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► **To cite this version:**

Binh-Minh Bui-Xuan, Michel Habib, Vincent Limouzy, Fabien de Montgolfier. A new tractable combinatorial decomposition. 2006. hal-00022957v1

HAL Id: hal-00022957

<https://hal.science/hal-00022957v1>

Preprint submitted on 16 Apr 2006 (v1), last revised 26 Jun 2007 (v3)

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Algorithmic Aspects of a General Modular Decomposition Theory

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Abstract. This paper introduces the ulmods, a generalisation of modules for the homogeneous relations. We first present some properties of the ulmod family, then show that, if the homogeneous relation fulfills some natural axioms, the ulmod family has a unique decomposition tree. Under a certain size assumption we show that this tree can be computed in polynomial time. We apply this theory to a new tournament decomposition and a graph decomposition. In both cases, the polynomial decomposition tree computing time becomes linear. Moreover, we characterise completely decomposable relations and give necessary and sufficient condition for testing if a given homogeneous relation corresponds to a graph or to a tournament, with a polynomial-time test. Finally we give polynomial-time algorithms for the maximal ulmods and the undecomposability of a relation, and conclude with further applications of homogeneous relations.

1 Introduction

Many authors have worked around modular decomposition [6,2,13,5] or its variations [3,10,14]. In the companion paper [1] was introduced an abstract framework for modular decomposition based on the so-called homogeneous relations, combinatorial structures more general than graphs. We study here a decomposition scheme which generalises both modular decomposition [6,13] and bi-join decomposition [14] from this viewpoint, and leads to a new tournament decomposition.

A module M of a graph is such that no exterior vertex can distinguish any pair of vertices belonging to M . Although the modules of homogeneous relations inherit many interesting properties from graph modules [1], they do not necessarily satisfy the following essential one. One can *shrink* a whole graph module M into one single vertex m : if some vertex of M distinguishes two exterior vertices, then so does every vertex of M and so does m . Actually, this property is the basis of many divide-and-conquer paradigms derived from the modular decomposition framework, such as the computations of weighted maximal stable set or clique set, and graph colouring [12,7]. That is what motivates us to study the family of *ulmods* – subsets of elements that fulfil by definition the above property.

Our central notion, namely the ulmods, fits in the “decomposition frame with the intersection and transitivity properties” of Cunningham [3] (also known under the different formalisms of “bipartitive families” [4] or “unrooted set families” [10]) as soon as two axioms, the *quotient* and *co-quotient* properties, are fulfilled. This has important structural consequences, both theoretical and algorithmic.

As an application of this new general decomposition scheme we exhibit a new decomposition of tournaments which greatly generalises modular decomposition and moreover we propose a linear time algorithm to handle this decomposition. Another direct application is the bi-join decomposition [14]. For some important cases of ulmodar decomposition (including the above bi-join decomposition and tournament decomposition) we prove that a nice transformation can be made in order to reduce the problem into a general modular decomposition as studied in [1]. This transformation is known as **Seidel switch** and it is an important non trivial transformation in graphs.

For the general ulmodar decomposition, we propose $O(|X|^3)$ time algorithms. More precisely, if the family has the *self-complemented* property, then it has a compact representation as a tree with $|X|$ leaves. This tree can be computed in $O(|X|^3)$ time. And if it has not this property, as no polynomial-size “basis” of the family is known, we can only compute the maximal ulmods in $O(|X|^3)$ time.

2 Ulmodar Decomposition Theory

Throughout this section X is a finite set. The family of all subsets of X is denoted by $\mathcal{P}(X)$. A *reflectless* triple is $(x, y, z) \subseteq X^3$ with $x \neq y$ and $x \neq z$. Reflectless triples will be denoted $(x|yz)$ instead of (x, y, z) since the first element does not play the same role. Let H be a relation over the reflectless triples of X . Given $s \in X$, the relation H_s is a binary relation on X defined as $H_s(x, y)$ if and only if $H(s|xy)$. In [1] are defined homogeneous relations along with their modules.

Definition 1 (Homogeneous relation). H is a homogeneous relation on X if, for all $s \in X$, H_s is an equivalence relation on $X \setminus \{s\}$:

- (Symmetry): $\forall s, x, y \in X, H(s|xy) \Leftrightarrow H(s|yx)$.
- (Reflexivity): $\forall s, x \in X, s \neq x \Rightarrow H(s|xx)$.
- (Transitivity): $\forall s, x, y, z \in X, H(s|xy) \wedge H(s|yz) \Rightarrow H(s|xz)$.

Definition 2 (Modules). Let H be a homogeneous relation on X . $M \subseteq X$ is a module of H if

$$\forall m, m' \in M \quad \forall x \in X \setminus M \quad H(x|mm')$$

The family of modules of H is denoted \mathcal{M}_H or \mathcal{M} if not ambiguous. M is a trivial module if $|M| \leq 1$ or $M = X$. H is prime w.r.t. modules, or M-prime, if \mathcal{M}_H is reduced to the trivial modules.

If $\neg H(x|yz)$ we say that x distinguishes y from z . The notions of homogeneity and distinction can be applied to graph. Indeed, there is a natural homogeneous relation associated to the case of (directed) graphs as follows.

Definition 3 (Standard homogeneous relation of digraphs). The homogeneous relation $H(G)$ of a directed graph $G = (X, E)$ is defined such that, for all $a, b, c \in X$, $H(a|bc)$ is true if and only if the two following conditions hold:

1. either both b and c or none of them are in-neighbours of a , and
2. either both b and c or none of them are out-neighbours of a .

In other word, $H(a|bc)$ tells if a “sees” b and c in the same way. Of course this definition also holds for undirected graphs, tournaments, and can also be extended to 2-structures [5].

Proposition 1. For a graph G , and H its standard homogeneous relation, the modules of H are the modules of G in the usual sense [6,13].

Let us now introduce the central notion of this paper, which, thanks to Proposition 2, can be seen as a proper generalisation of modules.

Definition 4 (Ulmods). A subset U of X is an ulmod of H if

$$\forall u, u' \in U \quad \forall x, x' \in X \setminus U \quad H(u|xx') \Leftrightarrow H(u'|xx')$$

The family of ulmods of H is denoted \mathcal{U}_H or \mathcal{U} if not ambiguous. U is a trivial ulmod if $|U| \leq 1$ or $|U| \geq |X| - 1$. H is prime w.r.t. ulmods, or U-prime, if \mathcal{U}_H is reduced to the trivial ulmods.

Proposition 2. Let H be a homogeneous relation. If M is a module then $X \setminus M$ is an ulmod.

Proof. No element of $X \setminus M$ can distinguish two elements of the module M . □

It should be noticed that the converse is not always true. The ulmods are in some sense the dual of the modules: a module is a set that is not distinguished by the outside, and an ulmod is a set that distinguishes its outside in the same way. The following proposition links ulmods to the 1-intersecting families framework as defined in [8].

Proposition 3. Let \mathcal{U} be the family of ulmods w.r.t. to H . For any two ulmods $U, U' \in \mathcal{U}$, if $U \cap U' \neq \emptyset$ then $U \cup U' \in \mathcal{U}$.

The ulmod family of an arbitrary homogeneous relation has thus, as far as we know, few structural properties. We shall now survey some homogeneous relations with additional properties, making the ulmod family behave in a more tractable manner.

Definition 5 (Quotient Property). The relation H has the quotient property if there are no $m, m', x, x' \in X$ such that

$$H(m|xx') \wedge H(m'|xx') \wedge H(x|mm') \wedge \neg H(x'|mm')$$

Proposition 4. If H fulfils the Quotient Property then every module of H is an ulmod.

Notice that the converse of Proposition 4 does not necessarily hold.

Definition 6 (Co-quotient Property). The relation H has the co-quotient property if there are no $m, m', x, x' \in X$ such that

$$\neg H(m|xx') \wedge \neg H(m'|xx') \wedge H(x|mm') \wedge \neg H(x'|mm')$$

Definition 7 (Self-complementation). A family \mathcal{F} of subsets of X is self-complemented if $A \in \mathcal{F}$ implies $X \setminus A \in \mathcal{F}$.

Proposition 5. If the relation H fullfils the Quotient property and the Co-quotient property then the family \mathcal{U} of ulmods of H is self-complemented.

Proof. Let us assume that U is an ulmod and $X \setminus U$ is not, i.e. there are two elements x and x' of $X \setminus U$, and two elements m and m' of X such that $H(x|mm')$ but $\neg H(x'|mm')$. As U is an ulmod, either both m and m' distinguish x from x' (i.e. $\neg H(m|xx')$ and $\neg H(m'|xx')$) or none of m and m' distinguish x from x' (i.e. $H(m|xx')$ and $H(m'|xx')$). The first case is a forbidden quadruplet for the co-quotient property, and the second for the quotient property. \square

3 Self-complemented Ulmodar Families

We have seen that ulmodar families are closed under union of intersecting subsets. But self-complemented ulmodar families have stronger properties that we shall now survey. Our two main applications, the decompositions on graphs and on tournaments described below, are indeed based on such ulmodar families.

3.1 Unique Decomposition Tree property

We shall now describe a general decomposition framework which allows us to prove the existence of a unique decomposition tree for ulmodar decomposition.

$\{X_i^1, X_i^2\}$ is a *bipartition* of X if $X_i^1 \cup X_i^2 = X$ and $X_i^1 \cap X_i^2 = \emptyset$. Two bipartitions $\{X_i^1, X_i^2\}$ and $\{X_j^1, X_j^2\}$ *overlap* if for all $a, b = 1, 2$ the four intersections $X_i^a \cap X_j^b$ are not empty. Let $\mathcal{B} = \{\{X_i^1, X_i^2\}_{i \in 1..k}\}$ be a family of k bipartitions of X . The *strong* bipartitions of X do not overlap any other bipartition. A bipartition is *trivial* if one part has only one element. Clearly, the trivial bipartitions contained in any family of bipartitions are strong.

The following results on bipartitions can be found in [3] under the name of “decomposition frame with the intersection and transitivity properties”, in [4] under the name of “bipartitive families” (the formalism used here) and in [10] under the name of “unrooted set families”.

Proposition 6. If \mathcal{B} contains all trivial bipartitions of X , then there exists a unique tree $T(\mathcal{B})$

- with $|X|$ leaves, each leaf being labelled by an element of X .
- such that each edge e of $T(\mathcal{B})$ correspond to a strong bipartition of \mathcal{B} : the leaf labels of the two connected components of $T - e$ are exactly the two parts of a strong bipartition, and the converse also holds.

Let N be a node of $T(\mathcal{B})$ of degree k . The labels of the leaves of the connected components of $T - N$ form a partition $X_1..X_k$ of X . For $I \subset \{1..k\}$ with $1 < |I| < k$, the bipartition $B(I)$ is $\{\bigcup_{i \in I} X_i, X \setminus \bigcup_{i \in I} X_i\}$.

Definition 8 (Bipartitive Family). *A family of bipartitions is a bipartitive family if it contains all the trivial bipartitions and if, for two overlapping bipartitions $\{X_i^1, X_i^2\}$ and $\{X_j^1, X_j^2\}$, the four bipartitions $\{X_i^a \cup X_j^b, X \setminus (X_i^a \cup X_j^b)\}$ (for all $a, b = 1, 2$) belong to \mathcal{B} .*

Theorem 1. *If \mathcal{B} is a bipartitive family, the nodes of $T(\mathcal{B})$ can be labelled complete, circular or prime, and the children of the circular nodes can be ordered in such a way that*

- *If N is a complete node, for any $I \subset \{1..k\}$ such that $1 < |I| < k$, $B(I) \in \mathcal{B}$*
- *If N is a circular node, for any interval $I = [a..b]$ of $\{1..k\}$ such that $1 < |b - a| < k$, $B(I) \in \mathcal{B}$*
- *If N is a prime node, for any element $I = \{a\}$ of $\{1..k\}$ $B(I) \in \mathcal{B}$*
- *There are no more bipartitions in \mathcal{B} than the ones described above*

For a bipartitive family \mathcal{B} , the labelled tree $T(\mathcal{B})$ is thus an $O(|X|)$ -sized representation of \mathcal{B} , while the family \mathcal{B} can have up to $2^{|X|-1} - 1$ bipartitions of $|X|$ elements each. Furthermore, it allows to perform some algorithmic operations efficiently on \mathcal{B} .

The self-complemented ulmodar families fit in this formalism. Indeed let us consider the family $\mathcal{U}'(H) = \{\{U, X \setminus U\} \mid U \text{ is an ulmod of } H\}$.

Theorem 2. *$\mathcal{U}'(H)$ is a bipartitive family.*

Proof. As $\mathcal{U}'(H)$ is self-complemented each part of a bipartition is an ulmod, furthermore the bipartitions $\{U, X \setminus U\}$ and $\{V, X \setminus V\}$ overlap (in the bipartition sense) iff U and V overlap (in the set sense). According to Proposition 3 if U and V overlap $U \cup V$ is an ulmod, and therefore $\{U \cup V, X \setminus (U \cup V)\} \in \mathcal{U}'(H)$. Self-complementation gives the results for the three other bipartitions needed. \square

Corollary 1. *There exists a unique $O(|X|)$ -sized decomposition tree that give a description of all possible ulmods of a homogeneous relation H .*

This tree is henceforth called *ulmodar decomposition tree*. Notice that it is an unrooted tree, unlike the modular decomposition tree.

3.2 Self-Complemented Relations of Local Congruency 2

Now we have seen that the self-complemented ulmodar families are well structured, we shall focus on a more specific class of homogeneous relation, namely Local Congruency 2 homogeneous relations, where there is a nice transformation from the ulmods to the modules of another relation.

Definition 9 (Local congruency). *Let H be a homogeneous relation on X . For $x \in X$, the congruency of x is the maximal number of elements that x pairwise distinguishes. In other words, it is the number of equivalence classes of H_x . The local congruency of H is the maximum congruency of the elements of X .*

It is not hard to be convinced that

Proposition 7. *The homogeneous relation of a directed graph has local congruency 4, and the homogeneous relations of undirected graphs or tournaments have local congruency 2.*

The Seidel switch at a given vertex x was defined in [15] for graphs as exchanging (or complementing) edges and non-edges between the neighbourhood of x and its non-neighbourhood. This can be extended to local congruency 2 homogeneous relations. For convenience, if H is a homogeneous relation on X and $s \in X$, we also refer to the equivalence classes of H_s as H_s^1, \dots, H_s^k .

Definition 10 (Seidel switch). Let H be a homogeneous relation of local congruency 2 on X , and s an element of X . The Seidel switch at s transforms H into the homogeneous relation $H(s)$ on $X \setminus \{s\}$ defined as follows.

$$\forall x \in X \setminus \{s\}, H(s)_x^1 = (H_x^1 \Delta H_s^1) \setminus \{s\} \text{ and } H(s)_x^2 = (H_x^2 \Delta H_s^1) \setminus \{s\}$$

where $A \Delta B$ denotes the symmetric difference of A and B .

Theorem 3. Let H be a homogeneous relation local congruency 2 on X such that \mathcal{U}_H is self-complemented. Let s be a member of X , and $U \subseteq X$ a subset containing s . Then, U is an ulmod of H if and only if $M = X \setminus U$ is a module of the Seidel switch $H(s)$.

Proof. Let $C = H_s^1 \cap M$ and $D = H_s^2 \cap M$. Since H is of local congruency 2, $\{C, D\}$ is a partition of M . Let $a \in U \setminus \{s\}$.

Suppose that U is an ulmod of H . Then, for all $y, z \notin U$, $H(a|yz)$ if and only if $H(s|yz)$. In other words, C is included in one class among H_a^1 and H_a^2 while D is included in the other class. As $C \subseteq H_s^1$ and $D \cap H_s^1 = \emptyset$, $C \cup D$ is included in one among the two classes $H(s)_a^i = H_a^i \Delta H_s^1$ ($i \in \{1, 2\}$). Hence, $M = C \cup D$ is a module of $H(s)$.

Conversely, if M is a module of $H(s)$, then $C \cup D$ is included in either $H(s)_a^1$ or $H(s)_a^2$. Moreover, the definition of the Seidel switch can also be written as $H_a^i = H(s)_a^i \Delta H_s^1$ for $i \in \{1, 2\}$. Therefore, C is included in one class among H_a^1 and H_a^2 while D is included in the other class. In other words, for all $a \in U \setminus \{s\}$, and $y, z \notin U$, $H(a|yz)$ if and only if $H(s|yz)$. This implies for all $a, b \in U$, and $y, z \notin U$, $H(a|yz)$ if and only if $H(b|yz)$ and U is therefore an ulmod. \square

Modular decomposition trees have well-known properties [2,13]. They are rooted trees whose leaves are in one-to-one correspondence with elements of X . A node of the modular decomposition tree is exactly a strong module, a module that overlap (in the set sense) no other modules. For a node N let $F_1..F_k$ be the leaf-sets of its k children in the tree. When the family of ulmods of H is bipartitive as it is the case in Theorem 3, the family of modules of any Seidel switch of H is a partitive set family [2], also known as *rooted set family* [10], we have:

Theorem 4. The nodes of a modular decomposition tree T can be labelled complete, linear or prime, and the children of the linear nodes can be ordered in such a way that

- If N is a complete node, for any $I \subset \{1..k\}$ such that $1 < |I| < k$, $\bigcup_{i \in I} F_i$ is a module
- If N is a linear node, for any interval $I = [a..b]$ of $\{1..k\}$ such that $1 < |b - a| < k$, $\bigcup_{i \in I} F_i$ is a module
- If N is a prime node, for any element $I = \{a\}$ of $\{1..k\}$ $\bigcup_{i \in I} F_i$ is a module
- There are no more modules than the ones described above

The relationships between the ulmodar decomposition tree of H and the modular decomposition tree of $H(v)$ are very tight:

Proposition 8. Let H be a homogeneous relation on X and s an element of X . The ulmodar decomposition tree \mathcal{T}_H of H is the same than the modular decomposition tree $\mathcal{T}_H(s)$ of the Seidel switch $H(s)$ of H at s , excepted that

- the two trees are exactly the same (same nodes and edges) excepted that the leaf with label s is missing in $\mathcal{T}_H(s)$ but present in \mathcal{T}_H
- The node of \mathcal{T}_H that is adjacent to the leaf s corresponds to the root of $\mathcal{T}_H(s)$
- A circular node of \mathcal{T}_U corresponds to a linear node of $\mathcal{T}_M(s)$. The orderings of the children are the same.

Proof. This is a consequence of Theorem 3. Every strong module of $H(s)$ give a strong bipartition of \mathcal{T}_H , and the converse is true. Then for a node N of the modular decomposition tree, for any union $\bigcup_{i \in I} F_i$ of leaf-sets of children there is a bipartition $\{\bigcup_{i \in I} F_i, X \setminus (\bigcup_{i \in I} F_i) = B(I)\}$ using the notations defined above. For each bipartition of ulmods of H , the part that contain s is dropped and the other part is put in the family of modules of $H(s)$. \square

A proof similar (and more detailed) with this result can be found in [10]. That article indeed describes the relationship between the consecutive-ones ordering and the circular-ones ordering of a boolean matrix, but the results (described in [10] as the transformation of a PQ-tree into a PC-tree) are the same. Notice that the modular decomposition tree of H can be trivial, while the one of its Seidel switch at s may be not.

Corollary 2. *The ulmodar decomposition tree of a self-complemented homogeneous relation of local congruency 2 on X can be computed in $O(|X|^3)$ time.*

Proof. This can be done using a Seidel switch and then the $O(|X|^3)$ -time modular decomposition algorithm for homogeneous relations of [1].

3.3 Composition operations and totally decomposable structures

We present here a generic operation to obtain every homogeneous relation totally decomposable provided that the relation fulfils the 2 local congruency property and the self-complemented property. *Totally decomposable* means that every subset of X with more than four elements contains a nontrivial ulmod.

Let A and B be two disjoint sets, and H_A and H_B be two totally decomposable homogeneous relations on (respectively) A and B . H_A and H_B are totally decomposable w.r.t. the ulmodar decomposition.

The generic operation will be noted $R = H_A \otimes H_B$. It is defined so that H_A and H_B are two ulmods of R . To do this, we can proceed as follows. Partition B into two arbitrary classes B_1 and B_2 ($B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \emptyset$). Then, for each $a_i \in A$ decide which "role" B_1 (resp. B_2) will play, i.e. in which class of H_A B_1 will belong.

As we will see later every totally decomposable homogeneous relation can be generated using this operation. We shall decompose this operation into three disjoint cases. An illustration of the following composition operations is given in figure 1.

First of all the \otimes_{1-1} operation consists in having only one class for each part (A and B). For example in tournaments, for H_A there is only one class of arcs connected to B : either all arcs are outgoing or all are incoming. Thus from H_A 's point of view every element of B lye in the same class. It is obviously the same thing from B 's point of view.

The \otimes_{1-2} operation means H_A has only one class linked to B , and H_B has two classes linked to A . To obtain this we partition A (resp B) into two classes A_1 and A_2 , and we choose if A_1 plays the first "role" for B or the converse, thus for B there are two classes containing A , and for A there is only one class containing elements of B . In clear for tournaments A_1 may be the set of arcs outgoing from B , or the set of arcs incoming to B .

And finally the \otimes_{2-2} operation is that both H_A and H_B have two classes. To do that, we partition B (w.l.o.g.) into two classes B_1 and B_2 . There exists at least two elements of A , say a_i and a_j , such that B_1 represents the first "role" for a_i , but the second "role" for a_j .

Remark 1 *Every operations described above are commutative, and associative. Furthermore the operations \otimes_{1-1} and \otimes_{1-2} generate all homogeneous relation totally decomposable w.r.t. modular decomposition.*

We can observe that the operation $H_A \otimes_{1-1} H_B$ produces two modules H_A and H_B , the operation $H_A \otimes_{1-2} H_B$ produces an ulmod on the subset A and a module on the subset B . Finally the operation $H_A \otimes_{2-2} H_B$ produces two ulmods.

Theorem 5. *With the the operations described above, we can generate any self-complemented homogeneous relation with local congruency two which is totally decomposable w.r.t. the ulmodar decomposition, starting with self complemented homogeneous relation with local congruency two of two and three elements.*

Proof. Let $\mathcal{H}_{2,3} = \{H, H \text{ self-complemented homogeneous relation with local congruency two on } X \text{ with } 2 \leq |X| \leq 3\}$ and let $\mathcal{I} = \{H, H \text{ self-complemented homogeneous relation with local congruency two totally decomposable}\}$, $\mathcal{C} = \{\otimes_{1-1}, \otimes_{1-2}, \otimes_{2-2}\}$, \mathcal{H} be the closure on $\mathcal{H}_{2,3}$ under \mathcal{C} operations. Let us prove that $\mathcal{I} = \mathcal{H}$. $\mathcal{H} \subseteq \mathcal{I}$ is given by definition. Conversely the sketch of proof for $\mathcal{I} \subseteq \mathcal{H}$ is as follows. Let H_X be

a self-complemented homogeneous relation with local congruency two totally decomposable. We proceed by induction on the size of the set X . When $2 \leq |X| \leq 3$ the property obviously holds. It suffices to apply the induction hypothesis on two disjoint smaller subrelations H_A and H_B , with $X = A \cup B$. A and B are nontrivial ulmods, because H_X fulfils the self-complemented property. Induction hypothesis on A and B implies $H_A, H_B \in \mathcal{H}$. As H_X fulfils the two local congruency property $H_X = H_A \otimes H_B$ with $\otimes \in C$ thus $H_X \in \mathcal{H}$. \square

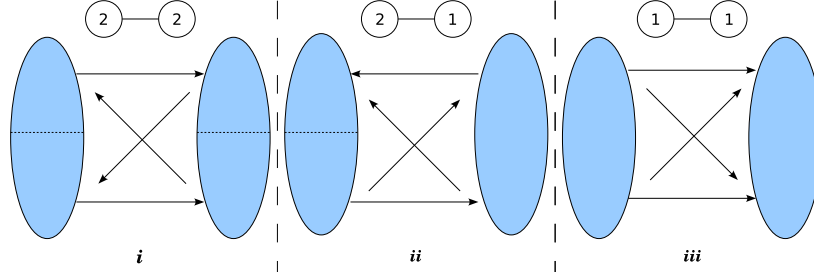


Fig. 1. Composition operations: example applied to tournaments, *i*) \otimes_{2-2} operation, *ii*) \otimes_{1-2} operation and *iii*) \otimes_{1-1} operation

4 Application: a New Tournament Decomposition

In this section we present a new tournament decomposition: the ulmodar decomposition. It is indeed the ulmodar decomposition of the standard homogeneous relation of the tournament. Actually this decomposition is more powerful than general decomposition, because every module of a tournament is an ulmod, while ulmodar decomposition is able to decompose M -prime tournaments (those having no nontrivial modules). It is well-known that

Proposition 9. – *If T is an M -prime tournament then M contains an induced cycle with 3 vertices*
– *If T is totally decomposable w.r.t. modular decomposition then it contains no induced cycle with 3 vertices (it is a transitive tournament)*

It allows us to deduce:

Proposition 10. – *If T is an U -prime tournament then M contains one of the induced subgraph described in Figure 2*
– *If T is totally decomposable w.r.t. ulmodar decomposition then it is free of the subtournaments described in Figure 2*

Proof. Thanks to Theorem 3, T is U -prime iff for any vertex v a Seidel switch at v gives an M -prime tournament. Thanks to Proposition 9, one just has to check all the four-vertices tournament where a Seidel switch on a vertex produces the cycle with 3 vertices. It is tedious but no hard. \square

We can deduce from Proposition 8 some very interesting properties of the ulmodar decomposition of tournaments. We have:

Corollary 3. *The ulmodar decomposition tree of a tournament has no complete node.*

Corollary 4. *There exists a circular permutation of the vertices of the tournament such that every ulmod of the tournament is a factor (interval) of this circular permutation*

Proof. Any traversal of the ulmodar decomposition tree, respecting the order of the sons of a circular node, orders the leaf labels into the desired circular permutation. \square

This result was already known for modular decomposition [4]: there exists a (not circular) permutation of the vertices whose every module of the tournament is a factor. It is called *factorizing permutation*.

Proposition 11. *The ulmodar decomposition tree of a tournament can be computed in $O(|X|^2)$ time*

Proof. Again Theorem 3 says that one just has to perform a Seidel switch on a arbitrarily chosen vertex, then to compute the modular decomposition of the tournament. This can be done in linear (in fact $O(|X|^2)$) time using the algorithm from [11] \square

5 Application: Bi-join decomposition of undirected graphs

We have investigated in the previous section the ulmods of tournaments, and seen that they lead to a nice decomposition. The same is true for undirected graphs. The decomposition, however, was already published in [14]. We resume here the main results of that paper and establish the link with ulmods.

Definition 11 (bi-join). *A bi-join of a graph $G = (X, E)$ is a bipartition $\{X^1, X^2\}$ of the vertex-set such that the edges between X^1 and X^2 form at most two disjoint complete bipartite graphs, and that for each $i, j = 1, 2$ every vertex of X^i is adjacent to a vertex of X^j .*

Proposition 12. *If $\{X^1, X^2\}$ is a bi-join of a graph then both X^1 and X^2 are ulmods of G*

Proof. Let A (resp. C) be the vertices of X^1 (resp. X^2) adjacent with the first complete bipartite graph, and B (resp. D) be the other vertices of X^1 (resp. X^2). Any vertex of X^1 distinguishes a vertex of C from a vertex of D , but can not distinguish two vertices from C , nor two vertices from D . X^1 is thus an ulmod, and a similar proof holds for X^2 . \square

In [14] the Seidel switch was used to derive most of the properties:

Proposition 13. *Let G be a graph. $\{X^1, X^2\}$ is a bi-join of G iff for every $v \in X^1$ (resp. X^2) X^2 (resp. X^1) is a module of the Seidel switch $G(v)$.*

It may be used to prove the converse of Proposition 12:

Proposition 14. *If U is an ulmod of a graph $G = (X, E)$ then $\{M, X \setminus U\}$ is a bi-join of G .*

This is because the homogeneous relation of a tournament has local congruency 2 and is self-complemented (see Section 3.2).

Corollary 5. *The ulmodar decomposition of a graph is exactly its bi-join decomposition*

Among the consequences exposed in [14] are that the bi-join (thus ulmodar) decomposition tree has no *circular* nodes (but has *complete* ones), a characterisation of totally decomposable graphs in term of forbidden induced subgraphs (they are exactly the $\{C_5, \text{bull}, \text{gem}, \text{co-gem}\}$ -free graphs), and a proof that computing the bi-join (thus ulmodar) decomposition tree can be performed in linear time, since modular decomposition can be computed in linear time.



Fig. 2. Minimal U-Prime Configurations in tournaments = forbidden subgraphs of a tournament totally decomposable w.r.t. ulmodar decomposition

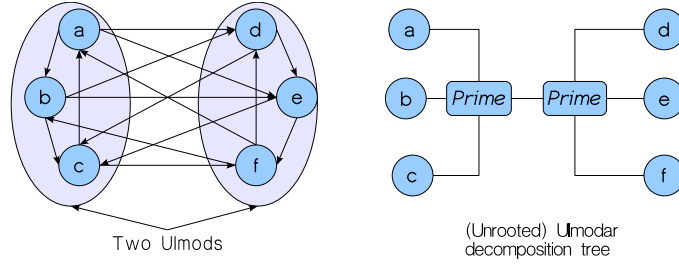


Fig. 3. An example of a prime (w.r.t. modular decomposition) tournament but not prime for the ulmodar decomposition together with its Unrooted Ulmodar Decomposition Tree associated.

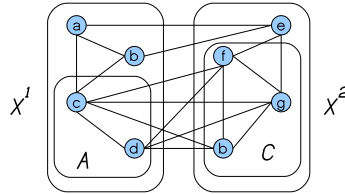


Fig. 4. Example of a bi-join of a graph

6 Characterisation of graphic homogeneous relations

Given a (directed) graph, a standard homogeneous relation is defined in Definition 3. This section focuses on the converse: given a homogeneous relation H , does it exist an undirected graph, or a tournament, admitting H as standard homogeneous relation ?

We define the restriction $H[A]$ of a ternary relation H on X to $A \subseteq X$ as $H[A] = H \cap (A \times A \times A)$. Let H be a homogeneous relation of local congruency 2. It means that any element has congruency 1 or 2, and for any $A \subset X$, $H[A]$ is of local congruency 1 or 2. We define H as a *graphic relation* if for any triple $\{a, b, c\}$, $H[\{a, b, c\}]$ has exactly 0 or 2 elements of congruency 2. H is defined as a *tournamental relation* if for any triple $\{a, b, c\}$, $H[\{a, b, c\}]$ has exactly 1 or 3 elements of congruency 2.

Theorem 6. H is the standard homogeneous relation of an undirected graph if and only if it is graphic. H is the standard homogeneous relation of a tournament if and only if it is tournamental.

Proof. It is straightforward to check that the standard homogeneous relation of any graph (resp. tournament) is graphic (resp. tournamental).

The converse for graphs can be proved as follows. Let H be a graphic homogeneous relation of local congruency 2 on X , and $x_1 \in X$. We define the matrix M as: $M(x_1, y) = 1$ if $y \in H_{x_1}^1$ and $M(x_1, y) = 0$ otherwise; for all $x \neq x_1$, $M(x, y) = 1$ if $y \in H_x^i$ and $M(x, y) = 0$ otherwise, where i is such that $x_1 \in H_x^i$. Suppose M not symmetric. Then, there exists $x \neq y$ both distinct to x_1 such that $M(x, y) = 1$ and $M(y, x) = 0$. But then $H[\{x_1, x, y\}]$ would have exactly 1 or 3 elements of congruency 2. Thus, M is a 0, 1 symmetric matrix and can be seen as the adjacency matrix of an undirected graph. It is then straightforward to verify that H is the standard relation of this graph.

The proof for tournaments is similar. We use the characterisation that the adjacency matrix of a tournament is anti-symmetric with no non-edges and no double arcs. \square

Corollary 6. It can be tested in $O(|X|^3)$ time if a self-complemented homogeneous relation H admits a graph G or a tournament T such that $H(G) = H$ or $H(T) = H$.

Proof. Just check for all triples the number of congruency one element of the restricted relation.

Notice that, for a graphic (resp. tournamental) relation H given as sets of (at most two) classes for each vertex, the corresponding graph (resp. tournament) can be built in $O(|X|)$ time. For graphs, one just has to decide which class of the first vertex v_1 is its neighbourhood. For any other vertex v , the class containing v_1 is its neighbourhood if v is a neighbour of v_1 , and else is its non-neighbourhood. Simply remove the non-neighbourhoods classes: the remaining ones are the adjacency lists of all vertices. A similar construction can be performed for tournaments, also in $O(|X|)$ time.

7 General Algorithms

We have seen that linear-time algorithms exist for computing the ulmodar decomposition of a undirected graph (see Section 5) or of a tournament (see Section 4), and an $O(|X|^3)$ -time algorithm for the more general case when the relation is self-complemented and of local congruency 2 (see Section 3.2). We shall now give algorithms for the general case. Since, as far as we know, we have no decomposition tree to compute, the only valuable objects to compute seem to be the maximal ulmods and the primality test.

7.1 Computation of maximal ulmods

Let S be a subset of X . As \mathcal{U} is closed under the union of overlapping elements, then the maximal (w.r.t. inclusion) ulmods included either in S or in $X \setminus S$ are a partition of X , denoted $MU(S)$. Notice that $MU(S) = MU(X \setminus S)$. $MU(S)$ gives an indication on how the ulmods are structured with respect to S , since an ulmod either is included in an ulmod of $MU(S)$ or properly intersects S or properly intersects $X \setminus S$ or is trivial.

Definition 12. Let H be a homogeneous relation over X and $E \subset X$. The binary relation R_E between two elements of E is defined as

$$R_E(x, y) \text{ if } \forall a, b \in (X \setminus E) \quad H(x|ab) \iff H(y|ab)$$

This relation clearly is an equivalence relation on E . Furthermore, E is an ulmod iff R_E contains only one equivalence class.

Definition 13. Let \mathcal{P} be a partition of X and C a part of \mathcal{P} . Let $C_1..C_k$ be the equivalence classes of R_C . $Refine(\mathcal{P}, C)$ is the partition \mathcal{P} where the part C is replaced with parts $C_1..C_k$.

Definition 14. A partition is unrefinable if for every part C of \mathcal{P} , we have $\mathcal{P} = Refine(\mathcal{P}, C)$. A refinement is proper if $\mathcal{P} \neq Refine(\mathcal{P}, C)$. Let \mathcal{P}_0 be a partition. A refinement sequence is a sequence $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k$ of partitions where $\mathcal{P}_{i+1} = Refine(\mathcal{P}_i, C_i)$ and C_i is a part of \mathcal{P}_i .

Lemma 1. In an unrefinable partition, all parts are ulmods.

The proof is immediate since C is an ulmod iff $\mathcal{P} = Refine(\mathcal{P}, C)$. [9] proves (for a more general case) that

Lemma 2. Let S be a subset of X . Any sequence of proper refinements, starting at partition $\mathcal{P}_0 = \{S, X \setminus S\}$, ends at the (unique unrefinable) partition $MU(S)$.

This lemma suggests a polynomial-time algorithm for computing $MU(S)$.

Theorem 7. It is possible to compute $MU(S)$ in $O(|X|^3)$ time.

Proof. We first show how to test for $R_C(x, y)$, then how to implement the $Refine(\mathcal{P}, C)$ procedure in $O(|X|^2)$ time, and then how to implement the whole algorithm in $O(|X|^3)$ time. Let us consider a partition \mathcal{P}_i and a class C of \mathcal{P}_i .

First compute, for every element x of C , a partition $\mathcal{H}(x, C) = \{P_x^1, \dots, P_x^{k(x)}\}$ of $X \setminus C$. It is the restriction of H_x to $X \setminus C$, i.e. $P_x^i = H_x^i \setminus C$. It is easy to build in $O(|X|)$ time for each element of C . Then we have $R_C(x, y)$ iff $H(x, C)$ is exactly the same partition than $H(y, C)$. It can be tested in $O(|X|)$ time, but performing this for each couple of elements of C would lead to an $O(|X|^3)$ time implementation of $Refine(\mathcal{P}, C)$.

Instead, we will use sorting in order to cluster the elements having the same partitions. Let L be the list of all parts of $H(x, C)$ (i.e. $L = (P_{x_1}^1, \dots, P_{x_1}^{k(x_1)}, P_{x_2}^1, \dots, P_{x_2}^{k(x_2)}, \dots, P_{x_{|C|}}^{k(x_{|C|})})$). Let $\{L_1 \dots L_k\}$ be the partition of L into duplicate classes (two parts P_x^i and P_y^j are equal iff they are in the same L_a). This partition can be computed by bucket sorting L using the first element of $X \setminus C$, then the second, and so on. This takes $O(|X|^2)$ time.

Let $\mathcal{Z}(s)$ be the partition of C such that x and y are in the same part iff the parts P_x^i of $H(x, C)$ containing s and the part P_y^j of $H(y, C)$ containing s are equal. Using the duplicate classes of $\{L_1 \dots L_k\}$ this equality can be tested in $O(1)$ time and thus $\mathcal{Z}(s)$ can be computed in $O(|X|)$ time.

Now we have $R_C(x, y)$ iff, for every s of $X \setminus C$, x and y are in the same class of $\mathcal{Z}(s)$. The partition of C into $C_1 \dots C_k$ computed by $Refine(\mathcal{P}, C)$ is thus the coarsest partition compatible with every partition $\mathcal{Z}(s)$. In other word, for each s each C_i is included in one part of $\mathcal{Z}(s)$, and C_i is maximal for this property. Using classical partition refining techniques [9], this can be computed in time linear in the size of the partitions, i.e. in $O(|X|^2)$ time. We just have computed $Refine(\mathcal{P}, C)$ in $O(|X|^2)$ time.

If $\mathcal{P} = Refine(\mathcal{P}, C)$ then C is an ulmod and the computation of $Refine(\mathcal{P}, C)$ is no more performed. This case occurs $|X|$ times at most. Else, C is split in new classes. As they are at most $|X|$ parts in the final partition, this case also occurs at most $|X|$ times. The whole algorithm thus runs in $O(|X|^3)$ time. \square

7.2 Testing for U-primality

By definition, if $MU(S)$ contains only trivial ulmod, then all nontrivial ulmods of X intersect both S and $X \setminus S$. This allow to test for primality of a homogeneous relation in polynomial time:

Theorem 8. *It is possible to check in $O(|X|^5)$ time if a homogeneous relation H is U-prime and, if not, to output a nontrivial ulmod.*

Proof. Just test if $MU(\{x, y\})$ is trivial for all pair of elements $\{x, y\}$. \square

Notice this is hard to do better than $O(|X|^5)$ using the MU algorithm as toolbox, since there exists homogeneous relations whose sole ulmod is $X \setminus \{x, y\}$.

8 Extensions and further developments

We have presented the ulmods and homogeneous relations focusing on graph theory field. But ulmods may be found in many other objects. Let us briefly present two of them.

8.1 Homogeneous relation based on a binary function

Let f be a binary function $X \times X \rightarrow Y$. The homogeneous relation *based on* f , written H_f , is defined as $H_f(s|ab)$ iff $f(s, a) = f(s, b)$ and $f(a, s) = f(b, s)$.

For instance on graphs f is the existence of an edge. On directed graph is the existence of an arc. And on a 2-structures $f(x, y)$ is the number of equivalence class of the couple (x, y) . It can also be seen as a colouring of the edge (x, y) .

Proposition 15. *The homogeneous relation H_f based on a binary function f has the quotient property.*

Proof. Let s, t, u and v be four elements of X . If $H_f(s|ab)$ and $H_f(t|ab)$ and $H_f(a|st)$ then $f(s, a) = f(s, b) = f(t, a) = f(t, b)$ and therefore $H_f(b|st)$. \square

Notice that weaker homogeneous relations can be defined from a binary function: the *left* homogeneous relation based on f, H_f^l , is defined as $H_f^l(s|ab)$ iff $f(s, a) = f(s, b)$. And the *right* homogeneous relation based on f, H_f^r , is defined as $H_f^r(s|ab)$ iff $f(a, s) = f(b, s)$. But these relation do not have the quotient properties, and have not the same ulmods. We have:

Proposition 16. *If M is an ulmod for H_f^r and for H_f^l then is an ulmod for H_f*

The proof is immediate from definition. Notice that the converse is not true. For instance for $X = \{a, b, c, d\}$ if $f(a, c) = f(a, d)$, $f(b, c) = f(b, d)$ and all other couples have pairwise different values, then $\{a, b\}$ is a ulmod for H_f but neither for H_f^l nor for H_f^r . If f is a symmetric function, then the three homogeneous relations of course are the same. This is true for graphs and for symmetric 2-structures, for instance.

8.2 Ulmods and monoid decompositions

If f is a law in some monoid X , henceforth denoted $x.y$, then a homogeneous relation H can be defined:

$$H(x|yz) \Leftrightarrow x.y = x.z$$

For instance, in $s \mathbb{Z}_6$ with the multiplication, $\{1, 5\}$ and $\{2, 4\}$ are ulmods.

Proposition 17. *The principal ideals of a ring are ulmods (w.r.t. its multiplication homogeneous relation)*

8.3 Further work

The ulmodar decomposition of directed graphs and 2-structures seems more difficult to survey since it is not a self-complemented family. We do not know for instance if a compact representation (like a tree) of this exponentially-sized family is possible. If not, this is a limitation on ulmodar algorithmics.

The U-primality test presented here is polynomial, but its asymptotic complexity can surely be reduced, especially when applied to particular combinatorial objects.

It seems that graphs and tournaments play a very similar role with respect to ulmodar decomposition. This could lead to some results in the tournament isomorphism problem.

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