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AUGMENTED LAGRANGIAN TECHNIQUES FOR ELLIPTIC STATE CONSTRAINED OPTIMAL CONTROL PROBLEMS

MAÏTINE BERGOUNIOUX¹ AND KARL KUNISCH²

Abstract. We propose augmented Lagrangian methods to solve state and control constrained optimal control problems. The approach is based on the Lagrangian formulation of nonsmooth convex optimization in Hilbert spaces developed in [6]. We investigate a linear optimal control problem with a boundary control function as in [1]. Both the equation and the constraints are augmented. The proposed methods are general and can be adapted to a much wider class of problems.

Key Words : State and Control Constrained Optimal Control Problems, Augmented Lagrangian, Elliptic Equations.

AMS Subject Qualification : 49J20, 49M29

1. Setting of the Problem. Let Ω be an open, bounded subset of \mathbb{R}^n , $n \leq 3$, with a smooth boundary Γ . We consider the following optimal control problem :

$$\min J(y, u) = \frac{1}{2} \int_{\Omega} (y - z_d)^2 dx + \frac{\alpha}{2} \int_{\Gamma} (u - u_d)^2 d\sigma, \quad (\mathcal{P})$$

$$(1.1) \quad Ay = f \text{ in } \Omega, \quad y = u \text{ on } \Gamma,$$

$$(1.2) \quad \Lambda_1 y \in K, \quad u \in U,$$

where

- $f, z_d \in L^2(\Omega)$, $u, u_d \in L^2(\Gamma)$ and either $\alpha > 0$ or U_{ad} is bounded in $L^2(\Gamma)$,
- A is an elliptic operator defined by :

$$(1.3) \quad \left\{ \begin{array}{l} Ay = - \sum_{i,j=1}^n \partial_{x_i} (a_{ij}(x) \partial_{x_j} y) + a_0(x)y \text{ with} \\ a_{ij}, a_0 \in \mathcal{C}^2(\bar{\Omega}) \text{ for } i, j = 1, \dots, n, \inf \{a_0(x) \mid x \in \bar{\Omega}\} > 0 \\ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \delta \sum_{i=1}^n \xi_i^2, \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n, \delta > 0, \end{array} \right.$$

- L is a Hilbert space (with dual L' identified with L) and $\Lambda_1 \in \mathcal{L}(W, L)$, (W is defined just below).
- K and U are nonempty, closed, convex subsets of L and $L^2(\Gamma)$ respectively.

System (1) is well-posed: for every $(u, f) \in L^2(\Gamma) \times L^2(\Omega)$ there exists a unique solution $y = \mathcal{T}(u, f)$ in W , where

$$W = \{ y \in L^2(\Omega) \mid Ay \in L^2(\Omega), y|_{\Gamma} \in L^2(\Gamma) \}.$$

Moreover \mathcal{T} is continuous from $L^2(\Gamma) \times L^2(\Omega)$ to W , when W is endowed with the graph norm :

$$|y|_W^2 = |y|_{\Omega}^2 + |Ay|_{\Omega}^2 + |y|_{\Gamma}^2.$$

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For more details, one may refer to Lions-Magenes [7], Vol.1, Chap. 2. From now on, when H is a Hilbert-space, we denote by $(\cdot, \cdot)_H$ (resp. $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_\Gamma$) the H (resp. $L^2(\Omega)$ and $L^2(\Gamma)$) inner products and by $|\cdot|_H$, $|\cdot|_\Omega$, $|\cdot|_\Gamma$, the H , $L^2(\Omega)$ and $L^2(\Gamma)$ -norms, respectively. We assume that the feasible domain

$$\mathcal{D} = \{ (y, u) \in W \times L^2(\Gamma) \mid Ay = f \text{ in } \Omega, y = u \text{ on } \Gamma, (\Lambda_1 y, u) \in K \times U \},$$

is nonempty. It is easy to see that problem (\mathcal{P}) has a unique solution (\bar{y}, \bar{u}) since the functional J is strictly convex and coercive and \mathcal{D} is convex, closed and nonempty. Our main purpose is to retrieve optimality conditions for such a problem, with a new ‘‘penalization’’ method and to use them as a basis for numerical algorithms. Indeed, this has been done via a penalization of the state-equation only in Bergounioux [1] where the existence of Lagrange multipliers for the state-equation has been proved, under appropriate qualification conditions. Here we use a different point of view, since we also use a penalization of the nonsmooth constraints ‘‘ $\Lambda_1 y \in K, u \in U$ ’’ with an augmented Lagrangian method as in Ito and Kunisch [6]. Optimality systems have been derived by other authors before. We mention, for instance, the work of Bonnans and Casas [3, 4] and the references given in [1]. In contrast to our work these contributions are not based on augmented Lagrangian formulations and they do not analyse algorithmic aspects. Similarly the algorithm we present in Section 4 is based on the augmentation of both the state-equation as well as **the state and control constraints**. The main contribution of this research is the elimination of these latter constraints from the set of explicit constraints by augmentation. Commonly augmented Lagrangian algorithms are based on the augmentation of the state-equation only. This is the case for instance for all the methods described in [5].

2. Augmented Lagrangian Formulation. In this section we use the framework of [6]. We denote

$$X = W \times L^2(\Gamma), \quad H = L \times L^2(\Gamma), \quad D = K \times U.$$

Let Λ be defined from X to H by $\Lambda(y, u) = (\Lambda_1 y, u)$ and let φ be the characteristic function of the convex set D . Then, following [6], we define (for any $c > 0$) the function $\varphi_c : H \times H \rightarrow \mathbb{R}$ by:

$$(2.1) \quad \varphi_c(x, \lambda) = \inf_{\xi \in H} \{ \varphi(x - \xi) + (\lambda, \xi)_H + \frac{c}{2} |\xi|_H^2 \},$$

where $x = (y, u)$.

Here $(\cdot, \cdot)_H$ is given by $(\lambda, \xi)_H = (\lambda_1, \xi_1)_\Omega + (\lambda_2, \xi_2)_\Gamma$, with $\lambda = (\lambda_1, \lambda_2)$ and $\xi = (\xi_1, \xi_2)$.

Let us give some properties of the function φ_c (for the proof we refer to [6]):

PROPOSITION 2.1. *For all $x \in H$, $\lambda \in H$, the infimum in (1) is attained at a unique point $\xi_c(x, \lambda)$. φ_c is convex, Lipschitz-continuously Fréchet-differentiable in x and*

$$(2.2) \quad \varphi'_c(x, \lambda) = \lambda + c \xi_c(x, \lambda).$$

Moreover $\lim_{c \rightarrow +\infty} \varphi_c(x, \lambda) = \varphi(x)$. ■

Now we compute φ_c for our case :

PROPOSITION 2.2. *For all $x = (y, u) \in H$ and $\lambda = (\lambda_1, \lambda_2) \in H$*

$$(2.3) \quad \begin{aligned} \varphi_c(x, \lambda) &= \frac{c}{2} |x - P_D(x + \frac{\lambda}{c})|_H^2 + \left(\lambda, x - P_D(x + \frac{\lambda}{c}) \right)_H \\ &= \frac{c}{2} |y - P_K(y + \frac{\lambda_1}{c})|_L^2 + \left(\lambda_1, y - P_K(y + \frac{\lambda_1}{c}) \right)_L + \\ &\quad \frac{c}{2} |u - P_U(u + \frac{\lambda_2}{c})|_\Gamma^2 + \left(\lambda_2, u - P_U(u + \frac{\lambda_2}{c}) \right)_\Gamma, \end{aligned}$$

$$(2.4) \quad \varphi'_c(x, \lambda) = c \left(y + \frac{\lambda_1}{c} - P_K(y + \frac{\lambda_1}{c}), u + \frac{\lambda_2}{c} - P_U(u + \frac{\lambda_2}{c}) \right),$$

where P_K (resp. P_U, P_D) is the L (resp. $L^2(\Gamma), H$) projection on K (resp. on U, D).

Proof.- Setting $\tilde{\xi} = x - \xi$ in (1) we obtain the equivalent representation

$$\varphi_c(x, \lambda) = \inf_{\tilde{\xi} \in H} \left\{ \varphi(\tilde{\xi}) + \left(\lambda, x - \tilde{\xi} \right)_H + \frac{c}{2} |x - \tilde{\xi}|_H^2 \right\}.$$

As $\left(\lambda, x - \tilde{\xi} \right)_H + \frac{c}{2} |x - \tilde{\xi}|_H^2 = \frac{c}{2} |\tilde{\xi} - (x + \frac{\lambda}{c})|_H^2 - \frac{1}{2c} |\lambda|_H^2$ and $\varphi(\tilde{\xi}) = +\infty$, if $\tilde{\xi} \notin D$, $\varphi(\tilde{\xi}) = 0$ else, it follows that

$$(2.5) \quad \varphi_c(x, \lambda) = \frac{c}{2} \left[\inf_{\tilde{\xi} \in D} |\tilde{\xi} - (x + \frac{\lambda}{c})|_H^2 \right] - \frac{1}{2c} |\lambda|_H^2.$$

The infimum is attained at $\tilde{\xi} = P_D(x + \frac{\lambda}{c})$. We define ξ_c so that $\tilde{\xi} = x - \xi_c$, that is

$$\xi_c = \left[y - P_K(y + \frac{\lambda_1}{c}), u - P_U(u + \frac{\lambda_2}{c}) \right].$$

and

$$\begin{aligned} \varphi_c(x, \lambda) &= \frac{c}{2} |P_D(x + \frac{\lambda}{c}) - (x + \frac{\lambda}{c})|_H^2 - \frac{1}{2c} |\lambda|_H^2, \\ &= \frac{c}{2} |x - P_D(x + \frac{\lambda}{c})|_H^2 + \left(\lambda, x - P_D(x + \frac{\lambda}{c}) \right)_H. \end{aligned}$$

Now we compute φ'_c with formula (2) of Proposition 2.1: $\varphi'_c(x, \lambda) = \lambda + c \xi_c(x, \lambda)$, and the desired result follows. \blacksquare

Next we consider the “augmented” problem :

$$\min \{ F_{c,\lambda}(y, u) \mid Ay = f \text{ in } \Omega, y = u \text{ on } \Gamma \} \quad (\mathcal{P}_{c,\lambda})$$

where $F_{c,\lambda}(y, u) = J(y, u) + \varphi_c(\Lambda(y, u), \lambda)$ is the augmented Lagrangian function of (\mathcal{P}) associated with the constraint $\Lambda(y, u) \in D$. We have first an asymptotic result :

THEOREM 2.1. *For all $\lambda \in H$ and $c > 0$, problem $(\mathcal{P}_{c,\lambda})$ has a unique solution $(y_{c,\lambda}, u_{c,\lambda})$. Moreover for every fixed $\lambda \in H$*

$$\lim_{c \rightarrow +\infty} y_{c,\lambda} = \bar{y} \text{ strongly in } W \text{ and } \lim_{c \rightarrow +\infty} u_{c,\lambda} = \bar{u} \text{ strongly in } L^2(\Gamma).$$

Proof.- Let $\lambda \in H$ be fixed. For convenience, we shall omit the subscript λ and write x_c for $x_{c,\lambda}$. Existence and uniqueness of a solution (y_c, u_c) to $(\mathcal{P}_{c,\lambda})$ follows easily since the feasible domain is nonempty, closed and convex, and $F_{c,\lambda}$ is strictly convex and coercive. We set $x_c = (\Lambda_1 y_c, u_c) \in H$. To prove convergence of (y_c, u_c) to the solution (\bar{y}, \bar{u}) of (\mathcal{P}) we first argue that $\{(y_c, u_c)\}_{c \geq c_0}$ is bounded, where $c_0 > 0$ is arbitrary and fixed. Since (\bar{y}, \bar{u}) is feasible for $(\mathcal{P}_{c,\lambda})$ we have

$$F_{c,\lambda}(y_c, u_c) \leq F_{c,\lambda}(\bar{y}, \bar{u}) \quad \text{for all } c > 0.$$

Observe from the definition of φ_c in (1) that $\varphi_c(\Lambda(\bar{y}, \bar{u}), \lambda) = 0$ for all c . Hence, using (5), we obtain

$$(2.6) \quad J(y_c, u_c) - \frac{1}{2c} |\lambda|_H^2 \leq J(y_c, u_c) + \varphi_c(\Lambda(y_c, u_c), \lambda) \leq J(\bar{y}, \bar{u}) \quad \text{for all } c > 0.$$

It follows that $\{(y_c, u_c)\}_{c \geq c_0}$ is bounded in $L^2(\Omega) \times L^2(\Gamma)$. Since $Ay_c = f$ for all $c > 0$ the set $\{(y_c, u_c)\}_{c \geq c_0}$ is bounded in X as well. Hence there exists $(\tilde{y}, \tilde{u}) \in X$ such that a subsequence of $\{(y_c, u_c)\}_{c > 0}$, denoted by the same symbol, converges weakly in X to (\tilde{y}, \tilde{u}) . Wellposedness of (1) in W implies that $A\tilde{y} = f$ in Ω and $\tilde{y} = \tilde{u}$ on Γ .

Due to Proposition 2.2 and (6)

$$|x_c - P_D(x_c + \frac{\lambda}{c})|_H^2 + \frac{2}{c} \left(\lambda, x_c - P_D(x_c + \frac{\lambda}{c}) \right)_H \leq \frac{1}{c} J(\bar{y}, \bar{u})$$

and consequently

$$(2.7) \quad \left| x_c + \frac{\lambda}{c} - P_D(x_c + \frac{\lambda}{c}) \right|_H^2 \leq \frac{2J(\tilde{y}, \tilde{u})}{c} + \frac{|\lambda|^2}{c^2} \quad \text{for all } c > 0.$$

Thus $\left(x_c + \frac{\lambda}{c} - P_D(x_c + \frac{\lambda}{c}) \right)$ converges strongly to 0 in H . As (y_c, u_c) converges weakly to (\tilde{y}, \tilde{u}) in X and Λ is linear continuous, $\left(x_c + \frac{\lambda}{c} \right)$ converges weakly to $\tilde{x} = (\Lambda_1 \tilde{y}, \tilde{u})$ in H . This yields that $P_D(x_c + \frac{\lambda}{c})$ converges weakly to \tilde{x} as well. Since D is closed in H and convex, it is also weakly closed and $\tilde{x} \in \tilde{D} = K \times U$. Thus (\tilde{y}, \tilde{u}) is a feasible pair for (\mathcal{P}) .

• Let us prove the strong convergence of (y_c, u_c) to (\tilde{y}, \tilde{u}) in X .

First we note that due to Proposition 2.1 $\lim_{c \rightarrow +\infty} \varphi_c(\tilde{x}, \lambda) = \varphi(\tilde{x}) = 0$. As (\tilde{y}, \tilde{u}) is a feasible pair for (\mathcal{P}) it is also a feasible pair for $(\mathcal{P}_{c,\lambda})$ for any $c > 0$ and we have

$$J(y_c, u_c) + \varphi_c(\Lambda(y_c, u_c), \lambda) \leq J(\tilde{y}, \tilde{u}) + \varphi_c(\Lambda(\tilde{y}, \tilde{u}), \lambda) \quad \text{for all } c > 0 \quad .$$

Relation (3) implies that : $\varphi_c(\Lambda(y_c, u_c), \lambda) \geq \left(\lambda, x_c - P_D(x_c + \frac{\lambda}{c}) \right)_H$, and consequently

$$(2.8) \quad J(y_c, u_c) + \left(\lambda, x_c - P_D(x_c + \frac{\lambda}{c}) \right)_H \leq J(\tilde{y}, \tilde{u}) + \varphi_c(\Lambda(\tilde{y}, \tilde{u}), \lambda) \quad \text{for all } c > 0.$$

We take the limes inferior in this relation.

With the strong convergence of $(x_c - P_D(x_c + \frac{\lambda}{c}))$ to 0 in H , we obtain

$$0 \leq J(\tilde{y}, \tilde{u}) \leq \liminf_{c \rightarrow +\infty} J(y_c, u_c) \leq J(\tilde{y}, \tilde{u}) + \lim_{c \rightarrow +\infty} \varphi_c(\Lambda(\tilde{y}, \tilde{u}), \lambda) = J(\tilde{y}, \tilde{u}) \quad .$$

Finally

$$(2.9) \quad \lim_{c \rightarrow +\infty} J(y_c, u_c) = J(\tilde{y}, \tilde{u}) \quad .$$

This implies that (y_c, u_c) converges strongly to (\tilde{y}, \tilde{u}) in $L^2(\Omega) \times L^2(\Gamma)$. Moreover $Ay_c = f = A\tilde{y}$, and therefore (y_c, u_c) converges to (\tilde{y}, \tilde{u}) strongly in X .

• It remains to prove that $(\tilde{y}, \tilde{u}) = (\bar{y}, \bar{u})$. We use relation (8) with (\bar{y}, \bar{u}) as a feasible pair for (\mathcal{P}) instead of (\tilde{y}, \tilde{u}) and obtain

$$J(y_c, u_c) + \left(\lambda, x_c - P_D(x_c + \frac{\lambda}{c}) \right)_H \leq J(\bar{y}, \bar{u}) + \varphi_c(\Lambda(\bar{y}, \bar{u}), \lambda) \quad \text{for all } c > 0 \quad .$$

Taking the limit as c tends to $+\infty$ we have : $0 \leq J(\tilde{y}, \tilde{u}) \leq J(\bar{y}, \bar{u})$ ($\leq J(\tilde{y}, \tilde{u})$) .

As (\bar{y}, \bar{u}) is the unique solution of (\mathcal{P}) we get the result. ■

The following section will be devoted to deriving optimality conditions. We first consider the augmented problem $\mathcal{P}_{c,\lambda}$ and we shall then pass to the limit with respect to c .

3. Optimality Conditions.

3.1. Penalized Optimality Conditions. We first write the necessary optimality conditions for problem $(\mathcal{P}_{c,\lambda})$. This problem can be expressed as :

$$\min \{ F_{c,\lambda}(y, u) \mid e(y, u) = 0 \} ,$$

where e is defined by

$$e : \begin{array}{ll} W \times L^2(\Gamma) & \rightarrow L^2(\Omega) \times L^2(\Gamma) \\ (y, u) & \mapsto (Ay - f, y|_\Gamma - u) . \end{array}$$

As the Fréchet derivative $e'(y_c, u_c)$ of e at (y_c, u_c) given by:

$$\begin{aligned} e'(y_c, u_c) : W \times L^2(\Gamma) &\rightarrow L^2(\Omega) \times L^2(\Gamma) \\ (y, u) &\mapsto (Ay, y|_\Gamma - u), \end{aligned}$$

is surjective, we may apply the general theory of Lagrange multipliers:

There exists $q_c \in L^2(\Omega)$ and $r_c \in L^2(\Gamma)$ such that the (generalized) Lagrange functional:

$$(3.1) \quad \mathcal{L}_{c,\lambda}(y, u, q, r) = J(y, u) + \varphi_c(\Lambda(y, u), \lambda) + (q, Ay - f)_\Omega + (r, y - u)_\Gamma,$$

satisfies the optimality condition

$$(3.2) \quad \nabla_{(y,u)} \mathcal{L}_{c,\lambda}(y_c, u_c, q_c, r_c) = 0.$$

Let us detail the above relation: we may decouple and obtain

$$(y_c - z_d, y)_\Omega + (q_c, Ay)_\Omega + (r_c, y)_\Gamma + (\mu_{1,c}, \Lambda_1 y)_L = 0 \quad \text{for all } y \in W,$$

$$\alpha (u_c - u_d, u)_\Gamma - (r_c, u)_\Gamma + (\mu_{2,c}, u)_\Gamma = 0 \quad \text{for all } u \in L^2(\Gamma),$$

where $\mu_{1,c} = \nabla_y \varphi_c(\Lambda(y_c, u_c), \lambda) = c \left[\Lambda_1 y_c + \frac{\lambda_1}{c} - P_K(\Lambda_1 y_c + \frac{\lambda_1}{c}) \right] \in L$, and

$$\mu_{2,c} = \nabla_u \varphi_c(\Lambda(y_c, u_c), \lambda) = c \left[u_c + \frac{\lambda_2}{c} - P_U(u_c + \frac{\lambda_2}{c}) \right] \in L^2(\Gamma).$$

We summarize these calculations in

THEOREM 3.1. *Let λ be fixed in H and let (y_c, u_c) be the optimal solution of $(\mathcal{P}_{c,\lambda})$. Then there exists $(\mu_{1,c}, \mu_{2,c}) \in H$ and $(q_c, r_c) \in L^2(\Omega) \times L^2(\Gamma)$ such that*

$$Ay_c = f \text{ in } \Omega, \quad y_c = u_c \text{ on } \Gamma,$$

$$(3.3) \quad (y_c - z_d, y)_\Omega + (q_c, Ay)_\Omega + (r_c, y)_\Gamma + (\mu_{1,c}, \Lambda_1 y)_L = 0 \quad \text{for all } y \in W$$

$$(3.4) \quad \alpha (u_c - u_d) - r_c + \mu_{2,c} = 0,$$

where

$$(3.5) \quad \begin{aligned} \mu_{1,c} &= c \left[\Lambda_1 y_c + \frac{\lambda_1}{c} - P_K(\Lambda_1 y_c + \frac{\lambda_1}{c}) \right] \in L, \\ \mu_{2,c} &= c \left[u_c + \frac{\lambda_2}{c} - P_U(u_c + \frac{\lambda_2}{c}) \right] \in L^2(\Gamma). \end{aligned}$$

3.2. Passage to the Limit. The approximate optimality systems of Theorem 3.1 were obtained without assumption beyond those that are required to ascertain existence of a solution to (\mathcal{P}) . To obtain an optimality system for (\mathcal{P}) itself we pass to the limit as c tends to $+\infty$ in (3)-(5). This requires a-priori estimates for q_c and r_c which depend upon qualification conditions. More precisely, let $V = V_1 \times V_2$ be a dense separable Banach subspace of $L^2(\Omega) \times L^2(\Gamma)$. We introduce the following assumption :

There exists a bounded (in $L^2(\Omega) \times L^2(\Gamma)$) subset \mathcal{M} of X such that $\Lambda(\mathcal{M}) \subset K \times U$ and $0 \in \text{Int}_V(\mathcal{V}(\mathcal{M}))$, where Int_V denotes the interior with respect to the V -topology and $\mathcal{V}(y, u) = (Ay - f, y|_\Gamma - u)$. (H)

We note that (H) is equivalent to

There exists an $L^2(\Omega) \times L^2(\Gamma)$ -bounded subset $\mathcal{M} \subset X$ and $\rho > 0$ such that for all $\xi = (\xi_1, \xi_2) \in B_V(0, 1)$, there exists $(y_\xi, u_\xi) \in \mathcal{M}$ satisfying $(\Lambda_1 y_\xi, u_\xi) \in K \times U$ and $Ay_\xi = f - \rho \xi_1$ in Ω , $y_\xi = u_\xi - \rho \xi_2$ on Γ .

Here $B_V(0, 1)$ denotes the unit ball in V . For $f \equiv 0$ and $V = L^2(\Omega) \times L^2(\Gamma)$ condition (\mathcal{H}) is satisfied, for example, if $0 \in \text{int}_{L \times L^2(\Gamma)}(K \times U)$.

Under this hypothesis we can pass to the limit in the previous optimality conditions to obtain the main result of this section:

THEOREM 3.2. *Let (\bar{y}, \bar{u}) be the optimal solution of (\mathcal{P}) and assume that (\mathcal{H}) holds. Then there exists $(\bar{q}, \bar{r}) \in V'_1 \times V'_2$ and $(\bar{\mu}_1, \bar{\mu}_2) \in L'_1 \times V'_2$ such that :*

$$A\bar{y} = f \text{ in } \Omega, \quad \bar{y} = \bar{u} \text{ on } \Gamma,$$

$$(3.6) \quad \begin{aligned} & (\bar{y} - z_d, y)_\Omega + \langle \bar{q}, Ay \rangle_{V'_1, V_1} + \\ & \langle \bar{r}, y \rangle_{V'_2, V_2} + \langle \bar{\mu}_1, \Lambda_1 y \rangle_{L'_1, L_1} = 0 \quad \text{for all } y \in W_{1,2} \end{aligned}$$

$$(3.7) \quad \alpha (\bar{u} - u_d) - \bar{r} + \bar{\mu}_2 = 0 \text{ in } V'_2,$$

$$(3.8) \quad \begin{aligned} & \langle \bar{\mu}_1, \Lambda_1(y - \bar{y}) \rangle_{L'_1, L_1} \leq 0 \quad \text{for all } y \in \{ \bar{y} + W_{1,2} \} \text{ such that } \Lambda_1 y \in K \\ & \langle \bar{\mu}_2, u - \bar{u} \rangle_{V'_2, V_2} \leq 0 \text{ for all } u \in U \cap \{ \bar{u} + V_2 \}, \end{aligned}$$

where $W_{1,2} = \{ y \in L^2(\Omega) \mid Ay \in V_1, y|_\Gamma \in V_2 \}$ endowed with the norm

$$|y|_{W_{1,2}}^2 = |y|_\Omega^2 + |Ay|_{V_1}^2 + |y|_\Gamma|_{V_2}^2,$$

$L_1 = \Lambda_1(W_{1,2})$ endowed with the graph norm and $\langle \cdot, \cdot \rangle_{V', V}$ denotes the duality product between V and V' .

Proof.-Throughout the proof we assume that $(\lambda_1, \lambda_2) \in L_1 \times V_2$. We first remark that (5) implies :

$$(3.9) \quad \begin{aligned} & (\mu_{1,c}, z - \Lambda_1 y_c)_L + \frac{|\mu_{1,c}|_L^2}{c} \leq \frac{1}{c} (\mu_{1,c}, \lambda_1)_L \quad \text{for all } z \in K \\ & (\mu_{2,c}, u - u_c)_\Gamma + \frac{|\mu_{2,c}|^2}{c} \leq \frac{1}{c} (\mu_{2,c}, \lambda_2)_\Gamma \quad \text{for all } u \in U. \end{aligned}$$

We just prove the first inequality (the second one may be proved quite similarly).

The projection $P_K(\Lambda_1 y_c + \frac{\lambda_1}{c})$ is characterized by :

$$\left(z - P_K(\Lambda_1 y_c + \frac{\lambda_1}{c}), \Lambda_1 y_c + \frac{\lambda_1}{c} - P_K(\Lambda_1 y_c + \frac{\lambda_1}{c}) \right)_L \leq 0 \quad \text{for all } z \in K,$$

and with (5) this yields :

$$(\mu_{1,c}, z - \Lambda_1 y_c)_L + \frac{|\mu_{1,c}|^2}{c} \leq \frac{1}{c} (\mu_{1,c}, \lambda_1)_L \quad \text{for all } z \in K.$$

Thus the first inequality in (9) is verified. We next note that (5) may be written as :

$$\frac{\mu_c}{c} = x_c + \frac{\lambda}{c} - P_D(x_c + \frac{\lambda}{c}),$$

where $x_c = \Lambda(y_c, u_c)$. We have seen in the proof of Theorem 2.1 that $x_c - P_D(x_c + \frac{\lambda}{c})$ converges strongly to 0 in H . Therefore $\frac{\mu_c}{c}$ converges strongly to 0 in H as well and there exists $c_o > 0$ and M such that :

$$(3.10) \quad \left(\frac{\mu_{1,c}}{c}, \lambda_1 \right)_L + \left(\frac{\mu_{2,c}}{c}, \lambda_2 \right)_\Gamma \leq M \quad \text{for all } c \geq c_o.$$

Now we may obtain estimates on q_c and r_c . Let ξ be in $B_V(0,1)$ and (y_ξ, u_ξ) be the associated pair given by (\mathcal{H}) . We add relations (3) and (4) used with the pair $(y_\xi - y_c, u_\xi - u_c)$ to obtain:

$$(y_c - z_d, y_\xi - y_c)_\Omega + (q_c, A(y_\xi - y_c))_\Omega + (r_c, y_\xi - y_c)_\Gamma + (\mu_{1,c}, \Lambda_1(y_\xi - y_c))_L \\ + \alpha (u_c - u_d, u_\xi - u_c)_\Gamma + (\mu_{2,c}, u_\xi - u_c)_\Gamma - (r_c, u_\xi - u_c)_\Gamma = 0 ,$$

and consequently

$$(3.11) \quad (q_c, \rho \xi_1)_\Omega + (r_c, \rho \xi_2)_\Gamma = (y_c - z_d, y_\xi - y_c)_\Omega + \alpha (u_c - u_d, u_\xi - u_c)_\Gamma \\ + (\mu_{1,c}, \Lambda_1(y_\xi - y_c))_L + (\mu_{2,c}, u_\xi - u_c)_\Gamma .$$

Furthermore, relations (9) and (10) imply that :

$$(\mu_{1,c}, \Lambda_1(y_\xi - y_c))_L + (\mu_{2,c}, u_\xi - u_c)_\Gamma \leq \left(\frac{\mu_{1,c}}{c}, \lambda_1 \right)_L + \left(\frac{\mu_{2,c}}{c}, \lambda_2 \right)_\Gamma \leq M \text{ for } c \geq c_0 .$$

The convergence properties of Theorem 2.1 and the boundedness assumption on \mathcal{M} in $L^2(\Omega) \times L^2(\Gamma)$ imply that $(y_c - z_d, y_\xi - y_c)_\Omega + \alpha (u_c - u_d, u_\xi - u_c)_\Gamma$ is uniformly bounded with respect to $c \geq c_0$. So we obtain with (11) the existence of $k > 0$ such that

$$\langle q_c, \xi_1 \rangle_{V'_1, V_1} + \langle r_c, \xi_2 \rangle_{V'_2, V_2} \leq \frac{k}{\rho} \text{ for all } \xi = (\xi_1, \xi_2) \in B_V(0,1) \text{ and } c \geq c_0 .$$

Therefore $\{q_c\}_{c \geq c_0}$ is bounded in V'_1 and a subsequence, again denoted by q_c converges weakly * to some \bar{q} in V'_1 . Similarly r_c is bounded in V'_2 and converges weakly * to some \bar{r} in V'_2 . As we have chosen $V_1 \subset L^2(\Omega)$ and $V_2 \subset L^2(\Gamma)$ we may apply to (3) “smooth” test functions in $W_{1,2}$. Let us consider the Gelfand triple

$$(3.12) \quad L_1 \subset \overline{\Lambda_1(W_{1,2})} \subset L'_1 ,$$

where $\overline{\Lambda_1(W_{1,2})}$ is considered as a subset of L and $\overline{\Lambda_1(W_{1,2})}$ denotes the closure of $\Lambda_1(W_{1,2})$ in L . Further let $\mu_{1,c}^P$ denote the projection of $\mu_{1,c}$ in L onto $\overline{\Lambda_1(W_{1,2})}$. It follows that

$$\langle \mu_{1,c}^P, \Lambda_1 y \rangle_{L'_1, L_1} = -(y_c - z_d, y)_\Omega - \langle q_c, Ay \rangle_{V'_1, V_1} - \langle r_c, y \rangle_{V'_2, V_2} \text{ for all } y \in W_{1,2} .$$

It follows that, $\mu_{1,c}^P$ is bounded in L'_1 . Moreover the separability of V_1 and V_2 implies the separability of L_1 (see Lemma 3.1 below). So a subsequence of $\mu_{1,c}^P$ converges weakly * to $\bar{\mu}_1$ in L'_1 . Taking the limit in the above equality gives :

$$(\bar{y} - z_d, y)_\Omega + \langle \bar{\mu}_1, \Lambda_1 y \rangle_{L'_1, L_1} + \langle \bar{q}, Ay \rangle_{V'_1, V_1} + \langle \bar{r}, y \rangle_{V'_2, V_2} = 0 \text{ for all } y \in W_{1,2} .$$

Similarly $\mu_{2,c} = r_c - \alpha(u_c - u_d)$ converges weakly to $\bar{\mu}_2 = \bar{r} - \alpha(\bar{u} - u_d)$ in V'_2 . Thus (6) and (7) are verified.

It remains to show (8). Let $y \in \bar{y} + W_{1,2}$ and $u \in \bar{u} + V_2$ be such that $\Lambda_1 y \in K$ and $u \in U$. Then we add (3) with $y - y_c = (y - \bar{y}) + (\bar{y} - y_c) \in W$, and the relation that results from taking the inner product of (4) with $u - u_c \in L^2(\Gamma)$:

$$(y_c - z_d, y - y_c)_\Omega + \alpha (u_c - u_d, u - u_c)_\Gamma + (q_c, A(y - y_c))_\Omega \\ + (r_c, (y - u) - (y_c - u_c))_\Gamma = -(\mu_{1,c}, \Lambda_1(y - y_c))_L - (\mu_{2,c}, u - u_c)_\Gamma .$$

As $Ay_c = f = A\bar{y}$ in Ω and $y_c = u_c$ on Γ , we get

$$(3.13) \quad (y_c - z_d, y - \bar{y})_\Omega + \alpha (u_c - u_d, u - \bar{u})_\Gamma + (q_c, A(y - \bar{y}))_\Omega \\ + (r_c, y - u)_\Gamma = -(\mu_{1,c}, \Lambda_1(y - y_c))_L - (\mu_{2,c}, u - u_c)_\Gamma \\ - (y_c - z_d, \bar{y} - y_c)_\Omega - \alpha (u_c - u_d, \bar{u} - u_c)_\Gamma .$$

Moreover relation (9) implies

$$(3.14) \quad \begin{aligned} -(\mu_{1,c}, \Lambda_1(y - y_c))_L &\geq -\frac{1}{c} (\mu_{1,c}, \lambda_1)_\Omega \quad \text{for all } y \in W \text{ such that } \Lambda_1 y \in K \\ -(\mu_{2,c}, u - u_c)_\Gamma &\geq -\frac{1}{c} (\mu_{2,c}, \lambda_2)_\Gamma \quad \text{for all } u \in U \quad . \end{aligned}$$

Thus (13) becomes :

$$(3.15) \quad \begin{aligned} (y_c - z_d, y - \bar{y})_\Omega + \alpha (u_c - u_d, u - \bar{u})_\Gamma + \\ (q_c, A(y - \bar{y}))_\Omega + (r_c, y - u)_\Gamma \geq \\ -\frac{1}{c} [(\mu_{1,c}, \lambda_1)_\Omega + (\mu_{2,c}, \lambda_2)_\Gamma] \\ - (y_c - z_d, \bar{y} - y_c)_\Omega - \alpha (u_c - u_d, \bar{u} - u_c)_\Gamma . \end{aligned}$$

Let us denote by σ_c the term on the right-hand side of (15). Since by assumption $(\lambda_1, \lambda_2) \in L_1 \times V_2$ it is easy to see that $\lim_{c \rightarrow +\infty} \sigma_c = 0$.

Next we set successively $u = \bar{u}$ and $y = \bar{y}$. First, we choose $u = \bar{u}$ so that inequality (15) becomes :

$$(y_c - z_d, y - \bar{y})_\Omega + (q_c, A(y - \bar{y}))_\Omega + (r_c, y - \bar{u})_\Gamma \geq \sigma_c ,$$

and consequently $(y_c - z_d, y - \bar{y})_\Omega + \langle q_c, A(y - \bar{y}) \rangle_{V'_1, V_1} + \langle r_c, y - \bar{y} \rangle_{V'_2, V_2} \geq \sigma_c$.

We may now pass to the limit in the previous expression to get :

$$(\bar{y} - z_d, y - \bar{y})_\Omega + \langle \bar{q}, A(y - \bar{y}) \rangle_{V'_1, V_1} + \langle \bar{r}, y - \bar{y} \rangle_{V'_2, V_2} \geq 0 .$$

With (6) we finally have :

$$\langle \bar{\mu}_1, y - \bar{y} \rangle_{W'_{1,2}, W_{1,2}} \leq 0 \quad \text{for all } y \in \{ \bar{y} + W_{1,2} \} \text{ such that } \Lambda_1 y \in K .$$

Now we choose $y = \bar{y}$ and inequality (15) gives :

$$\alpha (u_c - u_d, u - \bar{u})_\Gamma + (r_c, \bar{y} - u)_\Gamma \geq \sigma_c ,$$

$$\alpha (u_c - u_d, u - \bar{u})_\Gamma - \langle r_c, u - \bar{u} \rangle_{V'_2, V_2} \geq \sigma_c .$$

Once again, we may pass to the limit and we obtain :

$$\alpha (\bar{u} - u_d, u - \bar{u})_\Gamma - \langle \bar{r}, u - \bar{u} \rangle_{V'_2, V_2} \geq 0 .$$

Together with (7) this implies the second inequality in (8) and the proof is finished as soon as the following Lemma is proved.

LEMMA 3.1. L_1 is separable.

Proof.-As $L_1 = \Lambda_1(W_{1,2})$ with Λ_1 continuous, it is sufficient to prove that $W_{1,2}$ is separable. Let D_1 (resp. D_2) be dense countable subsets of V_1 (resp. V_2). Then the subset $D = \{ y \in L^2(\Omega) \mid Ay \in D_1, y|_\Gamma \in D_2 \}$ is a countable subset of $W_{1,2}$ (since \mathcal{T} defined in Section 1 is a bijection from $D_1 \times D_2$ onto D). Moreover, the linear operator \mathcal{T} is continuous from $V_1 \times V_2$ to $W_{1,2}$. We may therefore assert that D is dense because of the properties of V_i and the continuity of \mathcal{T} . \blacksquare

REMARK 3.1. Let us still denote by Λ_1 the restriction of Λ_1 to $W_{1,2}$ (i.e. from $W_{1,2}$ to L_1). Then the adjoint operator Λ_1^* is defined from L'_1 to $W'_{1,2}$ and satisfies:

$$\langle \mu, \Lambda_1 y \rangle_{L'_1, L_1} = \langle \Lambda_1^* \mu, y \rangle_{W'_{1,2}, W_{1,2}} \quad \text{for all } (\mu, y) \in L'_1 \times W_{1,2} .$$

Then relation (6) and the first part of relation (8) may be written as

$$(\bar{y} - z_d, y)_\Omega + \langle \bar{q}, Ay \rangle_{V'_1, V_1} + \langle \bar{r}, y \rangle_{V'_2, V_2} + \langle \bar{v}_1, y \rangle_{W'_{1,2}, W_{1,2}} = 0 , \text{ for all } y \in W_{1,2} ,$$

$$\text{and } \langle \bar{v}_1, y - \bar{y} \rangle_{W'_{1,2}, W_{1,2}} \leq 0 \quad \text{for all } y \in \{ \bar{y} + W_{1,2} \} \text{ such that } \Lambda_1 y \in K ,$$

where $\bar{v}_1 = \Lambda_1^* \bar{\mu}_1 \in W'_{1,2}$.

3.3. Example. To illustrate the previous abstract result, we give an example for a particular choice of spaces V and L . Here we set $V_1 = L^2(\Omega)$ and $V_2 = L^2(\Gamma)$ so that $V_1' = L^2(\Omega)$, $V_2' = L^2(\Gamma)$, $W_{1,2} = W$.

The previous theorem gives the following optimality system :

$$(3.16) \quad \begin{aligned} \bar{q} &\in L^2(\Omega) , \quad \bar{r} \in L^2(\Gamma) \quad \text{and} \quad \bar{\mu}_1 \in L_1' , \quad \bar{\mu}_2 \in L^2(\Gamma) , \\ A\bar{y} &= f \quad \text{in } \Omega , \quad \bar{y} = \bar{u} \quad \text{on } \Gamma , \\ (\bar{y} - z_d, y)_\Omega + (\bar{q}, Ay)_\Omega + (\bar{r}, y)_\Gamma + \langle \bar{\mu}_1, \Lambda_1 y \rangle_{L_1', L_1} &= 0 \quad \text{for all } y \in W , \\ \alpha(\bar{u} - u_d) &= \bar{r} - \bar{\mu}_2 \in L^2(\Gamma) \\ \langle \bar{\mu}_1, \Lambda_1(y - \bar{y}) \rangle_{L_1', L_1} &\leq 0 \quad \text{for all } y \in W \text{ such that } \Lambda_1 y \in K , \\ (\bar{\mu}_2, u - \bar{u})_\Gamma &\leq 0 \quad \text{for all } u \in U . \end{aligned}$$

If in addition L is finite dimensional we may identify the spaces L_1 , $\overline{\Lambda_1(W)}$ and L_1' of the Gelfand triple in (12). In this very case the optimality system becomes:

$$(3.17) \quad \bar{q} \in L^2(\Omega) , \quad \bar{r} \in L^2(\Gamma) \quad \text{and} \quad \bar{\mu}_1 \in \Lambda_1(W) , \quad \bar{\mu}_2 \in L^2(\Gamma) ,$$

$$(3.18) \quad A\bar{y} = f \quad \text{in } \Omega , \quad \bar{y} = \bar{u} \quad \text{on } \Gamma ,$$

$$(3.19) \quad (\bar{y} - z_d, y)_\Omega + (\bar{q}, Ay)_\Omega + (\bar{r}, y)_\Gamma + (\bar{\mu}_1, \Lambda_1 y)_L = 0 \quad \text{for all } y \in W ,$$

$$(3.20) \quad \alpha(\bar{u} - u_d) = \bar{r} - \bar{\mu}_2 \in L^2(\Gamma) ,$$

$$(3.21) \quad (\bar{\mu}_1, \Lambda_1(y - \bar{y}))_L \leq 0 \quad \text{for all } y \text{ such that } \Lambda_1 y \in K ,$$

$$(3.22) \quad (\bar{\mu}_2, u - \bar{u})_\Gamma \leq 0 \quad \text{for all } u \in U .$$

As a specific example, L can be chosen as the set of linear finite elements with respect to a triangulation of Ω and $\Lambda_1 : W \rightarrow L$ can be the L^2 -projection. (H^1 -projection is not admitted since the elements of W are not in general H^1 -smooth).

REMARK 3.2. *Let us still consider the case with $V_1 = L^2(\Omega)$, $V_2 = L^2(\Gamma)$, L finite dimensional and assume that $\Lambda_1^T(L) \subset L^2(\Omega)$, where $\Lambda_1^T : L \rightarrow W'$ denotes the transpose of Λ_1 . In this case we may introduce $\bar{p} \in H^2(\Omega) \cap H_o^1(\Omega)$ as the solution of*

$$(3.23) \quad A^* \bar{p} = -(\bar{y} - z_d + \Lambda_1^T \bar{\mu}_1) \text{ in } \Omega , \quad \bar{p} = 0 \quad \text{on } \Gamma ,$$

where A^* is the adjoint operator of A . Then with Green's formula relation (19) becomes :

$$(3.24) \quad (\bar{q} - \bar{p}, Ay)_\Omega + \left(\bar{r} - \frac{\partial \bar{p}}{\partial \nu_{A^*}}, y \right)_\Gamma = 0 \quad \text{for all } y \in W.$$

For all $z \in L^2(\Omega)$ there exists $y \in H^2(\Omega) \cap H_o^1(\Omega) \subset W$ such that $Ay = z$ in Ω . So (24) implies $(\bar{q} - \bar{p}, z)_\Omega = 0$ for all $z \in L^2(\Omega)$, that is $\bar{q} = \bar{p}$. Then (24) gives $\bar{r} = \frac{\partial \bar{p}}{\partial \nu_{A^*}}$. Thus we see that relations (19)-(20) are equivalent to

$$(3.25) \quad \begin{aligned} A^* \bar{p} + \bar{y} - z_d + \Lambda_1^T \bar{\mu}_1 &= 0, \quad \bar{p} \in H^2(\Omega) \cap H_o^1(\Omega) \\ \alpha(\bar{u} - u_d) - \frac{\partial \bar{p}}{\partial \nu_{A^*}} + \bar{\mu}_2 &= 0 . \end{aligned}$$

A specific case in which Λ_1 satisfies the assumption $\Lambda_1^T(L) \subset L^2(\Omega)$ is given if L is a finite demensional subspace of $L^2(\Omega)$ and Λ_1 is the L^2 -orthogonal projection onto L .

4. Lagrangian Algorithms. In this section we turn to the numerical realization of the constrained optimal control problem (\mathcal{P}). We shall combine the techniques from [1] and [6] augmenting the state equations **well as the constraints** characterizing the feasible set D , to obtain well performing algorithms.

First we recall an augmented Lagrangian algorithm based on the penalization of the state-equation (see [5], [1] and the references therein).

Algorithm \mathcal{A}_o

- Step 1. Initialization : Set $n = 0$, and choose $\gamma > 0$, $q_o \in L^2(\Omega)$, $r_o \in L^2(\Gamma)$.
- Step 2. Compute

$$(y_n, u_n) = \text{Arg min } \{ L_\gamma(y, v, q_n, r_n) \mid \Lambda(y, u) \in K \times U \},$$

where

$$L_\gamma(y, u, q, r) = J(y, u) + (q, Ay)_\Omega + (r, y - u)_\Gamma + \frac{\gamma}{2}|Ay - f|_\Omega^2 + \frac{\gamma}{2}|y - u|_\Gamma^2$$

is the augmented Lagrangian with respect to the **state-equation** constraint.

- Step 3. Set
 - $q_{n+1} = q_n + \rho_1 (Ay_n - f)$ where $\rho_1 \in (0, 2\gamma]$,
 - $r_{n+1} = r_n + \rho_2 (y_n|_\Gamma - u_n)$ where $\rho_2 \in (0, 2\gamma]$.

The analysis of this algorithm is rather standard, see [1, 5] and the references there.

THEOREM 4.1. *Let (\bar{y}, \bar{u}) be the solution to (\mathcal{P}) and suppose that (\mathcal{H}) holds with $V = L^2(\Omega) \times L^2(\Gamma)$. Then the iterates of Algorithm \mathcal{A}_o satisfy*

$$(4.1) \quad |y_n - \bar{y}|_\Omega^2 + \alpha|u_n - \bar{u}|_\Gamma^2 + \frac{1}{2\rho_1}|q_{n+1} - \bar{q}|_\Omega^2 + \frac{1}{2\rho_2}|r_{n+1} - \bar{r}|_\Gamma^2 + (\gamma - \frac{\rho_1}{2})|Ay_n - f|_\Omega^2 + (\gamma - \frac{\rho_2}{2})|y_n - u_n|_\Gamma^2 \leq \frac{1}{2\rho_1}|q_n - \bar{q}|_\Omega^2 + \frac{1}{2\rho_2}|r_n - \bar{r}|_\Gamma^2$$

for all $n = 0, 1, 2, \dots$.

With ρ_1 and ρ_2 given as in Step 3, this implies

$$(4.2) \quad \sum_{n=0}^{\infty} |y_n - \bar{y}|_\Omega^2 + \alpha \sum_{n=0}^{\infty} |u_n - \bar{u}|_\Gamma^2 + (\gamma - \frac{\rho_1}{2}) \sum_{n=0}^{\infty} |Ay_n - f|_\Omega^2 + (\gamma - \frac{\rho_2}{2}) \sum_{n=0}^{\infty} |y_n - u_n|_\Gamma^2 \leq \frac{1}{2\rho_1}|q_0 - \bar{q}|_\Omega^2 + \frac{1}{2\rho_2}|r_0 - \bar{r}|_\Gamma^2,$$

and in particular strong convergence of $(y_n, u_n) \rightarrow (\bar{y}, \bar{u})$ in $L^2(\Omega) \times L^2(\Gamma)$, and boundedness of $\{(q_n, r_n)\}$. If moreover $\rho_1 < 2\gamma$ and $\rho_2 < 2\gamma$ then $(y_n, u_n) \rightarrow (\bar{y}, \bar{u})$ in X , and every weak limit (\tilde{q}, \tilde{r}) of (q_n, r_n) has the property that $(\bar{y}, \bar{u}, \tilde{q}, \tilde{r})$ satisfies, for all $\Lambda(y, u) \in K \times U$

$$\left(J'(\bar{y}, \bar{u}), (y, u) - (\bar{y}, \bar{u}) \right)_{\Omega \times \Gamma} + \left(\tilde{q}, A(y - \bar{y}) \right)_\Omega + \left(\tilde{r}, y - \bar{y} - (u - \bar{u}) \right)_\Gamma \geq 0.$$

Proof.-We refer to [2] ■

The main remaining problem is the resolution of the auxilliary problem of Step 2 in Algorithms \mathcal{A}_o which can be written as :

$$(y_n, u_n) = \text{Arg min } \{ L_\gamma(y, u) \mid \Lambda(y, u) \in D \}.$$

To simplify the notation we omit to indicate the dependence of L_γ on q and r . During Step 2 these functions are fixed. We are going to use the following algorithm and a splitting variant to solve the auxiliary problem :

Algorithm \mathcal{A}_1

- Step 1. Initialization : Choose $\lambda^o \in H$ and $c > 0$.
- Step 2. Compute

$$(y^j, u^j) = \text{Arg min } \{ L_\gamma(y, u) + \varphi_c(\Lambda(y, u), \lambda^j) \mid \Lambda(y, u) \in X \},$$

where φ_c has been defined in the previous section.

- Step 3. Set (see 4)

$$\begin{aligned}\lambda^{j+1} &= \varphi'_c(\Lambda(y^j, u^j), \lambda^j) \\ &= c\left(\Lambda_1 y^j + \frac{\lambda^j}{c} - P_K(\Lambda_1 y^j + \frac{\lambda^j}{c}), u^j + \frac{\lambda^j}{c} - P_U(u^j + \frac{\lambda^j}{c})\right).\end{aligned}$$

The convergence of this algorithm under the assumption that L is finite dimensional follows from result in [6]. The assumption on finite dimensionality of L entails that the duality pairing between L_1 and L'_1 in (16) can be replaced by the inner product in L , see (19), which is necessary for the convergence proof. We now write the version where Algorithm \mathcal{A}_1 appears as an inner loop in algorithm \mathcal{A}_o :

Algorithm \mathcal{A}

- Step 1. Initialization : Set $n = 0$, and choose $\gamma > 0$, $c > 0$.
Choose $(q_o, r_o) \in L^2(\Omega) \times L^2(\Gamma)$ and $\lambda_o = (\lambda_{o1}, \lambda_{o2}) \in L \times L^2(\Gamma)$.
- Step 2. Choose $k_n \in \mathbb{N}$, set $\lambda_n^o = \lambda_n$, and for $j = 0, \dots, k_n$:

$$\begin{aligned}(y_n^j, u_n^j) &= \text{Arg min} \{ L_\gamma(y, u, q_n, r_n) + \varphi_c(\Lambda(y, u), \lambda_n^j) \mid (y, u) \in W \times L^2(\Gamma), \\ \lambda_n^{j+1} &= (\lambda_{n,1}^{j+1}, \lambda_{n,2}^{j+1}) \quad \text{with} \quad \begin{cases} \lambda_{n,1}^{j+1} = c[\Lambda_1 y_n^j + \frac{\lambda_{n,1}^j}{c} - P_K(\Lambda_1 y_n^j + \frac{\lambda_{n,1}^j}{c})], \\ \lambda_{n,2}^{j+1} = c[u_n^j + \frac{\lambda_{n,2}^j}{c} - P_U(u_n^j + \frac{\lambda_{n,2}^j}{c})]. \end{cases}\end{aligned}$$

End of the inner loop : $\lambda_{n+1} = \lambda_n^{k_n+1}$, $y_n = y_n^{k_n}$, $u_n = u_n^{k_n}$.

- Step 3. $q_{n+1} = q_n + \frac{\rho_1}{k_n + 1} \left(\sum_{j=0}^{k_n} A y_n^j - f \right)$ where $\rho_1 \in (0, 2\gamma]$,
- $$r_{n+1} = r_n + \frac{\rho_2}{k_n + 1} \left(\sum_{j=0}^{k_n} (y_n^j|_\Gamma - u_n^j) \right) \quad \rho_2 \in (0, 2\gamma].$$

THEOREM 4.2. *Let (\bar{y}, \bar{u}) be the solution to (\mathcal{P}) and suppose that (\mathcal{H}) holds with $V = L^2(\Omega) \times L^2(\Gamma)$ and that L is finite dimensional. Let $(\bar{q}, \bar{r}, \bar{\mu}) \in L^2(\Omega) \times L^2(\Gamma) \times L \times L^2(\Gamma)$ be an associated Lagrange multiplier. Then the iterates of Algorithm \mathcal{A} satisfy*

$$\begin{aligned}(4.3) \quad & |y_n - \bar{y}|_\Omega^2 + \alpha |u_n - \bar{u}|_\Gamma^2 + \frac{k_n + 1}{2\rho_1} |q_{n+1} - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_{n+1} - \bar{r}|_\Gamma^2 \\ & + (\gamma - \frac{\rho_1}{2}) |A y_n - f|_\Omega^2 + (\gamma - \frac{\rho_2}{2}) |u_n - y_n|_\Gamma^2 + \frac{1}{2c} |\lambda_{n+1} - \bar{\mu}|_{L \times L^2(\Gamma)}^2 \\ & \leq \frac{k_n + 1}{2\rho_1} |q_n - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_n - \bar{r}|_\Gamma^2 + \frac{1}{2c} |\lambda_n - \bar{\mu}|_{L \times L^2(\Gamma)}^2\end{aligned}$$

for all $n = 0, 1, 2, \dots$. If k_n is nonincreasing this implies

$$\begin{aligned}(4.4) \quad & \sum_{n=0}^{\infty} |y_n - \bar{y}|_\Omega^2 + \alpha \sum_{n=0}^{\infty} |u_n - \bar{u}|_\Gamma^2 + \\ & (\gamma - \frac{\rho_1}{2}) \sum_{n=0}^{\infty} |A y_n - f|_\Omega^2 + (\gamma - \frac{\rho_2}{2}) \sum_{n=0}^{\infty} |u_n - y_n|_\Gamma^2 \\ & \leq \frac{k_0 + 1}{2\rho_1} |q_0 - \bar{q}|_\Omega^2 + \frac{k_0 + 1}{2\rho_2} |r_0 - \bar{r}|_\Gamma^2 + \frac{1}{2c} |\lambda_0 - \bar{\mu}|_{L \times L^2(\Gamma)}^2\end{aligned}$$

and in particular strong convergence of $(y_n, u_n) \rightarrow (\bar{y}, \bar{u})$ in $L^2(\Omega) \times L^2(\Gamma)$, and boundedness of $\{(q_n, r_n, \lambda_n)\}$. If moreover $\rho_1 < 2\gamma$ and $\rho_2 < 2\gamma$ then $(y_n, u_n) \rightarrow (\bar{y}, \bar{u})$ in X and every weak limit $(\tilde{q}, \tilde{r}, \tilde{\lambda})$ of $\{(q_n, r_n, \lambda_n)\}$ has the property that $(\bar{y}, \bar{u}, \tilde{q}, \tilde{r}, \tilde{\lambda})$ satisfies (19), (20).

Proof.-See [2]. ■

REMARK 4.1. *The resolution of the unconstrained minimization problem occurring in algorithm \mathcal{A} is equivalent to the resolution of*

$$\nabla_{(y,u)} L_\gamma(y_n, u_n, q_n, r_n) + \varphi'_c(\Lambda(y_n, u_n), \lambda_n^j) = 0 ,$$

that is :

$$(4.5) \quad \begin{aligned} \nabla_y L_\gamma(y_n, u_n, q_n, r_n) + c \left[\Lambda_1 y_n + \frac{\lambda_{n,1}^j}{c} - P_K \left(\Lambda_1 y_n + \frac{\lambda_{n,1}^j}{c} \right) \right] &= 0 , \\ \nabla_u L_\gamma(y_n, u_n, q_n, r_n) + c \left[u_n + \frac{\lambda_{n,2}^j}{c} - P_U \left(u_n + \frac{\lambda_{n,2}^j}{c} \right) \right] &= 0 . \end{aligned}$$

This can be done with a Newton or a descent method for instance.

Our final goal is the analysis of Gauss-Seidel splitting techniques to solve the auxiliary problems. The splitting avoids the minimization of the auxiliary problem with respect to y and u simultaneously. The resulting algorithm is :

Algorithm \mathcal{A}_o^{GS}

- Step 1. Initialization : Set $n = 0$, choose $\gamma > 0$, $q_o \in L^2(\Omega)$, $r_o \in L^2(\Gamma)$, $u_{-1} \in U$.
- Step 2.

$$\begin{aligned} y_n &= \text{Arg min} \{ L_\gamma(y, u_{n-1}, q_n, r_n) \mid \Lambda_1 y \in K \} \\ u_n &= \text{Arg min} \{ L_\gamma(y_n, u, q_n, r_n) \mid u \in U \} . \end{aligned}$$

- Step 3.

$$\begin{aligned} q_{n+1} &= q_n + \rho_1 (A y_n - f) \quad \text{where } \rho_1 \in (0, 2\gamma] , \\ r_{n+1} &= r_n + \rho_2 (y_n|_\Gamma - u_n) \quad \text{where } \rho_2 \in (0, 2\gamma] . \end{aligned}$$

Once again, we may use algorithm \mathcal{A}_1 to solve the first sub-problem of Step 2. The second one is easily solved directly, see Remark 4.2 below. For convenience we shall henceforth delete the index 1 in the notation of the state component of the multiplier.

Algorithm \mathcal{A}^{GS}

- Step 1. Initialization : Set $n = 0$ and choose $\gamma > 0$, $c > 0$.
Choose $(q_o, r_o) \in L \times L^2(\Gamma)$, $\lambda_o \in L^2(\Omega)$ and $u_{-1} \in L^2(\Gamma)$.
- Step 2. Choose $k_n \in \mathbb{N}$, set $\lambda_n^o = \lambda_n$, $u_n^{-1} = u_{n-1}$ and for $j = 0, \dots, k_n$

$$\begin{aligned} y_n^j &= \text{Arg min} \{ L_\gamma(y, u_n^{j-1}, q_n, r_n) + \varphi_c(\Lambda(y, u_n^{j-1}), (\lambda_n^j, 0)) \mid y \in W \} \\ \lambda_n^{j+1} &= c \left[\Lambda_1 y_n^j + \frac{\lambda_n^j}{c} - P_K \left(\Lambda_1 y_n^j + \frac{\lambda_n^j}{c} \right) \right] , \\ u_n^j &= \text{Arg min} \{ L_\gamma(y_n^j, u, q_n, r_n) \mid u \in U \} . \end{aligned}$$

End of the inner loop : $\lambda_{n+1} = \lambda_n^{k_n+1}$, $y_n = y_n^{k_n}$, $u_n = u_n^{k_n}$.

- Step 3.

$$\begin{aligned} q_{n+1} &= q_n + \frac{\rho_1}{k_n + 1} \sum_{j=0}^{k_n} (A y_n^j - f), \quad \text{where } \rho_1 \in (0, 2\gamma] , \\ r_{n+1} &= r_n + \frac{\rho_2}{k_n + 1} \sum_{j=0}^{k_n} (y_n^j|_\Gamma - u_n^j), \quad \text{where } \rho_2 \in (0, \gamma] . \end{aligned}$$

REMARK 4.2. *We may solve the first unconstrained minimization problem occurring in the previous algorithm as it has been mentioned in Remark 4.1. The second minimization problem is indeed equivalent to*

$$u_n^j = \text{Arg min} \left\{ \left| u - \frac{\alpha u_d + r_n + \gamma y_n^j}{\alpha + \gamma} \right|_\Gamma : u \in U \right\} ,$$

that is u_n^j is the $L^2(\Gamma)$ -projection of $\frac{\alpha u_d + r_n + \gamma y_n^j}{\alpha + \gamma}$ on U .

We end this section with a convergence analysis for Algorithm \mathcal{A}^{GS} .

THEOREM 4.3. *Let (\bar{y}, \bar{u}) be the solution to (\mathcal{P}) and suppose that (\mathcal{H}) holds with $V = L^2(\Omega) \times L^2(\Gamma)$ and that L is finite dimensional. Let $(\bar{q}, \bar{r}, \bar{\mu}) \in L^2(\Omega) \times L^2(\Gamma) \times L \times L^2(\Gamma)$ be a Lagrange multiplier associated to the state-equation and the state constraint .*

Then the iterates (y_n, u_n, q_n, r_n) of Algorithm \mathcal{A}^{GS} satisfy

$$(4.6) \quad \begin{aligned} & |y_n - \bar{y}|_\Omega^2 + (\alpha + \frac{\gamma}{2})|u_n - \bar{u}|_\Gamma^2 + \frac{k_n + 1}{2\rho_1}|q_{n+1} - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2}|r_{n+1} - \bar{r}|_\Gamma^2 \\ & + (\gamma - \frac{\rho_1}{2})|Ay_n - f|_\Omega^2 + \frac{\gamma - \rho_2}{2}|u_n - y_n|_\Gamma^2 + \frac{1}{2c}|\lambda_{n+1} - \bar{\mu}|_L^2 \\ & \leq \frac{k_n + 1}{2\rho_1}|q_n - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2}|r_n - \bar{r}|_\Gamma^2 \\ & + \frac{1}{2c}|\lambda_n - \bar{\mu}|_L^2 + \frac{\gamma - \rho_2}{2}|u_{n-1} - y_{n-1}|_\Gamma^2 + \frac{\gamma}{2}|u_{n-1} - \bar{u}|_\Gamma^2 \end{aligned}$$

for all $n = 1, 2, \dots$. If k_n is nonincreasing this implies

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(|y_n - \bar{y}|_\Omega^2 + \alpha|u_n - \bar{u}|_\Gamma^2 + (\gamma - \frac{\rho_1}{2})|Ay_n - f|_\Omega^2 + \frac{\gamma}{2}|u_n - y_n|_\Gamma^2 \right) \leq \\ & \frac{k_1 + 1}{2\rho_1}|q_1 - \bar{q}|_\Omega^2 + \frac{k_1 + 1}{2\rho_2}|r_1 - \bar{r}|_\Gamma^2 + \frac{1}{2c}|\lambda_1 - \bar{\mu}|_L^2 + \frac{\gamma - \rho_2}{2}|y_o - u_o|_\Gamma^2 + \frac{\gamma}{2}|u_o - \bar{u}|_\Gamma^2 . \end{aligned}$$

Proof.-We use the optimality conditions issued from Step 2 of Algorithm \mathcal{A}^{GS} .

The iterates (y_n^j, u_n^j) of Step 2 satisfy, for $j = 0, \dots, k_n$, for all $y \in W$

$$(4.7) \quad \begin{aligned} & \left(J'_y(y_n^j, u_n^{j-1}), y \right)_\Omega + \left(q_n + \frac{\rho_1}{k_n + 1}(Ay_n^j - f), Ay \right)_\Omega + \\ & \left(\gamma - \frac{\rho_1}{k_n + 1} \right) (Ay_n^j - f, Ay)_\Omega + \left(r_n + \frac{\rho_2}{k_n + 1}(y_n^j - u_n^{j-1}), y \right)_\Gamma + \\ & \left(\gamma - \frac{\rho_2}{k_n + 1} \right) (y_n^j - u_n^{j-1}, y)_\Gamma + \left(\varphi'_{c,1}(\Lambda_1 y_n^j, \lambda_n^j), \Lambda_1 y \right)_L = 0, \end{aligned}$$

and for all $u \in U_{ad}$

$$(4.8) \quad \begin{aligned} & \left(J'_u(y_n^j, u_n^j), u - u_n^j \right)_\Gamma - \left(r_n + \frac{\rho_2}{k_n + 1}(y_n^j - u_n^j), u - u_n^j \right)_\Gamma \\ & - \left(\gamma - \frac{\rho_2}{k_n + 1} \right) (y_n^j - u_n^j, u - u_n^j)_\Gamma \geq 0 . \end{aligned}$$

From (19) and (20) it follows that

$$(4.9) \quad \begin{aligned} & \left(J'(\bar{y}, \bar{u}), (y, u - \bar{u}) \right)_{\Omega \times \Gamma} + \\ & \left(\bar{q}, Ay \right)_\Omega + \left(\bar{r}, y - (u - \bar{u}) \right)_\Gamma + \left(\bar{\mu}, \Lambda_1 y \right)_{L \times L^2(\Gamma)} \geq 0 \end{aligned}$$

for all $(y, u) \in W \times U_{ad}$. From [6] it is known that

$$(4.10) \quad \begin{aligned} & \left(\varphi'_c(\Lambda(y_n^j, u_n^j), \lambda_n^j) - \varphi'_c(\Lambda(\bar{y}, \bar{u}), \bar{\mu}), \Lambda(y_n^j, u_n^j) - \Lambda(\bar{y}, \bar{u}) \right) \\ & \geq \frac{1}{2c}|\lambda_n^{j+1} - \bar{\mu}|^2 - \frac{1}{2c}|\lambda_n^j - \bar{\mu}|^2 , \end{aligned}$$

for $j = 0, 1, \dots, k_n$. Combining (7)-(9) and (10) and setting

$$q_n^j = q_n + \frac{\rho_1}{k_n + 1} \sum_{i=0}^j (Ay_n^i - f) \text{ and } r_n^j = r_n + \frac{\rho_2}{k_n + 1} \sum_{i=0}^j (y_n^i - u_n^i) ,$$

for $j = 0 \cdots k_n$ and $q_n^{-1} := q_n$, $r_n^{-1} := r_n$ imply

$$\begin{aligned}
& |y_n^j - \bar{y}|_\Omega^2 + \alpha |u_n^j - \bar{u}|_\Gamma^2 + \frac{k_n + 1}{2\rho_1} |q_n^j - \bar{q}|_\Omega^2 - \frac{k_n + 1}{2\rho_1} |q_n^{j-1} - \bar{q}|_\Omega^2 + \\
& (\gamma - \frac{\rho_1}{2(k_n + 1)}) |Ay_n^j - f|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_n^j - \bar{r}|_\Gamma^2 - \frac{k_n + 1}{2\rho_2} |r_n^{j-1} - \bar{r}|_\Gamma^2 - \\
& \frac{\rho_2}{k_n + 1} \sum_{i=0}^{j-1} (y_n^i - u_n^i, y_n^j - u_n^j)_\Gamma - \frac{\rho_1}{k_n + 1} \sum_{i=0}^{j-1} (Ay_n^i - f, Ay_n^j - f)_\Omega - \\
& (\gamma - \frac{\rho_2}{2(k_n + 1)}) |y_n^j - u_n^j|_\Gamma^2 + \frac{1}{2c} (|\lambda_n^{j+1} - \bar{\mu}|_L^2 - |\lambda_n^j - \bar{\mu}|_L^2) + \\
& \gamma (u_n^j - u_n^{j-1}, y_n^j - \bar{y})_\Gamma \leq 0,
\end{aligned}$$

for $n, j = 0, 1, \dots, k_n$. Summing the above inequality over j and using the fact that

$$\sum_{j=1}^{k_n} \sum_{i=0}^{j-1} (a_i, a_j)_H \leq \frac{k_n}{2} \sum_{j=0}^{k_n} |a_j|^2$$

we arrive at

$$\begin{aligned}
& \sum_{j=0}^{k_n} (|y_n^j - \bar{y}|_\Omega^2 + \alpha |u_n^j - \bar{u}|_\Gamma^2) + \frac{k_n + 1}{2\rho_1} |q_n^{k_n} - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_n^{k_n} - \bar{r}|_\Gamma^2 + \\
& \frac{1}{2c} |\lambda_{n+1} - \bar{\mu}|_L^2 + (\gamma - \frac{\rho_1}{2}) \sum_{j=0}^{k_n} |Ay_n^j - f|_\Omega^2 + (\gamma - \frac{\rho_2}{2}) \sum_{j=0}^{k_n} |y_n^j - u_n^j|_\Gamma^2 + \\
& \gamma \sum_{j=0}^{k_n} (u_n^j - u_n^{j-1}, y_n^j - \bar{y})_\Gamma \leq \frac{k_n + 1}{2\rho_1} |q_n - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_n - \bar{r}|_\Gamma^2 + \frac{1}{2c} |\lambda_n^0 - \bar{\mu}|_L^2.
\end{aligned}$$

The estimation of $(u_n^j - u_n^{j-1}, y_n^j - \bar{y})_\Gamma$ is standard (see [5]) : we obtain, for $j = 1, 2, \dots$ and $n = 0, 1, \dots$

$$\gamma (u_n^{j-1} - u_n^j, \bar{y} - y_n^j)_\Gamma \geq \alpha |u_n^j - u_n^{j-1}|_\Gamma^2 - \frac{\gamma}{2} (|y_n^{j-1} - u_n^{j-1}|_\Gamma^2 + |u_n^{j-1} - \bar{u}|_\Gamma^2 - |u_n^j - \bar{u}|_\Gamma^2).$$

A similar calculus provides the estimation of $(u_n^o - u_n^{-1}, y_n^o - \bar{y})_\Gamma$ for $n = 1, 2, \dots$

$$(4.11) \quad \gamma (u_n^{-1} - u_n^o, \bar{y} - y_n^o)_\Gamma \geq (\alpha + \frac{\rho_2}{2}) |u_n^o - u_n^{-1}|_\Gamma^2 + \frac{\rho_2 - \gamma}{2} |y_{n-1} - u_n^{-1}|_\Gamma^2 + \frac{\gamma}{2} |u_n^o - \bar{u}|_\Gamma^2 - \frac{\gamma}{2} |u_n^{-1} - \bar{u}|_\Gamma^2.$$

We henceforth assume $n \geq 1$. We obtain

$$\begin{aligned}
(4.12) \quad & \gamma \sum_{j=0}^{k_n} (u_n^j - u_n^{j-1}, y_n^j - \bar{y})_\Gamma \geq (\alpha + \frac{\rho_2}{2}) |u_n^o - u_{n-1}|_\Gamma^2 + \frac{\rho_2 - \gamma}{2} |y_{n-1} - u_{n-1}|_\Gamma^2 \\
& + \frac{\gamma}{2} |u_n^o - \bar{u}|_\Gamma^2 - \frac{\gamma}{2} |u_{n-1} - \bar{u}|_\Gamma^2 + \alpha \sum_{j=1}^{k_n} |u_n^j - u_n^{j-1}|_\Gamma^2 \\
& - \frac{\gamma}{2} \sum_{j=1}^{k_n} (|y_n^{j-1} - u_n^{j-1}|_\Gamma^2 + |u_n^{j-1} - \bar{u}|_\Gamma^2 - |u_n^j - \bar{u}|_\Gamma^2).
\end{aligned}$$

We finally get for $k_n \geq 1$

$$\begin{aligned} & \sum_{j=0}^{k_n} (|y_n^j - \bar{y}|_\Omega^2 + \alpha |u_n^j - \bar{u}|_\Gamma^2) + \frac{k_n + 1}{2\rho_1} |q_{n+1} - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_{n+1} - \bar{r}|_\Gamma^2 + \\ & \frac{1}{2c} |\lambda_{n+1} - \bar{\mu}|_L^2 + (\gamma - \frac{\rho_1}{2}) \sum_{j=0}^{k_n} |Ay_n^j - f|_\Omega^2 + \frac{\gamma - \rho_2}{2} \sum_{j=0}^{k_n} |y_n^j - u_n^j|_\Gamma^2 + \\ & \frac{\gamma}{2} |u_n - \bar{u}|_\Gamma^2 \leq \frac{k_n + 1}{2\rho_1} |q_n - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_n - \bar{r}|_\Gamma^2 + \\ & \frac{1}{2c} |\lambda_n - \bar{\mu}|_L^2 + \frac{\gamma - \rho_2}{2} |y_{n-1} - u_{n-1}|_\Omega^2 + \frac{\gamma}{2} |u_{n-1} - \bar{u}|_\Gamma^2. \end{aligned}$$

Since $\rho_2 \leq \gamma$ we deduce that

$$\begin{aligned} & |y_n - \bar{y}|_\Omega^2 + \alpha |u_n - \bar{u}|_\Gamma^2 + \frac{k_n + 1}{2\rho_1} |q_{n+1} - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_{n+1} - \bar{r}|_\Gamma^2 \\ & + (\gamma - \frac{\rho_1}{2}) |Ay_n - f|_\Omega^2 + (\gamma - \frac{\rho_2}{2}) |y_n - u_n|_\Gamma^2 + \frac{1}{2c} |\lambda_{n+1} - \bar{\mu}|_\Omega^2 + \frac{\gamma}{2} |u_n - \bar{u}|_\Gamma^2 \\ & \leq \frac{k_n + 1}{2\rho_1} |q_n - \bar{q}|_\Omega^2 + \frac{k_n + 1}{2\rho_2} |r_n - \bar{r}|_\Gamma^2 + \frac{1}{2c} |\lambda_n - \bar{\mu}|_\Omega^2 \\ & \quad + \frac{\gamma - \rho_2}{2} |y_{n-1} - u_{n-1}|_\Gamma^2 + \frac{\gamma}{2} |u_{n-1} - \bar{u}|_\Gamma^2 \end{aligned}$$

if $k_n \geq 1$. Using (11) the same estimate follows for $k_n = 0$. The final claim again follows with a telescoping argument. \blacksquare

5. Numerical Experiments.

5.1. Implementation. Numerical experiments were carried out for one and two dimensional problems. Since Algorithm \mathcal{A}^{GS} is the simplest for implementation we have used it for our tests. The discretization of the problem was done with finite-differences discretization schemes. The size of the grid was $\frac{1}{N}$ so that $L = \mathbb{R}^{N+1}$ for the 1D-case and $L = \mathbb{R}^{2(N+1)}$ for the 2D-case. Λ was chosen as the discretization operator with respect to the given equidistant grid.

The main difficulty that remains in applying Algorithm \mathcal{A}^{GS} is given by the (unconstrained) minimization with respect to y . This was done via the adjoint state equation and results, for fixed u , q and r in the resolution of

$$(5.1) \quad \begin{aligned} A^*p &= y - z_d + c \left[y + \frac{\lambda}{c} - P_K \left(y + \frac{\lambda}{c} \right) \right] \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \Gamma, \\ Ay &= f - \frac{q+p}{\gamma} \quad \text{in } \Omega, \quad y = u - \frac{r}{\gamma} + \frac{1}{\gamma} \frac{\partial p}{\partial \nu_{A^*}} \quad \text{on } \Gamma, \end{aligned}$$

for p and y . Here $\frac{\partial p}{\partial \nu_{A^*}}$ denotes the conormal derivative of p with respect of A^* (which is the adjoint operator of A). The coupled system (1) was solved via a descent algorithm combined with a relaxation method. The control function was computed using the L^∞ -projection of $\frac{r + \alpha u_d + \gamma y}{\alpha + \gamma}$ on U_{ad} .

All numerical tests were carried out on an HP workstation using the MATLAB[®] package. For all examples that we report here the required accuracy and stopping criteria were set to 10^{-6} .

5.2. Examples.

1D-Example.- In this example we chose

- $\Omega =]0, 1[$ and $N = 30$; $A = -\Delta$ and $f(x) = -(x+2)\exp(x)$.
- $z_d \equiv -1$, $\alpha = 0.1$, $u_d(0) = -2$, $u_d(1) = 1$; $U_{ad} = [-3, 3]$ and $K = \{ Y \in L : -1.1 \leq Y \leq 1 \}$

Note that z_d is quite close to the boundary of K .

In fact, as can be seen from Figure 1, the lower bound on the state is active. The active set is a singleton. In view of the fact that the influence of the boundary control at $x = 0$ and $x = 1$ is restricted to the superposition of straight lines to the uncontrolled state, this is not surprising.

The numerical values for J and the control at the minimum are :

$$J = 1.5862 \cdot 10^{-1} \text{ and } \bar{u}(0) = -9.9573 \cdot 10^{-1}, \quad \bar{u}(1) = 2.6314 \cdot 10^{-2}.$$

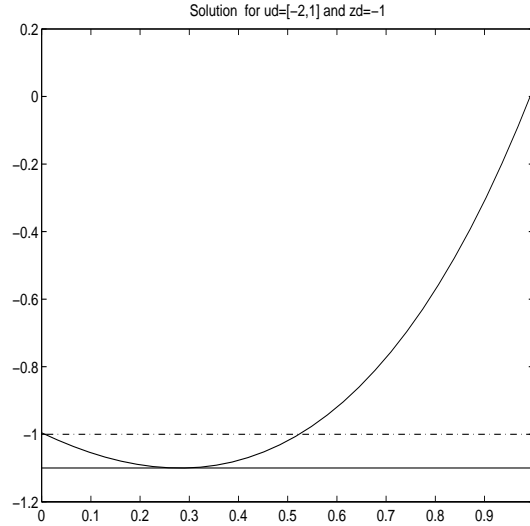


Figure 1

One of the main questions concerning the class of algorithms that we analysed is the choice of the parameters ρ_i , c and γ . From Table 1 we conclude that while the choice of the parameters certainly has an influence on the convergence properties of the algorithm, there is a wide range of parameters values for which convergence is achieved numerically, for this and other examples that we tested. In all calculations we chose $\rho_i = 1$. Some tests shows that the ratio $\frac{\gamma}{c} = \frac{1}{10}$ is a good one. For $(c, \gamma) = (1, 1)$, $(c, \gamma) = (100, 50)$, $(c, \gamma) = (1, 0.5)$ (all with $k_n = 10$ for all n), convergence is achieved but it is slower than for those pairs that are presented in Table 1. From that Table, as well as from other tests, it can also be seen that the auxiliary problem should be solved sufficiently accurately, before the Lagrange-multipliers (q, r) for the state equation and the boundary condition are updated (see $k_n \equiv 1$). The values $(c, \gamma) = (10, 0.1)$ still with $\rho_i = 1$ led to divergence. This is not unexpected in view of the result of Theorem 4.5 which requires $\rho_2 \leq \gamma$.

c	γ	k_n (constant)	$\ \Delta y + f\ _\infty$	$\ y - v\ _\infty$	n n	CPU units	$\min[y - (-1.1)]$
10	1	10	4.8 e-07	4. e-07	58	1	4. e-10
10	1	1	9.3 e-07	6. e-07	154	2.17	2.5 e-06
10	1	100	2.2 e-07	5. e-07	13	1.35	-2 e-09
100	10	10	6.2 e-07	9. e-07	95	1.01	-1.3 e-11

Table 1

2D-Example.- Now we consider

- $\Omega =]0, 1[\times]0, 1[$ and $N = 30$; $A = -\Delta$ and $f \equiv 20$.
- $z_d(x_1, x_2) = 5[x_1(x_1 - 1) * x_2(x_2 - 1)] + 3$, $u_d \equiv 0$ and $\alpha = 0.01$
- $U_{ad} = [-10, 10]$ and $K = \{ y \in L \mid 0 \leq y \leq 3.5 \}$

State

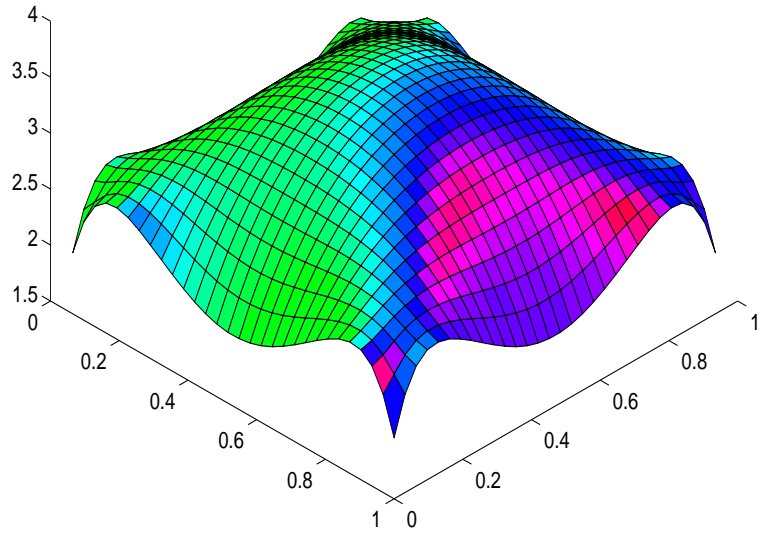


Figure 2

The results for selected values during the iteration procedure are shown in Table 2. The effect of the discretization is given in Table 3. : the CPU time is approximately a linear function of N^2 . The optimal state and control (on one side of the domain) are given in Figure 2 and Figure 3 respectively.

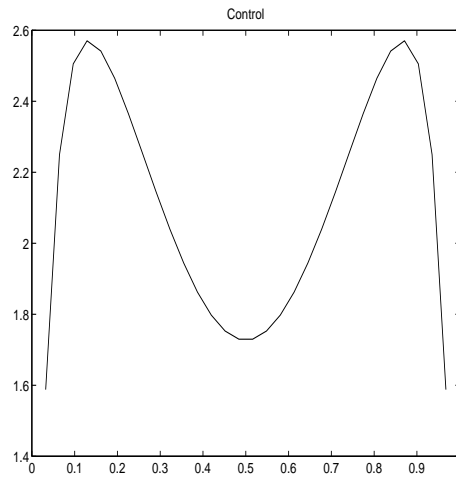


Figure 3

Iteration	$\ \Delta y + f\ _\infty$	$\ y - v\ _\infty$	J	$\min (3.5-y)$
0	4.688280e+00	1.223633e-02	2.414087e-01	-9.707107e-02
10	8.449125e-04	2.439992e-04	2.062097e-01	-1.030313e-02
50	2.819024e-05	3.966610e-06	2.083813e-01	-1.987233e-05
53	9.776863e-07	7.128897e-07	2.083877e-01	4.117863e-06

Table 2

N	10	20	30	40	50	60
$\mathbf{N}^2 \cdot 10^{-2}$	1	4	9	16	25	36
CPU units	1	3.14	7	13	21.8	35.4

Table 3

In this case the upper bound $y \leq 3.5$ is active, while the lower bound $y \geq 0$ is not, except in some corners of the domain. We must admit however, that the numerical values of y may not be accurate in the corners since no special treatment of the discontinuities of the conormals at the corners was incorporated in the code. The results were obtained with : $c = 10$, $\gamma = 1$ and $k_n = 10$ for all n .

6. Conclusion. The augmented Lagrangian algorithms with splitting into state and control variable can effectively be used to solve state and control constrained optimization problems. For the first order methods that are presented in this paper the auxiliary problems in the inner-loop must be solved sufficiently accurately before the Lagrange-multipliers of state equation and boundary condition are updated. Appropriate choices for the penalty parameters (here c and γ) and the step lengths ρ_i for the dual variables are easily determined since the algorithm are not particularly sensitive to them. It is our intention to also analyse second order methods for the same class of problems.

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