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Jianya Liu, Emmanuel Royer, Jie Wu. On a conjecture of Montgomery-Vaughan on extreme values of automorphic L-functions at 1. Jean-Marie de Koninck, Andrew Granville et Florian Luca. Anatomy of integers, CRM, pp.217 à 245, 2008, CRM Proceedings and Lecture Notes 46. hal-00022853v3

HAL Id: hal-00022853

<https://hal.science/hal-00022853v3>

Submitted on 25 Jun 2007

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On a conjecture of Montgomery-Vaughan on extreme values of automorphic L -functions at 1

J.-Y. LIU, E. ROYER & J. WU

Abstract. In this paper, we prove a weaker form of a conjecture of Montgomery-Vaughan on extreme values of automorphic L -functions at 1.

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§ 1. Introduction

The automorphic L -functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer N and even integer k , denote by $H_k^*(N)$ the set of all newforms of level N and of weight k . It is known that

$$(1.1) \quad |H_k^*(N)| = \frac{k-1}{12} \varphi(N) + O((kN)^{2/3}),$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute. Let $m \geq 1$ be an integer and let $L(s, \text{sym}^m f)$ be the m th symmetric power L -function of $f \in H_k^*(N)$ normalised so that the critical strip is given by $0 < \Re s < 1$. The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [18] showed that the Sato-Tate conjecture is equivalent to $L(1 + i\tau, \text{sym}^m f) \neq 0$ for all $m \in \mathbb{N}$ and $\tau \in \mathbb{R}$. The distribution of the values $L(1, \text{sym}^m f)$ has received attention of many authors, including Goldfeld, Hoffstein & Lieman [2], Hoffstein & Lockhart [7], Luo [12], Royer [14, 15], Royer & Wu [16, 17], Cogdell & Michel [1], Habsieger & Royer [5] and Lau & Wu [10, 11]. In particular, Lau & Wu ([10], [11]) proved the following results:

(i) For every fixed integer $m \geq 1$, there are four positive constants A_m^\pm and B_m^\pm such that for any newform $f \in H_k^*(1)$, under the Great Riemann Hypothesis (GRH) for $L(s, \text{sym}^m f)$, we

have, for $k \rightarrow \infty$,

$$(1.2) \quad \{1 + o(1)\}(2B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + o(1)\}(2B_m^+ \log_2 k)^{A_m^+}.$$

Here (and in the sequel) \log_j denotes the j -fold iterated logarithm. For most values of m , the constants A_m^\pm and B_m^\pm can be explicitly evaluated, for example,

$$\begin{cases} A_m^+ = m + 1, & B_m^+ = e^\gamma & (m \in \mathbb{N}), \\ A_m^- = m + 1, & B_m^- = e^\gamma \zeta(2)^{-1} & (\text{odd } m), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^-, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, γ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product ([10], Theorem 3).

(ii) In the opposite direction, it was shown unconditionally that for $m \in \{1, 2, 3, 4\}$ there are newforms $f_m^\pm \in \mathbf{H}_k^*(1)$ such that for $k \rightarrow \infty$ ([10], Theorem 2),

$$(1.3) \quad \begin{cases} L(1, \text{sym}^m f_m^+) \geq \{1 + o(1)\}(B_m^+ \log_2 k)^{A_m^+}, \\ L(1, \text{sym}^m f_m^-) \leq \{1 + o(1)\}(B_m^- \log_2 k)^{-A_m^-}. \end{cases}$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.3)), an almost all result was established. Let $\varepsilon > 0$ be an arbitrarily small positive number, $m \in \{1, 2, 3, 4\}$ and $2 \mid k$. Then there is a subset \mathbf{E}_k^* of $\mathbf{H}_k^*(1)$ such that $|\mathbf{E}_k^*| \ll \mathbf{H}_k^*(1)e^{-(\log k)^{1/2-\varepsilon}}$ and for each $f \in \mathbf{H}_k^*(1) \setminus \mathbf{E}_k^*$, we have, for $k \rightarrow \infty$,

$$(1.4) \quad \{1 + O(\varepsilon_k)\}(B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + O(\varepsilon_k)\}(B_m^+ \log_2 k)^{A_m^+},$$

where $\varepsilon_k := (\log k)^{-\varepsilon}$ and the implied constants depend on ε only ([11], Corollary 2).

By comparing (1.3) with (1.4), the extreme values of $L(1, \text{sym}^m f)$ seem to be given by (1.3). Clearly it is interesting to investigate further the size of exceptional set \mathbf{E}_k^* . In the case of quadratic characters L -functions, Montgomery & Vaughan [13] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [4]. As Cogdell & Michel indicated in [1], it would be interesting to try to get, as close as possible, the analogues of the conjectures of Montgomery-Vaughan for automorphic L -functions. The analogue of Montgomery-Vaughan's first conjecture for the automorphic symmetric power L -functions can be stated as follows.

Conjecture. *Let $m \geq 1$ be a fixed integer and*

$$F_k(t, \text{sym}^m) := \frac{1}{|\mathbf{H}_k^*(1)|} \sum_{f \in \mathbf{H}_k^*(1), L(1, \text{sym}^m f) \geq (B_m^+ t)^{A_m^+}} 1,$$

$$G_k(t, \text{sym}^m) := \frac{1}{|\mathbf{H}_k^*(1)|} \sum_{f \in \mathbf{H}_k^*(1), L(1, \text{sym}^m f) \leq (B_m^- t)^{-A_m^-}} 1.$$

Then there are positive constants $c_i = c_i(m)$ ($i = 1, 2$) such that for $k \rightarrow \infty$,

$$(1.5) \quad \begin{cases} e^{-c_1(\log k)/\log_2 k} \ll F_k(\log_2 k, \text{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}, \\ e^{-c_1(\log k)/\log_2 k} \ll G_k(\log_2 k, \text{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}. \end{cases}$$

The aim of this paper is to prove a weaker form of this conjecture for $m = 1$. In this case, we write, for simplification of notation,

$$L(s, f) = L(s, \text{sym}^1 f), \quad F_k(t) = F_k(t, \text{sym}^1), \quad G_k(t) = G_k(t, \text{sym}^1).$$

In view of the trace formula of Petersson ([8], Theorem 3.6), it is more convenient to consider the weighted arithmetic distribution function. As usual, denote by

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1} \|f\|}$$

the harmonic weight in modular forms theory and define the weighted arithmetic distribution functions

$$\begin{aligned} \tilde{F}_k(t) &:= \left(\sum_{f \in \mathbb{H}_k^*(1)} \omega_f \right)^{-1} \sum_{f \in \mathbb{H}_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f, \\ \tilde{G}_k(t) &:= \left(\sum_{f \in \mathbb{H}_k^*(1)} \omega_f \right)^{-1} \sum_{f \in \mathbb{H}_k^*(1), L(1, f) \leq (6\pi^{-2} e^\gamma t)^{-2}} \omega_f. \end{aligned}$$

By using (1.1), the classical estimate

$$(1.6) \quad \sum_{f \in \mathbb{H}_k^*(1)} \omega_f = 1 + O(k^{-5/6})$$

and the bound of Goldfeld, Hoffstein & Lieman [2]:

$$(1.7) \quad 1/(k \log k) \ll \omega_f \ll (\log k)/k,$$

we easily see that

$$(1.8) \quad \begin{cases} \tilde{F}_k(t)/\log k \ll F_k(t) \ll \tilde{F}_k(t) \log k, \\ \tilde{G}_k(t)/\log k \ll G_k(t) \ll \tilde{G}_k(t) \log k. \end{cases}$$

This shows that in order to prove (1.5) it is sufficient to establish corresponding estimates of the same quality for $\tilde{F}_k(t)$ and $\tilde{G}_k(t)$.

Our main result is the following one.

Theorem 1. *For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the estimate*

$$(1.9) \quad \tilde{F}_k(t) = \{1 + \Delta_k(t)\} \exp \left\{ -\frac{e^{t-\gamma_0}}{t} \left(1 + O\left(\frac{1}{t}\right) \right) \right\}$$

holds uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where γ_0 is given by (1.24) below, $|\theta| \leq 1$ and

$$(1.10) \quad \begin{cases} \Delta_k(t) := \theta e^{t-T(k)-C} (t/T(k))^{1/2} + O_A(e^{-ce^{t/5}} + (\log k)^{-A}), \\ T(k) := \log_2 k - \frac{5}{2} \log_3 k - \log_4 k - 3C. \end{cases}$$

In particular there are two positive constants c_1 and c_2 such that

$$(1.11) \quad e^{-c_1(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \ll F_k(T(k)) \ll e^{-c_2(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}}.$$

The similar estimates for $\tilde{G}_k(t)$ and $G_k(T(k))$ hold also.

Remark 1. The estimates (1.11) of Theorem 1 can be considered as a weaker form of Montgomery-Vaughan's conjecture (1.5) for $m = 1$, since $T(k) \sim \log_2 k$ as $k \rightarrow \infty$. Moreover, if we could take $T(k) = \log_2 k$ in (1.11) then (1.9) would lead to the Montgomery-Vaughan's conjecture (1.5). Hence we fail from a shift

$$\frac{5}{2} \log_3 k + \log_4 k + 3C.$$

It seems however to be rather difficult to resolve completely this conjecture. One of the main difficulties is that there are no analogues of the quadratic reciprocity law and Graham-Ringrose's estimates for short characters sums of friable moduli [3], which have been exploited by Granville & Soundararajan [4].

In order to prove Theorem 1, we need to introduce a probabilistic model as in [1]. Consider a probability space (Ω, μ) , with measure μ . Let $\mathrm{SU}(2)^\natural$ be the set of conjugacy classes of $\mathrm{SU}(2)$. The group $\mathrm{SU}(2)$ is endowed with its Haar measure μ_H and

$$\mathrm{SU}(2)^\natural = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi] \right\} / \sim$$

is endowed with the Sato-Tate measure $d\mu_{\mathrm{st}}(\theta) := (2/\pi) \sin^2 \theta d\theta$, *i.e.*, the direct image of μ_H by the canonical projection $\mathrm{SU}(2) \rightarrow \mathrm{SU}(2)^\natural$. On the space (Ω, μ) , define a sequence indexed by the prime numbers, $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$ of random matrices taking values in $\mathrm{SU}(2)^\natural$, given by

$$g_p^\natural(\omega) := \begin{pmatrix} e^{i\vartheta_p(\omega)} & 0 \\ 0 & e^{-i\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each function $g_p^\natural(\omega)$ is distributed according to the Sato-Tate measure. This means that, for each integrable function $\phi : \mathrm{SU}(2)^\natural \rightarrow \mathbb{R}$, the expected value of $\phi \circ g_p^\natural$ is

$$\mathbb{E}(\phi \circ g_p^\natural) := \int_{\Omega} \phi \circ g_p^\natural(\omega) d\mu(\omega) = \int_0^\pi \phi \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta d\theta.$$

Moreover, we assume that the sequence $g^\natural(\omega)$ is made of independent random variables. This means that, for any sequence of integrable functions $\{G_p : \mathrm{SU}(2)^\natural \rightarrow \mathbb{R}\}_p$, we have

$$\begin{aligned} (1.12) \quad \mathbb{E} \left(\prod_p G_p \circ g_p^\natural \right) &:= \int_{\Omega} \prod_p G_p \circ g_p^\natural(\omega) d\mu(\omega) \\ &= \prod_p \int_{\Omega} G_p \circ g_p^\natural(\omega) d\mu(\omega) \\ &= \prod_p \int_0^\pi G_p \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta d\theta. \end{aligned}$$

Let I be the identity matrix. Then for $\Re s > \frac{1}{2}$, the random Euler product

$$L(s, g^\natural(\omega)) := \prod_p \det(I - p^{-s} g_p^\natural(\omega))^{-1} =: \prod_p L_p(s, g^\natural(\omega))$$

turns out to be absolutely convergent a.s.

Now we define our probabilistic distribution functions

$$\begin{cases} \Phi(t) := \mathrm{Prob}(\{L(1, g^\natural(\cdot)) \geq (e^\gamma t)^2\}), \\ \Psi(t) := \mathrm{Prob}(\{L(1, g^\natural(\cdot)) \leq (6\pi^{-2} e^\gamma t)^{-2}\}). \end{cases}$$

We shall prove Theorem 1 in two steps. The first one is to compare $\tilde{F}_k(t)$ with $\Phi(t)$ (resp. $\tilde{G}_k(t)$ with $\Psi(t)$).

Theorem 2. For any $A \geq 1$ there are two positive constants $c = c(A)$ and $C = C(A)$ such that the asymptotic formulas

$$(1.13) \quad \tilde{F}_k(t) = \Phi(t)\{1 + \Delta_k(t)\} \quad \text{and} \quad \tilde{G}_k(t) = \Psi(t)\{1 + \Delta_k(t)\}$$

hold uniformly for $k \geq 16, 2 \mid k$ and $t \leq T(k)$, where $\Delta_k(t)$ and $T(k)$ are defined by (1.10).

The second step of the proof of Theorem 1 is the evaluation of $\Phi(t)$ (resp. $\Psi(t)$). For this, we consider a truncated random Euler product

$$L(s, g^{\natural}(\omega); y) := \prod_{p \leq y} L_p(s, g^{\natural}(\omega))$$

and the corresponding distribution functions

$$\begin{cases} \Phi(t, y) := \text{Prob}(\{L(1, g^{\natural}(\omega); y) \geq (e^{\gamma t})^2\}), \\ \Psi(t, y) := \text{Prob}(\{L(1, g^{\natural}(\omega); y) \leq (6\pi^{-2}e^{\gamma t})^{-2}\}). \end{cases}$$

We have

$$(1.14) \quad \Phi(t) = \Phi(t, \infty) \quad \text{and} \quad \Psi(t) = \Psi(t, \infty).$$

We shall use the saddle-point method (introduced by Hildebrand & Tenenbaum [6]) to evaluate $\Phi(t, y)$ and $\Psi(t, y)$. For this, we need to introduce some notation. For $s \in \mathbb{C}$ and $y \geq 2$, define

$$(1.15) \quad E(s, y) := \mathbb{E}(L(1, g^{\natural}(\omega); y)^s) \quad \text{and} \quad E(s) := E(s, \infty),$$

where $\mathbb{E}(\cdot)$ denotes the expected value. We define also

$$(1.16) \quad \phi(s, y) := \log E(s, y), \quad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geq 0).$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation

$$(1.17) \quad \phi_1(\kappa, y) = 2(\log t + \gamma)$$

has a unique positive solution $\kappa = \kappa(t, y)$ and for each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \dots, \gamma_J$ such that the asymptotic formula

$$(1.18) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J \left(\frac{1}{t^{J+1}} + \frac{e^t t}{y \log y} \right) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, the constant γ_0 beign given by (1.24) below.

Finally write $\sigma_n := \phi_n(\kappa, y)$.

Theorem 3. *We have*

$$\Phi(t, y) = \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\}$$

uniformly for $t \geq 1$ and $y \geq 2e^t$.

Theorem 4. *For each integer $J \geq 1$, we have*

$$(1.19) \quad \Phi(t, y) = \exp \left\{ -\kappa \left[\sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}$$

uniformly for $t \geq 1$ and $y \geq 2e^t$, where the error term $R_J(\kappa, y)$ is given by

$$(1.20) \quad R_J(\kappa, y) := \frac{1}{(\log \kappa)^{J+1}} + \frac{\kappa}{y \log y}$$

and

$$(1.21) \quad a_j := \int_0^\infty \left(\frac{h(u)}{u} \right)' (\log u)^{j-1} du$$

with

$$(1.22) \quad h(u) := \begin{cases} \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) & \text{if } 0 \leq u < 1, \\ \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) - 2u & \text{if } u \geq 1. \end{cases}$$

As a corollary of Theorem 4, we can obtain an asymptotic development for $\log \Phi(t, y)$ in t^{-1} . In particular we see that the probabilistic distribution function $\Phi(t)$ decays double exponentially as $t \rightarrow \infty$.

Corollary 5. *For each integer $J \geq 1$, there are computable constants a_1^*, \dots, a_J^* such that the asymptotic formula*

$$(1.23) \quad \Phi(t, y) = \exp \left\{ -e^{t-\gamma_0} \left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J(R_J(e^t, y)) \right] \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$. Further we have

$$(1.24) \quad \gamma_0 := \frac{1}{2} \int_0^\infty \frac{h'(u)}{u} du, \quad a_1^* := 1, \quad a_2^* := \gamma_0 - \frac{\gamma_0^2}{2} - \int_0^\infty \frac{h(u)}{u^2} (\log u) du.$$

In particular for each integer $J \geq 1$, we have

$$(1.25) \quad \Phi(t) = \exp \left\{ -e^{t-\gamma_0} \left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J\left(\frac{1}{t^{J+1}}\right) \right] \right\}$$

uniformly for $t \geq 1$.

Remark 2. (i) The same results hold also for $\Psi(t, y)$.

(ii) Taking $t = \log_2 k$ and $J = 1$ in (1.25) of Corollary 5, we see that the probabilistic distribution function $\Phi(t)$ (resp. $\Psi(t)$) verifies Montgomery-Vaughan's conjecture (1.5). But (1.13) is too weak to derive this conjecture for $F_k(t)$ (resp. $G_k(t)$). This means that we must take $T(k) = \log_2 k$ in Theorem 2, which seems to be rather difficult.

(iii) Our method can be generalized (with a little extra effort) to prove that Theorems 1 and 2 hold for $L(1, \text{sym}^m f)$ for $m \geq 1$ (unconditionally when $m = 1, 2, 3, 4$ and under Cogdell-Michel's hypothesis $\text{Sym}^m(f)$ and $\text{LSZ}^m(1)$ [1] when $m \geq 5$) and that Theorems 3, 4 and Corollary 5 are true for $L(1, \text{sym}^m g^{\natural}(\omega); y)$ when $m \geq 1$.

Acknowledgements. We began working on this paper in November 2004 during the visit of the first author to l'Institut Elie Cartan de Nancy, and finished in January 2006 when the third author visited School of Mathematics and System Sciences of Shandong University. We are indebted to both institutions for invitations and support. The second and third authors want to thank the CRM at Montréal for its invitation. Finally we would express our sincere gratitude to Y.-K. Lau of the University of Hong Kong for valuable discussion.

§ 2. Expression of $E(s, y)$ and existence of saddle-point

The aim of this section is to prove the existence of the saddle-point $\kappa(t, y)$, defined by equation (1.17). The first step is to give an explicit expression of $E(s, y)$, which is (1.24) of [1]. For the convenience of readers, we state it here as a lemma.

Lemma 2.1. For prime p , real θ and complex number s , we define

$$(2.1) \quad D_p(\theta) := \prod_{0 \leq j \leq 1} (1 - e^{i(1-2j)\theta} p^{-1})^{-1} \quad \text{and} \quad E_p(s) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^s \sin^2 \theta \, d\theta.$$

Then for all $s \in \mathbb{C}$ and $y \geq 2$, we have

$$(2.2) \quad E(s, y) = \prod_{p \leq y} E_p(s).$$

Proof. Taking

$$G_p(M^{\natural}) = \begin{cases} \det(I - p^{-s'} M^{\natural})^{-s} & \text{if } p \leq y \\ 1 & \text{otherwise} \end{cases}$$

in (1.12), we get

$$\begin{aligned} \mathbb{E}(L(s', g^{\natural}(\omega); y)^s) &= \prod_{p \leq y} \mathbb{E}(L_p(s', g_p^{\natural}(\omega))^s) \\ &= \prod_{p \leq y} \int_{\Omega} \det(1 - p^{-s'} g_p^{\natural}(\omega))^{-s} \, d\mu(\omega) \\ &= \prod_{p \leq y} \frac{2}{\pi} \int_0^\pi (1 - 2p^{-s'} \cos \theta + p^{-2s'})^{-s} \sin^2 \theta \, d\theta. \end{aligned}$$

Taking $s' = 1$ and noticing (1.15) and (2.1), we get the desired result. \square

Lemma 2.2. *For all p and $\sigma > 0$, we have*

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 > 0.$$

In particular for all $\sigma > 0$ and $y \geq 2$, we have $\phi_2(\sigma, y) > 0$.

Proof. By using the definition (2.1) of $E_p(\sigma)$, it is easy to see that

$$\begin{aligned} E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 &= \frac{4}{\pi^2} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \int_0^\pi D_p(\theta)^\sigma \sin^2 \theta \, d\theta \\ &\quad - \left(\frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \right)^2 \\ &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma (\log^2 D_p(\theta_1) - \log D_p(\theta_1) \log D_p(\theta_2)) \times \\ &\quad \times \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2. \end{aligned}$$

In view of the symmetry in θ_1 and θ_2 , the same formula holds if we exchange the roles of θ_1 and θ_2 . Thus it follows that

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2 \left(\frac{D_p(\theta_1)}{D_p(\theta_2)} \right) \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2.$$

This proves the first assertion and the second follows immediately. \square

Lemma 2.3. *There is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution in σ . Denoting by $\kappa(t, y)$ this solution, we have $\kappa(t, y) \asymp e^t$ uniformly for $t \geq 4 \log c$ and $y \geq ce^t$.*

Proof. According to Lemma 4.3 below with the choice of $J = 1$, we have

$$\phi_1(\sigma, y) = 2(\log_2 \sigma + \gamma) + O(1/\log \sigma)$$

for $y \geq \sigma \geq 2$. Thus

$$\begin{aligned} \phi(ce^t, y) &= 2 \log(t + \log c) + 2\gamma + O\left(\frac{1}{t + \log c}\right) \\ &> 2 \log t + 2\gamma \end{aligned}$$

and

$$\begin{aligned} \phi(c^{-1}e^t, y) &= 2 \log(t - \log c) + 2\gamma + O\left(\frac{1}{t - \log c}\right) \\ &< 2 \log t + 2\gamma, \end{aligned}$$

provided that c is a large constant and $t \geq 4 \log c$. On the other hand, in view of Lemma 2.2, we know that for any $y \geq 2$, $\phi_1(\sigma, y)$ is an increasing function of σ in $(0, \infty)$. Hence the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution $\kappa(t, y)$ and $c^{-1}e^t \leq \kappa(t, y) \leq ce^t$ for $t \geq 4 \log c$ and $y \geq ce^t$. This completes the proof. \square

§ 3. Preliminary lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

Lemma 3.1. *Let $j \geq 0$ be a fixed real number. Then we have*

$$(3.1) \quad \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta \asymp_j e^{2u} u^{-(j+3/2)} \quad (u \geq 1).$$

The implied constant depends on j only.

Proof. First we write

$$\begin{aligned} \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta &= \int_0^{\pi/2} (e^{2u \cos \theta} (1 - \cos \theta)^j + e^{-2u \cos \theta} (1 + \cos \theta)^j) \sin^2 \theta \, d\theta \\ &= \int_0^1 (e^{2ut} (1-t)^j + e^{-2ut} (1+t)^j) (1-t^2)^{1/2} \, dt \\ &\asymp \int_0^1 e^{2ut} (1-t)^{j+1/2} \, dt + \int_0^1 e^{-2ut} (1-t)^{1/2} \, dt. \end{aligned}$$

By the change of variables $u(1-t) = v$, it follows that

$$\begin{aligned} \int_0^1 e^{2ut} (1-t)^{j+1/2} \, dt &= e^{2u} u^{-(j+3/2)} \int_0^u e^{-2v} v^{j+1/2} \, dv \\ &\asymp e^{2u} u^{-(j+3/2)}, \\ \int_0^1 e^{-2ut} (1-t)^{1/2} \, dt &\leq \int_0^1 e^{-2ut} \, dt \ll u^{-1}. \end{aligned}$$

We obtain the desired result by insertion of these estimates into the preceding relation. \square

Lemma 3.2. *Let $j \geq 0$ be an integer and*

$$(3.2) \quad E_{p,j}(\sigma) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma (1 - \cos \theta)^j \sin^2 \theta \, d\theta.$$

(In particular $E_{p,0}(\sigma) = E_p(\sigma)$.) Then we have

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du$$

and the estimate

$$(3.3) \quad E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j$$

holds uniformly for all primes p and $\sigma > 0$. Further if $p \geq \sigma \geq 0$, we have

$$(3.4) \quad E_p(\sigma) \asymp 1.$$

The implied constant in (3.3) depends on j only and the one in (3.4) is absolute.

Proof. By the change of variables $u = \sin^2(\theta/2)$, a simple computation shows that the first assertion is true. Obviously (3.3) holds for $j = 0$.

Now assume that it is true for j . An integration by parts leads to

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} du \\ &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left\{ \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \right\} \times \\ &\quad \times \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} du. \end{aligned}$$

On the other hand, we have

$$0 < u < 1 \Rightarrow \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \geq \left(1 + \frac{1}{p}\right)^{-2} \frac{4\sigma}{p} \geq \frac{16\sigma}{9p}.$$

Inserting it into the preceding estimate, we see that

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^{j+1} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} du \\ &\asymp_j \left(\frac{\sigma}{p}\right)^{j+1} E_{p,j+1}(\sigma). \end{aligned}$$

Thus (3.3) holds also for $j+1$.

Since $(1+1/p)^{-2} \leq D_p(\theta) \leq (1-1/p)^{-2}$ for all primes p and any $\theta \in \mathbb{R}$, we have $D_p(\theta)^\sigma \asymp 1$ uniformly for $p \geq \sigma \geq 0$ and $\theta \in \mathbb{R}$. This implies (3.4). \square

Introduce the function

$$(3.5) \quad g(u) := \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) \quad (u \geq 0)$$

and let $h(u)$ be defined as in (1.22). Clearly we have

$$(3.6) \quad h(u) = \begin{cases} g(u) & \text{if } 0 \leq u < 1, \\ g(u) - 2u & \text{if } u \geq 1, \end{cases}$$

$$(3.7) \quad h'(u) = \begin{cases} g'(u) & \text{if } 0 \leq u < 1, \\ g'(u) - 2 & \text{if } u \geq 1, \end{cases}$$

$$(3.8) \quad h''(u) = g''(u) \quad (u \geq 0, u \neq 1).$$

Lemma 3.3. *We have*

$$(3.9) \quad h(u) \asymp \begin{cases} u^2 & \text{if } 0 \leq u < 1, \\ \log(2u) & \text{if } u \geq 1, \end{cases}$$

$$(3.10) \quad h'(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-1} & \text{if } u \geq 1, \end{cases}$$

$$(3.11) \quad h''(u) \asymp \begin{cases} 1 & \text{if } 0 \leq u < 1, \\ u^{-2} & \text{if } u \geq 1, \end{cases}$$

$$(3.12) \quad h'''(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-3} & \text{if } u \geq 1. \end{cases}$$

Proof. When $0 \leq u < 1$, we have

$$e^{2u \cos \theta} = \sum_{n=0}^{\infty} \frac{(u \cos \theta)^n}{n!}.$$

From this we deduce that

$$(3.13) \quad \begin{aligned} h(u) &= \log \left(\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_0^{\pi} (\cos \theta)^n \sin^2 \theta \, d\theta \right) \\ &= \log \left(1 + \sum_{\ell=1}^{\infty} \frac{2 \cdot (2\ell - 1)!!}{(2\ell)!(2\ell + 2)!!} u^{2\ell} \right), \end{aligned}$$

where we have used the following facts:

$$\int_0^{\pi} (\cos \theta)^{2\ell+1} \sin^2 \theta \, d\theta = 0$$

and

$$\frac{2}{\pi} \int_0^{\pi} (\cos \theta)^{2\ell} \sin^2 \theta \, d\theta = \begin{cases} 1 & \text{if } \ell = 0, \\ 2 \frac{(2\ell - 1)!!}{(2\ell + 2)!!} & \text{if } \ell \geq 1 \end{cases}$$

and where $n!!$ denotes the product of all positive integer from 1 to n having same parity than n . Now we easily deduce, from (3.13), the desired results (3.9)–(3.12) in the case of $0 \leq u < 1$.

The estimates of (3.9)–(3.12) for $u > 1$ are simple consequences of (3.1), by noticing the following relations

$$\begin{aligned} h'(u) &= -2 \frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta}, \\ h''(u) &= 4 \frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta)^2 \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta} - 4 \left(\frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta} \right)^2. \end{aligned}$$

This completes the proof. □

§ 4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of $\phi_n(\sigma, y)$ for $n = 0, 1, 2, 3, 4$.

Lemma 4.1. *For any fixed integer $J \geq 1$, we have*

$$(4.1) \quad \phi_0(\sigma, y) = \sigma \left\{ 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 3$, where $R_J(\sigma, y)$ is defined as in (1.20) and

$$(4.2) \quad b_{j,0} := \int_0^{\infty} \frac{h(u)}{u^2} (\log u)^{j-1} \, du.$$

Proof. By the definition (2.1) of $D_p(\theta)$ and the one of $E_p(\sigma)$, it is easy to see that for $p \geq \sigma^{1/2}$, we have

$$(4.3) \quad D_p(\theta)^\sigma = e^{2(\sigma/p)\cos\theta} \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\},$$

$$(4.4) \quad E_p(\sigma) = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p)\cos\theta} \sin^2\theta \, d\theta.$$

From these, we deduce that

$$(4.5) \quad \sum_{\sigma^{1/2} < p \leq y} \log E_p(\sigma) = \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}/\log\sigma)$$

where $g(u)$ is defined as in (3.5).

In order to treat the sum over $p \leq \sigma$, we write

$$E_p(\sigma) = (1 - 1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi} \int_0^\pi \left\{ 1 + \frac{2(1 - \cos\theta)}{p} \left(1 - \frac{1}{p}\right)^{-2} \right\}^{-\sigma} \sin^2\theta \, d\theta.$$

By using the change of variables $u = \sin^2(\theta/2)$, we have

$$\begin{aligned} E_p^*(\sigma) &= \frac{8}{\pi} \int_0^\pi \left\{ 1 + \frac{4}{p} \left(1 - \frac{1}{p}\right)^{-2} \sin^2(\theta/2) \right\}^{-\sigma} \sin^2(\theta/2) \cos^2(\theta/2) \, d\theta \\ &\geq \frac{8}{\pi} \int_0^{p/2\sigma} \left\{ 1 + \frac{4}{p} \left(1 - \frac{1}{p}\right)^{-2} u \right\}^{-\sigma} \sqrt{u(1-u)} \, du \\ &\geq \frac{8}{\pi} \left(1 + \frac{8}{\sigma}\right)^{-\sigma} \int_0^{p/2\sigma} \sqrt{u(1-u)} \, du \\ &\geq C \left(\frac{p}{\sigma}\right)^{3/2}, \end{aligned}$$

where $C > 0$ is a constant. On the other hand, we have trivially $E_p^*(\sigma) \leq 1$ for all p and $\sigma > 0$. Thus $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$ for $p \leq \sigma^{1/2}$ and

$$(4.6) \quad \sum_{p \leq \sigma^{1/2}} |\log E_p^*(\sigma)| \ll \sum_{p \leq \sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}.$$

Combining (4.5) and (4.6), we can write

$$\sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma^{1/2}} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2} < p \leq \sigma} (2\sigma \log(1 - 1/p)^{-1} - 2\sigma/p) \ll \sum_{\sigma^{1/2} < p \leq \sigma} \sigma/p^2 \ll \sigma^{1/2}/\log\sigma,$$

the preceding estimate can be written as

$$(4.7) \quad \sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} h(\sigma/p) + O(\sigma^{1/2}).$$

By using the prime number theorem in the form

$$(4.8) \quad \pi(t) := \sum_{p \leq t} 1 = \int_2^t \frac{dv}{\log v} + O\left(te^{-8\sqrt{\log t}}\right),$$

it follows that

$$(4.9) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt + O(R_0),$$

where

$$\begin{aligned} R_0 &:= h\left(\frac{\sigma}{y}\right)ye^{-8\sqrt{\log y}} + h(\sigma^{1/2})\sigma^{1/2}e^{-4\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^y (\sigma/t)|h'(\sigma/t)|e^{-8\sqrt{\log t}} dt \\ &\ll \frac{\sigma^2}{y}e^{-8\sqrt{\log y}} + \sigma^{1/2}e^{-2\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^{\sigma} e^{-2\sqrt{\log t}} dt + \sigma^2 \int_{\sigma}^y \frac{e^{-8\sqrt{\log t}}}{t^2} dt \\ &\ll \sigma e^{-\sqrt{\log \sigma}} \end{aligned}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables $u = \sigma/t$ to write

$$\begin{aligned} \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt &= \sigma \int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du \\ &= \sigma \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du + O(R'_0) \end{aligned}$$

where

$$\begin{aligned} R'_0 &:= \sigma \int_0^{\sigma/y} \frac{|h(u)|}{u^2 \log(\sigma/u)} du + \sigma \int_0^{\sigma^{-1/2}} \frac{|h(u)|}{u^2 \log(\sigma/u)} du \\ &\ll \frac{\sigma^2}{y \log y} + \frac{\sigma^{1/2}}{\log \sigma}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du &= \frac{1}{\log \sigma} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2(1 - (\log u)/\log \sigma)} du \\ &= \sum_{j=1}^J \frac{1}{(\log \sigma)^j} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du + O\left(\frac{1}{(\log \sigma)^{J+1}}\right). \end{aligned}$$

Extending the interval of integration $[\sigma^{-1/2}, \sigma^{1/2}]$ to $(0, \infty)$ and bounding the contributions of $(0, \sigma^{-1/2}]$ and $[\sigma^{1/2}, \infty)$ by using (3.9) of Lemma 3.3, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du = b_{j,0} + O\left(\frac{(\log \sigma)^j}{\sigma^{1/2}}\right).$$

Combining these estimates, we find that

$$(4.10) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \sigma \left\{ \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}.$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$(4.11) \quad \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} = \log_2 \sigma + \gamma + O\left(e^{-2\sqrt{\log \sigma}}\right).$$

This completes the proof. \square

Remark 3. In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma, y) = \sigma \left\{ \log(B_1^+ \log \sigma)^{A_1^+} + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 3$. In the case $\sigma < 0$, a similar asymptotic formula (with A_1^-, B_1^- and corresponding $b_{j,0}^-$ in place of A_1^+, B_1^+ and $b_{j,0}$) can be established uniformly for $y \geq -\sigma \geq 3$. As indicated in the introduction, Lemma 4.1 can be easily generalised to the general case $m \geq 1$. Thus we give an improvement and generalisation of Corollaries A and C of [15], of Theorem B of [5], and an improvement of Theorem 1.12 of [1]. It is worthy to indicate that our method seems to be simpler and more natural.

Lemma 4.2. *We have*

$$(4.12) \quad \frac{E'_p(\sigma)}{E_p(\sigma)} = \begin{cases} \log D_p(0) + O\left(\frac{1}{\sigma}\right) & \text{for all } p \text{ and } \sigma > 0, \\ \frac{1}{2}g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2} + \frac{\sigma}{p^3}\right) & \text{if } p \geq \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.13) \quad \begin{aligned} E'_p(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \\ &= E_p(\sigma) \log D_p(0) + R', \end{aligned}$$

where

$$(4.14) \quad R' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log \left(\frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\left| \log \left(\frac{D_p(\theta)}{D_p(0)} \right) \right| = \left| -\log \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) \right| \leq \frac{2p(1 - \cos \theta)}{(p-1)^2} \leq \frac{8(1 - \cos \theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with $j = 1$ that

$$\frac{R'}{E_p(\sigma)} \ll \frac{E_{p,1}(\sigma)}{pE_p(\sigma)} \ll \frac{1}{\sigma}$$

for all p and $\sigma > 0$. This implies, via (4.13), the first estimate of (4.12).

We have

$$\begin{aligned} \log D_p(\theta) &= (\cos \theta)(2/p) + O(1/p^2) \\ &= (\cos \theta) \log D_p(0) + O(1/p^2). \end{aligned}$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for $p \geq \sigma^{1/2}$,

$$\begin{aligned} E'_p(\sigma) &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \left\{ (\cos \theta) \log D_p(0) + O\left(\frac{1}{p^2}\right) \right\} \sin^2 \theta \, d\theta \\ &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} (\cos \theta) \sin^2 \theta \, d\theta \log D_p(0) + O\left(\frac{E_p(\sigma)}{p^2}\right). \end{aligned}$$

From this and (4.4), we deduce

$$\frac{E'_p(\sigma)}{E_p(\sigma)} = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{1}{2} g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2}\right),$$

which implies the second estimate of (4.12). This completes the proof. \square

Lemma 4.3. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_1(\sigma, y) = 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y))$$

uniformly for $y \geq \sigma \geq 3$, where the constant $b_{j,1}$ is given by

$$(4.15) \quad b_{j,1} := \int_0^\infty \frac{h'(u)}{u} (\log u)^{j-1} \, du$$

and $R_J(\sigma, y)$ is defined as in (1.20).

Proof. We have

$$\phi_1(\sigma, y) = \sum_{p \leq y} E'_p(\sigma) / E_p(\sigma).$$

Using the first relation of (4.12) for $p \leq \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leq y$, we obtain

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma^{2/3}} \log D_p(0) + \frac{1}{2} \sum_{\sigma^{2/3} < p \leq y} g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{\sigma^{1/3}}\right).$$

In view of (3.7), the preceding formula can be written as

$$(4.16) \quad \phi_1(\sigma, y) = \sum_{p \leq \sigma} \log D_p(0) + \sum_{\sigma^{2/3} < p \leq y} h'\left(\frac{\sigma}{p}\right) \log \left(1 - \frac{1}{p}\right)^{-1} + O\left(\frac{1}{\sigma^{1/3}}\right).$$

Similarly to (4.10), we can prove that

$$(4.17) \quad \sum_{\sigma^{2/3} < p \leq y} h'\left(\frac{\sigma}{p}\right) \log \left(1 - \frac{1}{p}\right)^{-1} = \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)),$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17). \square

Lemma 4.4. *We have*

$$(4.18) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O\left(\frac{1}{\sigma^2}\right) & \text{if } p \leq \sigma^{1/2}, \\ \frac{1}{p^2}g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2}\right\}\right) & \text{if } p > \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.19) \quad \begin{aligned} E_p''(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \\ &= E_p(\sigma) \log^2 D_p(0) + R'', \end{aligned}$$

where

$$R'' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \left(\log^2 D_p(\theta) - \log^2 D_p(0) \right) \sin^2 \theta \, d\theta.$$

Using (4.13) and (4.19), we can deduce

$$(4.20) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} - \left(\frac{R'}{E_p(\sigma)} \right)^2,$$

where R' is defined as in (4.14).

From the definitions of R' and R'' , a simple calculation shows that

$$R'' - 2R' \log D_p(0) = \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 \left(\frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\log^2 \left(\frac{D_p(\theta)}{D_p(0)} \right) = \log^2 \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = \frac{4(1 - \cos \theta)^2}{p^2} + O\left(\frac{(1 - \cos \theta)^2}{p^3}\right),$$

we have

$$R'' - 2R' \log D_p(0) = \frac{4}{p^2} E_{p,2}(\sigma) + O\left(\frac{E_{p,2}(\sigma)}{p^3}\right),$$

where $E_{p,j}(\sigma)$ is defined as in (3.2). By using (3.3) with the choice of $j = 2$ and the trivial estimate $E_{p,2}(\sigma) \leq 4E_p(\sigma)$, we deduce

$$(4.21) \quad \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} = \frac{4}{p^2} \frac{E_{p,2}(\sigma)}{E_p(\sigma)} + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{p^3}\right\}\right).$$

Similarly we have

$$\log \left(\frac{D_p(\theta)}{D_p(0)} \right) = -\log \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = -\frac{2(1 - \cos \theta)}{p} + O\left(\frac{(1 - \cos \theta)}{p^2}\right),$$

and therefore

$$R' = -\frac{2}{p} E_{p,1}(\sigma) + O\left(\frac{E_{p,1}(\sigma)}{p^2}\right).$$

Now (3.3) with $j = 1$ and the trivial estimate $E_{p,1}(\sigma) \leq 2E_p(\sigma)$ imply

$$(4.22) \quad \begin{aligned} \left(\frac{R'}{E_p(\sigma)} \right)^2 &= \frac{4}{p^2} \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left(\frac{E_{p,1}(\sigma)^2}{p^3 E_p(\sigma)^2} \right) \\ &= \frac{4}{p^2} \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right). \end{aligned}$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$(4.23) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{4}{p^2} h_p(\sigma) + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right)$$

for all p and $\sigma > 0$, where

$$h_p(\sigma) := \frac{E_{p,2}(\sigma)}{E_p(\sigma)} - \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2.$$

When $p \leq \sigma^{1/2}$, the inequality (3.3) of Lemma 3.2 implies that $h_p(\sigma) \ll (p/\sigma)^2$. From this and (4.23) we deduce the first estimate of (4.18).

If $p \geq \sigma^{1/2}$, we can use (4.3), (3.11) and (3.8) to write

$$\begin{aligned} 4h_p(\sigma) &= g''\left(\frac{\sigma}{p}\right) \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \\ &= g''\left(\frac{\sigma}{p}\right) + O\left(\min \left\{ \frac{\sigma}{p^2}, \frac{1}{\sigma} \right\} \right). \end{aligned}$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for $p \geq \sigma^{1/2}$,

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{1}{p^2} g''\left(\frac{\sigma}{p}\right) + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2} \right\} \right).$$

This completes the proof. □

Lemma 4.5. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 2$, where

$$b_{j,2} := \int_0^\infty h''(u) (\log u)^{j-1} du.$$

In particular $b_{1,2} = 2$.

Proof. From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned} \phi_2(\sigma, y) &= \sum_{p \leq y} \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{g''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma} \right) \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma} \right). \end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceding two estimates.

Finally

$$\begin{aligned} b_{1,2} &= \int_0^1 h''(u) du + \int_1^\infty h''(u) du \\ &= h'(1-) - h'(1+) = h'(1-) - (h'(1-) - 2) = 2. \end{aligned}$$

This completes the proof. \square

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

Lemma 4.6. *We have*

$$(4.24) \quad \phi_n(\sigma, y) \ll 1/(\sigma^{n-1} \log \sigma) \quad (n = 3, 4)$$

uniformly for $y \geq \sigma \geq 3$.

§ 5. Estimate of $|E(\kappa + i\tau, y)|$

Lemma 5.1. *For any $\delta \in (0, \frac{1}{4})$, there are two absolute positive constants c_1, c_2 and a positive constant $c_3 = c_3(\delta)$ such that for all $y \geq \sigma \geq 3$ we have*

$$(5.1) \quad \left| \frac{E(\sigma + i\tau, y)}{E(\sigma, y)} \right| \leq \begin{cases} 1 & \text{if } |\tau| \leq c_1 \sigma^{1/2} \log \sigma \text{ or } |\tau| \geq y^{1/\delta}, \\ e^{-c_2 \tau^2 / [\sigma (\log \sigma)^2]} & \text{if } c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma, \\ e^{-c_3 |\tau|^\delta} & \text{if } \sigma \leq |\tau| \leq y^{1/\delta}. \end{cases}$$

Proof. First we write

$$\begin{aligned} E_p(s) &= \frac{2}{\pi} \int_0^\pi (D_p(\theta)^{-1})^{-s} \sin^2 \theta d\theta \\ &= \frac{2}{\pi} \int_0^\pi \frac{\sin^2 \theta}{(1-s)(D_p(\theta)^{-1})'} d(D_p(\theta)^{-1})^{1-s}. \end{aligned}$$

Since $(D_p(\theta)^{-1})' = 2p^{-1} \sin \theta$, after a simplification and an integration by parts it follows that

$$\begin{aligned} E_p(s) &= \frac{p}{\pi(s-1)} \int_0^\pi D_p(\theta)^{s-1} \cos \theta d\theta \\ &= \frac{p}{\pi(s-1)} \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\} \cos \theta d\theta. \end{aligned}$$

This implies that

$$(5.2) \quad \left| \frac{E_p(s)}{E_p(\sigma)} \right| = \left| \frac{\sigma-1}{s-1} \right| \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right|$$

with

$$E_p^*(s) := \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi - \theta)^{s-1}\} \cos \theta \, d\theta.$$

1° Case of $\sigma^{1/\delta} < |\tau| \leq y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1} \{1 - \Delta_p(\theta)^{s-1}\} \cos \theta \, d\theta$$

with

$$\Delta_p(\theta) := \frac{1 - 2p^{-1} \cos \theta + p^{-2}}{1 + 2p^{-1} \cos \theta + p^{-2}}.$$

It is clear that for all p , the function $\theta \mapsto \Delta_p(\theta)$ is increasing on $[0, \pi/2]$. It follows that

$$\begin{aligned} E_p^*(\sigma) &\geq \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \{1 - \Delta_p(\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &\geq \{1 - \Delta_p(\pi/4)^{\sigma-1}\} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \end{aligned}$$

for all p and $\sigma \geq 1$. This implies that

$$(5.3) \quad \left| \frac{1}{E_p^*(\sigma)} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

Similarly since the function $\theta \mapsto D_p(\theta)^{\sigma-1} \cos \theta$ is decreasing on $[0, \pi/2]$ for all p and $\sigma \geq 2$, we can deduce, via (5.3), that

$$(5.4) \quad \left| \frac{1}{E_p^*(\sigma)} \int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

From (5.3) and (5.4), we deduce that

$$\left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \frac{2}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all $p \geq \sigma \geq 2$, we have

$$\Delta_p\left(\frac{\pi}{4}\right)^{\sigma-1} \leq \left(1 - \frac{\sqrt{2}}{p} + \frac{1}{p^2}\right)^{\sigma-1} \leq 1 - \frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left| \frac{E_p(s)}{E_p(\sigma)} \right| \leq \frac{8p}{|s-1|} \leq \frac{p^4}{|\tau|} \quad (p \geq \sigma).$$

By multiplying this inequality for $\sigma < p \leq |\tau|^\delta (\leq y)$ and the trivial inequality $|E_p(s)| \leq |E_p(\sigma)|$ for the others p , we deduce, via the prime number theorem, that

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma < p \leq |\tau|^\delta} \log |\tau| + 4 \sum_{\sigma < p \leq |\tau|^\delta} \log p \right\} \\ &\leq e^{-\{1/\delta - 4 + o(1)\}|\tau|^\delta}. \end{aligned}$$

2° Case of $c_1\sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma^{1/\delta}$

For $p \geq \sigma^{1/2} \geq 2$, we can write

$$\begin{aligned} |E_p^*(s)| &\leq \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p] \cos \theta} + e^{-2[(\sigma-1)/p] \cos \theta}) \cos \theta \, d\theta \end{aligned}$$

and

$$\begin{aligned} |E_p^*(\sigma)| &= \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p] \cos \theta} - e^{-2[(\sigma-1)/p] \cos \theta}) \cos \theta \, d\theta. \end{aligned}$$

From these, we deduce that

$$(5.5) \quad \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \left\{1 + O\left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\right)\right\} \quad (2 \leq \sigma^{1/2} \leq p \leq \sigma)$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p] \cos \theta} \cos \theta \, d\theta \gg e^{\sigma/p} \quad \text{and} \quad \int_0^{\pi/2} e^{-2[(\sigma-1)/p] \cos \theta} \cos \theta \, d\theta \ll 1.$$

Inserting (5.5) into (5.2), for $2 \leq \sigma^{1/2} \leq p \leq \sigma$ we obtain

$$\begin{aligned} \left| \frac{E_p(s)}{E_p(\sigma)} \right| &\leq \exp \left\{ -\log \left| \frac{s-1}{\sigma-1} \right| + C \left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}} \right) \right\} \\ &\leq \begin{cases} e^{-\tau^2/(2\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } 3 \leq |\tau| \leq \sigma, \\ e^{-\frac{1}{2} \log(1+\tau^2/\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } \sigma \leq |\tau| \leq \sigma^{1/\delta}, \end{cases} \end{aligned}$$

where $C > 0$ is an absolute constant.

Now by multiplying these inequalities for $\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)$ and the trivial inequality $|E_p(s)| \leq E_p(\sigma)$ for the other p , we get

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)} \left(\frac{\tau^2}{2\sigma^2} - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right) \right\} \\ &\leq \exp \left\{ - \left(\frac{\tau^2}{16\sigma(\log \sigma)^2} - 10C - \frac{10C}{\sigma \log \sigma} \right) \right\} \\ &\leq \exp \left\{ - \frac{c_2 \tau^2}{\sigma(\log \sigma)^2} \right\} \end{aligned}$$

if $c_1\sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma$, and

$$(5.6) \quad \begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)} \left[\frac{1}{2} \log \left(1 + \frac{\tau^2}{\sigma^2} \right) - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right] \right\} \\ &\leq \exp \left\{ - \left[\frac{\sigma}{8 \log \sigma} \log \left(1 + \frac{\tau^2}{\sigma^2} \right) - 10C - \frac{10C}{\sigma \log \sigma} \right] \right\} \\ &\leq \exp \left\{ - c_3 |\tau|^\delta \right\} \end{aligned}$$

if $\sigma \leq |\tau| \leq \sigma^{1/\delta}$. This completes the proof. \square

§ 6. Proof of Theorem 3

We follow the argument of Granville & Soundararajan [4] to prove Theorem 3. We shall divide the proof in several steps which are embodied in the following lemmas.

The first one is a classic integration formula (see [4], page 1019).

Lemma 6.1. *Let $c > 0$, $\lambda > 0$ and $N \in \mathbb{N}$. Then we have*

$$(6.1) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < e^{-\lambda N}, \\ \in [0, 1] & \text{if } e^{-\lambda N} \leq y < 1, \\ 1 & \text{if } y \geq 1. \end{cases}$$

The second one is an analogue for (3.6) and (3.7) of [4] (see also Lemma 3.1 of [20]).

Lemma 6.2. *Let $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Then we have*

$$(6.2) \quad \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s}} \frac{ds}{\lambda s} \leq \Phi(te^{-\lambda}, y),$$

$$(6.3) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s}} \frac{e^{2\lambda s} - e^{-2\lambda s}}{\lambda s} ds.$$

Proof. Denote by $\mathbf{1}_X(\omega)$ the characteristic function of the set $X \subset \Omega$. Then by Lemma 6.1 with $N = 1$ and $c = \kappa$, we have

$$\mathbf{1}_{\{\omega \in \Omega: L(1, g^{\mathfrak{h}}(\omega); y) > (e\gamma t)^2\}}(\omega) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\mathfrak{h}}(\omega); y)}{(e\gamma t)^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds.$$

Integrating over Ω and interchanging the order of integrations yield

$$\begin{aligned} \Phi(t, y) &\leq \int_{\Omega} \left\{ \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\mathfrak{h}}(\omega); y)}{(e\gamma t)^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds \right\} d\mu(\omega) \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s}} \frac{ds}{\lambda s^2}. \end{aligned}$$

This proves the first inequality of (6.2). The second can be treated by noticing that

$$\begin{aligned} \mathbf{1}_{\{\omega \in \Omega: L(1, g^{\mathfrak{h}}(\omega); y) > (e\gamma^{-\lambda} t)^2\}}(\omega) &= \mathbf{1}_{\{\omega \in \Omega: L(1, g^{\mathfrak{h}}(\omega); y) > (e\gamma t)^2\}}(\omega) \\ &\quad + \mathbf{1}_{\{\omega \in \Omega: (e\gamma t)^2 \geq L(1, g^{\mathfrak{h}}(\omega); y) > (e\gamma^{-\lambda} t)^2\}}(\omega) \\ &\geq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\mathfrak{h}}(\omega); y)}{(e\gamma t)^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds. \end{aligned}$$

From (6.2), we can deduce

$$\begin{aligned} \Phi(te^{-\lambda}, y) - \Phi(t, y) &\leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma^{-\lambda} t)^{2s}} \frac{ds}{\lambda s^2} - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma + \lambda t)^{2s}} \frac{ds}{\lambda s^2} \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s}} \frac{e^{2\lambda s} - e^{-2\lambda s}}{\lambda s^2} ds. \end{aligned}$$

This completes the proof. \square

Lemma 6.3. *Let $t \geq 1$, $y \geq 2e^t$ and $0 < \kappa\lambda \leq 1$. Then we have*

$$\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\kappa\lambda + \frac{\log \kappa}{\kappa}\right) \right\}.$$

Proof. First in view of (4.24) we write, for $s = \kappa + i\tau$ and $|\tau| \leq \kappa$,

$$E(s, y) = \exp \left\{ \sigma_0 + i\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - i\frac{\sigma_3}{6}\tau^3 + O(\sigma_4\tau^4) \right\}$$

and

$$\frac{e^{\lambda s} - 1}{\lambda s^2} = \frac{1}{\kappa} \left\{ 1 - \frac{i}{\kappa}\tau + O\left(\kappa\lambda + \frac{\tau^2}{\kappa^2}\right) \right\}.$$

Since $\sigma_1 = \log t + \gamma$, we have

$$\frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} = \frac{E(\kappa, y)}{\kappa (e^{\gamma t})^{2\kappa}} e^{-(\sigma_2/2)\tau^2} \left\{ 1 - \frac{i}{\kappa}\tau - i\frac{\sigma_3}{6}\tau^3 + O(R(\tau)) \right\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau| \leq \kappa$ to obtain

$$(6.4) \quad \frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds = \frac{E(\kappa, y)}{2\pi\kappa (e^{\gamma t})^{2\kappa}} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} \{1 + O(R(\tau))\} d\tau,$$

where we have used the fact that the integrals involving $(i/\kappa)\tau$ and $(i\sigma_3/6)\tau^3$ vanish.

On the other hand, using lemmas 4.5 and 4.6 we have

$$\begin{aligned} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} d\tau &= \sqrt{\frac{2\pi}{\sigma_2}} \left\{ 1 + O\left(\exp\left\{-\frac{1}{2}\kappa^2\sigma_2\right\}\right) \right\}, \\ \int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2} R(\tau) d\tau &\ll \frac{1}{\sqrt{\sigma_2}} \left(\kappa\lambda + \frac{1}{\kappa^2\sigma_2} + \frac{\sigma_3^2}{\sigma_2^3} + \frac{\sigma_4}{\sigma_2^2} \right) \\ &\ll \frac{1}{\sqrt{\sigma_2}} \left(\kappa\lambda + \frac{\log \kappa}{\kappa} \right). \end{aligned}$$

Inserting these into (6.4), we obtain the desired result. \square

Lemma 6.4. *Let δ and c_3 be two constants determined by Lemma 5.1. Then we have*

$$(6.5) \quad \int_{\kappa\pm i\kappa}^{\kappa\pm i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_1,$$

$$(6.6) \quad \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} (e^{2\lambda s} - e^{-2\lambda s}) ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_2,$$

uniformly for $t \geq 1$, $y \geq 2e^t$, $\kappa \geq 2$ and $0 < \lambda\kappa \leq 1$, where

$$\begin{aligned} R_1 &:= \lambda^{-1} e^{-c_3\kappa^\delta} + \lambda^{-1} (\kappa/\log \kappa)^{1/2} y^{-1/\delta}, \\ R_2 &:= \lambda\kappa (\log \kappa)^{1/2} + e^{-(c_3/2)\kappa^\delta} + \lambda^{-1} (\kappa/\log \kappa)^{1/2} y^{-1/\delta}. \end{aligned}$$

Proof. We split the integral in (6.5) into two parts according to $\kappa \leq |\tau| \leq y^{1/\delta}$ or $|\tau| \geq y^{1/\delta}$. Using Lemma 5.1 with $\sigma = \kappa$ and the inequality $(e^{\lambda s} - 1)/s^2 \ll 1/\tau^2$, the integral in (6.5) is

$$\ll \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa} \lambda} \left(\frac{e^{-c_3 \kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}} \right),$$

which implies (6.5), in view of Lemma 4.5 with $J = 1$.

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leq c_1 \kappa^{1/2} \log \kappa, \quad c_1 \kappa^{1/2} \log \kappa < |\tau| \leq \kappa, \quad \kappa < |\tau| \leq y^{1/\delta}, \quad |\tau| \geq y^{1/\delta}.$$

By Lemma 5.1 with $\sigma = \kappa$ and the inequalities

$$(e^{\lambda s} - 1)/\lambda s \ll \min\{1, 1/(\lambda|\tau|)\}, \\ (e^{2\lambda s} - e^{-2\lambda s})/s \ll \min\{\lambda, 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa}} \left(\lambda \kappa^{1/2} \log \kappa + e^{-c_3 \kappa^\delta} + \lambda^{-1} y^{-1/\delta} \right),$$

which implies (6.6), as before. \square

Now we are ready to complete the proof of Theorem 3. Lemma 6.3 and (6.5) of Lemma 6.4 give

$$(6.7) \quad \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \{1 + O(R')\}$$

where

$$R' := \frac{\log \kappa}{\kappa} + \kappa \lambda + \frac{e^{-c_3 \kappa^\delta} + (\kappa/\log \kappa)^{1/2} y^{-1/\delta}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geq 2e^t \asymp \kappa$ and $1/\delta > 4$, we deduce

$$(6.8) \quad R' \ll t/e^t.$$

Combining (6.7) and (6.8) with (6.2), we obtain

$$(6.9) \quad \Phi(t, y) \leq \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\} \leq \Phi(te^{-\lambda}, y)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$.

On the other hand, (6.3) of Lemma 6.2 and (6.6) of Lemma 6.4 imply

$$\Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} + \frac{(\kappa/\log \kappa)^{1/2}}{\lambda y^{1/\delta}} \right) \\ \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} \right)$$

when $y^{-1/(2\delta)} \kappa^{-1/2} (\log \kappa)^{-1} \leq \lambda \leq \kappa^{-1}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t, y)$ is a non-decreasing function of λ , we deduce

$$(6.10) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} + \frac{\kappa (\log \kappa)^{1/2}}{y^{1/(2\delta)}} \right)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 3. \square

§ 7. Proof of Theorem 4

Using Lemmas 4.1 and 4.5, we can write

$$\begin{aligned} \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} &= \exp \left\{ \phi(\kappa, y) - 2\kappa(\gamma + \log t) + O(\log \kappa) \right\} \\ &= \exp \left\{ \kappa \left(2 \log_2 \kappa - 2 \log t + \sum_{j=1}^J \frac{b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right) \right\}. \end{aligned}$$

On the other hand, Lemma 4.3 and (1.17) imply that

$$2 \log_2 \kappa + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) = 2(\log t + \gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} = \exp \left\{ -\kappa \left[\sum_{j=1}^J \frac{b_{j,1} - b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}.$$

In view of (1.21), (4.2) and (4.15), we have $b_{j,1} - b_{j,0} = a_j$. This completes the proof. \square

§ 8. Proof of Corollary 5

We first prove an asymptotic development of $\kappa(t, y)$ in t .

Lemma 8.1. *For each integer $J \geq 1$, there are computable constants $\gamma_0, \gamma_1, \dots, \gamma_J$ such that the asymptotic formula*

$$(8.1) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J(R_J^*(t, y)) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, where

$$R_N^*(t, y) := \frac{1}{t^{N+1}} + \frac{e^t t}{y \log y}.$$

Further γ_0 is given by (1.24) and $\gamma_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$.

Proof. By Lemma 4.3 and (1.17), we have

$$(8.2) \quad 2 \log t = 2 \log_2 \kappa + \sum_{j=1}^{J+1} \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)),$$

where $R_J(\kappa, y)$ is defined as in (1.20). From (8.2), we easily deduce that

$$\begin{aligned} t &= (\log \kappa) \prod_{j=1}^{J+1} \exp \left\{ \frac{b_{j,1}}{2(\log \kappa)^j} \right\} \exp \{O_J(R_{J+1}(\kappa, y))\} \\ &= (\log \kappa) \prod_{j=1}^{J+1} \left\{ \sum_{m_j=0}^{J+1} \frac{1}{m_j!} \left(\frac{b_{j,1}}{2(\log \kappa)^j} \right)^{m_j} + O_J(R_{J+1}(\kappa, y)) \right\}. \end{aligned}$$

Developping the product, we get

$$t = (\log \kappa) \left\{ \sum_{j=0}^{J+1} \frac{b'_j}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)) \right\},$$

where

$$\begin{aligned} b'_j &:= \sum_{\substack{m_1 \geq 0, \dots, m_{J+1} \geq 0 \\ m_1 + 2m_2 + \dots + (J+1)m_{J+1} = j}} \frac{b_{1,1}^{m_1} \dots b_{J+1,1}^{m_{J+1}}}{(2m_1)!! \dots (2m_{J+1})!!} \\ &= \sum_{\substack{m_1 \geq 0, \dots, m_j \geq 0 \\ m_1 + 2m_2 + \dots + jm_j = j}} \frac{b_{1,1}^{m_1} \dots b_{j,1}^{m_j}}{(2m_1)!! \dots (2m_j)!!}. \end{aligned}$$

Since $b'_0 = 1$ and $b'_1 = b_{1,1}/2 = \gamma_0$, the preceding asymptotic formula can be written as

$$(8.3) \quad t = \log \kappa + \gamma_0 + \sum_{j=1}^J \frac{b'_{j+1}}{(\log \kappa)^j} + O_J(R_J^*(t, y)),$$

where we have used the fact that $\kappa(t, y) \asymp e^t$ (see Lemma 2.3) and $(\log \kappa)R_{J+1}(\kappa, y) \asymp R_J^*(t, y)$.

With the help of (8.3), a simple recurrence argument shows that there are constants γ'_n such that

$$(8.4) \quad t = \log \kappa + \sum_{j=0}^J \frac{\gamma'_j}{t^j} + O_J(R_J^*(t, y)).$$

In fact taking $J = 0$ in (8.3), we see that (8.4) holds for $J = 0$. Suppose that it holds for $0, \dots, J-1$, i.e.

$$t = \log \kappa + \sum_{i=0}^{J-j-1} \frac{\gamma'_i}{t^i} + O(R_{J-j-1}^*(t, y)) \quad (j = 0, \dots, J-1),$$

which is equivalent to

$$(8.5) \quad \log \kappa = t \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma'_{i-1}}{t^i} + O\left(\frac{R_{J-j-1}^*(t, y)}{t}\right) \right\} \quad (j = 0, \dots, J-1).$$

This holds also for $j = J$ if we use the convention:

$$\sum_{i=0}^{-1} = 0 \quad \text{and} \quad R_{-1}^*(t, y) := 1,$$

since $\log \kappa = t + O(1)$. Inserting it into (8.3), we easily see that (8.4) holds also for J . In particular we have

$$\gamma'_1 = b'_2 = \frac{1}{8}b_{1,1}^2 + \frac{1}{4}b_{2,1}.$$

Now (8.1) is an immediate consequence of (8.4) with

$$\gamma_j := \sum_{\substack{m_1 \geq 0, \dots, m_J \geq 0 \\ m_1 + 2m_2 + \dots + Jm_J = j}} (-1)^{m_1 + \dots + m_J} \frac{\gamma'_1{}^{m_1} \cdots \gamma'_J{}^{m_J}}{m_1! \cdots m_J!}.$$

This completes the proof. \square

Now we are ready to prove Corollary 5.

Using (8.5), we have

$$(8.6) \quad \begin{aligned} \sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} &= \sum_{j=1}^J \frac{a_j}{t^j} \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma'_{i-1}}{t^i} + O_N \left(\frac{R_{J-j-1}^*(t, y)}{t} \right) \right\}^{-j} \\ &= \sum_{j=1}^J \frac{\rho_j}{t^j} + O_J \left(\frac{R_{J-2}^*(t, y)}{t^2} \right), \end{aligned}$$

where the ρ_n are constants. In particular we have $\rho_1 = a_1 = 1$ and $\rho_2 = \gamma_0 + a_2$.

Now Theorem 4, (8.1) and (8.6) imply the result of Corollary with

$$a_1^* = \rho_1 = 1, \quad a_j^* = \rho_j + \sum_{i=1}^{j-1} \gamma_i \rho_{j-i} \quad (j \geq 2).$$

This completes the proof of Corollary 5. \square

§ 9. Proof of Theorem 2

For each $\eta \in (0, \frac{1}{2})$, define

$$\mathbb{H}_k^+(1; \eta) := \{f \in \mathbb{H}_k^*(1) : L(s, f) \neq 0, s \in \mathcal{S}\},$$

where $\mathcal{S} := \{s := \sigma + i\tau : \sigma \geq 1 - \eta, |\tau| \leq 100k^\eta\} \cup \{s := \sigma + i\tau : \sigma \geq 1, \tau \in \mathbb{R}\}$, and

$$\mathbb{H}_k^-(1; \eta) := \mathbb{H}_k^*(1) \setminus \mathbb{H}_k^+(1; \eta).$$

Then we have (see [10], (1.11))

$$(9.1) \quad |\mathbb{H}_k^-(1; \eta)| \ll_\eta k^{31\eta}.$$

Our starting point in the proof of Theorem 2 is the evaluation of the moments of $L(1, f)$. For this, we recall a particular case of Proposition 6.1 of [10].

Lemma 9.1. *Let $\eta \in (0, \frac{1}{31})$ be fixed. There are two positive constants $c_i = c_i(\eta)$ ($i = 4, 5$) such that*

$$(9.2) \quad \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^s = E(s) + O_\eta(e^{-c_4 \log k / \log_2 k})$$

uniformly for

$$(9.3) \quad k \geq 16, \quad 2 \mid k \quad \text{and} \quad |s| \leq 2T_k$$

with

$$T_k := c_5 \log k / (\log_2 k \log_3 k).$$

Here $E(s)$ is defined by (1.15).

Let $\kappa(t, y)$ be the saddle-point determined by (1.17) and $\kappa_t := \kappa(t, \infty)$. For $k \geq 16, 2 \mid k, \lambda > 0, N \in \mathbb{N}$ and $t > 0$, introduce the two integrals

$$I_1(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f \left(\frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}$$

and

$$I_2(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \frac{E(s)}{(e^\gamma t)^{2s}} \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}.$$

Lemma 9.2. *Let $\eta \in (0, \frac{1}{200}]$ be fixed. Then we have*

$$(9.4) \quad \tilde{F}_k(t) + O_\eta(k^{-5/6}) \leq I_1(k, t; \lambda, N) \leq \tilde{F}_k(te^{-\lambda N}) + O_\eta(k^{-5/6}),$$

$$(9.5) \quad \Phi(t) \leq I_2(k, t; \lambda, N) \leq \Phi(te^{-\lambda N})$$

uniformly for $k \geq 16, 2 \mid k, \lambda > 0, N \in \mathbb{N}$ and $t > 0$. The implied constants depend on η only.

Proof. By exchanging the order of summation and by using Lemma 6.1 with $c = \kappa_t$, we obtain

$$\begin{aligned} I_1(k, t; \lambda, N) &= \sum_{f \in \mathbb{H}_k^+(1; \eta)} \frac{\omega_f}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left(\frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}, \\ &\geq \sum_{f \in \mathbb{H}_k^+(1; \eta), L(1, f) \geq (e^\gamma t)^2} \omega_f. \end{aligned}$$

In view of the second estimate of (1.7) and of (9.1), we reintroduce the missing forms

$$\begin{aligned} I_1(k, t; \lambda, N) &\geq \sum_{f \in \mathbb{H}_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f + O\left(\sum_{f \in \mathbb{H}_k^* \setminus \mathbb{H}_k^+(1; \eta)} \omega_f \right) \\ &\geq \sum_{f \in \mathbb{H}_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f + O(k^{-1+31\eta} \log k). \end{aligned}$$

Clearly this implies the first inequality of (9.4), thanks to (1.6) and (1.7).

Similarly, using Lemma 6.1 with $c = \kappa_t$, we find

$$\begin{aligned} I_1(k, t; \lambda, N) &\leq \sum_{\substack{f \in \mathbb{H}_k^+(1; \eta) \\ L(1, f) \geq (e^\gamma t)^2}} \omega_f + \sum_{\substack{f \in \mathbb{H}_k^+(1; \eta) \\ (e^\gamma te^{-\lambda N})^2 \leq L(1, f) < (e^\gamma t)^2}} \omega_f \\ &= \sum_{\substack{f \in \mathbb{H}_k^+(1; \eta) \\ L(1, f) \geq (e^\gamma te^{-\lambda N})^2}} \omega_f. \end{aligned}$$

As before, we can easily show that the last sum is $\leq \tilde{F}_k(te^{-\lambda N}) + O(k^{-5/6})$.

The estimates (9.5) can be proved in the same way as (6.2). \square

Lemma 9.3. *Let $\eta \in (0, \frac{1}{200}]$ be fixed and c_4 be the positive constant given by Lemma 9.1. Then we have*

$$(9.6) \quad |I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)| \ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda\kappa_t})^{2N} \log T_k}{(e^{\gamma t})^{2\kappa_t}} \\ + \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{N(e^{\gamma t})^{2\kappa_t}} \left(\frac{1 + e^{\lambda\kappa_t}}{\lambda T_k} \right)^{2N}$$

uniformly for $\lambda > 0$, $N \in \mathbb{N}$, $k \geq 16$, $2 \mid k$ and $t \leq T(k)$, where $T(k)$ is given by (1.10). The implied constant depends on η only.

Proof. By the definitions of I_1 and I_2 , we can write

$$I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N) \\ = \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left(\sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^s - E(s) \right) \left(\frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s(e^{\gamma t})^{2s}}.$$

In order to estimate the last integral, we split it into two parts according to $|\tau| \leq T_k$ or $|\tau| > T_k$.

In view of (1.18), it is easy to see that $\kappa_t \leq T_k$ for $t \leq T(k)$. Thus we may apply (9.2) of Lemma 9.1 for $s = \kappa_t + i\tau$ with $|\tau| \leq T_k$. Note that $|(e^{\lambda s} - 1)/(\lambda s)| \leq 1 + e^{\lambda\kappa_t}$ for $s = \kappa_t + i\tau$, which is easily seen by looking at the cases $|\lambda s| \leq 1$ and $|\lambda s| > 1$. The contribution of $|\tau| \leq T_k$ to $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$ is

$$(9.7) \quad \ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda\kappa_t})^{2N} \log T_k}{(e^{\gamma t})^{2\kappa_t}}.$$

Since $\kappa_t \leq T_k$ for $t \leq T(k)$, we can apply (9.2) of Lemma 9.1 to write, for $s = \kappa_t + i\tau$ with $\tau \in \mathbb{R}$,

$$\left| \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^s - E(s) \right| \leq \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^{\kappa_t} + E(\kappa_t) \\ \leq 2E(\kappa_t) + O(e^{-c_4(\log k)/\log_2 k}).$$

Thus the contribution of $|\tau| > T_k$ to $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$ is

$$(9.8) \quad \ll \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{(e^{\gamma t})^{2\kappa_t}} \int_{|\tau| \geq T_k} \left(\frac{1 + e^{\lambda\kappa_t}}{\lambda |\tau|} \right)^{2N} \frac{d\tau}{|\tau|} \\ \ll \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{N(e^{\gamma t})^{2\kappa_t}} \left(\frac{1 + e^{\lambda\kappa_t}}{\lambda T_k} \right)^{2N}.$$

Combining (9.7) and (9.8) yields to the required estimate. \square

End of the proof of Theorem 2

For simplicity of notation, we write

$$I_j := I_j(k, t; \lambda, N) \quad \text{and} \quad I_j^+ := I_j(k, te^{\lambda N}; \lambda, N) \quad (j = 1, 2).$$

By using Lemma 9.2, we have

$$(9.9) \quad \tilde{F}_k(t) \leq I_1 + O(k^{-5/6}) \\ = I_2 + O(|I_1 - I_2| + k^{-5/6}) \\ \leq \Phi(te^{-\lambda N}) + O(|I_1 - I_2| + k^{-5/6}) \\ \leq \Phi(t) + |\Phi(te^{-\lambda N}) - \Phi(t)| + O(|I_1 - I_2| + k^{-5/6})$$

and

$$\begin{aligned}
 \tilde{F}_k(t) &\geq I_1^+ + O(k^{-5/6}) \\
 &= I_2^+ + O(|I_1^+ - I_2^+| + k^{-5/6}) \\
 (9.10) \quad &\geq \Phi(te^{\lambda N}) + O(|I_1^+ - I_2^+| + k^{-5/6}) \\
 &\geq \Phi(t) - |\Phi(t) - \Phi(te^{\lambda N})| + O(|I_1^+ - I_2^+| + k^{-5/6}).
 \end{aligned}$$

In view of (6.10) and Theorem 3, we have

$$|\Phi(t) - \Phi(te^{-\lambda N})| \ll \Phi(t) \{ \lambda N \kappa_t (\log \kappa_t)^{1/2} + e^{-(c_3/2)\kappa_t^\delta} \}$$

for $\lambda N \leq e^{-t}$. Take

$$(9.11) \quad \lambda = e^{5A}/T_k \quad \text{and} \quad N = \lfloor \log_2 k \rfloor.$$

Since $T_k = e^{T(k) + \frac{3}{2} \log_3 k + 2C + \log c_5}$, it is easy to see that

$$\lambda N \leq e^{-T(k) - 2C} T(k)^{-1/2} \quad \text{and} \quad \kappa_t \asymp e^t.$$

Inserting these estimates into the preceding inequality, a simple calculation shows that

$$(9.12) \quad |\Phi(t) - \Phi(te^{-\lambda N})| \leq \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \},$$

provided the constant C is suitably large, where $c_6 = c_6(\eta, \delta)$ is a positive constant.

Similarly by using (6.10) with $te^{\lambda N}$ in place of t , we have

$$|\Phi(t) - \Phi(te^{\lambda N})| \ll \Phi(te^{\lambda N}) \{ \lambda N \kappa_{te^{\lambda N}} (\log \kappa_{te^{\lambda N}})^{1/2} + e^{-(c_3/2)\kappa_{te^{\lambda N}}^\delta} \}.$$

Since for $t \leq T(k)$ we have

$$te^{\lambda N} = t + O((\log_2 k)^3 (\log_3 k) / \log k) \quad \text{and} \quad \kappa_{te^{\lambda N}} \asymp e^{te^{\lambda N}} \asymp e^t,$$

the preceding estimate can be written as

$$\begin{aligned}
 |\Phi(t) - \Phi(te^{\lambda N})| &\leq \frac{1}{4} \Phi(te^{\lambda N}) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \} \\
 &\leq \frac{1}{4} \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \} \\
 &\quad + \frac{1}{4} |\Phi(t) - \Phi(te^{\lambda N})| \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \},
 \end{aligned}$$

from which we deduce that

$$(9.13) \quad |\Phi(t) - \Phi(te^{\lambda N})| \leq \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \}.$$

By using Lemma 9.3 with $te^{\lambda N}$ in place of t , we have

$$\begin{aligned}
 |I_1^+ - I_2^+| &\ll e^{-c_4(\log k) / \log_2 k} \frac{(1 + e^{\lambda \kappa_{te^{\lambda N}}})^{2N} \log T_k}{(e^{\gamma te^{\lambda N}})^{2\kappa_{te^{\lambda N}}}} \\
 &\quad + \frac{E(\kappa_{te^{\lambda N}}) + e^{-c_4(\log k) / \log_2 k}}{N(e^{\gamma te^{\lambda N}})^{2\kappa_{te^{\lambda N}}}} \left(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k} \right)^{2N}.
 \end{aligned}$$

On the other hand, by using Theorem 3 and (1.25), it is easy to see that there is a positive constant c such that

$$\Phi(te^{\lambda N}) \asymp \Phi(t) \sim \frac{E(\kappa_t)}{\kappa_t \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa_t}} \gg e^{-c_8 e^t/t} \gg e^{-c_9 (\log k)/[(\log_2 k)^{7/2} \log_3 k]}$$

for $t \leq T(k)$. Thanks to Lemma 4.5, the previous estimate can be written as

$$(9.14) \quad |I_1^+ - I_2^+| \ll \Phi(t) \frac{1}{N} \left(\frac{\kappa_{te^{\lambda N}}}{\log \kappa_{te^{\lambda N}}} \right)^{1/2} \left(\frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k} \right)^{2N} \ll \frac{\Phi(t)}{(\log k)^A}.$$

Similarly we can prove (even more easily)

$$(9.15) \quad |I_1 - I_2| \ll \Phi(t)/(\log k)^A.$$

Inserting (9.12) and (9.16) into (9.9) and (9.13) and (9.15) into (9.10), we obtain

$$\tilde{F}_k(t) \leq \Phi(t) \{1 + e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}} + (\log k)^{-A})\}$$

and

$$\tilde{F}_k(t) \geq \Phi(t) \{1 - e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}} + (\log k)^{-A})\}.$$

This implies the first asymptotic formula of (1.13) by taking $\eta = \frac{1}{200}$ and $\delta = \frac{1}{5}$.

The second can be established similarly. This completes the proof of Theorem 2. \square

§ 10. Proof of Theorem 1

The formula (1.9) is an immediate consequence of Theorem 2 and (1.25).

Taking $t = T(k)$ in (1.9), we find that

$$(10.1) \quad e^{-c'_1 (\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \ll \tilde{F}_k(T(k)) \ll e^{-c'_2 (\log k)/\{(\log_2 k)^{7/2} \log_3 k\}},$$

where c'_1 and c'_2 are two positive constants. Clearly (10.1) and (1.8) imply (1.11).

The related results on $\tilde{G}_k(t)$ and $G_k(T(k))$ can be proved similarly. This completes the proof of Theorem 1. \square

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