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# On a conjecture of Montgomery-Vaughan on extreme values of automorphic $L$ -functions at 1

J.-Y. LIU, E. ROYER & J. WU

**Abstract.** In this paper, we prove a weaker form of a conjecture of Montgomery-Vaughan on extreme values of automorphic  $L$ -functions at 1.

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## § 1. Introduction

The automorphic  $L$ -functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer  $N$  and even integer  $k$ , denote by  $H_k^*(N)$  the set of all newforms of level  $N$  and of weight  $k$ . It is known that

$$(1.1) \quad |H_k^*(N)| = \frac{k-1}{12} \varphi(N) + O((kN)^{2/3}),$$

where  $\varphi(N)$  is the Euler function and the implied constant is absolute. Let  $m \geq 1$  be an integer and let  $L(s, \text{sym}^m f)$  be the  $m$ th symmetric power  $L$ -function of  $f \in H_k^*(N)$  normalised so that the critical strip is given by  $0 < \Re s < 1$ . The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [18] showed that the Sato-Tate conjecture is equivalent to  $L(1 + i\tau, \text{sym}^m f) \neq 0$  for all  $m \in \mathbb{N}$  and  $\tau \in \mathbb{R}$ . The distribution of the values  $L(1, \text{sym}^m f)$  has received attention of many authors, including Goldfeld, Hoffstein & Lieman [2], Hoffstein & Lockhart [7], Luo [12], Royer [14, 15], Royer & Wu [16, 17], Cogdell & Michel [1], Habsieger & Royer [5] and Lau & Wu [10, 11]. In particular, Lau & Wu (Theorems 2 and 3 of [10], and Corollary 2 of [11]) proved the following results:

(i) For every fixed integer  $m \geq 1$ , there are four positive constants  $A_m^\pm$  and  $B_m^\pm$  such that for any newform  $f \in H_k^*(1)$ , under the Great Riemann Hypothesis (GRH) for  $L(s, \text{sym}^m f)$ , we have, for  $k \rightarrow \infty$ ,

$$(1.2) \quad \{1 + o(1)\} (2B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + o(1)\} (2B_m^+ \log_2 k)^{A_m^+}.$$

Here (and in the sequel)  $\log_j$  denotes the  $j$ -fold iterated logarithm. For most values of  $m$ , the constants  $A_m^\pm$  and  $B_m^\pm$  can be explicitly evaluated, for example,

$$\begin{cases} A_m^+ = m + 1, & B_m^+ = e^\gamma & (m \in \mathbb{N}), \\ A_m^- = m + 1, & B_m^- = e^\gamma \zeta(2)^{-1} & (\text{odd } m), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^-, \end{cases}$$

where  $\zeta(s)$  is the Riemann zeta-function,  $\gamma$  denotes the Euler constant and  $B_4'^-$  is a positive constant given by a rather complicated Euler product (cf. [10], (1.16)).

(ii) In the opposite direction, it was shown unconditionally that for  $m \in \{1, 2, 3, 4\}$  there are newforms  $f_m^\pm \in \mathbf{H}_k^*(1)$  such that for  $k \rightarrow \infty$ ,

$$(1.3) \quad \begin{cases} L(1, \text{sym}^m f_m^+) \geq \{1 + o(1)\} (B_m^+ \log_2 k)^{A_m^+}, \\ L(1, \text{sym}^m f_m^-) \leq \{1 + o(1)\} (B_m^- \log_2 k)^{-A_m^-}. \end{cases}$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.3)), an almost all result was established. Let  $\varepsilon > 0$  be an arbitrarily small positive number,  $m \in \{1, 2, 3, 4\}$  and  $2 \mid k$ . Then there is a subset  $\mathbf{E}_k^*$  of  $\mathbf{H}_k^*(1)$  such that  $|\mathbf{E}_k^*| \ll \mathbf{H}_k^*(1) e^{-(\log k)^{1/2-\varepsilon}}$  and for each  $f \in \mathbf{H}_k^*(1) \setminus \mathbf{E}_k^*$ , we have, for  $k \rightarrow \infty$ ,

$$(1.4) \quad \{1 + O(\varepsilon_k)\} (B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + O(\varepsilon_k)\} (B_m^+ \log_2 k)^{A_m^+},$$

where  $\varepsilon_k := (\log k)^{-\varepsilon}$  and the implied constants depend on  $\varepsilon$  only.

By comparing (1.3) with (1.4), the extreme values of  $L(1, \text{sym}^m f)$  seem to be given by (1.3). Clearly it is interesting to investigate further the size of exceptional set  $\mathbf{E}_k^*$ . In the case of quadratic characters  $L$ -functions, Montgomery & Vaughan [13] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [4]. As Cogdell & Michel indicated in [1], it would be interesting to try to get, as close as possible, the analogues of the conjectures of Montgomery-Vaughan for automorphic  $L$ -functions. The analogue of Montgomery-Vaughan's first conjecture for the automorphic symmetric power  $L$ -functions can be stated as follows.

**Conjecture.** *Let  $m \geq 1$  be a fixed integer and*

$$\begin{aligned} F_k(t, \text{sym}^m) &:= |\mathbf{H}_k^*(1)|^{-1} \sum_{f \in \mathbf{H}_k^*(1), L(1, \text{sym}^m f) \geq (B_m^+ t)^{A_m^+}} 1, \\ G_k(t, \text{sym}^m) &:= |\mathbf{H}_k^*(1)|^{-1} \sum_{f \in \mathbf{H}_k^*(1), L(1, \text{sym}^m f) \leq (B_m^- t)^{-A_m^-}} 1. \end{aligned}$$

Then there are positive constants  $c_i = c_i(m)$  ( $i = 1, 2$ ) such that for  $k \rightarrow \infty$ ,

$$(1.5) \quad \begin{cases} e^{-c_1(\log k)/\log_2 k} \ll F_k(\log_2 k, \text{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}, \\ e^{-c_1(\log k)/\log_2 k} \ll G_k(\log_2 k, \text{sym}^m) \ll e^{-c_2(\log k)/\log_2 k}. \end{cases}$$

The aim of this paper is to prove a weaker form of this conjecture for  $m = 1$ . In this case, we write, for simplification of notation,

$$L(s, f) = L(s, \text{sym}^1 f), \quad F_k(t) = F_k(t, \text{sym}^1), \quad G_k(t) = G_k(t, \text{sym}^1).$$

In view of the trace formula of Petersson ([8], Theorem 3.6), it is more convenient to consider the weighted arithmetic distribution function. As usual, denote by

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1} \|f\|}$$

the harmonic weight in modular forms theory and define the weighted arithmetic distribution functions

$$\begin{aligned} \tilde{F}_k(t) &:= \left( \sum_{f \in \mathbb{H}_k^*(1)} \omega_f \right)^{-1} \sum_{f \in \mathbb{H}_k^*(1), L(1,f) \geq (e\gamma t)^2} \omega_f, \\ \tilde{G}_k(t) &:= \left( \sum_{f \in \mathbb{H}_k^*(1)} \omega_f \right)^{-1} \sum_{f \in \mathbb{H}_k^*(1), L(1,f) \leq (6\pi^{-2}e\gamma t)^{-2}} \omega_f. \end{aligned}$$

By using (1.1), the classical estimate

$$(1.6) \quad \sum_{f \in \mathbb{H}_k^*(1)} \omega_f = 1 + O(k^{-5/6})$$

and the bound of Goldfeld, Hoffstein & Lieman [2]:

$$(1.7) \quad 1/(k \log k) \ll \omega_f \ll (\log k)/k,$$

we easily see that

$$(1.8) \quad \begin{cases} \tilde{F}_k(t)/\log k \ll F_k(t) \ll \tilde{F}_k(t) \log k, \\ \tilde{G}_k(t)/\log k \ll G_k(t) \ll \tilde{G}_k(t) \log k. \end{cases}$$

This shows that in order to prove (1.5) it is sufficient to establish corresponding estimates of the same quality for  $\tilde{F}_k(t)$  and  $\tilde{G}_k(t)$ .

Our main result is the following one.

**Theorem 1.** *For any  $A \geq 1$  there are two positive constants  $c = c(A)$  and  $C = C(A)$  such that the estimate*

$$(1.9) \quad \tilde{F}_k(t) = \{1 + \Delta_k(t)\} \exp \left\{ -\frac{e^{t-\gamma_0}}{t} \left( 1 + O\left(\frac{1}{t}\right) \right) \right\}$$

holds uniformly for  $k \geq 16, 2 \mid k$  and  $t \leq T(k)$ , where  $\gamma_0$  is given by (1.24) below,  $|\theta| \leq 1$  and

$$(1.10) \quad \begin{cases} \Delta_k(t) := \theta e^{t-T(k)-C} (t/T(k))^{1/2} + O_A(e^{-ce^{t/5}} + (\log k)^{-A}), \\ T(k) := \log_2 k - \frac{5}{2} \log_3 k - \log_4 k - 3C. \end{cases}$$

In particular there are two positive constants  $c_1$  and  $c_2$  such that

$$(1.11) \quad e^{-c_1(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \ll F_k(T(k)) \ll e^{-c_2(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}}.$$

The similar estimates for  $\tilde{G}_k(t)$  and  $G_k(T(k))$  hold also.

**Remark 1.** The estimates (1.11) of Theorem 1 can be considered as a weaker form of Montgomery-Vaughan's conjecture (1.5) for  $m = 1$ , since  $T(k) \sim \log_2 k$  as  $k \rightarrow \infty$ . Moreover,

if we could take  $T(k) = \log_2 k$  in (1.11) then (1.9) would lead to the Montgomery-Vaughan's conjecture (1.5). Hence we fail from a shift

$$\frac{5}{2} \log_3 k + \log_4 k + 3C.$$

It seems however to be rather difficult to resolve completely this conjecture. One of the main difficulties is that there are no analogues of the quadratic reciprocity law and Graham-Ringrose's estimates for short characters sums of friable moduli [3], which have been exploited by Granville & Soundararajan [4].

In order to prove Theorem 1, we need to introduce a probabilistic model as in [1]. Consider a probability space  $(\Omega, \mu)$ , with measure  $\mu$ . Let  $\mathrm{SU}(2)^\natural$  be the set of conjugacy classes of  $\mathrm{SU}(2)$ . The group  $\mathrm{SU}(2)$  is endowed with its Haar measure  $\mu_H$  and

$$\mathrm{SU}(2)^\natural = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi] \right\} / \sim$$

is endowed with the Sato-Tate measure  $d\mu_{\mathrm{st}}(\theta) := (2/\pi) \sin^2 \theta d\theta$ , *i.e.*, the direct image of  $\mu_H$  by the canonical projection  $\mathrm{SU}(2) \rightarrow \mathrm{SU}(2)^\natural$ . On the space  $(\Omega, \mu)$ , define a sequence indexed by the prime numbers,  $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$  of random matrices taking values in  $\mathrm{SU}(2)^\natural$ , given by

$$g_p^\natural(\omega) := \begin{pmatrix} e^{i\vartheta_p(\omega)} & 0 \\ 0 & e^{-i\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each function  $g_p^\natural(\omega)$  is distributed according to the Sato-Tate measure. This means that, for each integrable function  $\phi : \mathrm{SU}(2)^\natural \rightarrow \mathbb{R}$ , the expected value of  $\phi \circ g_p^\natural$  is

$$\mathbb{E}(\phi \circ g_p^\natural) := \int_{\Omega} \phi \circ g_p^\natural(\omega) d\mu(\omega) = \int_0^\pi \phi \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta d\theta.$$

Moreover, we assume that the sequence  $g^\natural(\omega)$  is made of independent random variables. This means that, for any sequence of integrable functions  $\{G_p : \mathrm{SU}(2)^\natural \rightarrow \mathbb{R}\}_p$ , we have

$$\begin{aligned} (1.12) \quad \mathbb{E} \left( \prod_p G_p \circ g_p^\natural \right) &:= \int_{\Omega} \prod_p G_p \circ g_p^\natural(\omega) d\mu(\omega) \\ &= \prod_p \int_{\Omega} G_p \circ g_p^\natural(\omega) d\mu(\omega) \\ &= \prod_p \int_0^\pi G_p \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta d\theta. \end{aligned}$$

Let  $I$  be the identity matrix. Then for  $\Re s > \frac{1}{2}$ , the random Euler product

$$L(s, g^\natural(\omega)) := \prod_p \det (I - p^{-s} g_p^\natural(\omega))^{-1} =: \prod_p L_p(s, g^\natural(\omega))$$

turns out to be absolutely convergent a.s.

Now we define our probabilistic distribution functions

$$\begin{cases} \Phi(t) := \mathrm{Prob}(\{L(1, g^\natural(\cdot)) \geq (e^{\gamma t})^2\}), \\ \Psi(t) := \mathrm{Prob}(\{L(1, g^\natural(\cdot)) \leq (6\pi^{-2} e^{\gamma t})^{-2}\}). \end{cases}$$

We shall prove Theorem 1 in two steps. The first one is to compare  $\tilde{F}_k(t)$  with  $\Phi(t)$  (resp.  $\tilde{G}_k(t)$  with  $\Psi(t)$ ).

**Theorem 2.** For any  $A \geq 1$  there are two positive constants  $c = c(A)$  and  $C = C(A)$  such that the asymptotic formulas

$$(1.13) \quad \tilde{F}_k(t) = \Phi(t)\{1 + \Delta_k(t)\} \quad \text{and} \quad \tilde{G}_k(t) = \Psi(t)\{1 + \Delta_k(t)\}$$

hold uniformly for  $k \geq 16, 2 \mid k$  and  $t \leq T(k)$ , where  $\Delta_k(t)$  and  $T(k)$  are defined by (1.10).

The second step of the proof of Theorem 1 is the evaluation of  $\Phi(t)$  (resp.  $\Psi(t)$ ). For this, we consider a truncated random Euler product

$$L(s, g^{\natural}(\omega); y) := \prod_{p \leq y} L_p(s, g^{\natural}(\omega))$$

and the corresponding distribution functions

$$\begin{cases} \Phi(t, y) := \text{Prob}(\{L(1, g^{\natural}(\omega); y) \geq (e^{\gamma t})^2\}), \\ \Psi(t, y) := \text{Prob}(\{L(1, g^{\natural}(\omega); y) \leq (6\pi^{-2}e^{\gamma t})^{-2}\}). \end{cases}$$

We have

$$(1.14) \quad \Phi(t) = \Phi(t, \infty) \quad \text{and} \quad \Psi(t) = \Psi(t, \infty).$$

We shall use the saddle-point method (introduced by Hildebrand & Tenenbaum [6]) to evaluate  $\Phi(t, y)$  and  $\Psi(t, y)$ . For this, we need to introduce some notation. For  $s \in \mathbb{C}$  and  $y \geq 2$ , define

$$(1.15) \quad E(s, y) := \mathbb{E}(L(1, g^{\natural}(\omega); y)^s) \quad \text{and} \quad E(s) := E(s, \infty),$$

where  $\mathbb{E}(\cdot)$  denotes the expected value. We define also

$$(1.16) \quad \phi(s, y) := \log E(s, y), \quad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geq 0).$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant  $c \geq 2$  such that for  $t \geq 4 \log c$  and  $y \geq ce^t$ , the equation

$$(1.17) \quad \phi_1(\kappa, y) = 2(\log t + \gamma)$$

has a unique positive solution  $\kappa = \kappa(t, y)$  and for each integer  $J \geq 1$ , there are computable constants  $\gamma_0, \gamma_1, \dots, \gamma_J$  such that the asymptotic formula

$$(1.18) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J \left( \frac{1}{t^{J+1}} + \frac{e^t t}{y \log y} \right) \right\}$$

holds uniformly for  $t \geq 1$  and  $y \geq 2e^t$ , the constant  $\gamma_0$  beign given by (1.24) below.

Finally write  $\sigma_n := \phi_n(\kappa, y)$ .

**Theorem 3.** *We have*

$$\Phi(t, y) = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\}$$

uniformly for  $t \geq 1$  and  $y \geq 2e^t$ .

**Theorem 4.** *For each integer  $J \geq 1$ , we have*

$$(1.19) \quad \Phi(t, y) = \exp \left\{ -\kappa \left[ \sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}$$

uniformly for  $t \geq 1$  and  $y \geq 2e^t$ , where the error term  $R_J(\kappa, y)$  is given by

$$(1.20) \quad R_J(\kappa, y) := \frac{1}{(\log \kappa)^{J+1}} + \frac{\kappa}{y \log y}$$

and

$$(1.21) \quad a_j := \int_0^\infty \left( \frac{h(u)}{u} \right)' (\log u)^{j-1} du$$

with

$$(1.22) \quad h(u) := \begin{cases} \log \left( \frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) & \text{if } 0 \leq u < 1, \\ \log \left( \frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) - 2u & \text{if } u \geq 1. \end{cases}$$

As a corollary of Theorem 4, we can obtain an asymptotic development for  $\log \Phi(t, y)$  in  $t^{-1}$ . In particular we see that the probabilistic distribution function  $\Phi(t)$  decays double exponentially as  $t \rightarrow \infty$ .

**Corollary 5.** *For each integer  $J \geq 1$ , there are computable constants  $a_1^*, \dots, a_J^*$  such that the asymptotic formula*

$$(1.23) \quad \Phi(t, y) = \exp \left\{ -e^{t-\gamma_0} \left[ \sum_{j=1}^J \frac{a_j^*}{t^j} + O_J(R_J(e^t, y)) \right] \right\}$$

holds uniformly for  $t \geq 1$  and  $y \geq 2e^t$ . Further we have

$$(1.24) \quad \gamma_0 := \frac{1}{2} \int_0^\infty \frac{h'(u)}{u} du, \quad a_1^* := 1, \quad a_2^* := \gamma_0 - \frac{\gamma_0^2}{2} - \int_0^\infty \frac{h(u)}{u^2} (\log u) du.$$

In particular for each integer  $J \geq 1$ , we have

$$(1.25) \quad \Phi(t) = \exp \left\{ -e^{t-\gamma_0} \left[ \sum_{j=1}^J \frac{a_j^*}{t^j} + O_J\left(\frac{1}{t^{J+1}}\right) \right] \right\}$$

uniformly for  $t \geq 1$ .

**Remark 2.** (i) The same results hold also for  $\Psi(t, y)$ .

(ii) Taking  $t = \log_2 k$  and  $J = 1$  in (1.25) of Corollary 5, we see that the probabilistic distribution function  $\Phi(t)$  (resp.  $\Psi(t)$ ) verifies Montgomery-Vaughan's conjecture (1.5). But (1.13) is too weak to derive this conjecture for  $F_k(t)$  (resp.  $G_k(t)$ ). This means that we must take  $T(k) = \log_2 k$  in Theorem 2, which seems to be rather difficult.

(iii) Our method can be generalized to prove that Theorems 1 and 2 hold for  $L(1, \text{sym}^m f)$  when  $m = 1, 2, 3, 4$  and that Theorems 3, 4 and Corollary 5 are true for  $L(1, \text{sym}^m g^{\natural}(\omega); y)$  when  $m \geq 1$ .

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## § 2. Expression of $E(s, y)$ and existence of saddle-point

The aim of this section is to prove the existence of the saddle-point  $\kappa(t, y)$ , defined by equation (1.17). The first step is to give an explicit expression of  $E(s, y)$ , which is (1.24) of [1]. For the convenience of readers, we state it here as a lemma.

**Lemma 2.1.** *For prime  $p$ , real  $\theta$  and complex number  $s$ , we define*

$$(2.1) \quad D_p(\theta) := \prod_{0 \leq j \leq 1} (1 - e^{i(1-2j)\theta} p^{-1})^{-1} \quad \text{and} \quad E_p(s) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^s \sin^2 \theta \, d\theta.$$

Then for all  $s \in \mathbb{C}$  and  $y \geq 2$ , we have

$$(2.2) \quad E(s, y) = \prod_{p \leq y} E_p(s).$$

*Proof.* Taking

$$G_p(M^{\natural}) = \begin{cases} \det(I - p^{-s'} M^{\natural})^{-s} & \text{if } p \leq y \\ 1 & \text{otherwise} \end{cases}$$

in (1.12), we get

$$\begin{aligned} \mathbb{E}(L(s', g^{\natural}(\omega); y)^s) &= \prod_{p \leq y} \mathbb{E}(L_p(s', g_p^{\natural}(\omega))^s) \\ &= \prod_{p \leq y} \int_{\Omega} \det(1 - p^{-s'} g_p^{\natural}(\omega))^{-s} \, d\mu(\omega) \\ &= \prod_{p \leq y} \frac{2}{\pi} \int_0^\pi (1 - 2p^{-s'} \cos \theta + p^{-2s'})^{-s} \sin^2 \theta \, d\theta. \end{aligned}$$

Taking  $s' = 1$  and noticing (1.15) and (2.1), we get the desired result.  $\square$

**Lemma 2.2.** *For all  $p$  and  $\sigma > 0$ , we have*

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 > 0.$$



In particular for all  $\sigma > 0$  and  $y \geq 2$ , we have  $\phi_2(\sigma, y) > 0$ .

*Proof.* By using the definition (2.1) of  $E_p(\sigma)$ , it is easy to see that

$$\begin{aligned} E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 &= \frac{4}{\pi^2} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \int_0^\pi D_p(\theta)^\sigma \sin^2 \theta \, d\theta \\ &\quad - \left( \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \right)^2 \\ &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma (\log^2 D_p(\theta_1) - \log D_p(\theta_1) \log D_p(\theta_2)) \times \\ &\quad \times \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2. \end{aligned}$$

In view of the symmetry in  $\theta_1$  and  $\theta_2$ , the same formula holds if we exchange the roles of  $\theta_1$  and  $\theta_2$ . Thus it follows that

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2 \left( \frac{D_p(\theta_1)}{D_p(\theta_2)} \right) \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2.$$

This proves the first assertion and the second follows immediately.  $\square$

**Lemma 2.3.** *There is an absolute constant  $c \geq 2$  such that for  $t \geq 4 \log c$  and  $y \geq ce^t$ , the equation  $\phi_1(\sigma, y) = 2(\log t + \gamma)$  has a unique positive solution in  $\sigma$ . Denoting by  $\kappa(t, y)$  this solution, we have  $\kappa(t, y) \asymp e^t$  uniformly for  $t \geq 4 \log c$  and  $y \geq ce^t$ .*

*Proof.* According to Lemma 4.3 below with the choice of  $J = 1$ , we have

$$\phi_1(\sigma, y) = 2(\log_2 \sigma + \gamma) + O(1/\log \sigma)$$

for  $y \geq \sigma \geq 2$ . Thus

$$\begin{aligned} \phi(ce^t, y) &= 2 \log(t + \log c) + 2\gamma + O\left(\frac{1}{t + \log c}\right) \\ &> 2 \log t + 2\gamma, \\ \phi(c^{-1}e^t, y) &= 2 \log(t - \log c) + 2\gamma + O\left(\frac{1}{t - \log c}\right) \\ &< 2 \log t + 2\gamma, \end{aligned}$$

provided that  $c$  is a large constant and  $t \geq 4 \log c$ . On the other hand, in view of Lemma 2.2, we know that for any  $y \geq 2$ ,  $\phi_1(\sigma, y)$  is an increasing function of  $\sigma$  in  $(0, \infty)$ . Hence the equation  $\phi_1(\sigma, y) = 2(\log t + \gamma)$  has a unique positive solution  $\kappa(t, y)$  and  $c^{-1}e^t \leq \kappa(t, y) \leq ce^t$  for  $t \geq 4 \log c$  and  $y \geq ce^t$ . This completes the proof.  $\square$

### § 3. Preliminary lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

**Lemma 3.1.** *Let  $j \geq 0$  be a fixed real number. Then we have*

$$(3.1) \quad \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta \asymp_j e^{2u} u^{-(j+3/2)} \quad (u \geq 1).$$

The implied constant depends on  $j$  only.

*Proof.* First we write

$$\begin{aligned} \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta \, d\theta &= \int_0^{\pi/2} (e^{2u \cos \theta} (1 - \cos \theta)^j + e^{-2u \cos \theta} (1 + \cos \theta)^j) \sin^2 \theta \, d\theta \\ &= \int_0^1 (e^{2ut} (1-t)^j + e^{-2ut} (1+t)^j) (1-t^2)^{1/2} \, dt \\ &\asymp \int_0^1 e^{2ut} (1-t)^{j+1/2} \, dt + \int_0^1 e^{-2ut} (1-t)^{1/2} \, dt. \end{aligned}$$

By the change of variables  $u(1-t) = v$ , it follows that

$$\begin{aligned} \int_0^1 e^{2ut} (1-t)^{j+1/2} \, dt &= e^{2u} u^{-(j+3/2)} \int_0^u e^{-2v} v^{j+1/2} \, dv \\ &\asymp e^{2u} u^{-(j+3/2)}, \\ \int_0^1 e^{-2ut} (1-t)^{1/2} \, dt &\leq \int_0^1 e^{-2ut} \, dt \ll u^{-1}. \end{aligned}$$

We obtain the desired result by insertion of these estimates into the preceding relation.  $\square$

**Lemma 3.2.** *Let  $j \geq 0$  be an integer and*

$$(3.2) \quad E_{p,j}(\sigma) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma (1 - \cos \theta)^j \sin^2 \theta \, d\theta.$$

(In particular  $E_{p,0}(\sigma) = E_p(\sigma)$ .) Then we have

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi} \int_0^1 \left[ \left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du$$

and the estimate

$$(3.3) \quad E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j$$

holds uniformly for all primes  $p$  and  $\sigma > 0$ . Further if  $p \geq \sigma \geq 0$ , we have

$$(3.4) \quad E_p(\sigma) \asymp 1.$$

The implied constant in (3.3) depends on  $j$  only and the one in (3.4) is absolute.

*Proof.* By the change of variables  $u = \sin^2(\theta/2)$ , a simple computation shows that the first assertion is true. Obviously (3.3) holds for  $j = 0$ .

Now assume that it is true for  $j$ . An integration by parts leads to

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left[ \left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du \\ &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left\{ \left[ \left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \right\} \times \\ &\quad \times \left[ \left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} \, du. \end{aligned}$$

On the other hand, we have

$$0 < u < 1 \Rightarrow \left[ \left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \geq \left(1 + \frac{1}{p}\right)^{-2} \frac{4\sigma}{p} \geq \frac{16\sigma}{9p}.$$

Inserting it into the preceding estimate, we see that

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^{j+1} \int_0^1 \left[ \left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} du \\ &\asymp_j \left(\frac{\sigma}{p}\right)^{j+1} E_{p,j+1}(\sigma). \end{aligned}$$

Thus (3.3) holds also for  $j+1$ .

Since  $(1+1/p)^{-2} \leq D_p(\theta) \leq (1-1/p)^{-2}$  for all primes  $p$  and any  $\theta \in \mathbb{R}$ , we have  $D_p(\theta)^\sigma \asymp 1$  uniformly for  $p \geq \sigma \geq 0$  and  $\theta \in \mathbb{R}$ . This implies (3.4).  $\square$

Introduce the function

$$(3.5) \quad g(u) := \log \left( \frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) \quad (u \geq 0)$$

and let  $h(u)$  be defined as in (1.22). Clearly we have

$$(3.6) \quad h(u) = \begin{cases} g(u) & \text{if } 0 \leq u < 1, \\ g(u) - 2u & \text{if } u \geq 1, \end{cases}$$

$$(3.7) \quad h'(u) = \begin{cases} g'(u) & \text{if } 0 \leq u < 1, \\ g'(u) - 2 & \text{if } u \geq 1, \end{cases}$$

$$(3.8) \quad h''(u) = g''(u) \quad (u \geq 0, u \neq 1).$$

**Lemma 3.3.** *We have*

$$(3.9) \quad h(u) \asymp \begin{cases} u^2 & \text{if } 0 \leq u < 1, \\ \log(2u) & \text{if } u \geq 1, \end{cases}$$

$$(3.10) \quad h'(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-1} & \text{if } u \geq 1, \end{cases}$$

$$(3.11) \quad h''(u) \asymp \begin{cases} 1 & \text{if } 0 \leq u < 1, \\ u^{-2} & \text{if } u \geq 1, \end{cases}$$

$$(3.12) \quad h'''(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-3} & \text{if } u \geq 1. \end{cases}$$

*Proof.* The last three relations are trivial. We will prove only the first four estimates.

When  $0 \leq u < 1$ , we have

$$e^{2u \cos \theta} = \sum_{n=0}^{\infty} \frac{(u \cos \theta)^n}{n!}.$$

From this we deduce that

$$\begin{aligned} (3.13) \quad h(u) &= \log \left( \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_0^\pi (\cos \theta)^n \sin^2 \theta d\theta \right) \\ &= \log \left( 1 + \sum_{\ell=1}^{\infty} \frac{2 \cdot (2\ell-1)!!}{(2\ell)!(2\ell+2)!!} u^{2\ell} \right), \end{aligned}$$

where we have used the following facts:

$$\int_0^\pi (\cos \theta)^{2\ell+1} \sin^2 \theta \, d\theta = 0$$

and

$$\frac{2}{\pi} \int_0^\pi (\cos \theta)^{2\ell} \sin^2 \theta \, d\theta = \begin{cases} 1 & \text{if } \ell = 0, \\ 2 \frac{(2\ell-1)!!}{(2\ell+2)!!} & \text{if } \ell \geq 1. \end{cases}$$

Now we easily deduce, from (3.13), the desired results in the case of  $0 \leq u < 1$ .

The estimates of (3.9)–(3.12) for  $u > 1$  are simple consequences of (3.1), by noticing the following relations

$$h'(u) = -2 \frac{\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta},$$

$$h''(u) = 4 \frac{\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^2 \sin^2 \theta \, d\theta}{\int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta} - 4 \left( \frac{\int_0^\pi e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta} \right)^2.$$

This completes the proof. □

#### § 4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of  $\phi_n(\sigma, y)$  for  $n = 0, 1, 2, 3, 4$ .

**Lemma 4.1.** *For any fixed integer  $J \geq 1$ , we have*

$$(4.1) \quad \phi_0(\sigma, y) = \sigma \left\{ 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for  $y \geq \sigma \geq 3$ , where  $R_J(\sigma, y)$  is defined as in (1.20) and

$$(4.2) \quad b_{j,0} := \int_0^\infty \frac{h(u)}{u^2} (\log u)^{j-1} \, du.$$

*Proof.* By the definition (2.1) of  $D_p(\theta)$  and the one of  $E_p(\sigma)$ , it is easy to see that for  $p \geq \sigma^{1/2}$ , we have

$$(4.3) \quad D_p(\theta)^\sigma = e^{2(\sigma/p) \cos \theta} \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\},$$

$$(4.4) \quad E_p(\sigma) = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \sin^2 \theta \, d\theta.$$

From these, we deduce that

$$(4.5) \quad \sum_{\sigma^{1/2} < p \leq y} \log E_p(\sigma) = \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}/\log \sigma)$$

where  $g(u)$  is defined as in (3.5).

In order to treat the sum over  $p \leq \sigma$ , we write

$$E_p(\sigma) = (1 - 1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi} \int_0^\pi \left\{ 1 + \frac{2(1 - \cos \theta)}{p} \left( 1 - \frac{1}{p} \right)^{-2} \right\}^{-\sigma} \sin^2 \theta \, d\theta.$$

By using the change of variables  $u = \sin^2(\theta/2)$ , we have

$$\begin{aligned} E_p^*(\sigma) &= \frac{8}{\pi} \int_0^\pi \left\{ 1 + \frac{4}{p} \left( 1 - \frac{1}{p} \right)^{-2} \sin^2(\theta/2) \right\}^{-\sigma} \sin^2(\theta/2) \cos^2(\theta/2) \, d\theta \\ &\geq \frac{8}{\pi} \int_0^{p/2\sigma} \left\{ 1 + \frac{4}{p} \left( 1 - \frac{1}{p} \right)^{-2} u \right\}^{-\sigma} \sqrt{u(1-u)} \, du \\ &\geq \frac{8}{\pi} \left( 1 + \frac{8}{\sigma} \right)^{-\sigma} \int_0^{p/2\sigma} \sqrt{u(1-u)} \, du \\ &\geq C \left( \frac{p}{\sigma} \right)^{3/2}, \end{aligned}$$

where  $C > 0$  is a constant. On the other hand, we have trivially  $E_p^*(\sigma) \leq 1$  for all  $p$  and  $\sigma > 0$ . Thus  $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$  for  $p \leq \sigma^{1/2}$  and

$$(4.6) \quad \sum_{p \leq \sigma^{1/2}} |\log E_p^*(\sigma)| \ll \sum_{p \leq \sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}.$$

Combining (4.5) and (4.6), we can write

$$\sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma^{1/2}} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2} < p \leq \sigma} (2\sigma \log(1 - 1/p)^{-1} - 2\sigma/p) \ll \sum_{\sigma^{1/2} < p \leq \sigma} \sigma/p^2 \ll \sigma^{1/2}/\log \sigma,$$

the preceding estimate can be written as

$$(4.7) \quad \sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} h(\sigma/p) + O(\sigma^{1/2}).$$

By using the prime number theorem in the form

$$(4.8) \quad \pi(t) := \sum_{p \leq t} 1 = \int_2^t \frac{dv}{\log v} + O\left( t e^{-8\sqrt{\log t}} \right),$$

it follows that

$$(4.9) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} \, dt + R_0,$$

where

$$\begin{aligned} R_0 &:= h\left(\frac{\sigma}{y}\right)e^{-8\sqrt{\log y}} + h(\sigma^{1/2})e^{-4\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^y \sigma \frac{|h'(\sigma/t)|}{t^2} e^{-8\sqrt{\log t}} dt \\ &\ll \frac{\sigma^2}{y^2} e^{-8\sqrt{\log y}} + e^{-2\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^{\sigma} \frac{e^{-2\sqrt{\log t}}}{t} dt + \sigma^2 \int_{\sigma}^y \frac{e^{-8\sqrt{\log t}}}{t^3} dt \\ &\ll e^{-\sqrt{\log \sigma}} \end{aligned}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables  $u = \sigma/t$  to write

$$\begin{aligned} \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt &= \sigma \int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du \\ &= \sigma \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du + O(R'_0) \end{aligned}$$

where

$$\begin{aligned} R'_0 &:= \sigma \int_0^{\sigma/y} \frac{|h(u)|}{u^2 \log(\sigma/u)} du + \sigma \int_0^{\sigma^{-1/2}} \frac{|h(u)|}{u^2 \log(\sigma/u)} du \\ &\ll \frac{\sigma^2}{y \log y} + \frac{\sigma^{1/2}}{\log \sigma}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du &= \frac{1}{\log \sigma} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 (1 - (\log u)/\log \sigma)} du \\ &= \sum_{j=1}^J \frac{1}{(\log \sigma)^j} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du + O\left(\frac{1}{(\log \sigma)^{J+1}}\right). \end{aligned}$$

Extending the interval of integration  $[\sigma^{-1/2}, \sigma^{1/2}]$  to  $(0, \infty)$  and bounding the contributions of  $(0, \sigma^{-1/2}]$  and  $[\sigma^{1/2}, \infty)$  by using (3.9) of Lemma 3.3, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du = b_{j,0} + O\left(\frac{(\log \sigma)^j}{\sigma^{1/2}}\right).$$

Combining these estimates, we find that

$$(4.10) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \sigma \left\{ \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}.$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$(4.11) \quad \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} = \log_2 \sigma + \gamma + O\left(e^{-2\sqrt{\log \sigma}}\right).$$

This completes the proof.  $\square$

**Remark 3.** In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma, y) = \sigma \left\{ \log(B_1^+ \log \sigma)^{A_1^+} + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for  $y \geq \sigma \geq 3$ . In the case  $\sigma < 0$ , a similar asymptotic formula (with  $A_1^-, B_1^-$  and corresponding  $b_{j,0}^-$  in place of  $A_1^+, B_1^+$  and  $b_{j,0}$ ) can be established uniformly for  $y \geq -\sigma \geq 3$ . As indicated in the introduction, Lemma 4.1 can be easily generalised to the general case  $m \geq 1$ . Thus we give an improvement and generalisation of Corollaries A and C of [15], of Theorem B of [5], and an improvement of Theorem 1.12 of [1]. It is worthy to indicate that our method seems to be simpler and more natural.

**Lemma 4.2.** *We have*

$$(4.12) \quad \frac{E'_p(\sigma)}{E_p(\sigma)} = \begin{cases} \log D_p(0) + O\left(\frac{1}{\sigma}\right) & \text{for all } p \text{ and } \sigma > 0, \\ \frac{1}{2}g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2} + \frac{\sigma}{p^3}\right) & \text{if } p \geq \sigma^{1/2}, \end{cases}$$

where  $g(u)$  is defined as in (3.5).

*Proof.* First we write

$$(4.13) \quad \begin{aligned} E'_p(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \\ &= E_p(\sigma) \log D_p(0) + R', \end{aligned}$$

where

$$(4.14) \quad R' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log \left( \frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\left| \log \left( \frac{D_p(\theta)}{D_p(0)} \right) \right| = \left| -\log \left( 1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) \right| \leq \frac{2p(1 - \cos \theta)}{(p-1)^2} \leq \frac{8(1 - \cos \theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with  $j = 1$  that

$$\frac{R'}{E_p(\sigma)} \ll \frac{E_{p,1}(\sigma)}{pE_p(\sigma)} \ll \frac{1}{\sigma}$$

for all  $p$  and  $\sigma > 0$ . This implies, via (4.13), the first estimate of (4.12).

We have

$$\begin{aligned} \log D_p(\theta) &= (\cos \theta)(2/p) + O(1/p^2) \\ &= (\cos \theta) \log D_p(0) + O(1/p^2). \end{aligned}$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for  $p \geq \sigma^{1/2}$ ,

$$\begin{aligned} E'_p(\sigma) &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \left\{ (\cos \theta) \log D_p(0) + O\left(\frac{1}{p^2}\right) \right\} \sin^2 \theta \, d\theta \\ &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} (\cos \theta) \sin^2 \theta \, d\theta \log D_p(0) + O\left(\frac{E_p(\sigma)}{p^2}\right). \end{aligned}$$

From this and (4.4), we deduce

$$\frac{E'_p(\sigma)}{E_p(\sigma)} = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{1}{2}g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2}\right),$$

which implies the second estimate of (4.12). This completes the proof.  $\square$

**Lemma 4.3.** *Let  $J \geq 1$  be a fixed integer. Then we have*

$$\phi_1(\sigma, y) = 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y))$$

uniformly for  $y \geq \sigma \geq 3$ , where the constant  $b_{j,1}$  is given by

$$(4.15) \quad b_{j,1} := \int_0^\infty \frac{h'(u)}{u} (\log u)^{j-1} du$$

and  $R_J(\sigma, y)$  is defined as in (1.20).

*Proof.* We have

$$\phi_1(\sigma, y) = \sum_{p \leq y} E'_p(\sigma) / E_p(\sigma).$$

Using the first relation of (4.12) for  $p \leq \sigma^{2/3}$  and the second for  $\sigma^{2/3} < p \leq y$ , we obtain

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma^{2/3}} \log D_p(0) + \frac{1}{2} \sum_{\sigma^{2/3} < p \leq y} g' \left( \frac{\sigma}{p} \right) \log D_p(0) + O \left( \frac{1}{\sigma^{1/3}} \right).$$

In view of (3.7), the preceding formula can be written as

$$(4.16) \quad \phi_1(\sigma, y) = \sum_{p \leq \sigma} \log D_p(0) + \sum_{\sigma^{2/3} < p \leq y} h' \left( \frac{\sigma}{p} \right) \log \left( 1 - \frac{1}{p} \right)^{-1} + O \left( \frac{1}{\sigma^{1/3}} \right).$$

Similarly to (4.10), we can prove that

$$(4.17) \quad \sum_{\sigma^{2/3} < p \leq y} h' \left( \frac{\sigma}{p} \right) \log \left( 1 - \frac{1}{p} \right)^{-1} = \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)),$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17).  $\square$

**Lemma 4.4.** *We have*

$$(4.18) \quad \frac{E''_p(\sigma)E_p(\sigma) - E'_p(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O \left( \frac{1}{\sigma^2} \right) & \text{if } p \leq \sigma^{1/2}, \\ \frac{1}{p^2} g'' \left( \frac{\sigma}{p} \right) + O \left( \min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2} \right\} \right) & \text{if } p > \sigma^{1/2}, \end{cases}$$

where  $g(u)$  is defined as in (3.5).

*Proof.* First we write

$$(4.19) \quad \begin{aligned} E''_p(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta d\theta \\ &= E_p(\sigma) \log^2 D_p(0) + R'', \end{aligned}$$

where

$$R'' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \left( \log^2 D_p(\theta) - \log^2 D_p(0) \right) \sin^2 \theta d\theta.$$



Using (4.13) and (4.19), we can deduce

$$(4.20) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} - \left( \frac{R'}{E_p(\sigma)} \right)^2,$$

where  $R'$  is defined as in (4.14).

From the definitions of  $R'$  and  $R''$ , a simple calculation shows that

$$R'' - 2R' \log D_p(0) = \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 \left( \frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\log^2 \left( \frac{D_p(\theta)}{D_p(0)} \right) = \log^2 \left( 1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = \frac{4(1 - \cos \theta)^2}{p^2} + O\left( \frac{(1 - \cos \theta)^2}{p^3} \right),$$

we have

$$R'' - 2R' \log D_p(0) = \frac{4}{p^2} E_{p,2}(\sigma) + O\left( \frac{E_{p,2}(\sigma)}{p^3} \right),$$

where  $E_{p,j}(\sigma)$  is defined as in (3.2). By using (3.3) with the choice of  $j = 2$  and the trivial estimate  $E_{p,2}(\sigma) \leq 4E_p(\sigma)$ , we deduce

$$(4.21) \quad \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} = \frac{4}{p^2} \frac{E_{p,2}(\sigma)}{E_p(\sigma)} + O\left( \min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right).$$

Similarly we have

$$\log \left( \frac{D_p(\theta)}{D_p(0)} \right) = -\log \left( 1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = -\frac{2(1 - \cos \theta)}{p} + O\left( \frac{(1 - \cos \theta)}{p^2} \right),$$

and therefore

$$R' = -\frac{2}{p} E_{p,1}(\sigma) + O\left( \frac{E_{p,1}(\sigma)}{p^2} \right).$$

Now (3.3) with  $j = 1$  and the trivial estimate  $E_{p,1}(\sigma) \leq 2E_p(\sigma)$  imply

$$(4.22) \quad \begin{aligned} \left( \frac{R'}{E_p(\sigma)} \right)^2 &= \frac{4}{p^2} \left( \frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left( \frac{E_{p,1}(\sigma)^2}{p^3 E_p(\sigma)^2} \right) \\ &= \frac{4}{p^2} \left( \frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left( \min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right). \end{aligned}$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$(4.23) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{4}{p^2} h_p(\sigma) + O\left( \min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right)$$

for all  $p$  and  $\sigma > 0$ , where

$$h_p(\sigma) := \frac{E_{p,2}(\sigma)}{E_p(\sigma)} - \left( \frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2.$$

When  $p \leq \sigma^{1/2}$ , the inequality (3.3) of Lemma 3.2 implies that  $h_p(\sigma) \ll (p/\sigma)^2$ . From this and (4.23) we deduce the first estimate of (4.18).

If  $p \geq \sigma^{1/2}$ , we can use (4.3), (3.11) and (3.8) to write

$$\begin{aligned} 4h_p(\sigma) &= g'' \left( \frac{\sigma}{p} \right) \left\{ 1 + O\left( \frac{\sigma}{p^2} \right) \right\} \\ &= g'' \left( \frac{\sigma}{p} \right) + O\left( \min \left\{ \frac{\sigma}{p^2}, \frac{1}{\sigma} \right\} \right). \end{aligned}$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for  $p \geq \sigma^{1/2}$ ,

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{1}{p^2} g'' \left( \frac{\sigma}{p} \right) + O\left( \min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2} \right\} \right).$$

This completes the proof.  $\square$

**Lemma 4.5.** *Let  $J \geq 1$  be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for  $y \geq \sigma \geq 2$ , where

$$b_{j,2} := \int_0^\infty h''(u)(\log u)^{j-1} du.$$

In particular  $b_{1,2} = 2$ .

*Proof.* From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned} \phi_2(\sigma, y) &= \sum_{p \leq y} \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{g''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma}\right) \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma}\right). \end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceding two estimates.

Finally

$$\begin{aligned} b_{1,2} &= \int_0^1 h''(u) du + \int_1^\infty h''(u) du \\ &= h'(1-) - h'(1+) = h'(1-) - (h'(1-) - 2) = 2. \end{aligned}$$

This completes the proof.  $\square$

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

**Lemma 4.6.** *We have*

$$(4.24) \quad \phi_n(\sigma, y) \ll 1/(\sigma^{n-1} \log \sigma) \quad (n = 3, 4)$$

uniformly for  $y \geq \sigma \geq 3$ .

## § 5. Estimate of $|E(\kappa + i\tau, y)|$

**Lemma 5.1.** *For any  $\delta \in (0, \frac{1}{4})$ , there are two absolute positive constants  $c_1, c_2$  and a positive constant  $c_3 = c_3(\delta)$  such that for all  $y \geq \sigma \geq 3$  we have*

$$(5.1) \quad \left| \frac{E(\sigma + i\tau, y)}{E(\sigma, y)} \right| \leq \begin{cases} 1 & \text{if } |\tau| \leq c_1 \sigma^{1/2} \log \sigma \text{ or } |\tau| \geq y^{1/\delta}, \\ e^{-c_2 \tau^2 / [\sigma (\log \sigma)^2]} & \text{if } c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma, \\ e^{-c_3 |\tau|^\delta} & \text{if } \sigma \leq |\tau| \leq y^{1/\delta}. \end{cases}$$

*Proof.* First we write

$$\begin{aligned} E_p(s) &= \frac{2}{\pi} \int_0^\pi (D_p(\theta)^{-1})^{-s} \sin^2 \theta \, d\theta \\ &= \frac{2}{\pi} \int_0^\pi \frac{\sin^2 \theta}{(1-s)(D_p(\theta)^{-1})'} d(D_p(\theta)^{-1})^{1-s}. \end{aligned}$$

Since  $(D_p(\theta)^{-1})' = 2p^{-1} \sin \theta$ , after a simplification and an integration by parts it follows that

$$\begin{aligned} E_p(s) &= \frac{p}{\pi(s-1)} \int_0^\pi D_p(\theta)^{s-1} \cos \theta \, d\theta \\ &= \frac{p}{\pi(s-1)} \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\} \cos \theta \, d\theta. \end{aligned}$$

This implies that

$$(5.2) \quad \left| \frac{E_p(s)}{E_p(\sigma)} \right| = \left| \frac{\sigma-1}{s-1} \right| \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right|$$

with

$$E_p^*(s) := \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\} \cos \theta \, d\theta.$$

1° Case of  $\sigma^{1/\delta} < |\tau| \leq y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1} \{1 - \Delta_p(\theta)^{s-1}\} \cos \theta \, d\theta$$

with

$$\Delta_p(\theta) := \frac{1 - 2p^{-1} \cos \theta + p^{-2}}{1 + 2p^{-1} \cos \theta + p^{-2}}.$$

It is clear that for all  $p$ , the function  $\theta \mapsto \Delta_p(\theta)$  is increasing on  $[0, \pi/2]$ . It follows that

$$\begin{aligned} E_p^*(\sigma) &\geq \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \{1 - \Delta_p(\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &\geq \{1 - \Delta_p(\pi/4)^{\sigma-1}\} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \end{aligned}$$

for all  $p$  and  $\sigma \geq 1$ . This implies that

$$(5.3) \quad \left| \frac{1}{E_p^*(\sigma)} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

Similarly since the function  $\theta \mapsto D_p(\theta)^{\sigma-1} \cos \theta$  is decreasing on  $[0, \pi/2]$  for all  $p$  and  $\sigma \geq 2$ , we can deduce, via (5.3), that

$$(5.4) \quad \left| \frac{1}{E_p^*(\sigma)} \int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

From (5.3) and (5.4), we deduce that

$$\left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \frac{2}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all  $p \geq \sigma \geq 2$ , we have

$$\Delta_p \left( \frac{\pi}{4} \right)^{\sigma-1} \leq \left( 1 - \frac{\sqrt{2}}{p} + \frac{1}{p^2} \right)^{\sigma-1} \leq 1 - \frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left| \frac{E_p(s)}{E_p(\sigma)} \right| \leq \frac{8p}{|s-1|} \leq \frac{p^4}{|\tau|} \quad (p \geq \sigma).$$

By multiplying this inequality for  $\sigma < p \leq |\tau|^\delta$  ( $\leq y$ ) and the trivial inequality  $|E_p(s)| \leq |E_p(\sigma)|$  for the others  $p$ , we deduce, via the prime number theorem, that

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma < p \leq |\tau|^\delta} \log |\tau| + 4 \sum_{\sigma < p \leq |\tau|^\delta} \log p \right\} \\ &\leq e^{-\{1/\delta - 4 + o(1)\}|\tau|^\delta}. \end{aligned}$$

2° Case of  $c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma^{1/\delta}$

For  $p \geq \sigma^{1/2} \geq 2$ , we can write

$$\begin{aligned} |E_p^*(s)| &\leq \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p] \cos \theta} + e^{-2[(\sigma-1)/p] \cos \theta}) \cos \theta \, d\theta \end{aligned}$$

and

$$\begin{aligned} |E_p^*(\sigma)| &= \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p] \cos \theta} - e^{-2[(\sigma-1)/p] \cos \theta}) \cos \theta \, d\theta. \end{aligned}$$

From these, we deduce that

$$(5.5) \quad \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \left\{ 1 + O\left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\right) \right\} \quad (2 \leq \sigma^{1/2} \leq p \leq \sigma)$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p] \cos \theta} \cos \theta \, d\theta \gg e^{\sigma/p} \quad \text{and} \quad \int_0^{\pi/2} e^{-2[(\sigma-1)/p] \cos \theta} \cos \theta \, d\theta \ll 1.$$

Inserting (5.5) into (5.2), for  $2 \leq \sigma^{1/2} \leq p \leq \sigma$  we obtain

$$\begin{aligned} \left| \frac{E_p(s)}{E_p(\sigma)} \right| &\leq \exp \left\{ - \log \left| \frac{s-1}{\sigma-1} \right| + C \left( \frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}} \right) \right\} \\ &\leq \begin{cases} e^{-\tau^2/\sigma^2 + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } 3 \leq |\tau| \leq \sigma, \\ e^{-\frac{1}{2} \log(1+\tau^2/\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } \sigma \leq |\tau| \leq \sigma^{1/\delta}, \end{cases} \end{aligned}$$

where  $C > 0$  is an absolute constant.

Now by multiplying these inequalities for  $\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)$  and the trivial inequality  $|E_p(s)| \leq E_p(\sigma)$  for the other  $p$ , we get

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)} \left( \frac{\tau^2}{\sigma^2} - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right) \right\} \\ &\leq \exp \left\{ - \left( \frac{\tau^2}{8\sigma(\log \sigma)^2} - 10C - \frac{10C}{\sigma \log \sigma} \right) \right\} \\ &\leq \exp \left\{ - \frac{c_2 \tau^2}{\sigma(\log \sigma)^2} \right\} \end{aligned}$$

if  $c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma$ , and

$$\begin{aligned} (5.6) \quad \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)} \left[ \frac{1}{2} \log \left( 1 + \frac{\tau^2}{\sigma^2} \right) - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right] \right\} \\ &\leq \exp \left\{ - \left[ \frac{\sigma}{8 \log \sigma} \log \left( 1 + \frac{\tau^2}{\sigma^2} \right) - 10C - \frac{10C}{\sigma \log \sigma} \right] \right\} \\ &\leq \exp \left\{ - c_3 |\tau|^\delta \right\} \end{aligned}$$

if  $\sigma \leq |\tau| \leq \sigma^{1/\delta}$ . This completes the proof.  $\square$

## § 6. Proof of Theorem 3

We follow the argument of Granville & Soundararajan [4] to prove Theorem 3. We shall divide the proof in several steps which are embodied in the following lemmas.

The first one is a classic integration formula (see [4], page 1019).

**Lemma 6.1.** *Let  $c > 0$ ,  $\lambda > 0$  and  $N \in \mathbb{N}$ . Then we have*

$$(6.1) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < y < e^{-\lambda N}, \\ \in [0, 1] & \text{if } e^{-\lambda N} \leq y < 1, \\ 1 & \text{if } y \geq 1. \end{cases}$$

The second one is an analogue for (3.6) and (3.7) of [4] (see also Lemma 3.1 of [20]).

**Lemma 6.2.** *Let  $t \geq 1$ ,  $y \geq 2e^t$  and  $0 < \lambda \leq e^{-t}$ . Then we have*

$$(6.2) \quad \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s}} \frac{ds}{\lambda s} \leq \Phi(te^{-\lambda}, y),$$

$$(6.3) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - e^{-\lambda s}}{\lambda s} \frac{ds}{s}.$$

*Proof.* Denote by  $\mathbf{1}_X(\omega)$  the characteristic function of the set  $X \subset \Omega$ . Then by Lemma 6.1 with  $N = 1$  and  $c = \kappa$ , we have

$$\mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega); y) > (e^{\gamma t})^2\}}(\omega) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left( \frac{L(1, g^{\natural}(\omega); y)}{(e^{\gamma t})^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds.$$

Integrating over  $\Omega$  and interchanging the order of integrations yield

$$\begin{aligned}\Phi(t, y) &\leq \int_{\Omega} \left\{ \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left( \frac{L(1, g^{\natural}(\omega); y)}{(e^{\gamma t})^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds \right\} d\mu(\omega) \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds.\end{aligned}$$

This proves the first inequality of (6.2). The second can be treated by noticing that

$$\begin{aligned}\mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega); y) > (e^{\gamma-\lambda t})^2\}}(\omega) &= \mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega); y) > (e^{\gamma t})^2\}}(\omega) \\ &\quad + \mathbf{1}_{\{\omega \in \Omega: (e^{\gamma t})^2 \geq L(1, g^{\natural}(\omega); y) > (e^{\gamma-\lambda t})^2\}}(\omega) \\ &\geq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left( \frac{L(1, g^{\natural}(\omega); y)}{(e^{\gamma t})^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds.\end{aligned}$$

From (6.2), we can deduce

$$\begin{aligned}\Phi(te^{-\lambda}, y) - \Phi(t, y) &\leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma-\lambda t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma+\lambda t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds \\ &= \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} (e^{\lambda s} - e^{-\lambda s}) ds.\end{aligned}$$

This completes the proof.  $\square$

**Lemma 6.3.** *Let  $t \geq 1$ ,  $y \geq 2e^t$  and  $0 < \kappa\lambda \leq 1$ . Then we have*

$$\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\kappa\lambda + \frac{\log \kappa}{\kappa}\right) \right\}.$$

*Proof.* First we write, for  $s = \kappa + i\tau$  and  $|\tau| \leq \kappa$ ,

$$\begin{aligned}E(s, y) &= \exp \left\{ \sigma_0 + i\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - i\frac{\sigma_3}{6}\tau^3 + O(\sigma_4\tau^4) \right\}, \\ \frac{e^{\lambda s} - 1}{\lambda s^2} &= \frac{1}{\kappa} \left\{ 1 - \frac{i}{\kappa}\tau + O\left(\kappa\lambda + \frac{\tau^2}{\kappa^2}\right) \right\}.\end{aligned}$$

Since  $\sigma_1 = \log t + \gamma$ , we have

$$\frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} = \frac{E(\kappa, y)}{\kappa(e^{\gamma t})^{2\kappa}} e^{-(\sigma_2/2)\tau^2} \left\{ 1 - \frac{i}{\kappa}\tau - i\frac{\sigma_3}{6}\tau^3 + O(R(\tau)) \right\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over  $|\tau| \leq \kappa$  to obtain

$$(6.4) \quad \frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{2\pi\kappa(e^{\gamma t})^{2\kappa}} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} \{1 + O(R(\tau))\} d\tau,$$

where we have used the fact that the integrals involving  $(i/\kappa)\tau$  and  $(i\sigma_3/6)\tau^3$  vanish.

On the other hand, using lemmas 4.5 and 4.6 we have

$$\begin{aligned}\int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} d\tau &= \sqrt{\frac{2\pi}{\sigma_2}} \left\{ 1 + O\left(\exp\left\{-\frac{1}{2}\kappa^2\sigma_2\right\}\right) \right\}, \\ \int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2} R(\tau) d\tau &\ll \frac{1}{\sqrt{\sigma_2}} \left( \kappa\lambda + \frac{1}{\kappa^2\sigma_2} + \frac{\sigma_3^2}{\sigma_2^2} + \frac{\sigma_4}{\sigma_2} \right) \\ &\ll \frac{1}{\sqrt{\sigma_2}} \left( \kappa\lambda + \frac{\log \kappa}{\kappa} \right).\end{aligned}$$

Inserting these into (6.4), we obtain the desired result.  $\square$

**Lemma 6.4.** *Let  $\delta$  and  $c_3$  be two constants determined by Lemma 5.1. Then we have*

$$(6.5) \quad \int_{\kappa \pm i\kappa}^{\kappa \pm i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_1,$$

$$(6.6) \quad \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} (e^{\lambda s} - e^{-\lambda s}) ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_2,$$

uniformly for  $t \geq 1$ ,  $y \geq 2e^t$ ,  $\kappa \geq 2$  and  $0 < \lambda\kappa \leq 1$ , where

$$R_1 := \lambda^{-1} e^{-c_3 \kappa^\delta} + \lambda^{-1} (\kappa / \log \kappa)^{1/2} y^{-1/\delta},$$

$$R_2 := \lambda \kappa (\log \kappa)^{1/2} + e^{-(c_3/2)\kappa^\delta} + \lambda^{-1} (\kappa / \log \kappa)^{1/2} y^{-1/\delta}.$$

*Proof.* We split the integral in (6.5) into two parts according to  $\kappa \leq |\tau| \leq y^{1/\delta}$  or  $|\tau| \geq y^{1/\delta}$ . Using Lemma 5.1 with  $\sigma = \kappa$  and the inequality  $(e^{\lambda s} - 1)/s^2 \ll 1/\tau^2$ , the integral in (6.5) is

$$\ll \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa} \lambda} \left( \frac{e^{-c_3 \kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}} \right),$$

which implies (6.5), in view of Lemma 4.5 with  $J = 1$ .

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leq c_1 \kappa^{1/2} \log \kappa, \quad c_1 \kappa^{1/2} \log \kappa < |\tau| \leq \kappa, \quad \kappa < |\tau| \leq y^{1/\delta}, \quad |\tau| \geq y^{1/\delta}.$$

By Lemma 5.1 with  $\sigma = \kappa$  and the inequalities

$$(e^{\lambda s} - 1)/\lambda s \ll \min\{1, 1/(\lambda|\tau|)\}, \quad (e^{\lambda s} - e^{-\lambda s})/s \ll \min\{\lambda, 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa}} \left( \lambda \kappa^{1/2} \log \kappa + e^{-c_3 \kappa^\delta} + \lambda^{-1} y^{-1/\delta} \right),$$

which implies (6.6), as before.  $\square$

Now we are ready to complete the proof of Theorem 3. Lemma 6.3 and (6.5) of Lemma 6.4 give

$$(6.7) \quad \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \{1 + O(R')\}$$

where

$$R' := \frac{\log \kappa}{\kappa} + \kappa \lambda + \frac{e^{-c_3 \kappa^\delta} + (\kappa / \log \kappa)^{1/2} y^{-1/\delta}}{\lambda}.$$

Taking  $\lambda = \kappa^{-2}$  and noticing  $y \geq 2e^t \asymp \kappa$  and  $1/\delta > 4$ , we deduce

$$(6.8) \quad R' \ll t/e^t.$$

Combining (6.7) and (6.8) with (6.2), we obtain

$$(6.9) \quad \Phi(t, y) \leq \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\} \leq \Phi(te^{-\lambda}, y)$$

uniformly for  $t \geq 1$ ,  $y \geq 2e^t$  and  $0 < \lambda \leq e^{-t}$ .

On the other hand, (6.3) of Lemma 6.2 and (6.6) of Lemma 6.3 imply

$$\begin{aligned} \Phi(te^{-\lambda}, y) - \Phi(t, y) &\ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^{\gamma t})^{2\kappa}} \left( \lambda\kappa(\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{(\kappa/\log \kappa)^{1/2}}{\lambda y^{1/\delta}} \right) \\ &\ll_\varepsilon \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^{\gamma t})^{2\kappa}} \left( \lambda\kappa(\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3\kappa^\delta}} \right) \end{aligned}$$

when  $y^{-1/(2\delta)}\kappa^{-1/2}(\log \kappa)^{-1} \leq \lambda \leq \kappa^{-1}$ . Since  $\Phi(te^{-\lambda}, y) - \Phi(t, y)$  is a non-decreasing function of  $\lambda$ , we deduce

$$(6.10) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa\sqrt{\sigma_2}(e^{\gamma t})^{2\kappa}} \left( \lambda\kappa(\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3\kappa^\delta}} + \frac{\kappa(\log \kappa)^{1/2}}{y^{1/(2\delta)}} \right)$$

uniformly for  $t \geq 1$ ,  $y \geq 2e^t$  and  $0 < \lambda \leq e^{-t}$ . Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 3.  $\square$

### § 7. Proof of Theorem 4

Using Lemmas 4.1 and 4.5, we can write

$$\begin{aligned} \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} &= \exp \left\{ \phi(\kappa, y) - 2\kappa(\gamma + \log t) + O(\log \kappa) \right\} \\ &= \exp \left\{ \kappa \left( 2\log_2 \kappa - 2\log t + \sum_{j=1}^J \frac{b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right) \right\}. \end{aligned}$$

On the other hand, Lemma 4.3 and (1.17) imply that

$$2\log_2 \kappa + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) = 2(\log t + \gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa}} = \exp \left\{ -\kappa \left[ \sum_{j=1}^J \frac{b_{j,1} - b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}.$$

In view of (1.21), (4.2) and (4.15), we have  $b_{j,1} - b_{j,0} = a_j$ . This completes the proof.  $\square$

### § 8. Proof of Corollary 5

We first prove an asymptotic developpment of  $\kappa(t, y)$  in  $t$ .

**Lemma 8.1.** *For each integer  $J \geq 1$ , there are computable constants  $\gamma_0, \gamma_1, \dots, \gamma_J$  such that the asymptotic formula*

$$(8.1) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{\gamma_j}{t^j} + O_J(R_J^*(t, y)) \right\}$$



holds uniformly for  $t \geq 1$  and  $y \geq 2e^t$ , where

$$R_N^*(t, y) := \frac{1}{t^{N+1}} + \frac{e^t t}{y \log y}.$$

Further  $\gamma_0$  is given by (1.24) and  $\gamma_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$ .

*Proof.* By Lemma 4.3 and (1.17), we have

$$(8.2) \quad 2 \log t = 2 \log_2 \kappa + \sum_{j=1}^{J+1} \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)),$$

where  $R_J(\kappa, y)$  is defined as in (1.20). From (8.2), we easily deduce that

$$\begin{aligned} t &= (\log \kappa) \prod_{j=1}^{J+1} \exp \left\{ \sum_{j=1}^J \frac{b_{j,1}}{2(\log \kappa)^j} \right\} \exp \{O_J(R_{J+1}(\kappa, y))\} \\ &= (\log \kappa) \prod_{j=1}^{J+1} \left\{ \sum_{m_j=0}^{J+1} \frac{1}{m_j!} \left( \frac{b_{j,1}}{2(\log \kappa)^j} \right)^{m_j} + O_J(R_{J+1}(\kappa, y)) \right\}. \end{aligned}$$

Developping the product, we get

$$t = (\log \kappa) \left\{ \sum_{j=0}^{J+1} \frac{b'_j}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)) \right\},$$

where

$$\begin{aligned} b'_j &:= \sum_{\substack{m_1 \geq 0, \dots, m_{J+1} \geq 0 \\ m_1 + 2m_2 + \dots + (J+1)m_{J+1} = j}} \frac{b_{1,1}^{m_1} \dots b_{J+1,1}^{m_{J+1}}}{(2m_1)! \dots (2m_{J+1})!} \\ &= \sum_{\substack{m_1 \geq 0, \dots, m_j \geq 0 \\ m_1 + 2m_2 + \dots + jm_j = j}} \frac{b_{1,1}^{m_1} \dots b_{j,1}^{m_j}}{(2m_1)! \dots (2m_j)!}. \end{aligned}$$

Since  $b'_0 = 1$  and  $b'_1 = b_{1,1}/2 = \gamma_0$ , the preceding asymptotic formula can be written as

$$(8.3) \quad t = \log \kappa + \gamma_0 + \sum_{j=1}^J \frac{b'_{j+1}}{(\log \kappa)^j} + O_J(R_J^*(t, y)),$$

where we have used the fact that  $\kappa(t, y) \asymp e^t$  (see Lemma 2.3) and  $(\log \kappa)R_{J+1}(\kappa, y) \asymp R_J^*(t, y)$ .

With the help of (8.3), a simple recurrence argument shows that there are constants  $\gamma'_n$  such that

$$(8.4) \quad t = \log \kappa + \sum_{j=0}^J \frac{\gamma'_j}{t^j} + O_J(R_J^*(t, y)).$$

In fact taking  $J = 0$  in (8.3), we see that (8.4) holds for  $J = 0$ . Suppose that it holds for  $0, \dots, J-1$ , i.e.

$$t = \log \kappa + \sum_{i=0}^{J-j-1} \frac{\gamma'_i}{t^i} + O(R_{J-j-1}^*(t, y)) \quad (j = 0, \dots, J-1),$$

which is equivalent to

$$(8.5) \quad \log \kappa = t \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma'_{i-1}}{t^i} + O\left(\frac{R_{J-j-1}^*(t, y)}{t}\right) \right\} \quad (j = 0, \dots, J-1).$$

This holds also for  $j = J$  if we use the convention:

$$\sum_{i=0}^{-1} = 0 \quad \text{and} \quad R_{-1}^*(t, y) := 1,$$

since  $\log \kappa = t + O(1)$ . Inserting it into (8.3), we easily see that (8.4) holds also for  $J$ . In particular we have  $\gamma'_1 = b'_2 = b_{1,1}^2/8 + b_{2,1}/4$ .

Now (8.1) is an immediate consequence of (8.4) with

$$\gamma_j := \sum_{\substack{m_1 \geq 0, \dots, m_J \geq 0 \\ m_1 + 2m_2 + \dots + Jm_J = j}} (-1)^{m_1 + \dots + m_J} \frac{\gamma_1'^{m_1} \cdots \gamma_J'^{m_J}}{m_1! \cdots m_J!}.$$

This completes the proof. □

Now we are ready to prove Corollary 5.

Using (8.5), we have

$$(8.6) \quad \begin{aligned} \sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} &= \sum_{j=1}^J \frac{a_j}{t^j} \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma'_{i-1}}{t^i} + O_N\left(\frac{R_{J-j-1}^*(t, y)}{t}\right) \right\}^{-j} \\ &= \sum_{j=1}^J \frac{\rho_j}{t^j} + O_J\left(\frac{R_{J-2}^*(t, y)}{t^2}\right), \end{aligned}$$

where the  $\rho_n$  are constants. In particular we have  $\rho_1 = a_1 = 1$  and  $\rho_2 = \gamma_0 + a_2$ .

Now Theorem 4, (8.1) and (8.6) imply the result of Corollary with

$$a_1^* = \rho_1 = 1, \quad a_j^* = \rho_j + \sum_{i=1}^{j-1} \gamma_i \rho_{j-i} \quad (j \geq 2).$$

This completes the proof of Corollary 5. □

## § 9. Proof of Theorem 2

For each  $\eta \in (0, \frac{1}{2})$ , define

$$\mathbf{H}_k^+(1; \eta) := \{f \in \mathbf{H}_k^*(1) : L(s, f) \neq 0, s \in \mathcal{S}\},$$

where  $\mathcal{S} := \{s := \sigma + i\tau : \sigma \geq 1 - \eta, |\tau| \leq 100k^\eta\} \cup \{s := \sigma + i\tau : \sigma \geq 1, \tau \in \mathbb{R}\}$ , and

$$\mathbf{H}_k^-(1; \eta) := \mathbf{H}_k^*(1) \setminus \mathbf{H}_k^+(1; \eta).$$

Then we have (see [10], (1.11))

$$(9.1) \quad |\mathbf{H}_k^-(1; \eta)| \ll_\eta k^{31\eta}.$$

Our starting point in the proof of Theorem 2 is the evaluation of the moments of  $L(1, f)$ . For this, we recall a particular case of Proposition 6.1 of [10].

**Lemma 9.1.** *Let  $\eta \in (0, \frac{1}{31})$  be fixed. There are two positive constants  $c_i = c_i(\eta)$  ( $i = 4, 5$ ) such that*

$$(9.2) \quad \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^s = E(s) + O_\eta(e^{-c_4 \log k / \log_2 k})$$

uniformly for

$$(9.3) \quad k \geq 16, \quad 2 \mid k \quad \text{and} \quad |s| \leq 2T_k$$

with

$$T_k := c_5 \log k / (\log_2 k \log_3 k).$$

Here  $E(s)$  is defined by (1.15).

Let  $\kappa(t, y)$  be the saddle-point determined by (1.17) and  $\kappa_t := \kappa(t, \infty)$ . For  $k \geq 16, 2 \mid k, \lambda > 0, N \in \mathbb{N}$  and  $t > 0$ , introduce the two integrals

$$I_1(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f \left( \frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s},$$

$$I_2(k, t; \lambda, N) := \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \frac{E(s)}{(e^\gamma t)^{2s}} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s}.$$

**Lemma 9.2.** *Let  $\eta \in (0, \frac{1}{200}]$  be fixed. Then we have*

$$(9.4) \quad \tilde{F}_k(t) + O_\eta(k^{-5/6}) \leq I_1(k, t; \lambda, N) \leq \tilde{F}_k(te^{-\lambda N}) + O_\eta(k^{-5/6}),$$

$$(9.5) \quad \Phi(t) \leq I_2(k, t; \lambda, N) \leq \Phi(te^{-\lambda N})$$

uniformly for  $k \geq 16, 2 \mid k, \lambda > 0, N \in \mathbb{N}$  and  $t > 0$ . The implied constants depend on  $\eta$  only.

*Proof.* By exchanging the order of summation and by using Lemma 6.1 with  $c = \kappa_t$ , we obtain

$$I_1(k, t; \lambda, N) = \sum_{f \in \mathbb{H}_k^+(1; \eta)} \frac{\omega_f}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left( \frac{L(1, f)}{(e^\gamma t)^2} \right)^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s},$$

$$\geq \sum_{f \in \mathbb{H}_k^+(1; \eta), L(1, f) \geq (e^\gamma t)^2} \omega_f.$$

In view of the second estimate of (1.7) and of (9.1), we reintroduce the missing forms

$$I_1(k, t; \lambda, N) \geq \sum_{f \in \mathbb{H}_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f + O\left( \sum_{f \in \mathbb{H}_k^* \setminus \mathbb{H}_k^+(1; \eta)} \omega_f \right)$$

$$\geq \sum_{f \in \mathbb{H}_k^*(1), L(1, f) \geq (e^\gamma t)^2} \omega_f + O(k^{-1+31\eta} \log k).$$

Clearly this implies the first inequality of (9.4), thanks to (1.6) and (1.7).

Similarly, using Lemma 6.1 with  $c = \kappa_t$ , we find

$$I_1(k, t; \lambda, N) \leq \sum_{\substack{f \in \mathbb{H}_k^+(1; \eta) \\ L(1, f) \geq (e^\gamma t)^2}} \omega_f + \sum_{\substack{f \in \mathbb{H}_k^+(1; \eta) \\ (e^\gamma te^{-\lambda N})^2 \leq L(1, f) < (e^\gamma t)^2}} \omega_f$$

$$= \sum_{\substack{f \in \mathbb{H}_k^+(1; \eta) \\ L(1, f) \geq (e^\gamma te^{-\lambda N})^2}} \omega_f.$$

As before, we can easily show that the last sum is  $\leq \tilde{F}_k(te^{-\lambda N}) + O(k^{-5/6})$ .

The estimates (9.5) can be proved in the same way as (6.2).  $\square$

**Lemma 9.3.** Let  $\eta \in (0, \frac{1}{200}]$  be fixed and  $c_4$  be the positive constant given by Lemma 9.1. Then we have

$$(9.6) \quad |I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)| \ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda\kappa_t})^{2N} \log T_k}{(e^{\gamma t})^{2\kappa_t}} + \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{N(e^{\gamma t})^{2\kappa_t}} \left( \frac{1 + e^{\lambda\kappa_t}}{\lambda T_k} \right)^{2N}$$

uniformly for  $\lambda > 0$ ,  $N \in \mathbb{N}$ ,  $k \geq 16$ ,  $2 \mid k$  and  $t \leq T(k)$ , where  $T(k)$  is given by (1.10). The implied constant depends on  $\eta$  only.

*Proof.* By the definitions of  $I_1$  and  $I_2$ , we can write

$$\begin{aligned} & I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N) \\ &= \frac{1}{2\pi i} \int_{\kappa_t - i\infty}^{\kappa_t + i\infty} \left( \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^s - E(s) \right) \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^{2N} \frac{ds}{s(e^{\gamma t})^{2s}}. \end{aligned}$$

In order to estimate the last integral, we split it into two parts according to  $|\tau| \leq T_k$  or  $|\tau| > T_k$ .

In view of (1.18), it is easy to see that  $\kappa_t \leq T_k$  for  $t \leq T(k)$ . Thus we may apply (9.2) of Lemma 9.1 for  $s = \kappa_t + i\tau$  with  $|\tau| \leq T_k$ . Note that  $|(e^{\lambda s} - 1)/(\lambda s)| \leq 1 + e^{\lambda\kappa_t}$  for  $s = \kappa_t + i\tau$ , which is easily seen by looking at the cases  $|\lambda s| \leq 1$  and  $|\lambda s| > 1$ . The contribution of  $|\tau| \leq T_k$  to  $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$  is

$$(9.7) \quad \ll e^{-c_4(\log k)/\log_2 k} \frac{(1 + e^{\lambda\kappa_t})^{2N} \log T_k}{(e^{\gamma t})^{2\kappa_t}}.$$

Since  $\kappa_t \leq T_k$  for  $t \leq T(k)$ , we can apply (9.2) of Lemma 9.1 to write, for  $s = \kappa_t + i\tau$  with  $\tau \in \mathbb{R}$ ,

$$\begin{aligned} \left| \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^s - E(s) \right| &\leq \sum_{f \in \mathbb{H}_k^+(1; \eta)} \omega_f L(1, f)^{\kappa_t} + E(\kappa_t) \\ &\leq 2E(\kappa_t) + O(e^{-c_4(\log k)/\log_2 k}). \end{aligned}$$

Thus the contribution of  $|\tau| > T_k$  to  $|I_1(k, t; \lambda, N) - I_2(k, t; \lambda, N)|$  is

$$(9.8) \quad \begin{aligned} &\ll \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{(e^{\gamma t})^{2\kappa_t}} \int_{|\tau| \geq T_k} \left( \frac{1 + e^{\lambda\kappa_t}}{\lambda |\tau|} \right)^{2N} \frac{d\tau}{|\tau|} \\ &\ll \frac{E(\kappa_t) + e^{-c_4(\log k)/\log_2 k}}{N(e^{\gamma t})^{2\kappa_t}} \left( \frac{1 + e^{\lambda\kappa_t}}{\lambda T_k} \right)^{2N}. \end{aligned}$$

Combining (9.7) and (9.8) yields to the required estimate.  $\square$

*End of the proof of Theorem 2*

For simplicity of notation, we write

$$I_j := I_j(k, t; \lambda, N) \quad \text{and} \quad I_j^+ := I_j(k, te^{\lambda N}; \lambda, N) \quad (j = 1, 2).$$

By using Lemma 9.2, we have

$$(9.9) \quad \begin{aligned} \tilde{F}_k(t) &\leq I_1 + O(k^{-5/6}) \\ &= I_2 + O(|I_1 - I_2| + k^{-5/6}) \\ &\leq \Phi(te^{-\lambda N}) + O(|I_1 - I_2| + k^{-5/6}) \\ &\leq \Phi(t) + |\Phi(te^{-\lambda N}) - \Phi(t)| + O(|I_1 - I_2| + k^{-5/6}) \end{aligned}$$

and

$$\begin{aligned}
(9.10) \quad \tilde{F}_k(t) &\geq I_1^+ + O(k^{-5/6}) \\
&= I_2^+ + O(|I_1^+ - I_2^+| + k^{-5/6}) \\
&\geq \Phi(te^{\lambda N}) + O(|I_1^+ - I_2^+| + k^{-5/6}) \\
&\geq \Phi(t) - |\Phi(t) - \Phi(te^{\lambda N})| + O(|I_1^+ - I_2^+| + k^{-5/6}).
\end{aligned}$$

In view of (6.10) and Theorem 3, we have

$$|\Phi(t) - \Phi(te^{-\lambda N})| \ll \Phi(t) \{ \lambda N \kappa_t (\log \kappa_t)^{1/2} + e^{-(c_3/2)\kappa_t^\delta} \}$$

for  $\lambda N \leq e^{-t}$ . Take

$$(9.11) \quad \lambda = e^{5A}/T_k \quad \text{and} \quad N = \lfloor \log_2 k \rfloor.$$

Since  $T_k = e^{T(k) + \frac{3}{2} \log_3 k + 2C + \log c_5}$ , it is easy to see that

$$\lambda N \leq e^{-T(k) - 2C} T(k)^{-1/2} \quad \text{and} \quad \kappa_t \asymp e^t.$$

Inserting these estimates into the preceding inequality, a simple calculation shows that

$$(9.12) \quad |\Phi(t) - \Phi(te^{-\lambda N})| \leq \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \},$$

provided the constant  $C$  is suitably large, where  $c_6 = c_6(\eta, \delta)$  is a positive constant.

Similarly by using (6.10) with  $te^{\lambda N}$  in place of  $t$ , we have

$$|\Phi(t) - \Phi(te^{\lambda N})| \ll \Phi(te^{\lambda N}) \{ \lambda N \kappa_{te^{\lambda N}} (\log \kappa_{te^{\lambda N}})^{1/2} + e^{-(c_3/2)\kappa_{te^{\lambda N}}^\delta} \}.$$

Since for  $t \leq T(k)$  we have

$$te^{\lambda N} = t + O((\log_2 k)^3 (\log_3 k) / \log k) \quad \text{and} \quad \kappa_{te^{\lambda N}} \asymp e^{te^{\lambda N}} \asymp e^t,$$

the preceding estimate can be written as

$$\begin{aligned}
|\Phi(t) - \Phi(te^{\lambda N})| &\leq \frac{1}{4} \Phi(te^{\lambda N}) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \} \\
&\leq \frac{1}{4} \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \} \\
&\quad + \frac{1}{4} |\Phi(t) - \Phi(te^{\lambda N})| \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \},
\end{aligned}$$

from which we deduce that

$$(9.13) \quad |\Phi(t) - \Phi(te^{\lambda N})| \leq \Phi(t) \{ e^{t-T(k)-C} (t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}}) \}.$$

By using Lemma 9.3 with  $te^{\lambda N}$  in place of  $t$ , we have

$$\begin{aligned}
|I_1^+ - I_2^+| &\ll e^{-c_4(\log k) / \log_2 k} \frac{(1 + e^{\lambda \kappa_{te^{\lambda N}}})^{2N} \log T_k}{(e^{\gamma te^{\lambda N}})^{2\kappa_{te^{\lambda N}}}} \\
&\quad + \frac{E(\kappa_{te^{\lambda N}}) + e^{-c_4(\log k) / \log_2 k}}{N(e^{\gamma te^{\lambda N}})^{2\kappa_{te^{\lambda N}}}} \left( \frac{1 + e^{\lambda \kappa_{te^{\lambda N}}}}{\lambda T_k} \right)^{2N}.
\end{aligned}$$

On the other hand, by using Theorem 3 and (1.25), it is easy to see that there is a positive constant  $c$  such that

$$\Phi(te^{\lambda N}) \asymp \Phi(t) \sim \frac{E(\kappa_t)}{\kappa_t \sqrt{2\pi\sigma_2}(e^{\gamma t})^{2\kappa_t}} \gg e^{-c_8 e^t/t} \gg e^{-c_9(\log k)/[(\log_2 k)^{7/2} \log_3 k]}$$

for  $t \leq T(k)$ . Thanks to Lemma 4.5, the previous estimate can be written as

$$(9.14) \quad \begin{aligned} |I_1^+ - I_2^+| &\ll \Phi(t) \frac{1}{N} \left( \frac{\kappa_t e^{\lambda N}}{\log \kappa_t e^{\lambda N}} \right)^{1/2} \left( \frac{1 + e^{\lambda \kappa_t e^{\lambda N}}}{\lambda T_k} \right)^{2N} \\ &\ll \frac{\Phi(t)}{(\log k)^A}. \end{aligned}$$

Similarly we can prove (even more easily)

$$(9.15) \quad |I_1 - I_2| \ll \Phi(t)/(\log k)^A.$$

Inserting (9.12) and (9.16) into (9.9) and (9.13) and (9.15) into (9.10), we obtain

$$\begin{aligned} \tilde{F}_k(t) &\leq \Phi(t) \{1 + e^{t-T(k)-C}(t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}} + (\log k)^{-A})\}, \\ \tilde{F}_k(t) &\geq \Phi(t) \{1 - e^{t-T(k)-C}(t/T(k))^{1/2} + O(e^{-c_6 e^{\delta t}} + (\log k)^{-A})\}. \end{aligned}$$

This implies the first asymptotic formula of (1.13) by taking  $\eta = \frac{1}{200}$  and  $\delta = \frac{1}{5}$ .

The second can be established similarly. This completes the proof of Theorem 2.  $\square$

## § 10. Proof of Theorem 1

The formula (1.9) is an immediate consequence of Theorem 2 and (1.25).

Taking  $t = T(k)$  in (1.9), we find that

$$(10.1) \quad e^{-c'_1(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}} \ll \tilde{F}_k(T(k)) \ll e^{-c'_2(\log k)/\{(\log_2 k)^{7/2} \log_3 k\}},$$

where  $c'_1$  and  $c'_2$  are two positive constants. Clearly (10.1) and (1.8) imply (1.11).

The related results on  $\tilde{G}_k(t)$  and  $G_k(T(k))$  can be proved similarly. This completes the proof of Theorem 1.  $\square$

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