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► To cite this version:

Maïtine Bergounioux. Optimal Control of Abstract Elliptic Variational Inequalities with State Constraints. SIAM Journal on Control and Optimization, 1998, 36, n°1, pp.273-289. <hal-00022987>

HAL Id: hal-00022987

<https://hal.science/hal-00022987v1>

Submitted on 18 Apr 2006

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OPTIMAL CONTROL OF PROBLEMS GOVERNED BY ABSTRACT ELLIPTIC VARIATIONAL INEQUALITIES WITH STATE CONSTRAINTS

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Abstract. In this paper we investigate optimal control problems governed by elliptic variational inequalities with additional state-constraints. We present a relaxed formulation for the problem. With penalization methods and approximation techniques we give qualification conditions to get first order optimality conditions.

Key Words : State and Control Constrained Optimal Control Problems, Variational Inequalities, Optimality Conditions.

AMS Subject Qualification : 49J20, 49M29

1. Introduction. In this paper we investigate optimal control problems governed by elliptic variational inequalities with additional state-constraints. This topic has been widely studied by many authors. We mainly could mention Barbu [1, 2, 3], Friedman [11, 12], Mignot [13], Mignot & Puel [14], Tiba [15] or Bermudez & Saguez [8]. Most of these contributions (for example [1, 2, 3]) study the problem via the penalization of the state (in)equation. On the other hand Mignot & Puel [14] (for instance) give an equivalent formulation of the variational inequality via the associated lagrange multiplier for the obstacle problem example. We have followed this point of view; our purpose is to set optimality conditions for such a problem that could easily be used from the numerical point of view. This paper is the generalization of the case of the obstacle problem that we have been studying in [5]. We deal here with quite abstract variational inequalities. Following our previous work, we first present a relaxed form of the original problem which can be considered as a good “approximation” of this problem. Then using both Moreau-Yosida approximation techniques and a penalization method we are able to set optimality conditions. We end the paper with the example of the obstacle problem.

2. Setting the Problem. Let V and H be a pair of real Hilbert spaces such that V is a dense subset of H and $V \subset H \subset V'$ algebraically and topologically (V' denotes the dual of V). We suppose in addition that

(2.1) the injection $V \subset H$ is compact

so that $H \subset V'$ is compact too. (For example one may choose $V = H_o^1(\Omega)$ and $H = L^2(\Omega)$ where Ω is an open bounded “regular” subset of \mathbb{R}^3 .) We denote $\langle \cdot, \cdot \rangle$ the pairing between V and V' , $(\cdot, \cdot)_H$ the H -scalar product and $|\cdot|_V$ the norm of V . We call $\Lambda_V : V \rightarrow V'$ the canonical isomorphism. Let U be another Hilbert-space (such that $U = U'$); we consider the variational inequality

$$(2.2) \quad Ay + \partial\Phi(y) \ni Bu + f ,$$

where

- $A : V \rightarrow V'$ is a linear, continuous operator satisfying the coercivity condition

$$(2.3) \quad \exists \omega > 0, \forall v \in V \quad \langle Av, v \rangle \geq \omega |v|_V^2 ;$$

•

$$(2.4) \quad \begin{array}{l} \Phi \text{ is a convex, proper, lower semi-continuous (lsc) function} \\ \text{from } V \text{ to } \mathbb{R} \cup \{+\infty\} ; \end{array}$$

We denote

$$\text{dom } \Phi = \{ y \in V \mid \Phi(y) < +\infty \}$$

the domain of Φ (which is convex and V -closed). We recall that the subdifferential $\partial\Phi(y_o)$ of Φ at $y_o \in V$ is :

$$\partial\Phi(y_o) = \{ z^* \in V' \mid \forall y \in V \quad \Phi(y) - \Phi(y_o) - \langle z^*, y - y_o \rangle \geq 0 \} ,$$

and that $\text{dom } \Phi = \overline{\text{dom } \partial\Phi}$.

- $f \in V'$ and

$$(2.5) \quad B \text{ is a linear, compact operator from } U \text{ to } V' .$$

Let us recall some general results about solutions of (2.2) (see [2, 3] for example).

THEOREM 2.1. (*Barbu [2] p 40*) *Under assumptions (2.3)-(2.4) and for all $\psi \in V'$ the variational inequality*

$$Ay + \partial\Phi(y) \ni \psi$$

has a unique solution $y(\psi) \in V$ and the mapping $\psi \mapsto y(\psi)$ is Lipschitz from V' to V .

COROLLARY 2.1. (*Barbu [2] p 63*) *With the assumptions of the previous theorem, for all $u \in U$, there exists a unique $y(u) \in V$ solution of (2.2) and the mapping $u \mapsto y(u)$ is weakly-strongly continuous from U to V . ■*

In order to get some regularity results, we suppose from now that

$$(2.6) \quad f \in H \text{ and } B \in \mathcal{L}(U, H) ,$$

so that (2.5) is fulfilled and we may use in addition the following result (Barbu [2] p 42) :let us denote $A_H : H \rightarrow H$ the operator

$$(2.7) \quad A_H(y) = Ay \text{ for all } y \in D(A_H) = \{ y \in V \mid Ay \in H \} .$$

This operator is maximal monotone in $H \times H$ and we have

THEOREM 2.2. *Assume (2.3) and suppose in addition that there exists $z \in H$ and $c \in \mathbb{R}$ such that*

$$(2.8) \quad \forall y \in V, \forall \lambda > 0 \quad \Phi((I + \lambda A_H)^{-1}(y + \lambda z)) \leq \Phi(y) + c\lambda .$$

Then for every $\psi \in H$ the solution $y(\psi)$ of $Ay + \partial\Phi(y) \ni \psi$ belongs to $D(A_H)$ and

$$|Ay(\psi)|_H \leq c(1 + |\psi|_H) .$$

■

From now we suppose that (2.8) is ensured. This is the case for example for the obstacle problem given as an example in the last section of this paper, where $V = H_o^1(\Omega)$, $H = L^2(\Omega)$ and $D(A_H) = H^2(\Omega) \cap H_o^1(\Omega)$.

Applying this regularity result to our case we get that for all $u \in U$, $f + Bu \in H$ so that the solution of (2.2) y belongs to $D(A_H) \subset V$, that is $Ay \in H$.

REMARK 2.1. *One could think that this regularity assumption is not really necessary. Indeed, it is not useful to prove the results of next section. Nevertheless, when we investigate penalized problems, then we shall need some “strong” convergence for the penalized solutions, that is with the compactness assumptions “weak” convergence in the pivot-space H .*

Now, we investigate the following optimal control problem

$$(P) \quad \begin{cases} \min & g(y) + h(u) \\ & Ay + \partial\Phi(y) \ni Bu + f , \\ & (y, u) \in K \times U_{ad} , \end{cases}$$

where

- g is convex from H to \mathbb{R} , finite everywhere ($\text{dom}(g) = H$) and continuous. This implies ([4] proposition 1.9 p.85) that

$$(2.9) \quad \exists (a_g, c_g) \in H \times \mathbb{R} \quad \text{such that} \quad \forall y \in H \quad g(y) \geq (a_g, y)_H + c_g$$

(because g is lsc) and g is everywhere subdifferentiable.

- h is convex from U to \mathbb{R} , finite everywhere ($\text{dom}(h) = U$), continuous and coercive :

$$(2.10) \quad \lim_{|u|_U \rightarrow +\infty} \frac{h(u)}{|u|_U} = +\infty ;$$

• U_{ad} (resp. K) is a closed, convex, non empty subset of U (resp. V). We note that $y \in \text{dom } \partial\Phi \subset \text{dom } \Phi$ so that one may always suppose that

$$(2.11) \quad K \subset \text{dom } \Phi .$$

REMARK 2.2. *From now, we always suppose that these assumptions are satisfied. They are not of course optimal. To get more information one can refer to Barbu [3] p. 150.*

We end this section with an existence result for (\mathcal{P}) .

THEOREM 2.3. *Under assumptions (2.3),(2.4),(2.9),(2.10), problem (\mathcal{P}) has (at least) one optimal solution.*

Proof - The proof is quite similar to the one given in Barbu [3] p 151. The main difference is the addition of the state constraint $y \in K$ which does not modify the proof. ■

3. “Relaxation” of the Problem. We denote $\Phi^* : V' \rightarrow \mathbb{R}$ the conjugate function of Φ ; it is also a convex, proper, lsc and we know that (see [4, 10])

$$(3.1) \quad z \in \partial\Phi(y) \Leftrightarrow y \in \partial\Phi^*(z) \Leftrightarrow \Phi(y) + \Phi^*(z) = \langle y, z \rangle .$$

Because of the regularity result we always have $Bu + f - Ay \in H$, so that $z = Bu + f - Ay \in \partial\Phi(y) \cap H$ and (3.1) is equivalent to

$$z \in \partial\Phi(y) \Leftrightarrow y \in \partial\Phi^*(z) \Leftrightarrow \Phi(y) + \Phi^*(z) = (y, z)_H .$$

REMARK 3.1. *In addition such an element z belongs to $\text{dom } \partial\Phi^* \subset \text{dom } \Phi^*$ so that the condition “ $z \in \text{dom } \Phi^*$ ” is implicitly included in relation (3.1). Finally, problem (\mathcal{P}) is equivalent to*

$$(\tilde{\mathcal{P}}) \quad \begin{cases} \min & g(y) + h(u) \\ Ay = Bu + f - z \in H , \\ \Phi(y) + \Phi^*(z) - (y, z)_H = 0 , \\ y \in D(A_H) \cap K, (u, z) \in U_{ad} \times (\text{dom } \Phi^* \cap H) . \end{cases}$$

$w = (u, z)$ is now considered as a new control variable. Problem $(\tilde{\mathcal{P}})$ is a state-constrained optimal control problem with a non-convex (because of the bilinear term) constraint coupling the state y and the control w . This constraint is quite difficult to deal with. It is not convex and the equality constraint makes the interior of the feasible domain empty in a very strong sense. So as we have done in [5] for the particular case of the obstacle problem we had rather study a “relaxed” problem. More precisely we consider

$$(\mathcal{P}_\alpha^R) \quad \begin{cases} \min & g(y) + h(u) \\ Ay = Bu + f - z , \\ \Phi(y) + \Phi^*(z) - (y, z)_H \leq \alpha , \\ y \in K, (u, z) \in U_{ad} \times B_R^* , \end{cases}$$

where $\alpha > 0$, $R > 0$, $B_R^* = B_H(0, R) \cap \text{dom } \Phi^*$ and $B_H(0, R)$ is the H -ball of radius R . B_R^* is convex and H -closed (since $\text{dom } \Phi^*$ is convex and V' -closed).

REMARK 3.2. Let us comment this “relaxed” form for problem $(\tilde{\mathcal{P}})$. First we know that $\Phi(y) + \Phi^*(z) - (y, z)_H$ is always nonnegative. So

$$(3.2) \quad \Phi(y) + \Phi^*(z) - (y, z)_H \leq \alpha$$

is equivalent to $|\Phi(y) + \Phi^*(z) - (y, z)_H| \leq \alpha$. This is the relaxed term : we have replaced the equality “ $= 0$ ” with the inequality “ $\leq \alpha$ ”, where α may be as small as wanted. This is quite realistic from the numerical point of view where equalities are indeed inequalities up to α .

On the other hand, if we do not add the constraint “ $z \in B_H(0, R)$ ” the relaxed problem is not coercive and so in general it has no solution. Moreover, by virtue of assumptions (2.8) and (2.10) the optimal solutions (y, u) and $A\bar{y}$ remain in a bounded set of $H \times U$ and H and the constant R has to be chosen accordingly (that is large enough); in particular R is greater than $|A\bar{y} - f - B\bar{u}|_H$ for any (\bar{y}, \bar{u}) solution of (\mathcal{P}) , so that the feasible domain of (\mathcal{P}_α^R) is non empty.

Anyway, this additional condition is not very restrictive. One could instead add a regularization term (as $|z|_H^2/R$) to the cost functional, which would have exactly the same effect. One may also add adapted penalization terms to this cost functional as $|y - \bar{y}|_V^2$ or $|z - \bar{z}|_H^2$.

From now we fix R so that we always omit the index R in the notations and (\mathcal{P}_α^R) becomes (\mathcal{P}_α) .

THEOREM 3.1. For every $\alpha > 0$, (\mathcal{P}_α) has at least one optimal solution denoted $(y_\alpha, u_\alpha, z_\alpha)$. Moreover, when $\alpha \rightarrow 0$, y_α strongly converges to \bar{y} in V , u_α weakly converges to \bar{u} in U and z_α weakly converges to \bar{z} in H where (\bar{y}, \bar{u}) is a solution of (\mathcal{P}) and $\bar{z} = A\bar{y} - B\bar{u} - f \in H$.

Proof - Let $\alpha > 0$; we have chosen R such that the feasible domain of (\mathcal{P}_α) is always non-empty. We first prove that that $d_\alpha = \inf (\mathcal{P}_\alpha) \in \mathbb{R}$. The coercivity and continuity assumptions on A yield that A is an isomorphism from V to V' . Let (y_n, u_n, z_n) be a minimizing sequence : $y_n = A^{-1}(Bu_n + f - z_n)$, $|z_n|_H \leq R$, $u_n \in U_{ad}$, $\Phi(y_n) + \Phi^*(z_n) - (y_n, z_n)_H \leq \alpha$ and $g(y_n) + h(u_n) \rightarrow d_\alpha$. Because of (2.9) we have

$$\begin{aligned} g(y_n) + h(u_n) &\geq (a_g, A^{-1}(Bu_n + f - z_n))_H + h(u_n) + c_g \\ &\geq (a_g, A^{-1}Bu_n)_H + h(u_n) - (a_g, A^{-1}z_n)_H + c_g + (a_g, A^{-1}f)_H. \end{aligned}$$

As $|z_n|_H \leq R$ then $-(a_g, A^{-1}z_n)_H + c_g + (a_g, A^{-1}f)_H$ is bounded from below. If $d_\alpha = -\infty$ then $(a_g, A^{-1}Bu_n)_H + h(u_n) \rightarrow -\infty$. If (u_n) were bounded in U then (extracting a subsequence) u_n would be weakly convergent to some \tilde{u} in U ; as B is continuous Bu_n would be convergent to $B\tilde{u}$ weakly in H and strongly in V' . Therefore $A^{-1}Bu_n$ would be strongly convergent to $A^{-1}B\tilde{u}$ in V and $(a_g, A^{-1}Bu_n)_H \rightarrow (a_g, A^{-1}B\tilde{u})_H$. Moreover h is lsc, so that $-\infty <$

$h(\tilde{u}) \leq \liminf_{n \rightarrow +\infty} h(u_n)$; so we get a contradiction. This means that (u_n) is unbounded. Coercivity assumption (2.10) implies that

$$\lim_{n \rightarrow +\infty} \frac{h(u_n)}{|u_n|_U} = +\infty .$$

Moreover Cauchy-Schwartz inequality shows that

$$\left| \frac{(a_g, A^{-1}Bu_n)_H}{|u_n|_U} \right| \leq c_o \frac{|u_n|_U}{|u_n|_U} ;$$

So

$$\lim_{n \rightarrow +\infty} (a_g, A^{-1}Bu_n)_H + h(u_n) = |u_n|_U \left[\frac{(a_g, A^{-1}Bu_n)_H}{|u_n|_U} + \frac{h(u_n)}{|u_n|_U} \right] = +\infty ,$$

and we get a contradiction.

- As $|z_n|_H \leq R$ one may extract a subsequence (still denoted z_n) weakly convergent in H to $z_\alpha \in B_R^*$ (since B_R^* is weakly closed). As $d_\alpha > -\infty$, $h(u_n)$ is bounded , and by coercivity (u_n) is bounded in U ; so (extracting a subsequence) u_n weakly converges to $u_\alpha \in U_{ad}$ (U_{ad} is weakly closed in U). The continuity of B yields that Bu_n converges to Bu_α weakly in H . So $Ay_n = Bu_n + f - z_n$ converges to $Bu_\alpha + f - z_\alpha$ weakly in H and strongly in V' . As A is an isomorphism from V to V' , y_n converges to $y_\alpha = A^{-1}(Bu_\alpha + f - z_\alpha)$ strongly in V . Moreover $y_\alpha \in K$ since K is closed.

- Let us prove that $(y_\alpha, u_\alpha, z_\alpha)$ is feasible for (\mathcal{P}_α) . It remains to show that $\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H \leq \alpha$. Φ and Φ^* are convex and lower semi-continuous so they are weakly lower semi-continuous and we have

$$\Phi(y_\alpha) + \Phi^*(z_\alpha) \leq \liminf_{n \rightarrow +\infty} \Phi(y_n) + \liminf_{n \rightarrow +\infty} \Phi^*(z_n) \leq \liminf_{n \rightarrow +\infty} [\Phi(y_n) + \Phi^*(z_n)] .$$

Moreover the strong convergence of y_n to y_α in H and the weak convergence of z_n to z_α in H gives

$$\lim_{n \rightarrow +\infty} (y_n, z_n)_H = (y_\alpha, z_\alpha)_H .$$

Finally

$$\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H \leq \liminf_{n \rightarrow +\infty} [\Phi(y_n) + \Phi^*(z_n) - (y_n, z_n)_H] \leq \alpha .$$

Therefore $(y_\alpha, u_\alpha, z_\alpha)$ is feasible for (\mathcal{P}_α) and $g(y_\alpha) + h(u_\alpha) \geq d_\alpha$. As g and h are lsc we have also

$$g(y_\alpha) + h(u_\alpha) \leq \liminf_{n \rightarrow +\infty} g(y_n) + \liminf_{n \rightarrow +\infty} h(u_n) \leq \liminf_{n \rightarrow +\infty} [g(y_n) + h(u_n)] = d_\alpha .$$

Finally $g(y_\alpha) + h(u_\alpha) = d_\alpha$ and $(y_\alpha, u_\alpha, z_\alpha)$ is an optimal solution for (\mathcal{P}_α) .

• It remains to prove the convergence of $(y_\alpha, u_\alpha, z_\alpha)$ to an optimal solution of (\mathcal{P}) . Let (y_o, u_o, z_o) be an optimal solution of (\mathcal{P}) such that $z_o \in B_R^*$ (remember that we have chosen R to ensure the existence of such a solution). It is also a feasible triple for (\mathcal{P}_α) , for any $\alpha > 0$. So

$$\forall \alpha > 0 \quad -\infty < d_\alpha = g(y_\alpha) + h(u_\alpha) \leq g(y_o) + h(u_o) = d_o .$$

So d_α is bounded from above in \mathbb{R} . If it were not bounded from below then we could find a sequence $\alpha_n \rightarrow 0$ such that $d_{\alpha_n} \rightarrow -\infty$. The same proof as before shows that it is impossible. So $h(u_\alpha)$ is bounded (independently of α) and by coercivity (u_α) is bounded in U as well. Similarly (z_α) is bounded in H ($z_\alpha \in B_R^*$). Then one can show (as we have proved the existence of $(y_\alpha, u_\alpha, z_\alpha)$) that

$$y_\alpha \rightarrow \bar{y} \text{ strongly in } V, \quad u_\alpha \rightharpoonup \bar{u} \text{ weakly in } U \text{ and } z_\alpha \rightharpoonup \bar{z} \text{ weakly in } H,$$

where (\bar{y}, \bar{u}) is a solution of (\mathcal{P}) with $\bar{z} = A\bar{y} - B\bar{u} - f$ and that

$$\lim_{\alpha \rightarrow 0} [g(y_\alpha) + h(u_\alpha)] = g(\bar{y}) + h(\bar{u}) .$$

■

4. Penalization of (\mathcal{P}_α) .

4.1. The approximated-penalized problem. From now, we fix also $\alpha > 0$ as small as wanted and we shall omit the index α most of time. We are going to approximate and penalize the state-equation of (\mathcal{P}_α) to get an approximated problem $(\mathcal{P}_\alpha^\varepsilon)$. Then we shall derive optimality conditions for this problem and set qualification conditions allowing us to pass to the limit with respect to ε . For $\varepsilon > 0$, we consider the following problem :

$$(\mathcal{P}_\alpha^\varepsilon) \quad \begin{cases} \min J_\varepsilon(y, u, z) \\ (y, u, z) \in K \times U_{ad} \times B_R^* , \end{cases}$$

where

$$\begin{aligned} J_\varepsilon(y, u, z) = & g_\varepsilon(y) + h_\varepsilon(u) \\ & + \frac{1}{2\varepsilon} |Ay - Bu - f + z|_{V'}^2 + \frac{1}{2\varepsilon} [\Phi_\varepsilon(y) + \Phi_\varepsilon^*(z) - (y, z)_H - \alpha]_+^2 \\ & + \frac{1}{2} |y - y_\alpha|_V^2 + \frac{1}{2} |u - u_\alpha|_U^2 + \frac{1}{2} |z - z_\alpha|_H^2 . \end{aligned}$$

Here $g_+ = \max(0, g)$, and g_ε , h_ε , Φ_ε and Φ_ε^* are the Moreau-Yosida approximations of g , h , Φ and Φ^* .

First, we briefly recall some useful properties of Moreau-Yosida approximation of convex functions. Let φ be a convex, proper, lower semi continuous function from \mathcal{H} to $\mathbb{R} \cup \{+\infty\}$ where \mathcal{H} is an Hilbert space (not necessarily identified with its dual). The Moreau-Yosida approximation of φ is defined by

$$\varphi_\varepsilon(x) = \inf \left\{ \frac{|x - y|_{\mathcal{H}}^2}{2\varepsilon} + \varphi(y) , \quad y \in \mathcal{H} \right\}$$

and we have the following properties ([3] pp.49-55) :

THEOREM 4.1. Let us call $I_\varepsilon = (\Lambda_{\mathcal{H}} + \varepsilon D)^{-1}$ the proximal mapping with $D = \partial\varphi$, $\Lambda_{\mathcal{H}}$ the canonical isomorphism from \mathcal{H} to \mathcal{H}' , and $D_\varepsilon = -\varepsilon^{-1}\Lambda_{\mathcal{H}}(I_\varepsilon - I)$.

i. I_ε is single valued and non expansive.

ii. $D_\varepsilon x \in \partial\varphi(I_\varepsilon x)$ for all $x \in \mathcal{H}$, and for all $x \in \text{dom}(\partial\varphi)$ $\lim_{\varepsilon \rightarrow 0} D_\varepsilon x = D^0 x \in \partial\varphi(x)$ (strongly in \mathcal{H}') where $D^0(x)$ is the element of minimal norm of $\partial\varphi(x)$. ■

iii. For all $x \in \text{dom } \varphi$ $I_\varepsilon x$ converges strongly in \mathcal{H} towards x .

iv. If $\varepsilon_n \rightarrow 0$, $x_{\varepsilon_n} \rightarrow x_o$ strongly in \mathcal{H} and $D_{\varepsilon_n} x_{\varepsilon_n} \rightharpoonup y_o$ weakly in \mathcal{H}' then $y_o \in \partial\varphi(x_o)$.

v. φ_ε is Fréchet-differentiable and $\varphi'_\varepsilon = D_\varepsilon$ is Lipschitz (so that φ_ε is \mathcal{C}^1).

vi. For all $x \in \mathcal{H}$ and $\varepsilon > 0$ $\varphi(I_\varepsilon x) \leq \varphi_\varepsilon(x) \leq \varphi(x)$.

Moreover, for all $x \in \mathcal{H}$, $\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon(x) = \varphi(x)$. ■

In addition, as we need sharper convergence results we set some further assumptions about the function φ and we suppose that

$$(4.1) \quad \begin{aligned} &\forall (x_\varepsilon) \in \text{dom } \varphi \text{ strongly convergent (in } \mathcal{H} \text{) to } x \in \text{dom } \varphi \\ &\text{then } \varphi'_\varepsilon(x_\varepsilon) \text{ is bounded in } \mathcal{H}' \text{ (with respect to } \varepsilon \text{).} \end{aligned}$$

Then we have the following useful theorem

THEOREM 4.2. For any convex, proper, lsc function φ ,

i. If x_ε strongly converges to some x in \mathcal{H} then $\lim_{\varepsilon \rightarrow 0} I_\varepsilon x_\varepsilon = x$ (strongly in \mathcal{H}).

ii. If x_ε weakly converges to some $x \in \text{dom } \partial\varphi$ in \mathcal{H} , then $\liminf_{\varepsilon \rightarrow 0} \varphi_\varepsilon(x_\varepsilon) \geq \varphi(x)$. ■

iii. If φ satisfies condition (4.1) and if $x_\varepsilon \in \text{dom } \varphi$ strongly converges to some $x \in \text{dom } \varphi$ then $\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon(x_\varepsilon) = \varphi(x)$ and $x \in \text{dom } \partial\varphi$.

Proof - i. and ii. are direct consequences of Theorem 4.1. To prove iii. we use the relation (2.18) given in Barbu [3] p.66 :

$$\forall z, y \in \mathcal{H}, \forall \varepsilon > 0, \quad \varphi_\varepsilon(y) - \varphi_\varepsilon(z) \leq \langle \varphi'_\varepsilon(y), y - z \rangle_{\mathcal{H}', \mathcal{H}}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}', \mathcal{H}}$ denotes the pairing between \mathcal{H} and \mathcal{H}' .

We use it first with $z = x_\varepsilon$ and $y = x$ and then with $y = x_\varepsilon$ and $y = x$; this gives

$$|\varphi_\varepsilon(x_\varepsilon) - \varphi_\varepsilon(x)| \leq \max(|\varphi'_\varepsilon(x_\varepsilon)|_{\mathcal{H}'}, |\varphi'_\varepsilon(x)|_{\mathcal{H}'}) |x - x_\varepsilon|_{\mathcal{H}}.$$

With Theorem 4.1 ii., assumption (4.1) and the strong convergence of x_ε to x , we get the strong convergence of $\varphi_\varepsilon(x_\varepsilon)$ to $\varphi_\varepsilon(x)$. We conclude with Theorem 4.1 vi. ■

REMARK 4.1. The above property is satisfied for any convex, proper, lsc function φ as soon as $x \in \text{int}(\text{dom } \varphi)$ since $\partial\varphi(x)$ is locally bounded in this case (see [4] p. 60). Anyway, here it may happen that $\text{int}(\text{dom } \varphi)$ is empty and this result cannot be used.

We precise now the hypotheses on functions Φ and Φ^* ; from now we assume that

$$(4.2) \quad \Phi \text{ satisfies (4.1) with } \mathcal{H} = V \text{ and } \Phi^* \text{ satisfies (4.1) with } \mathcal{H} = V'.$$

This assumption is not so restrictive since it allows to consider a wide class of convex functions; let us give some examples :

EXAMPLE 4.1. **Convex functions satisfying (4.1).**

- Any continuous, convex function defined on the whole space V satisfies (4.1) since $\text{int}(\text{dom } \varphi) = V$ so that $\partial\varphi(x)$ is locally bounded for any x (we use also Theorem 4.2 i., Theorem 4.1 ii. and a result of Barbu & Precupanu [4] p. 60).
- Any indicator function $\varphi = 1_C$ of a convex, closed, non empty subset C of \mathcal{H} satisfies (4.1) also; we recall that

$$1_C(y) = \begin{cases} 0 & \text{if } y \in C \\ +\infty & \text{else} \end{cases}$$

and $I_\varepsilon(x) = P_C(x)$ where P_C is the \mathcal{H} -projection on C . Then

$$\varphi_\varepsilon(x) = \frac{|x - P_C(x)|_{\mathcal{H}}^2}{2\varepsilon} \text{ and } \varphi'_\varepsilon(x) = \Lambda_{\mathcal{H}} \left(\frac{x - P_C(x)}{\varepsilon} \right) .$$

If x_ε strongly converges to x in $C = \text{dom } \varphi$, then $\varphi'_\varepsilon(x_\varepsilon) = 0$ for all ε and so remains bounded in \mathcal{H}' . ■

EXAMPLE 4.2. **Convex functions satisfying (4.2).**

- If Φ is the indicator function of a convex closed cone C of V then $\Phi^* = 1_{C^*}$ where C^* is the polar cone of C in V' ; so with Example 4.1. we see that (4.2) is ensured.

This case involves the obstacle problem or the Signorini problem.

- If $\Phi(x) = |x|_V$ is continuous and $\text{dom } (\Phi) = H$ then Φ^* is the indicator function of the unit ball of V' .
- $\Phi(x) = \frac{1}{p}|x|_V^p$ then $\Phi^*(x) = \frac{1}{p'}|x|_{V'}^{p'}$, where $p, p' \in]1, +\infty[$ are conjugate numbers (see Ekeland-Temam [10]). This leads to a **semilinear** state-equation. ■

REMARK 4.2. *The approximation process concerns the functions g , h , Φ and Φ^* which are not necessarily Fréchet-differentiable and are replaced by their Moreau-Yosida approximations. This method provides C^1 functions.*

We have also added two kinds of penalization terms : the state-equation and the inequality (non convex) constraint are penalized in a standard way. The other terms are adapted penalization terms which ensure the strong convergence of the penalized solution towards the desired solution (when uniqueness does not hold).

First we have an existence and convergence result for $(\mathcal{P}_\alpha^\varepsilon)$.

THEOREM 4.3. *For all $\varepsilon > 0$, problem $(\mathcal{P}_\alpha^\varepsilon)$ has (at least) a solution $(y_\varepsilon, u_\varepsilon, z_\varepsilon)$. Moreover, when $\varepsilon \rightarrow 0$, $(y_\varepsilon, u_\varepsilon, z_\varepsilon) \rightarrow (y_\alpha, u_\alpha, z_\alpha)$ strongly in $V \times U \times H$.*

Proof - We first prove the existence of a solution for $(\mathcal{P}_\alpha^\varepsilon)$. We notice that $(y_\alpha, u_\alpha, z_\alpha)$ is a feasible triple for $(\mathcal{P}_\alpha^\varepsilon)$ so that the feasible domain of $(\mathcal{P}_\alpha^\varepsilon)$ is non empty and we may find a minimizing sequence $(y_\varepsilon^n, u_\varepsilon^n, z_\varepsilon^n)$ converging to

$d_\varepsilon = \inf(\mathcal{P}_\alpha^\varepsilon)$. Setting $I_{\varepsilon,g} = (I_H + \varepsilon \partial g)^{-1}$ and $I_{\varepsilon,h} = (I_U + \varepsilon \partial h)^{-1}$ we get

$$J_\varepsilon(y, u, z) \geq g_\varepsilon(y) + h_\varepsilon(u) \geq g(I_{\varepsilon,g}(y)) + h(I_{\varepsilon,h}(u)) ,$$

and

$$\inf_{V \times U \times H} J_\varepsilon(y, u, z) \geq \inf_{V \times U} g(I_{\varepsilon,g}(y)) + h(I_{\varepsilon,h}(u)) \geq \gamma > -\infty$$

because of the properties of g and h . So $d_\varepsilon \in \mathbb{R}$ and the end of the proof is standard (see Theorem 3.1).

Now we prove the convergence result. Since $(y_\alpha, u_\alpha, z_\alpha)$ is a feasible triple for $(\mathcal{P}_\alpha^\varepsilon)$ we have

$$(4.3) \quad d_\varepsilon = J_\varepsilon(y_\varepsilon, u_\varepsilon, z_\varepsilon) \leq g(y_\alpha) + h(u_\alpha) = d_\alpha .$$

We have just seen that d_ε is lower bounded (with respect to ε) so that y_ε , u_ε and z_ε are bounded in V , U and H . Extracting a subsequence, we get the weak convergence of $(y_\varepsilon, u_\varepsilon, z_\varepsilon)$ to $(\tilde{y}, \tilde{u}, \tilde{z})$ in $V \times U \times H$; in particular, this yields that $Ay_\varepsilon - Bu_\varepsilon - f + z_\varepsilon$ converges to $A\tilde{y} - B\tilde{u} - f + \tilde{z}$ weakly in V' . Moreover $Ay_\varepsilon - Bu_\varepsilon - f + z_\varepsilon$ converges to 0 strongly in V' so that $A\tilde{y} - B\tilde{u} - f + \tilde{z} = 0$. In addition, as U_{ad} , K and B_R^* are weakly closed we get $\tilde{u} \in U_{ad}$, $\tilde{y} \in K$ and $\tilde{z} \in B_R^*$.

The injection of V in H is compact, so $y_\varepsilon \rightarrow \tilde{y}$ strongly in H ; as $z_\varepsilon \rightharpoonup \tilde{z}$ weakly in H we get the convergence of $(y_\varepsilon, z_\varepsilon)_H$ to $(\tilde{y}, \tilde{z})_H$. Moreover Theorem 4.2 gives

$$\liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(y_\varepsilon) \geq \Phi(\tilde{y}) \text{ and } \liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon^*(z_\varepsilon) \geq \Phi^*(\tilde{z}) .$$

So we get

$$[\Phi(\tilde{y}) + \Phi^*(\tilde{z}) - (\tilde{y}, \tilde{z})_H - \alpha]_+ \leq \liminf_{\varepsilon \rightarrow 0} [\Phi_\varepsilon(y_\varepsilon) + \Phi_\varepsilon^*(z_\varepsilon) - (y_\varepsilon, z_\varepsilon)_H - \alpha]_+ ;$$

Since $\lim_{\varepsilon \rightarrow 0} [\Phi_\varepsilon(y_\varepsilon) + \Phi_\varepsilon^*(z_\varepsilon) - (y_\varepsilon, z_\varepsilon)_H - \alpha]_+^2 = 0$ this yields

$$[\Phi(\tilde{y}) + \Phi^*(\tilde{z}) - (\tilde{y}, \tilde{z})_H - \alpha]_+ = 0 .$$

So $(\tilde{y}, \tilde{u}, \tilde{z})$ is feasible for (\mathcal{P}_α) . Now relation (4.3) gives

$$g_\varepsilon(y_\varepsilon) + h_\varepsilon(u_\varepsilon) + \frac{1}{2}|y_\varepsilon - y_\alpha|_V^2 + \frac{1}{2}|u_\varepsilon - u_\alpha|_U^2 + \frac{1}{2}|z_\varepsilon - z_\alpha|_H^2 \leq g(y_\alpha) + h(u_\alpha) .$$

Passing to the inf-limit in the above relation we get

$$g(\tilde{y}) + h(\tilde{u}) + \frac{1}{2}|\tilde{y} - y_\alpha|_V^2 + \frac{1}{2}|\tilde{u} - u_\alpha|_U^2 + \frac{1}{2}|\tilde{z} - z_\alpha|_H^2 \leq g(y_\alpha) + h(u_\alpha) \leq g(\tilde{y}) + h(\tilde{u}) ,$$

since $(\tilde{y}, \tilde{u}, \tilde{z})$ is feasible for (\mathcal{P}_α) . So $\tilde{y} = y_\alpha$, $\tilde{u} = u_\alpha$ and $\tilde{z} = z_\alpha$. Furthermore $\lim_{\varepsilon \rightarrow 0} |y_\varepsilon - y_\alpha|_V = 0$, $\lim_{\varepsilon \rightarrow 0} |u_\varepsilon - u_\alpha|_U = 0$ and $\lim_{\varepsilon \rightarrow 0} |z_\varepsilon - z_\alpha|_H = 0$ and we get the strong convergence. ■

COROLLARY 4.1. *There exists $(y^*, z^*) \in \partial\Phi^*(z_\alpha) \times \partial\Phi(y_\alpha)$ such that $\Phi'_\varepsilon(y_\varepsilon) \rightharpoonup z^*$ weakly in V' and $\Phi_\varepsilon^{*'}(z_\varepsilon) \rightharpoonup y^*$ weakly in V (and strongly in H). Moreover*

$$(4.4) \quad \lim_{\varepsilon \rightarrow 0} \langle \Phi'_\varepsilon(y_\varepsilon), y_\varepsilon \rangle = \langle z^*, y_\alpha \rangle \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} (\Phi_\varepsilon^{*'}(z_\varepsilon), z_\varepsilon)_H = (y^*, z_\alpha)_H .$$

Proof - As $y_\varepsilon \in K \subset \text{dom}(\Phi)$ strongly converges to y_α in V , we use assumption (4.2) to infer that $\Phi'_\varepsilon(y_\varepsilon)$ is bounded in V' . So we may extract a subsequence (denoted similarly) such that $\Phi'_\varepsilon(y_\varepsilon)$ weakly converges in V' to z^* . Theorem 4.1 iv. implies that $z^* \in \partial\Phi(y_\alpha)$. Similarly, we may prove that $\Phi_\varepsilon^{*'}(z_\varepsilon) \rightharpoonup y^* \in \partial\Phi^*(z_\alpha)$ weakly in V and strongly in H , since z_ε strongly converges to z_α in V' . Relations (4.4) are obvious. \blacksquare

4.2. Optimality Conditions for $(\mathcal{P}_\alpha^\varepsilon)$. Now, we want to derive optimality conditions for $(\mathcal{P}_\alpha^\varepsilon)$. J_ε is \mathcal{C}^1 and the feasible domain of $(\mathcal{P}_\alpha^\varepsilon)$ is convex, so using convex variations we have

$$(4.5) \quad \forall (y, u, z) \in K \times U_{ad} \times B_R^* \quad \nabla J_\varepsilon(y_\varepsilon, u_\varepsilon, z_\varepsilon)(y - y_\varepsilon, u - u_\varepsilon, z - z_\varepsilon) \geq 0 .$$

This leads to the following penalized optimality system :

THEOREM 4.4. *For all $\varepsilon > 0$ (small enough), there exist $q_\varepsilon \in V$ and $\lambda_\varepsilon \in \mathbb{R}^+$ such that*

$$(4.6) \quad \begin{aligned} & \forall y \in K \\ & (g'_\varepsilon(y_\varepsilon), y - y_\varepsilon)_H + (y_\varepsilon - y_\alpha, y - y_\varepsilon)_V + \langle A^* q_\varepsilon + \lambda_\varepsilon [\Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon], y - y_\varepsilon \rangle \geq 0 \end{aligned}$$

$$(4.7) \quad \forall u \in U_{ad} \quad (h'_\varepsilon(u_\varepsilon) - B^* q_\varepsilon + u_\varepsilon - u_\alpha, u - u_\varepsilon)_U \geq 0 ,$$

$$(4.8) \quad \forall z \in B_R^* \quad \left(q_\varepsilon + \lambda_\varepsilon [\Phi_\varepsilon^{*'}(z_\varepsilon) - y_\varepsilon] + z_\varepsilon - z_\alpha, z - z_\varepsilon \right)_H \geq 0 ,$$

where A^* and B^* are the adjoint operators of A and B .

Proof - Relation (4.5) may be decoupled to obtain

$$(4.9) \quad \forall y \in K \quad \nabla_y J_\varepsilon(y_\varepsilon, u_\varepsilon, z_\varepsilon)(y - y_\varepsilon) \geq 0 ,$$

$$(4.10) \quad \forall u \in U_{ad} \quad \nabla_u J_\varepsilon(y_\varepsilon, u_\varepsilon, z_\varepsilon)(u - u_\varepsilon) \geq 0 ,$$

$$(4.11) \quad \forall z \in B_R^* \quad \nabla_z J_\varepsilon(y_\varepsilon, u_\varepsilon, z_\varepsilon)(z - z_\varepsilon) \geq 0 .$$

Let us precise these relations : setting $q_\varepsilon = \Lambda_V^{-1}(s_\varepsilon) \in V$, and

$$s_\varepsilon = \frac{Ay_\varepsilon - Bu_\varepsilon - f + z_\varepsilon}{\varepsilon} \in H \subset V', \quad \lambda_\varepsilon = \frac{[\Phi_\varepsilon(y_\varepsilon) + \Phi_\varepsilon^*(z_\varepsilon) - (y_\varepsilon, z_\varepsilon)_H - \alpha]_+}{\varepsilon} \in \mathbb{R}^+ \quad \blacksquare$$

relation (4.9) gives, for all $y \in K$

$$\begin{aligned} & (g'_\varepsilon(y_\varepsilon), y - y_\varepsilon)_H + (y_\varepsilon - y_\alpha, y - y_\varepsilon)_V + \\ & \langle \lambda_\varepsilon [\Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon], y - y_\varepsilon \rangle + \langle q_\varepsilon, A(y - y_\varepsilon) \rangle \geq 0 ; \end{aligned}$$

introducing the adjoint operator A^* of A we get (4.6). The other relations are obtained similarly. \blacksquare

REMARK 4.3. Equation (4.8) (and (4.7) as well) can be reformulated using the normal cone to B_R^* . Indeed, as B_R^* is convex this normal cone is characterized with

$$N_{B_R^*}(z_\varepsilon) = \{ \xi \in H \mid (\xi, z_\varepsilon - z)_H \geq 0 \quad \forall z \in B_R^* \}$$

(see for instance Clarke [9] p.53), so that relation (4.8) is equivalent to

$$(4.12) \quad -[q_\varepsilon + \lambda_\varepsilon(\Phi_\varepsilon^*(z_\varepsilon) - y_\varepsilon)] \in z_\varepsilon - z_\alpha + N_{B_R^*}(z_\varepsilon) .$$

5. Optimality Conditions for (\mathcal{P}_α) . In order to pass to the limit (with respect to ε) in the previous relations we need further estimations on the multipliers q_ε and λ_ε .

5.1. Estimations of the penalized multipliers. Let be $(y, u, z) \in K \times U_{ad} \times B_R^*$ and let us add relations (4.6)-(4.8). This gives

$$\begin{aligned} & (g'_\varepsilon(y_\varepsilon), y - y_\varepsilon)_H + (h'_\varepsilon(u_\varepsilon), u - u_\varepsilon)_U + \\ & (y_\varepsilon - y_\alpha, y - y_\varepsilon)_V + (u_\varepsilon - u_\alpha, u - u_\varepsilon)_U + (z_\varepsilon - z_\alpha, z - z_\varepsilon)_H + \\ & \langle q_\varepsilon, Ay - Bu + z - f - (Ay_\varepsilon - Bu_\varepsilon + z_\varepsilon - f) \rangle + \\ & \lambda_\varepsilon \left[\langle \Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon, y - y_\varepsilon \rangle + \left(\Phi_\varepsilon^*(z_\varepsilon) - y_\varepsilon, z - z_\varepsilon \right)_H \right] \geq 0 . \end{aligned}$$

Using the definition of q_ε and that $\varepsilon \langle q_\varepsilon, \Lambda_V q_\varepsilon \rangle \geq 0$ we get

$$\begin{aligned} & \langle -q_\varepsilon, Ay - Bu + z - f \rangle - \lambda_\varepsilon \left[\langle \Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon, y - y_\varepsilon \rangle + \left(\Phi_\varepsilon^*(z_\varepsilon) - y_\varepsilon, z - z_\varepsilon \right)_H \right] \\ & \leq \\ & (g'_\varepsilon(y_\varepsilon), y - y_\varepsilon)_H + (h'_\varepsilon(u_\varepsilon), u - u_\varepsilon)_U \\ & + (y_\varepsilon - y_\alpha, y - y_\varepsilon)_V + (u_\varepsilon - u_\alpha, u - u_\varepsilon)_U + (z_\varepsilon - z_\alpha, z - z_\varepsilon)_H . \end{aligned}$$

The right-hand side term is bounded since $(y_\varepsilon, u_\varepsilon, z_\varepsilon) \rightarrow (y_\alpha, u_\alpha, z_\alpha)$ strongly in $V \times U \times H$, and $(g'_\varepsilon(y_\varepsilon), h'_\varepsilon(u_\varepsilon)) \rightarrow (y_\alpha^g, u_\alpha^h) \in \partial g(y_\alpha) \times \partial h(u_\alpha)$ weakly in $H \times U$ (because of Theorem 4.2 iii. and the continuity of g and h). The bounding constant σ depends only of (y, u, z) . So we have, for all $\varepsilon > 0$ small enough,

$$(5.1) \quad \begin{aligned} & -\lambda_\varepsilon \left[\langle \Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon, y - y_\varepsilon \rangle + \left(\Phi_\varepsilon^*(z_\varepsilon) - y_\varepsilon, z - z_\varepsilon \right)_H \right] \\ & - \langle q_\varepsilon, Ay - Bu + z - f \rangle \leq \sigma(y, u, z) . \end{aligned}$$

We first estimate the real number λ_ε . If the solution $(y_\alpha, u_\alpha, z_\alpha)$ is such that the non-convex constraint is inactive, i.e.

$$\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H - \alpha = G(y_\alpha, z_\alpha) < 0 ,$$

then convergence results yield

$$\exists \varepsilon_o > 0, \forall \varepsilon \leq \varepsilon_o, \Phi_\varepsilon(y_\varepsilon) + \Phi_\varepsilon^*(z_\varepsilon) - (y_\varepsilon, z_\varepsilon)_H - \alpha < 0 ,$$

as well and $\lambda_\varepsilon = 0$; hence the limit $\lambda_\alpha = 0$.

Now, we investigate the case when the constraint is active that is

$$\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H - \alpha = G(y_\alpha, z_\alpha) = 0 .$$

Let us assume the following condition (H_1) :

$$(H_1) \quad \begin{aligned} & \forall \alpha \text{ such that } G(y_\alpha, z_\alpha) = 0, \forall (y^*, z^*) \in \partial\Phi^*(z_\alpha) \times \partial\Phi(y_\alpha) \\ & \exists (\tilde{y}, \tilde{u}, \tilde{z}) \in K \times U_{ad} \times B_R^* \text{ such that } A\tilde{y} = B\tilde{u} + f - \tilde{z} \text{ and} \\ (5.2) & [\Phi(y_\alpha) - \Phi(y^*) - \langle y_\alpha - y^*, \tilde{z} \rangle] + [\Phi^*(z_\alpha) - \Phi^*(z^*) - \langle z_\alpha - z^*, \tilde{y} \rangle] < 2\alpha. \end{aligned}$$

REMARK 5.1. *Relation (5.2) is indeed equivalent to*

$$\langle y_\alpha - y^*, \tilde{z} - z_\alpha \rangle + \langle z_\alpha - z^*, \tilde{y} - y_\alpha \rangle > 0$$

as we shall prove it later. Moreover, in our very case, $\langle y_\alpha - y^*, \tilde{z} \rangle = (y_\alpha - y^*, \tilde{z})_H$ since $\tilde{z} \in H$.

THEOREM 5.1. *Assume (H_1) ; then λ_ε is bounded by a constant independent of ε and we may extract a subsequence converging to $\lambda_\alpha \in \mathbb{R}^+$.*

Proof - If α is such that $G(y_\alpha, z_\alpha) < 0$ we have already seen that $\lambda_\alpha = 0$.

If $G(y_\alpha, z_\alpha) = 0$, we use (H_1) . Let be $(y^*, z^*) \in \partial\Phi^*(z_\alpha) \times \partial\Phi(y_\alpha) \subset V \times V'$ given by Corollary 4.1. Let us apply relation (5.1) with the triple $(\tilde{y}, \tilde{u}, \tilde{z})$, given by (H_1) . We get

$$(5.3) \quad \lambda_\varepsilon \left[\langle z_\varepsilon - \Phi'_\varepsilon(y_\varepsilon), \tilde{y} - y_\varepsilon \rangle + \left(y_\varepsilon - \Phi_\varepsilon^{*'}(z_\varepsilon), \tilde{z} - z_\varepsilon \right)_H \right] \leq \tilde{C}$$

and

$$\Phi(y_\alpha) - \Phi(y^*) - (y_\alpha - y^*, \tilde{z})_H + \Phi^*(z_\alpha) - \Phi^*(z^*) - \langle z_\alpha - z^*, \tilde{y} \rangle < 2\alpha.$$

As $y^* \in \partial\Phi^*(z_\alpha)$ and $z^* \in \partial\Phi(y_\alpha)$ we have $\Phi(y^*) + \Phi^*(z_\alpha) = (y^*, z_\alpha)_H$ and $\Phi^*(z^*) + \Phi(y_\alpha) = \langle z^*, y_\alpha \rangle$.

Moreover we are in the case where $\Phi(y_\alpha) + \Phi^*(z_\alpha) = (y_\alpha, z_\alpha)_H + \alpha$, so that (5.2) is equivalent to

$$\rho = (y_\alpha - y^*, \tilde{z} - z_\alpha)_H + \langle z_\alpha - z^*, \tilde{y} - y_\alpha \rangle > 0 ,$$

as mentioned in Remark 5.1.

Convergence results given in Theorem 4.3 and Corollary 4.1 imply that

$$\lim_{\varepsilon \rightarrow 0} \left(y_\varepsilon - \Phi_\varepsilon^{*'}(z_\varepsilon), \tilde{z} - z_\varepsilon \right)_H + \langle z_\varepsilon - \Phi'_\varepsilon(y_\varepsilon), \tilde{y} - y_\varepsilon \rangle = \rho > 0 .$$

So, there exists $\varepsilon_o > 0$ such that for all $0 < \varepsilon < \varepsilon_o$ we have

$$\left(y_\varepsilon - \Phi_\varepsilon^*(z_\varepsilon), \tilde{z} - z_\varepsilon \right)_H + \langle z_\varepsilon - \Phi'_\varepsilon(y_\varepsilon), \tilde{y} - y_\varepsilon \rangle \geq \frac{\rho}{2} .$$

Then relation (5.3) gives

$$\forall \varepsilon < \varepsilon_o \quad 0 \leq \frac{\rho}{2} \lambda_\varepsilon \leq \tilde{C} .$$

So λ_ε is bounded and converges to some $\lambda_\alpha \in \mathbb{R}^+$ (extracting a subsequence).

■

It remains to bound q_ε . Following [7] we assume the (qualification) condition (H_2) :

$$(H_2) \quad \left\{ \begin{array}{l} \exists W \text{ separable Banach subspace such that} \\ \quad W \subset V' \text{ continuously and densely ,} \\ \exists \mathcal{M} \subset K \times U_{ad} \times B_R^* \text{ bounded in } V \times U \times H, \text{ such that} \\ \quad 0 \in \text{Int}_W \mathcal{T}(\mathcal{M}) \text{ in } W\text{-topology} \\ \quad \text{where } \mathcal{T}(y, u, z) = Ay - Bu - f + z . \end{array} \right.$$

More precisely, $0 \in \text{Int}_W \mathcal{T}(\mathcal{M})$ means the existence of $\rho > 0$ such that

$$\forall \xi \in W, \quad |\xi|_W \leq 1, \quad \exists (y_\xi, u_\xi, z_\xi) \in \mathcal{M} \text{ such that } Ay_\xi = Bu_\xi + f - z_\xi + \rho \xi .$$

THEOREM 5.2. *Assume (H_1) and (H_2) ; then q_ε is bounded in W' and one may extract a subsequence converging weak* to q_α in W' .*

Proof - Let be $\rho > 0$ given by (H_2) and $\xi \in W$ such that $|\xi|_W \leq 1$. We use relation (5.1) with (y_ξ, u_ξ, z_ξ) and we get

$$\langle -q_\varepsilon, \rho \xi \rangle \leq C_1 + C_2 \lambda_\varepsilon ,$$

where C_1 and C_2 are constants dependent only of (y_ξ, u_ξ, z_ξ) . Assumption (H_1) provides a bound for λ_ε and \mathcal{M} is bounded. So there exists a constant C (depending only of \mathcal{M}) such that

$$\forall \xi \in W, \quad |\xi|_W \leq 1 \quad \langle q_\varepsilon, \xi \rangle_{W', W} \leq C ;$$

(as $W \subset V'$ and $q_\varepsilon \in V$ then $q_\varepsilon \in W'$). Thus q_ε is bounded in W' . ■

Now, we are able to pass to the limit in the penalized optimality system with respect to ε .

5.2. Optimality Conditions for (\mathcal{P}_α) .

THEOREM 5.3. *Let be $\alpha > 0$ and assume (H_1) and (H_2) ; then there exists $(y_\alpha^g, u_\alpha^h, z_\alpha^*, y_\alpha^*) \in \partial g(y_\alpha) \times \partial h(u_\alpha) \times \partial \Phi(y_\alpha) \times \partial \Phi^*(z_\alpha) \subset H \times U \times V' \times V$ and $(q_\alpha, \lambda_\alpha) \in W' \times \mathbb{R}^+$ such that*

$$(5.4) \quad \begin{array}{l} \forall y \in K \text{ such that } A(y - y_\alpha) \in W \\ (y_\alpha^g, y - y_\alpha)_H + \langle q_\alpha, A(y - y_\alpha) \rangle_{W', W} + \lambda_\alpha \langle z_\alpha^* - z_\alpha, y - y_\alpha \rangle_{V', V} \geq 0 , \end{array}$$

$$(5.5) \quad \forall u \in U_{ad} \text{ such that } B(u - u_\alpha) \in W \\ (u_\alpha^h, u - u_\alpha)_U - \langle q_\alpha, B(u - u_\alpha) \rangle_{W', W} \geq 0 ,$$

$$(5.6) \quad \forall z \in B_R^* \text{ such that } z - z_\alpha \in W \\ \langle q_\alpha, z - z_\alpha \rangle_{W', W} + \lambda_\alpha (y_\alpha^* - y_\alpha, z - z_\alpha)_H \geq 0 ,$$

$$(5.7) \quad \lambda_\alpha [\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H - \alpha] = 0 .$$

Proof - Let be $y \in K$ such that $A(y - y_\alpha) \in W$, $u \in U_{ad}$ such that $B(u - u_\alpha) \in W$ and $z \in B_R^*$ such that $z - z_\alpha \in W$. We use relations (4.6-4.8) with these test functions and add them to get :

$$(g'_\varepsilon(y_\varepsilon), y - y_\varepsilon)_H + (h'_\varepsilon(u_\varepsilon), u - u_\varepsilon)_U + \\ (y_\varepsilon - y_\alpha, y - y_\varepsilon)_V + (u_\varepsilon - u_\alpha, u - u_\varepsilon)_U + (z_\varepsilon - z_\alpha, z - z_\varepsilon)_H + \\ \langle q_\varepsilon, Ay - Bu + z - f - (Ay_\varepsilon - Bu_\varepsilon + z_\varepsilon - f) \rangle + \\ \lambda_\varepsilon \left[\langle \Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon, y - y_\varepsilon \rangle + \left(\Phi_{\varepsilon}^{*'}(z_\varepsilon) - y_\varepsilon, z - z_\varepsilon \right)_H \right] \geq 0 ,$$

that is, as subsection 2.1,

$$(g'_\varepsilon(y_\varepsilon), y - y_\varepsilon)_H + (h'_\varepsilon(u_\varepsilon), u - u_\varepsilon)_U + \\ (u_\varepsilon - u_\alpha, u - u_\varepsilon)_U + (z_\varepsilon - z_\alpha, z - z_\varepsilon)_H + \\ \langle q_\varepsilon, A(y - y_\alpha) - B(u - u_\alpha) + z - z_\alpha \rangle_{W', W} + \\ \lambda_\varepsilon \left[\langle \Phi'_\varepsilon(y_\varepsilon) - z_\varepsilon, y - y_\varepsilon \rangle + \left(\Phi_{\varepsilon}^{*'}(z_\varepsilon) - y_\varepsilon, z - z_\varepsilon \right)_H \right] \geq 0 .$$

As g and h are continuous and finite everywhere, $(g'_\varepsilon(y_\varepsilon), h'_\varepsilon(u_\varepsilon))$ converges towards some $(y_\alpha^g, u_\alpha^h) \in \partial g(y_\alpha) \times \partial h(y_\alpha)$. Then we may pass to the limit in the above relation to infer

$$(y_\alpha^g, y - y_\alpha)_H + (u_\alpha^h, u - u_\alpha)_U + \\ \langle q_\alpha, A(y - y_\alpha) - B(u - u_\alpha) + z - z_\alpha \rangle_{W', W} + \\ \lambda_\alpha [\langle z_\alpha^* - z_\alpha, y - y_\alpha \rangle + (y_\alpha^* - y_\alpha, z - z_\alpha)_H] \geq 0 .$$

Taking in turn $y = y_\alpha$, $u = u_\alpha$ and $z = z_\alpha$ we obtain relations (5.4)-(5.6).

Finally if $\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H - \alpha < 0$ we have seen that $\lambda_\alpha = 0$. So relation (5.7) is satisfied. \blacksquare

REMARK 5.2. As we already mentioned it in Remark 4.3, equation (5.6) is equivalent to

$$(5.8) \quad -q_\alpha - \lambda_\alpha (y_\alpha^* - y_\alpha) \in N_{B_R^* \cap (z_\alpha + W)}(z_\alpha) .$$

COROLLARY 5.1. With assumptions of Theorem 5.3, there exists $(y_\alpha^g, u_\alpha^h) \in \partial g(y_\alpha) \times \partial h(u_\alpha) \subset H \times U$ and $(q_\alpha, \lambda_\alpha) \in W' \times \mathbb{R}^+$ such that

$$(5.9) \quad \forall y \in K \text{ s.t. } A(y - y_\alpha) \in W \\ (y_\alpha^g, y - y_\alpha)_H + \langle q_\alpha, A(y - y_\alpha) \rangle_{W', W} + \\ \lambda_\alpha [\Phi(y) + \Phi^*(z_\alpha) - (y, z_\alpha)_H - \alpha] \geq 0 ,$$

$$\begin{aligned}
(5.10) \quad & \forall u \in U_{ad} \text{ s.t. } B(u - u_\alpha) \in W \quad \left(u_\alpha^h, u - u_\alpha \right)_U - \langle q_\alpha, B(u - u_\alpha) \rangle_{W', W} \geq 0, \\
& \forall z \in B_R^* \text{ s.t. } z - z_\alpha \in W \\
& \langle q_\alpha, z - z_\alpha \rangle_{W', W} + \lambda_\alpha [\Phi(y_\alpha) + \Phi^*(z) - (y_\alpha, z)_H - \alpha] \geq 0, \\
& \lambda_\alpha [\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H - \alpha] = 0.
\end{aligned}$$

Proof - Theorem 5.3 gives $(z_\alpha^*, y_\alpha^*) \in \partial\Phi(y_\alpha) \times \partial\Phi^*(z_\alpha) \subset V' \times V$ such that relations (5.4) and (5.6) are satisfied.

As $z_\alpha^* \in \partial\Phi(y_\alpha)$ we get, for all $y \in K$ such that $A(y - y_\alpha) \in W$

$$\Phi(y) - \Phi(y_\alpha) \geq \langle z_\alpha^*, y - y_\alpha \rangle,$$

so that relation (5.4) becomes

$$(y_\alpha^g, y - y_\alpha)_H + \langle q_\alpha, A(y - y_\alpha) \rangle_{W', W} + \lambda_\alpha [\Phi(y) - \Phi(y_\alpha) - (z_\alpha, y - y_\alpha)_H] \geq 0.$$

Using (5.7) we obtain relation (5.9). We can show similarly relation (5.10). \blacksquare

COROLLARY 5.2. *With assumptions of the previous theorem and if g and h are Gâteaux-differentiable, there exists $(q_\alpha, \lambda_\alpha) \in W' \times \mathbb{R}^+$ such that*

$$\begin{aligned}
& \forall y \in K \text{ s.t. } A(y - y_\alpha) \in W \\
& (g'(y_\alpha), y - y_\alpha)_H + \langle q_\alpha, A(y - y_\alpha) \rangle_{W', W} + \lambda_\alpha [\Phi(y) + \Phi^*(z_\alpha) - (z_\alpha, y)_H - \alpha] \geq 0,
\end{aligned}$$

$$\forall u \in U_{ad} \text{ s.t. } B(u - u_\alpha) \in W \quad (h'(u_\alpha), u - u_\alpha)_U - \langle q_\alpha, B(u - u_\alpha) \rangle_{W', W} \geq 0,$$

$$\begin{aligned}
& \forall z \in B_R^* \text{ s.t. } z - z_\alpha \in W \\
& \langle q_\alpha, z - z_\alpha \rangle_{W', W} + \lambda_\alpha [\Phi(y_\alpha) + \Phi^*(z) - (z, y_\alpha)_H - \alpha] \geq 0, \\
& \lambda_\alpha [\Phi(y_\alpha) + \Phi^*(z_\alpha) - (y_\alpha, z_\alpha)_H - \alpha] = 0.
\end{aligned}$$

REMARK 5.3. *The natural idea would be now to study the asymptotic behavior of the previous optimality system when $\alpha \rightarrow 0$. Unfortunately, we would have to set an “(H1)-like” assumption with $\alpha = 0$, to be able to pass to the limit in the α -optimality system. This is impossible since the interior of the feasible domain of \mathcal{P} is empty because of the non convex equality constraint and an assumption like (H1) with $\alpha = 0$ would be never ensured. However, as we have already mentioned, this relaxed approach is sufficient for numerical applications.*

6. Example of The Obstacle Problem. In this section, we study an example where the variational inequality leads to an obstacle problem.

Let Ω be an open, bounded subset of \mathbb{R}^n with a smooth boundary $\partial\Omega$. We consider a bilinear form $a(\cdot, \cdot)$ defined on $H_o^1(\Omega) \times H_o^1(\Omega)$ and A the continuous linear operator from $H_o^1(\Omega)$ to $H^{-1}(\Omega)$ associated to a such that

$$(6.1) \quad \left\{ \begin{array}{l} Ay = - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j}y) + a_0(x)y \text{ with} \\ a_{ij}, a_0 \in C^2(\bar{\Omega}) \text{ for } i, j = 1, \dots, n, \inf \{a_0(x) \mid x \in \bar{\Omega}\} > 0 \\ \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta \sum_{i=1}^n \xi_i^2, \forall x \in \bar{\Omega}, \forall \xi \in \mathbb{R}^n, \delta > 0, \end{array} \right.$$

We shall denote $\| \cdot \|$, the $L^2(\Omega)$ -norm, (\cdot, \cdot) the $L^2(\Omega)$ -scalar product and $\langle \cdot, \cdot \rangle$ any duality-product. We set

$$V = H_o^1(\Omega), \quad H = L^2(\Omega), \quad D_H(A) = H^2(\Omega) \cap H_o^1(\Omega), \quad U = L^2(\Omega) \text{ and } B = Id_{L^2(\Omega)}. \blacksquare$$

Let us set also

$$K = V \text{ and } C^+ = \{y \mid y \in H_o^1(\Omega), y \geq 0 \text{ a.e. in } \Omega\}.$$

The convex function Φ is the indicator function I_+ of C^+ , Then Φ^* is the indicator function I_- of the negative cone C^- of H^{-1} and we have already mentioned that Φ and Φ^* satisfy condition (4.2). Then we get as a state equation

$$(6.2) \quad Ay = f + v - z \text{ in } \Omega, y = 0 \text{ on } \Gamma,$$

with f , v and z belong to $L^2(\Omega)$ (because of the regularity result mentioned in Section 1.) The constraint $z \in \partial\Phi(y)$ becomes

$$y \geq 0, \quad z \leq 0, \quad (y, z) = 0,$$

and the α -inequality constraint $\Phi(y) + \Phi^*(z) - (y, z) \leq \alpha$ gives :

$$y \geq 0, \quad z \leq 0, \quad (y, -z) \leq \alpha.$$

We set $\xi = -z$, so that the original control problem is defined as follows (see [5])

$$\min \left\{ J(y, v) = \frac{1}{2} \int_{\Omega} (y - z_d)^2 dx + \frac{M}{2} \int_{\Omega} v^2 dx \right\}, \quad (\mathcal{P})$$

$$(6.3) \quad Ay = f + v + \xi \text{ in } \Omega, y = 0 \text{ on } \Gamma,$$

$$(6.4) \quad (y, v, \xi) \in \mathcal{D},$$

where

$$\mathcal{D} = \{(y, v, \xi) \in H_o^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mid v \in U_{ad}, y \geq 0, \xi \geq 0, (y, \xi) = 0\}$$

and $z_d \in L^2(\Omega)$; The relaxed problem is

$$(\mathcal{P}^\alpha) \quad \min J(y, v),$$

$$(6.5) \quad Ay = v + \xi \text{ in } \Omega, \ y \in H_o^1(\Omega),$$

$$(6.6) \quad (y, v, \xi) \in \mathcal{D}_\alpha^R,$$

where

$$\mathcal{D}_\alpha^R = \{(y, v, \xi) \in H_o^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \mid v \in U_{ad}, \ y \geq 0, \ \xi \geq 0, \ \|\xi\| \leq R, \ (y, \xi) \leq \alpha\}. \blacksquare$$

The results of the previous section may be applied with $W = L^p(\Omega)$ and we get :

THEOREM 6.1. *Assume*

$$\begin{aligned} & \forall \alpha \text{ such that } \langle y_\alpha, \xi_\alpha \rangle = \alpha, \\ & \exists (\tilde{y}, \tilde{v}, \tilde{\xi}) \in C^+ \times U_{ad} \times B_R^* \text{ such that} \\ & A\tilde{y} = \tilde{v} + \tilde{\xi} \quad \text{and} \quad (\tilde{y}, \xi_\alpha) + (y_\alpha, \tilde{\xi}) < 2\alpha. \end{aligned} \quad (\mathcal{H}_1)$$

and

$$\begin{aligned} & \exists p \in [1, +\infty[, \ \exists \rho > 0, \ \forall \chi \in L^p(\Omega), \ \|\chi\|_{L^p(\Omega)} \leq 1, \\ & \exists (y_\chi, v_\chi, \xi_\chi) \text{ bounded in } C^+ \times U_{ad} \times B_R^* \text{ (independently of } \chi), \\ & \text{such that } Ay_\chi = v_\chi + \xi_\chi + \rho\chi \text{ in } \Omega. \end{aligned} \quad (\mathcal{H}_2)$$

and let $(y_\alpha, v_\alpha, \xi_\alpha)$ be a solution of (\mathcal{P}^α) ; then a lagrange multiplier $(q_\alpha, \lambda_\alpha) \in L^{p'}(\Omega) \times \mathbb{R}^+$ exists, such that

$$(6.7) \quad \begin{aligned} & \forall y \in C^+ \text{ such that } A(y - y_\alpha) \in L^p(\Omega) \\ & (y_\alpha - z_d, y - y_\alpha) + \langle q_\alpha, A(y - y_\alpha) \rangle + \lambda_\alpha (\xi_\alpha, y - y_\alpha) \geq 0, \end{aligned}$$

$$(6.8) \quad \forall v \in U_{ad}, \ v - v_\alpha \in L^p(\Omega) \quad \langle Mv_\alpha - q_\alpha, v - v_\alpha \rangle \geq 0,$$

$$(6.9) \quad \forall \xi \in B_R^*, \ \xi - \xi_\alpha \in L^p(\Omega) \quad \langle \lambda_\alpha y_\alpha - q_\alpha, \xi - \xi_\alpha \rangle \geq 0,$$

$$(6.10) \quad \lambda_\alpha ((y_\alpha, \xi_\alpha) - \alpha) = 0.$$

For more details one can refer to [5]. We just mention that assumptions (\mathcal{H}_1) and (\mathcal{H}_2) are satisfied for instance if $U_{ad} = L^2(\Omega)$ or $U_{ad} = \{v \in L^2(\Omega) \mid v \geq \psi \geq 0 \text{ a.e. in } \Omega\}$.

7. Conclusion. As already mentioned at the beginning of this paper, we have in mind the numerical aspects of the question: that is why we have underlined that the “relaxed” problem \mathcal{P}_α is a good approximation of the original problem. Now, we think that the main tool for a good numerical approach for such problems is the (necessary) optimality conditions that we have obtained in Theorem 5.3. They allow to interpret the optimal solution as the first argument of the saddle-point of a linearized lagrangian function, though the problem is not convex. We have developed this point of view and presented some algorithms in [6], for the case of the obstacle problem. The numerical behavior of these methods is quite nice.

On the other hand, though we have not tested methods using Yosida approximation, we believe that the use of penalization is not helpful for numerics. It seems to be too much unstable (because of the suitable choice of the parameter ε) and we think it is only a theoretical tool.

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