3).

Let $\mathcal{B}$ be a $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}$ ball around some (v_o, ξ_o, q_o, r_o) of radius ρ. Let us choose (v_1, ξ_1, q_1, r_1) and (v_2, ξ_2, q_2, r_2) in this ball and set $y_i = \varphi_1(v_i, \xi_i, q_i, r_i)$ for $i = 1, 2$.

Writing the first order optimality condition for the minimization problem (4.28) (the objective function is Gâteaux-differentiable) we get for all y in K and $i = 1, 2$

$$\begin{array}{l}(y_i - z_d, y - y_i) + (q_i, A(y - y_i)) + (r_i\xi_i, y - y_i) \\ +c\,(Ay_i - v_i - \xi_i - f, A(y - y_i)) + c\Psi_i\,(\xi_i, y - y_i) \quad \geq 0\ ,\end{array}$$

where we note $\Psi_i = \Psi(y_i, \xi_i) = [(y_i, \xi_i) - \alpha]_+$.
Taking successively $i = 1,\ y = y_2$ and $i = 2,\ y = y_1$ and adding the above inequalities we get

$$\begin{array}{rcl}\|y_1 - y_2\|^2 + c\|A(y_1 - y_2)\|^2 & \leq & (q_1 - q_2, A(y_2 - y_1)) + (r_1\xi_1 - r_2\xi_2, y_2 - y_1) + \\ & & c\,(v_2 - v_1 + \xi_2 - \xi_1, A(y_2 - y_1)) + \\ & & c\,(\Psi_1\xi_1 - \Psi_2\xi_2, y_2 - y_1)\,.\end{array}$$

$$\begin{array}{rcl}\|y_1 - y_2\|^2 + c\|A(y_1 - y_2)\|^2 & \leq & \|q_1 - q_2\|\ \|A(y_2 - y_1)\| + \\ & & [|r_1 - r_2|\ \|\xi_1\| + |r_2|\ \|\xi_1 - \xi_2\|]\ \ \|y_2 - y_1\| + \\ & & c\,[\ \|v_2 - v_1\| + \|\xi_2 - \xi_1\|\ ]\ \|A(y_2 - y_1)\| + \\ & & c\,[\ |\Psi_1 - \Psi_2|\ \|\xi_1\| + |\Psi_2|\ \|\xi_1 - \xi_2\|\ ]\ \|y_2 - y_1\|\ .\end{array}$$

The definition of Ψ_i gives

$$|\Psi_1 - \Psi_2| \leq |\,(y_1, \xi_1) - (y_2, \xi_2)\,| \leq \|y_2\|\ \|\xi_1 - \xi_2\| + \|y_1 - y_2\|\ \|\xi_1\|\ .$$

With $|r_i| \leq \rho + |r_o|,\ \|\xi_i\| \leq \rho + \|\xi_o\|$ and setting $\tilde{\rho} = \max(\rho + |r_o|, \rho + \|\xi_o\|)$ we obtain

$$\begin{array}{rl}\|y_1 - y_2\|^2 + c\|A(y_1 - y_2)\|^2 \leq & \|q_1 - q_2\|\ \|A(y_2 - y_1)\| + \\ & \tilde{\rho}\,[\ |r_1 - r_2| + \|\xi_1 - \xi_2\|\ ]\ \|y_2 - y_1\| + \\ & c\,[\ \|v_2 - v_1\| + \|\xi_2 - \xi_1\|]\ \|A(y_2 - y_1)\| + \\ & c\tilde{\rho}\,[\,\|y_2\|\,\|\xi_1 - \xi_2\| + \tilde{\rho}\|y_1 - y_2\| + |\Psi_2|\,\|\xi_1 - \xi_2\|]\ \|y_2 - y_1\|.\end{array}$$

Setting $N_c(y_1 - y_2) = \left(\|y_1 - y_2\|^2 + c\|A(y_1 - y_2)\|^2\right)^{1/2}$ and using

$$\|A(y_1 - y_2)\| \leq c^{-1/2}N_c(y_1 - y_2)\ \ ,\ \ \|y_1 - y_2\| \leq N_c(y_1 - y_2)$$

yields

$$\begin{array}{rcl}N_c(y_1 - y_2) & \leq & c^{-1/2}\|q_1 - q_2\| + \tilde{\rho}\,[\ |r_1 - r_2| + \|\xi_1 - \xi_2\|\ ] \\ & & +c^{1/2}\,[\ \|v_2 - v_1\| + \|\xi_2 - \xi_1\|\ ] \\ & & +c\tilde{\rho}\,[\ \|y_2\|\ \|\xi_1 - \xi_2\| + \tilde{\rho}\|y_1 - y_2\| + |\Psi_2|\ \|\xi_1 - \xi_2\|\ ]\ .\end{array}$$

$$\begin{array}{rcl}N_c(y_1 - y_2) & \leq & c^{-1/2}\|q_1 - q_2\| + \tilde{\rho}\ |r_1 - r_2| + c^{1/2}\ \|v_2 - v_1\| \\ & & + \left[\tilde{\rho} + c^{1/2} + c\rho\|y_2\| + c\tilde{\rho}|\Psi_2|\right]\|\xi_1 - \xi_2\| + c\tilde{\rho}^2\ \|y_1 - y_2\|\ .\end{array} \tag{4.30}$$

The above inequality used with $y_2 = y_o = \varphi_1(v_o, \xi_o, q_o, r_o)$ gives

$$\|y_1 - y_o\| \leq N_c(y_1 - y_o)\ \leq c_1 + c\tilde{\rho}^2\ \|y_1 - y_o\|\ .$$

Then if $c < 1/\tilde{\rho}^2$ we have

$$\forall (v_i, \xi_i, q_i, r_i) \in \mathcal{B} \quad \|y_i\| \leq c_1/(1 - c\tilde{\rho}^2) + \|y_o\| = c_2 \text{ and } |\Psi_i| \leq c_3$$

where the constants $c_i = C_i(\mathcal{B}, c)$ only depend on c and $\mathcal{B}$; Finally with (4.30) we get

$$\begin{aligned} N_c(y_1 - y_2) &\leq c^{-1/2}\|q_1 - q_2\| + \tilde{\rho}\ |r_1 - r_2| + +c^{1/2}\|v_2 - v_1\| \\ &\quad + \left[\ \tilde{\rho} + c^{1/2} + c(c_1 + c_3)\tilde{\rho}\ \right] \|\xi_1 - \xi_2\| + c\tilde{\rho}^2\ \|y_1 - y_2\| \\ &\leq \kappa\ \|(v_2, \xi_2, q_2, r_2) - (v_1, \xi_1, q_1, r_1)\|_{L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times R} + c\tilde{\rho}^2\ \|y_1 - y_2\|\ , \end{aligned}$$

where $\kappa = \kappa(c, \mathcal{B}) = \max(c^{-1/2}, \tilde{\rho}, c^{1/2}, \tilde{\rho} + c^{1/2} + c(c_1 + c_3))$; with $c < 1/\tilde{\rho}^2$ we get in turn

$$\|y_1 - y_2\| \leq N_c(y_1 - y_2) \leq [\kappa/(1 - c\tilde{\rho}^2)]\ \|(v_2, \xi_2, q_2, r_2) - (v_1, \xi_1, q_1, r_1)\|_{L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times R}\ ,$$

and

$$\|A(y_1 - y_2)\| \leq \kappa^* \|(v_2, \xi_2, q_2, r_2) - (v_1, \xi_1, q_1, r_1)\|_{L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)\times R}\ ,$$

where $\kappa^* = \kappa\ \left(c^{-1/2} + c^{1/2}\tilde{\rho}^2/(1 - c\tilde{\rho}^2)\right)$.

As N_c is equivalent to the $H^2(\Omega) \cap H^1_o(\Omega)$-norm this proves that φ_1 is locally Lipschitz-continuous provided $c < 1/(\max(\rho + |r_o|, \rho + \|\xi_o\|)^2$. □

Remark 4.2 *As the previous proof shows it, it is quite difficult to give a precise estimation of the Lipschitz constant of Ψ. Anyway our feeling is that an appropriate choice of c could make this constant strictly less than 1, if ρ is small enough.*

It remains to prove that the fixed point of Φ (whenever it exists) is a stationary point, i.e a solution of the optimality system.

Theorem 4.3 *Any solution $(\tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})$ of (4.29) satisfies the relations (2.11)-(2.12)-(2.13) of the optimality system of Theorem 2.2. In addition, if the inequality constraint is active the whole optimality system of Theorem 2.2 is satisfied.*

Proof - Let us call $(\tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})$ such a point and $\tilde{y} = \varphi_1(\tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})$. The definition of Ψ yields

$$\tilde{v} = \varphi_2(\tilde{y}, \tilde{q}, \tilde{\xi})\ , \tag{4.31}$$

$$\tilde{\xi} = \varphi_3(\tilde{y}, \tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})\ , \tag{4.32}$$

$$(\tilde{q}, \tilde{r}) = \varphi_4(\tilde{y}, \tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})\ . \tag{4.33}$$

Relation (4.33) gives :

$$\tilde{q} = \tilde{q} + \rho_1(A\tilde{y} - \tilde{v} - f - \tilde{\xi})\ ,$$

and

$$\tilde{r} = \tilde{r} + \rho_2[\left(\tilde{y}, \tilde{\xi}\right) - \alpha]_+\ ,$$

so that

$$A\tilde{y} - \tilde{v} - f - \tilde{\xi} = 0 \quad \text{and} \quad \left(\tilde{y}, \tilde{\xi}\right) \leq \alpha\ . \tag{4.34}$$

As $\tilde{y} \in K,\ \tilde{v} \in U_{ad}$ and $\tilde{\xi} \in V_{ad}$, this means that $(\tilde{y}, \tilde{v}, \tilde{\xi})$ is feasible for the problem $(\mathcal{P}_\alpha)$.

Now we write successively the optimality systems related to the definitions of $\varphi_1,\ \varphi_2$ and φ_3. From the definition of φ_1 we get for all $y \in K$

$$\begin{aligned}&(\tilde{y} - z_d, y - \tilde{y}) + (\tilde{q}, A(y - \tilde{y})) + \tilde{r}\left(\tilde{\xi}, y - \tilde{y}\right) + \\ &c\left(A\tilde{y} - \tilde{v} - \tilde{\xi}, A(y - \tilde{y})\right) + c[\left(\tilde{y}, \tilde{\xi}\right) - \alpha]_+ \left(\tilde{\xi}, y - \tilde{\xi}\right) \quad \geq 0\end{aligned}$$

and with (4.34)

$$(\tilde{y} - z_d, y - \tilde{y}) + (\tilde{q}, A(y - \tilde{y})) + \tilde{r}\left(\tilde{\xi}, y - \tilde{y}\right) \geq 0 \ .$$

This is exactly relation (2.11) with $(\tilde{y}, \tilde{\xi}, \tilde{q}, \tilde{r})$ instead of $(y_\alpha, \xi_\alpha, q_\alpha, r_\alpha)$.
Similarly, one can show that relations (2.12) and (2.13) are ensured for $(\tilde{y}, \tilde{v}, \tilde{\xi}, \tilde{q}, \tilde{r})$. □

5 Numerical Experiments

5.1 Implementation and Example

We have tested both algorithms A1 and A2. For numerical reasons that we are going to explain we have also tested algorithm A2 with the Gauss-Seidel splitting already used for A1. For all these methods, the main difficulty lies in the resolution of Step 2, that is the resolution of a problem of the following type

$$\min\{\ \varphi(x) \mid x \geq 0\ \} \ ,$$

where φ is not twice differentiable. We have tested a classical primal-dual method which was very slow because of the high number of unknowns ; finally, since the function φ is "almost" quadratic we have chosen a quite efficient active set method exposed by Ito and Kunisch in Ref.15, coupled with a Newton iteration.

We present the numerical results on a example given in Bermudez and Saguez (Ref.16). We take $\Omega =]0,1[\times]0,1[\subset \mathbb{R}^2,\ A = -\Delta$ the Laplacian operator ($\Delta y = \partial^2 y/\partial x_1^2 + \partial^2 y/\partial x_2^2$). The discretization is done via finite-differences and the size of the grid is given by $h = 1/N$ on each side of the domain. We set $U_{ad} = L^2(\Omega),\ \nu = 100$

$$z_d = \begin{cases} 200\ x_1 x_2\ (x_1 - \frac{1}{2})^2\ (1 - x_2) & \text{if } 0 < x_1 \leq 1/2\ , \\ 200\ x_2\ (x_1 - 1)(x_1 - \frac{1}{2})^2\ (1 - x_2) & \text{if } 1/2 < x_1 \leq 1\ , \end{cases} \tag{5.35}$$

and

$$f = \begin{cases} 200\ [2x_1(x_1 - \frac{1}{2})^2\ - x_2\ (6x_1 - 2)\ (1 - x_2)] & \text{if } 0 < x_1 \leq 1/2\ , \\ 200\ (\frac{1}{2} - x_1) & \text{if } 1/2 < x_1 \leq 1\ . \end{cases} \tag{5.36}$$

Moreover , we put $\alpha = 10^{-3}$. This example is constructed such that the null control $v^* \equiv 0$ is the optimal control for the original problem $(\mathcal{P})$ and

$$y^* = \begin{cases} z_d & \text{if } 0 < x_1 \leq 1/2\ , \\ 0 & \text{if } 1/2 < x_1 \leq 1\ , \end{cases} \tag{5.37}$$

and $J^* = J(y^*, v^*) = 25/504 \simeq 0.496\ 10^{-1}$. We have tested different initial values for the data ; we shall discuss them in the forthcoming subsections. The experimentation has been done with the MATLAB© software.

5.2 Algorithm A1

The first tests have shown that the convergence was effective but slow. This comes from the fact that the global minimization problem with (respect to $y,\ v$ and ξ) issued from Uzawa-method, which has been decoupled with a Gauss-Seidel splitting in Step 2. is not solved accurately enough with only one Gauss-Seidel iteration. So, following Fortin and Glowinski (Ref.7) we have introduced a longer "splitting loop" so that A1 becomes :

Algorithm A1'

- Step 1. Initialization : Set $n = 0$, choose $c > 0,\ q_o \in L^2(\Omega),\ r_o \in \mathbb{R}^+,\ (v_{-1}, \xi_{-1}) \in U_{ad} \times V_{ad}$.
- Step 2. $q_n,\ r_n,\ v_{n-1}$ and ξ_{n-1} are know ; set $k_n > 0,\ j = 0,\ v_n^o = v_{n-1}$ and $\xi_n^o = \xi_{n-1}$ Begin the splitting loop : for $j = 1, \cdots, k_n$ compute

$$\begin{aligned} y_n^j &= \underset{y \in K}{\arg\min} \quad \tfrac{1}{2}\|y - z_d\|^2 + (q_n, Ay) + r_n\left(y, \xi_n^{j-1}\right) \\ & \qquad\qquad +(c/2)\ \left[\|Ay - v_n^{j-1} - f - \xi_n^{j-1}\|^2 + [\left(y, \xi_n^{j-1}\right) - \alpha]_+^2\right] \\ v_n^j &= P_{U_{ad}}([q_n + c\ (Ay_n^j - f - \xi_n^{j-1})]/[\nu + c]) \\ \xi_n^j &= \underset{\xi \in V_{ad}}{\arg\min}\ \left(r_n y_n^j - q_n, \xi\right) + (c/2)\ \left[\|Ay_n^j - v_n^j - f - \xi\|^2 + [\left(y_n^j, \xi\right) - \alpha]_+^2\right] \end{aligned}$$

  At the end of loop, set $y_n = y_n^{k_n},\ v_n = v_n^{k_n},\ \xi_n = \xi_n^{k_n}$.
- Step 3. Compute

$$q_{n+1} = q_n + \rho_1\ (Ay_n - v_n - f - \xi_n) \quad \text{where} \quad \rho_1 \geq \rho_o > 0\ ,$$

$$r_{n+1} = r_n + \rho_2\ \left[(y_n, \xi_n) - \alpha\right]_+ \quad \text{where} \quad \rho_2 \geq \rho_o > 0\ .$$

Algorithm A1' has been tested with a constant length of the Gauss-Seidel loop : $k_n \equiv k_{gs}$. Tests have shown that a good choice for $N = 15$ is $k_{gs} = 5$. A bigger value of k_{gs} decreases the number of iterations but as we have to take into account the Gauss Seidel iterations, the global number of iterations is approximately the same. This algorithm converges, but it is very delicate to choose the parameters c and ρ. It seems to be quite sensitive to this choice. Of course the convergence rate depends on these parameters but we may also have convergence at the beginning and then oscillations or convergence towards a solution which is not very accurate. Anyway oscillations occurring for "bad" values of c and ρ may be "killed" if k_{gs} is increased, so that Step 2 is solved more precisely.
The choice of initial values is not very influent, except if they are chosen too far from the solution (for example $y_o = 1,\ v_o = -100,\ \xi_o = 10$) because of Newton iterations (once again a greater value of k_{gs} gives better results).
We get $(y, \xi) = 0$ after a small number of iterations ; there is a jump : (y, ξ) is "suddenly" set to 0 after a gentle decreasing to 0. We note that the jump occurs at the same iteration if we set α to 1.e-03 or 1.e-06 ($\alpha = 1$ gives a bad solution..) So the constraint $(y, \xi) \leq \alpha$ is inactive at the solution and the analysis of Section 3. is valided a-posteriori. The following table shows the influence of the parameters on the convergence process.

We set $k_{gs} = 5,\ y_o = 0,\ v_o = 0,\ \xi_o = 0\ , q_o = 0$ and $N = 15$. It is the number of iterations, so that the global number is $It * k_{gs}$. For $N = 15$ we have kept $c = 10,\ \rho = 5$ and $k_{gs} = 5$ as a good choice.

Table 1. Influence of parameters for Algorithm A1'

c	ρ	r_o	$\|Ay - f - v - \xi\|$	(y, ξ)	It	$J \times 10^2$	Comments
10	5	1	7.3 e-05	0	41	4.956	
10	5	0	7 e-05	0	41	"	
10	1	1	9 e-02	4.28	50	"	very slow
1	0.5	1					non convergence
1	1	1					non convergence

5.3 Algorithm A2

Step 2 of Algorithm A2 is solved directly and (we could say) "exactly". This is the most expensive step of the method. The first iteration "pushes" the iterate very near the solution and one can see (with Table 2.) that the convergence is quite fast during the first iterations. Then it is much slower. Moreover, as in the previous method we can see that constraint $(y, \xi) < \alpha$ is inactive at the solution and we have $(y, \xi) = 0$ very quickly (most of time from the first iteration, especially if $r_o \neq 0$).

Table 2. Convergence of Algorithm A2

and c=1, ρ =0.5 , N =10, $y_o = v_o = \xi_o = q_o = 0$, $r_o = 1$, accuracy 10^{-4}

Iteration	$\|Ay - f - v - \xi\|$	$J \times 10^2$
1	8.1966 e-04	4.9489
2	4.1005 e-04	"
3	2.0514 e-04	"
4	1.0262 e-04	"
5	6 e-05	"

This algorithm is convergent in any case ; the different parameters (c, ρ or the initialization of data) have few influence on the convergence itself but only on the convergence rate. We present some results in Table 3. As CPU time depends on the machine and has no absolute signification, we have normalized this time setting the smallest to 1, since only relative values are interesting (to give an idea, for this case, the unit CPU time on a HP - Work station is 63 s.)

Table 3. Influence of parameters for Algorithm A2

and $N = 10$, $y_o = v_o = \xi_o = q_o = 0$, accuracy 10^{-4}

c	ρ	r_o	$\|Ay - f - v - \xi\|$	N IT	CPU Units
1	0.5	1	9 e-05	5	1.5
0.1	0.01	1		>15	>10
1	1	1	4 e-07	2	1.4[2]
1	0.1	1	9 e-05	21	3
10	5	1	8 e-05	1	1
10	5	0	4 e-05	1	1.1
100	50	1	8 e-06	1	1.4

Thus, this method is quite satisfying since it is robust and needs few iterations (the first one is the most "expensive"). Anyway, during the resolution of Step 2 one has to assemble a matrix which size is $(3 * N^2)$ (each variable y, v or ξ is represented by a N^2 vector). Even the use of sparse matrix cannot avoid a full N^2-matrix. This resolution is quite expensive in time and memory and we had to restrain our tests to $N \leq 15$. It would not be wise to use it for a grid size of 50×50 for example, even on a powerful Work-Station. Even if the size of allocated memory would be sufficient the computational time would very long...

So we have also considered a Gauss-Seidel splitting which allows to decouple the unknowns and make the subproblems "smaller". Step 2 of algorithm A2 is modified ; we introduce a splitting loop of length k_n and we obtain the relaxed algorithm A2' described below.

Algorithm A2'

- Step 1. Initialization : Set $n = 0$, choose $c > 0$, $q_o \in L^2(\Omega)$, $r_o \in \mathbb{R}^+$, $\xi_{-1} \in L^2(\Omega)$ and $v_{-1} \in L^2(\Omega)$.
- Step 2. q_n, r_n, ξ_{n-1} and v_{n-1} are known ; k_n is given and $j = 0$, $v_n^o = v_{n-1}$, $\xi_n^o = \xi_{n-1}$.
  Begin the splitting loop : for $j = 1, \cdots, k_n$ compute

$$y_n^j = \arg\min \{ \hat{L}_c^\alpha(y, v_n^{j-1}, \xi_n^{j-1}, q_n, r_n) \mid y \geq 0 \} ,$$

$$\xi_n^j = \arg\min \{ \hat{L}_c^\alpha(y_n^j, v_n^{j-1}, \xi, q_n, r_n) \mid \xi \geq 0 \} ,$$

$$v_n^j = P_{U_{ad}}([\nu v_d - c\ (Ay_n^j - \xi_n^j - f + q_n/c)]/[\nu + c]) .$$

  At the end of splitting loop, set $y_n = y_n^{k_n}, \xi_n = \xi_n^{k_n}$, $v_n = v_n^{k_n}$.
- Step 3. Compute

$$q_{n+1} = q_n + \rho_1\ (Ay_n - v_n - f - \xi_n) \quad \text{where} \quad \rho_1 \in]0, c[,$$

$$r_{n+1} = r_n + \rho_2\ \max([(y_n, \xi_n) - \alpha], -r_n/c) \quad \text{where} \quad \rho_2 \in]0, c[.$$

The initial value for $(y_{-1}, v_{-1}, \xi_{-1})$ has been set to 0 ; anyway, it seems that it has not a great influence on the convergence ; for example the choice of $(y_{-1} \equiv 1, v_{-1} \equiv 100, \xi_{-1} \equiv 10)$ gives the same solution with the same CPU Time. We have not tested negative values

[2] In this case the value of $\|Ay - f - v - \xi\|$ is 8 10^{-4} at the first iteration and 4 10^{-7} at the second

for y_{-1} and ξ_{-1} since we tried to start from a feasible point. We have set $k_n \equiv k_{gs}$. Tests with $k_{gs} = 1$ show that convergence is effective but very slow : indeed, once again, it is quite important to solve the subproblem of Step 2 very accurately (it is of course the case for algorithm A2). A good value of k_{gs} for $N = 15$ is 5 (we have already mentioned why during the study of A1'). When $r_o \neq 0$ we can see that we get $(y, \xi) = 0$ very quickly ; the choice of $r_o = 0$ makes the convergence of (y, ξ) towards 0 slower, but anyway, the constraint (at the solution) is inactive. Once again, this justifies the analysis of Section 4. We summarize in Table 4. some tests about the influence of parameters.

Table 4. Parameter Influence on Algorithm A2' (N=15)

Initialization $y_o = v_o = \xi_o = 0$ and	c	ρ	k_{gs}	$\|Ay - f - v - \xi\|$	$J \times 10^2$	Iterations	Comments
$q_o = 0,\ r_o = 1$	0.1	0.01	5	Oscillating iterates			Bad convergence
"	1	0.1	5	5 e-02	4.9561	24	
"	1	1	5	5 e-02	"	10	
"	1	0.5	5	7 e-02	"	20	
"	1	0.5	5	2 e-03	"	25	$5 \times 25 = 125$ [3]
"	1	0.5	10	2 e-03	"	16	$10 \times 16 = 160$ [3]
"	1	0.5	1				Bad convergence
$q_o = 0,\ r_o = 0$	1	0.5	5	4 e-02	4.9567	12	$(y, \xi) = 3\ e - 02$
$q_o = 0,\ r_o = 1$	10	5	5	1 e-03	4.9684	21	$(y, \xi) = e - 01$
$(y_o, v_o, \xi_o) = (1, 100, 10)$ $q_o = 0,\ r_o = 1$	1	0.1	5	2 e-03	4.9563	11	$(y, \xi) = 4\ e - 02$

For $N = 15$, one could say that a good choice of parameters is $c = 1,\ \rho = 0.5$ and $k_{gs} = 5$. Table 5. gives indications on the convergence for these parameters and $y_o = 0,\ v_o = 0,\ \xi_o = 0$, $q_o = 0$, $r_o = 1$.

Table 5. Convergence of Algorithm A2'

Iteration	$\|Ay - f - v - \xi\|$	(y, ξ)	$J \times 10^2$
1	1.4630e+00	2.6870e+00	5.0598
2	1.4621e+00	6.4095e-01	5.0576
3	1.1458e+00	2.9751e-01	4.9606
4	6.0735e-01	2.8411e-01	4.9587
5	3.0226e-01	2.6581e-01	4.9578
10	9.9310e-03	1.5777e-01	4.9567
15	1.8642e-03	6.4416e-02	4.9563
19	**4.7422e-03**	**3.4128e-03**	**4.9562**
20	**7.0212e-02**	**0**	**4.9561**
25	2.2657e-03	0	4.9561
29	9.6108e-05	0	4.9561

[3] Global number of iterations : It * k_{gs}.

To spare time tests for algorithms A1' and A2 have been done with N=15. Some other tests have been performed with N =30. This is much slower of course because of the size of the grid. Moreover parameters have to be adjusted again. In particular k_{gs} must be increased. So the computing time is not proportional to the size of the grid.

6 Conclusions

Algorithm A2 seems to be better than A1' because the convergence is effective in any case and there is no sensitivity with respect to the different parameters. Algorithm A2' allows to consider a fine finite-difference grid (that is a great number of unknowns) and most of time faster than A2. Moreover all the non-sensitivity with respect to the parameters properties are preserved so that it is quite robust. All these reasons make A2' quite efficient to solve the problem we were interested in.

To conclude we may say that augmented Lagrangian methods with splitting are quite useful to solve (numerically) optimal control problems governed by variational inequalities. This not really surprising since it is known that they are quite efficient in the treatment of many nonlinear problems occurring in mechanics as Fortin and Glowinski (Ref.7) and Glowinski and Le Tallec (Ref.12) have already pointed it out.

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List of figures

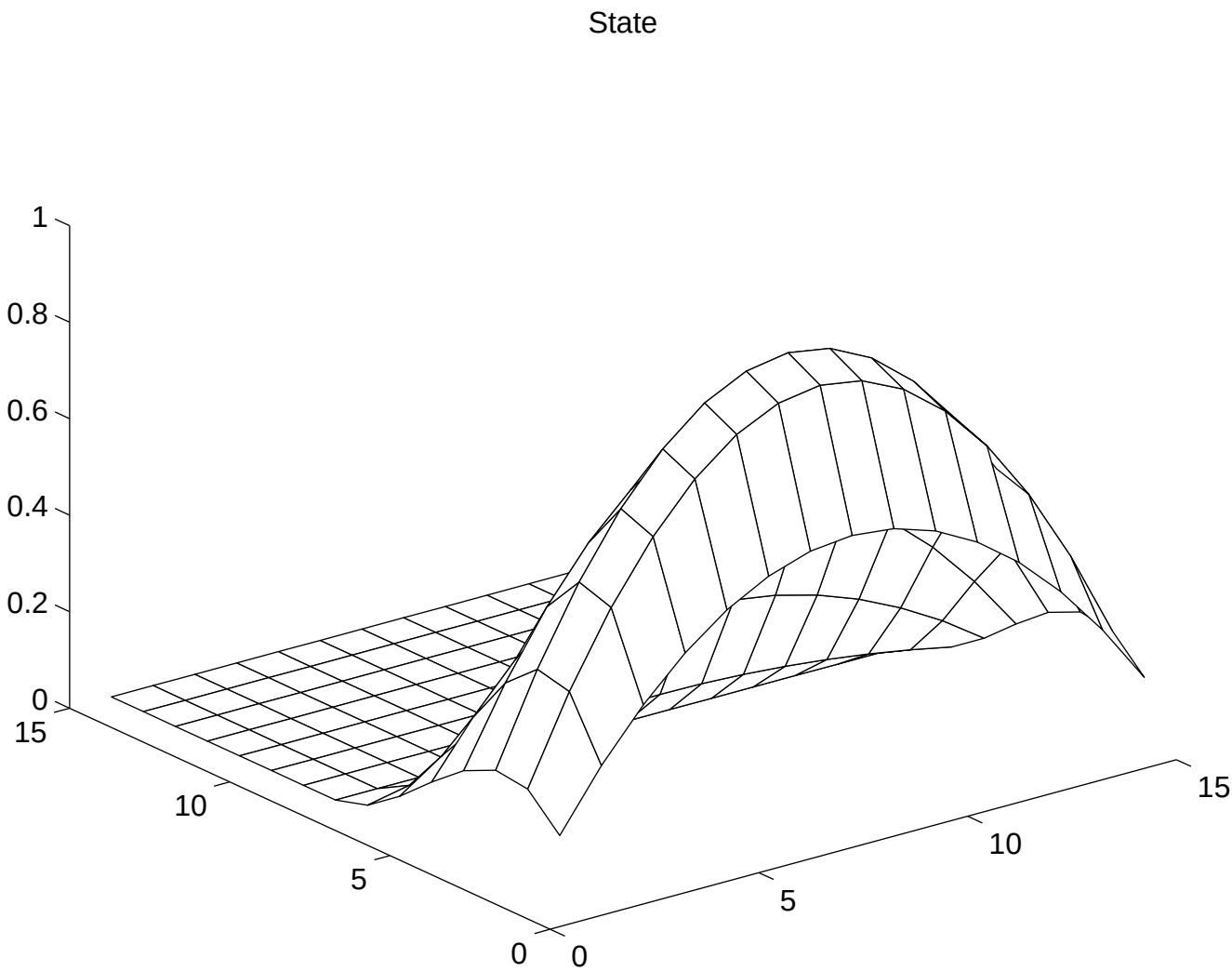

Figure 1. : Computed Optimal State for N=15

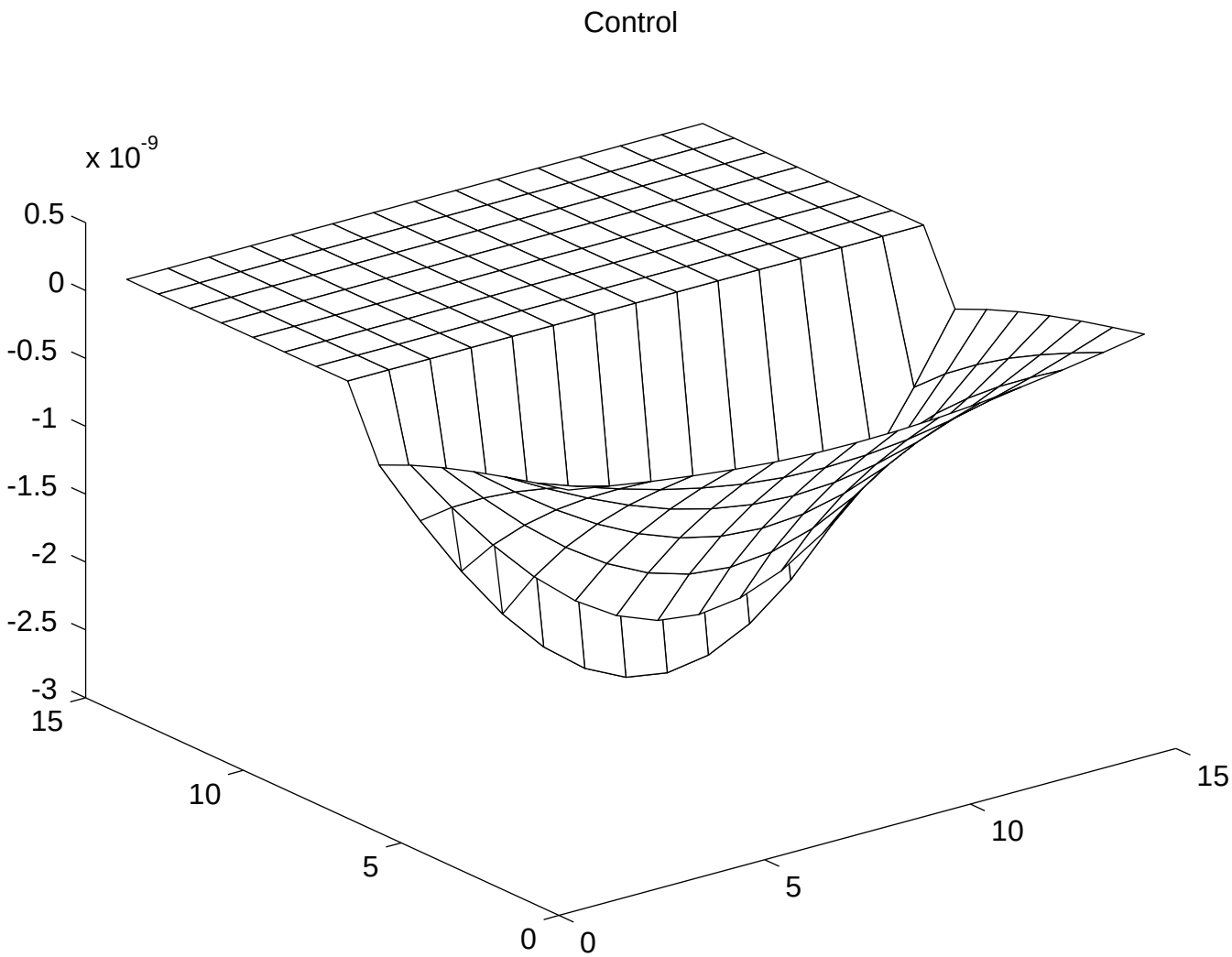

Figure 2. : Computed Optimal Control for N=15

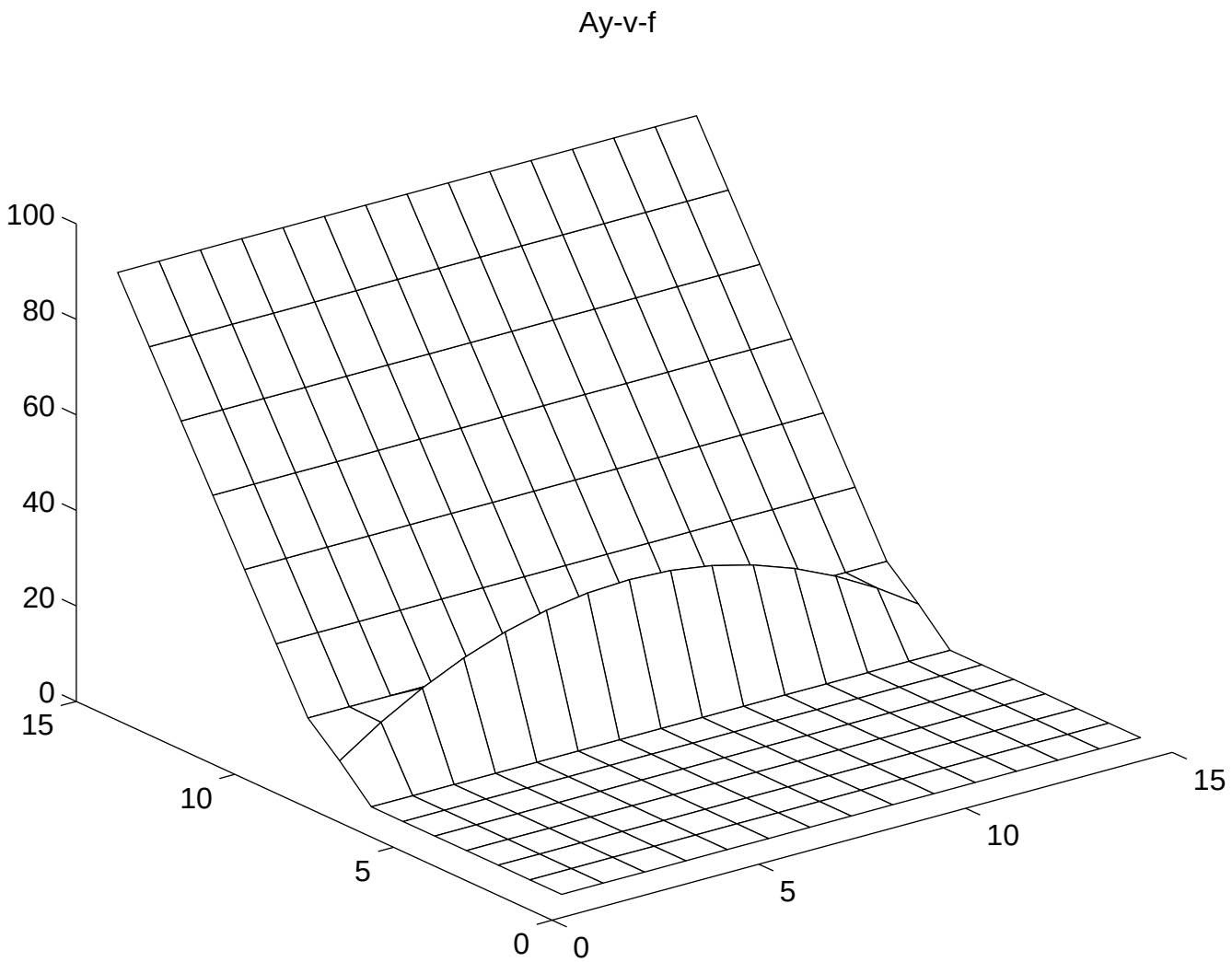

Figure 3. : Computed Solution for N=15 : $-\Delta y - f - v$