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Distribution function of values of random Euler products at 1

J.-Y. LIU, E. ROYER & J. WU

Abstract. In the aim of attacking the first conjecture of Montgomery-Vaughan on the distribution of values of the automorphic symmetric power L -functions at 1, we study the distribution function of the truncated random Euler products

$$L(1, \text{sym}^m g^{\natural}(\omega), y) := \prod_{p \leq y} \det(I - p^{-1} \text{sym}^m g_p^{\natural}(\omega))^{-1},$$

where $g^{\natural}(\omega) := \{g_p^{\natural}(\omega)\}_p$ is a sequence of independent random variables, defined in probability space Ω with values in the set of conjugacy classes of $\text{SU}(2)$ endowed with the Sato-Tate measure μ_{st} .

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§ 1. Introduction

The automorphic L -functions constitute a powerful tool for studying arithmetic, algebraic or geometric objects. For squarefree integer N and even integer k , denote by $\mathbb{H}_k^*(N)$ the set of all newforms of level N and of weight k . It is known that

$$(1.1) \quad |\mathbb{H}_k^*(N)| = \frac{k-1}{12} \varphi(N) + O((kN)^{2/3}),$$

where $\varphi(N)$ is the Euler function and the implied constant is absolute. Let $m \geq 1$ be an integer and let $L(s, \text{sym}^m f)$ be the m th symmetric power L -function of $f \in \mathbb{H}_k^*(N)$ normalised so that the critical strip is given by $0 < \Re s < 1$. The values of these functions at the edge of the critical strip contain information of great interest. For example, Serre [16] shew that the Sato-Tate conjecture is equivalent to $L(1 + i\tau, \text{sym}^m f) \neq 0$ for all $m \in \mathbb{N}$ and $\tau \in \mathbb{R}$. The distribution

of values $L(1, \text{sym}^m f)$ has received attention of many authors, such as Goldfeld, Hoffstein & Lieman [2], Hoffstein & Lockhart [6], Luo [10], Royer [12, 13], Royer & Wu [14, 15], Cogdell & Michel [1], Habsieger & Royer [4] and Lau & Wu [8, 9]. In particular, Lau & Wu (Theorems 2 and 3 of [8], and Corollary 2 of [9]) proved the following results:

(i) For every fixed integer $m \geq 1$, there are four positive constants A_m^\pm and B_m^\pm such that for any newform $f \in \mathbb{H}_k^*(1)$, under the Great Riemann Hypothesis (GRH) for $L(s, \text{sym}^m f)$, we have, for $k \rightarrow \infty$,

$$(1.2) \quad \{1 + o(1)\}(2B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + o(1)\}(2B_m^+ \log_2 k)^{A_m^+}.$$

Here (and in the sequel) \log_j denotes the j -fold iterated logarithm. For most values of m , the constants A_m^\pm and B_m^\pm can be explicitly evaluated, for example,

$$(1.3) \quad \begin{cases} A_m^+ = m + 1, & B_m^+ = e^\gamma & (m \in \mathbb{N}), \\ A_m^- = m + 1, & B_m^- = e^\gamma \zeta(2)^{-1} & (\text{odd } m), \\ A_2^- = 1, & B_2^- = e^\gamma \zeta(2)^{-2}, \\ A_4^- = \frac{5}{4}, & B_4^- = e^\gamma B_4'^-, \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, γ denotes the Euler constant and $B_4'^-$ is a positive constant given by a rather complicated Euler product (cf. [8], (1.16)).

(ii) In the opposite direction, it was shown unconditionally that for $m \in \{1, 2, 3, 4\}$ there are newforms $f_m^\pm \in \mathbb{H}_k^*(1)$ such that for $k \rightarrow \infty$,

$$(1.4) \quad L(1, \text{sym}^m f_m^+) \geq \{1 + o(1)\}(B_m^+ \log_2 k)^{A_m^+},$$

$$(1.5) \quad L(1, \text{sym}^m f_m^-) \leq \{1 + o(1)\}(B_m^- \log_2 k)^{-A_m^-}.$$

(iii) In the aim of removing GRH and closing up the gap coming from the factor 2 in (1.2) (comparing it with (1.4) and (1.5)), an almost all result was established. Let $\varepsilon > 0$ be an arbitrarily small positive number, $m \in \{1, 2, 3, 4\}$ and $2 \mid k$. Then there is a subset E_k^* of $\mathbb{H}_k^*(1)$ such that

$$|E_k^*| \ll k e^{-(\log k)^{1/2-\varepsilon}}$$

and for each $f \in \mathbb{H}_k^*(1) \setminus E_k^*$, we have, for $k \rightarrow \infty$,

$$(1.6) \quad \{1 + O(\varepsilon_k)\}(B_m^- \log_2 k)^{-A_m^-} \leq L(1, \text{sym}^m f) \leq \{1 + O(\varepsilon_k)\}(B_m^+ \log_2 k)^{A_m^+}.$$

where $\varepsilon_k := (\log k)^{-\varepsilon}$ and the implied constants depend on ε only.

By comparing (1.4) and (1.5) with (1.6), the extreme values of $L(1, \text{sym}^m f)$ seem to be given by (1.4) and (1.5). Clearly it is interesting to investigate further the size of exceptional set E_k^* . In the Dirichlet L -function case, Montgomery & Vaughan [11] proposed, based on a probabilistic model, three conjectures on the size of exceptional set. The first one has been proved recently by Granville & Soundararajan [3]. The analogue of Montgomery-Vaughan's first conjecture for the automorphic symmetric power L -functions can be stated as follows.

Conjecture. Let $m \geq 1$ be a fixed integer. Then there are positive constants $c_i = c_i(m)$ ($i = 1, 2$) such that for $k \rightarrow \infty$,

$$(1.7) \quad e^{-c_1(\log k)/\log_2 k} \ll \frac{1}{|\mathbb{H}_k^*(1)|} \sum_{\substack{f \in \mathbb{H}_k^*(1) \\ L(1, \text{sym}^m f) \geq (B_m^+ \log_2 k)^{A_m^+}}} 1 \ll e^{-c_2(\log k)/\log_2 k},$$

$$(1.8) \quad e^{-c_1(\log k)/\log_2 k} \ll \frac{1}{|\mathbb{H}_k^*(1)|} \sum_{\substack{f \in \mathbb{H}_k^*(1) \\ L(1, \text{sym}^m f) \leq (B_m^- \log_2 k)^{-A_m^-}}} 1 \ll e^{-c_2(\log k)/\log_2 k}.$$

A standard method for attacking this conjecture is to compare related arithmetic distribution function and probabilistic distribution function and to evaluate the later. In fact Granville & Soundararajan followed these lines to establish the first conjecture of Montgomery-Vaughan for Dirichlet L -functions.

Denote by

$$\omega_f := \frac{12\zeta(2)}{(k-1)L(1, \text{sym}^2 f)}$$

the usual harmonic weight in modular form theory (see [7]) and define the weighted arithmetic distribution function

$$(1.9) \quad F_k(x, \text{sym}^m) := \frac{1}{\sum_{f \in \mathbb{H}_k^*(1)} \omega(f)} \sum_{\substack{f \in \mathbb{H}_k^*(1) \\ L(1, \text{sym}^m f) \leq e^x}} \omega(f).$$

In view of (1.1), the classical estimate

$$\sum_{f \in \mathbb{H}_k^*(1)} \omega(f) \asymp 1$$

and the bound of Goldfeld, Hoffstein & Lieman [2]:

$$1/(k \log k) \ll \omega_f \ll (\log k)/k,$$

we easily see that

$$F_k(x, \text{sym}^m)/\log k \ll \frac{1}{|\mathbb{H}_k^*(1)|} \sum_{\substack{f \in \mathbb{H}_k^*(1) \\ L(1, \text{sym}^m f) \leq e^x}} 1 \ll F_k(x, \text{sym}^m) \log k.$$

This show that in order to prove (1.8) of Conjecture it is sufficient to establish corresponding estimates of the same quality for $F_k(x, \text{sym}^m)$.

Consider a probability space (Ω, μ) , with measure μ . Let $\text{SU}(2)^\natural$ be the set of conjugacy classes of $\text{SU}(2)$. The group $\text{SU}(2)$ is endowed with its Haar measure μ_H and

$$\text{SU}(2)^\natural = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi] \right\} / \sim$$

is endowed with the Sato-Tate measure $d\mu_{\text{st}}(\theta) := (2/\pi) \sin^2 \theta d\theta$, *i.e.* the direct image of μ_H by the canonical projection $\text{SU}(2) \rightarrow \text{SU}(2)^\natural$. On the space (Ω, μ) , define a sequence indexed by the prime numbers, $g^\natural(\omega) = \{g_p^\natural(\omega)\}_p$ of random matrices taking values in $\text{SU}(2)^\natural$, given by

$$g_p^\natural(\omega) := \begin{pmatrix} e^{i\vartheta_p(\omega)} & 0 \\ 0 & e^{-i\vartheta_p(\omega)} \end{pmatrix}^\natural.$$

We assume that each random matrix is distributed according to the Sato-Tate measure. This means that, for each integrable function $\phi : \mathrm{SU}(2)^{\natural} \rightarrow \mathbb{R}$, the expected value of $\phi \circ g_p^{\natural}$ is

$$\mathbb{E}(\phi \circ g_p^{\natural}) = \int_0^\pi \phi \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta \, d\theta.$$

Moreover, we assume that the sequence $g^{\natural}(\omega)$ is made of independent random variables. This means that, for any set of integrable functions $\{G_p : \mathrm{SU}(2)^{\natural} \rightarrow \mathbb{R}\}_p$, we have

$$(1.10) \quad \begin{aligned} \mathbb{E} \left(\prod_p G_p \circ g_p^{\natural} \right) &:= \int_{\Omega} \prod_p G_p \circ g_p^{\natural}(\omega) \, d\mu(\omega) \\ &= \prod_p \int_0^\pi G_p \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \cdot (2/\pi) \sin^2 \theta \, d\theta. \end{aligned}$$

Then for $\Re s > \frac{1}{2}$, the random Euler product

$$L(s, \mathrm{sym}^m g^{\natural}(\omega)) := \prod_p \det (I - p^{-s} \mathrm{sym}^m g_p^{\natural}(\omega))^{-1} =: \prod_p L_p(s, \mathrm{sym}^m g^{\natural}(\omega))$$

turns out to be absolutely convergent a.s., where I is the unit matrix and

$$\mathrm{sym}^m g_p^{\natural}(\omega) := \begin{pmatrix} e^{im\vartheta_p(\omega)} & 0 & \cdots & 0 \\ 0 & e^{i(m-2)\vartheta_p(\omega)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-im\vartheta_p(\omega)} \end{pmatrix}.$$

We denote by $F(x, \mathrm{sym}^m)$ the distribution function of $\log L(1, \mathrm{sym}^m g^{\natural}(\omega))$:

$$(1.11) \quad F(x, \mathrm{sym}^m) := \mathrm{Prob}(\{L(1, \mathrm{sym}^m g^{\natural}(\omega)) \leq e^x\}).$$

Similarly to Corollary 1.16 of [1], it is not difficult to prove, by using Proposition 6.1 of [8], that for $m \in \{1, 2, 3, 4\}$ and $k \rightarrow \infty$, we have

$$(1.12) \quad F_k(x, \mathrm{sym}^m) = F(x, \mathrm{sym}^m) + O\left(\frac{(\log_2 k)(\log_3 k)}{\log k}\right)$$

uniformly for $x \in \mathbb{R}$. This allows us to compare the weighted arithmetic distribution function $F_k(x, \mathrm{sym}^m)$ and the probabilistic distribution function $F(x, \mathrm{sym}^m)$.

The aim of this paper is to investigate the asymptotic comportement of $F(x, \mathrm{sym}^m)$. For this, we consider a more general truncated random Euler product

$$L(s, \mathrm{sym}^m g^{\natural}(\omega), y) := \prod_{p \leq y} L_p(s, \mathrm{sym}^m g^{\natural}(\omega)).$$

Since we are interested in the extreme values of $L(1, \mathrm{sym}^m f)$, it is more convenient to introduce, in view of (1.3) and (1.4) the functions

$$(1.13) \quad \begin{cases} \Phi_m(t, y) := \mathrm{Prob}(\{L(1, \mathrm{sym}^m g^{\natural}(\omega), y) \geq (B_m^+ t)^{A_m^+}\}), \\ \Psi_m(t, y) := \mathrm{Prob}(\{L(1, \mathrm{sym}^m g^{\natural}(\omega), y) \leq (B_m^- t)^{-A_m^-}\}), \end{cases}$$

for investigating the large and small values of $L(1, \text{sym}^m g^{\natural}(\omega), y)$, respectively. Writing

$$(1.14) \quad \Phi_m(t) := \Phi_m(t, \infty) \quad \text{and} \quad \Psi_m(t) := \Psi_m(t, \infty),$$

we have $F(x, \text{sym}^m) = \Psi_m(t)$ with $x = \log(B_m^- t)^{-A_m^-}$.

The saddle-point method was firstly applied to number theory by Hildebrand & Tenenbaum [5], in problem of friable integers. Interested readers are referred to [17] for an excellent paradigm. Here we shall use the saddle-point method to evaluate $\Phi_m(t, y)$ and $\Psi_m(t, y)$. To avoid notational complications, we shall treat only the case of $m = 1$. In this case, we have $\text{sym}^1 g^{\natural}(\omega) = g^{\natural}(\omega)$. All the results of this paper can be generalized to general case $m \geq 1$ without difficulty. For simplicity, write

$$\Phi(t, y) = \Phi_1(t, y), \quad \Phi(t) = \Phi_1(t), \quad \Psi(t, y) = \Psi_1(t, y), \quad \Psi(t) = \Psi_1(t).$$

For $s \in \mathbb{C}$ and $y \geq 2$, define

$$(1.15) \quad E(s, y) := \mathbb{E}(L(1, g^{\natural}(\omega), y)^s),$$

where $\mathbb{E}(\cdot)$ denotes the expected value. We define also

$$(1.16) \quad \phi(s, y) := \log E(s, y), \quad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geq 0).$$

According to Lemmas 2.3 and 8.1 below, there is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation

$$(1.17) \quad \phi_1(\kappa, y) = 2(\log t + \gamma)$$

has a unique positive solution $\kappa = \kappa(t, y)$ and for each integer $J \geq 1$ there are computable constants c_1, \dots, c_J such that the asymptotic formula

$$(1.18) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{c_j}{t^j} + O_J \left(\frac{1}{t^{J+1}} + \frac{e^t t}{y \log y} \right) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, where the constant γ_0 is given by (1.24) below.

Finally write $\sigma_n := \phi_n(\kappa, y)$.

Our first result is an asymptotic formula for $\Phi(t, y)$, which has the same form as Theorem 1 in [5], Theorem 3.1 in [3] and Theorem 1 in [19].

Theorem 1. *We have*

$$\Phi(t, y) = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2}(e^{\gamma}t)^{2\kappa}} \left\{ 1 + O \left(\frac{t}{e^t} \right) \right\}$$

uniformly for $t \geq 1$ and $y \geq 2e^t$.

Comparing $\zeta(s, y) := \prod_{p \leq y} (1 - p^{-s})^{-1}$ in [5] and (1.12) in [19] with (2.2) below, the expression of our $E(s, y)$ seems to be more complicated. This will cause some supplementary technical difficulties in the proof of Theorem 1. From it, we can derive an asymptotic development for $\log \Phi(t, y)$ in $(\log \kappa)^{-1}$.

Theorem 2. For each integer $J \geq 1$, we have

$$(1.19) \quad \Phi(t, y) = \exp \left\{ -\kappa \left[\sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}$$

uniformly for $t \geq 1$ and $y \geq 2e^t$, where the error term $R_J(\kappa, y)$ is given by

$$(1.20) \quad R_J(\kappa, y) := \frac{1}{(\log \kappa)^{J+1}} + \frac{\kappa}{y \log y}$$

and

$$(1.21) \quad a_j := \int_0^\infty \left(\frac{h(u)}{u} \right)' (\log u)^{j-1} du$$

with

$$(1.22) \quad h(u) := \begin{cases} \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) & \text{if } 0 \leq u < 1, \\ \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta d\theta \right) - 2u & \text{if } u \geq 1. \end{cases}$$

As a corollary of Theorem 2, we can obtain an asymptotic development for $\log \Phi(t, y)$ in t^{-1} . In particular we see that the probabilistic distribution function $\Phi(t)$ decays double exponentially as $t \rightarrow \infty$.

Corollary 3. For each integer $J \geq 1$, there are computable constants a_1^*, \dots, a_J^* such that the asymptotic formula

$$(1.23) \quad \Phi(t, y) = \exp \left\{ -e^{t-\gamma_0} \left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J(R_J(e^t, y)) \right] \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$. Further we have

$$(1.24) \quad \gamma_0 := \frac{1}{2} \int_0^\infty \frac{h'(u)}{u} du, \quad a_1^* := 1, \quad a_2^* := \gamma_0 - \frac{\gamma_0^2}{2} - \int_0^\infty \frac{h(u)}{u^2} (\log u) du.$$

In particular for each integer $J \geq 1$, we have

$$(1.25) \quad \Phi(t) = \exp \left\{ -e^{t-\gamma_0} \left[\sum_{j=1}^J \frac{a_j^*}{t^j} + O_J \left(\frac{1}{t^{J+1}} \right) \right] \right\}$$

uniformly for $t \geq 1$.

Remark 1. (i) The same results hold also for $\Psi(t, y)$.

(ii) Taking $t = \log_2 k$ and $J = 1$ in (1.25) of Corollary 3, we see that the random Euler product $L(s, g^{\mathfrak{h}}(\omega))$ verifies Montgomery-Vaughan's conjecture. But (1.12) is too weak to derive this conjecture for the automorphic L -function $L(s, \text{sym}^1 f)$. We plan to return to this delicate problem in the future and prove this conjecture.

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§ 2. Expression of $E(s, y)$ and existence of saddle-point

The aim of this section is to prove the existence of the saddle-point $\kappa(t, y)$, defined by equation (1.17). The first step is to give an explicite expression of $E(s, y)$, which is (1.24) of [1]. For the convenience of readers, we state it here as a lemma.

Lemma 2.1. *For prime p , real θ and complex number s , we define*

$$(2.1) \quad D_p(\theta) := \prod_{0 \leq j \leq 1} (1 - e^{i(1-2j)\theta} p^{-1})^{-1} \quad \text{and} \quad E_p(s) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^s \sin^2 \theta \, d\theta.$$

Then for all $s \in \mathbb{C}$ and $y \geq 2$, we have

$$(2.2) \quad E(s, y) = \prod_{p \leq y} E_p(s).$$

Proof. Taking

$$G_p(M^\natural) = \begin{cases} \det(I - p^{-s'} M^\natural)^{-s} & \text{if } p \leq y \\ 1 & \text{otherwise} \end{cases}$$

in (1.10), we get

$$\begin{aligned} \mathbb{E}(L(s', g^\natural(\omega), y)^s) &= \prod_{p \leq y} \mathbb{E}(L_p(s', g_p^\natural(\omega))^s) \\ &= \prod_{p \leq y} \int_{\Omega} \det(1 - p^{-s'} g_p^\natural(\omega))^{-s} \, d\mu(\omega) \\ &= \prod_{p \leq y} \frac{2}{\pi} \int_0^\pi (1 - 2p^{-s'} \cos \theta + p^{-2s'})^{-s} \sin^2 \theta \, d\theta. \end{aligned}$$

Taking $s' = 1$ and noticing (1.15) and (2.1), we get the desired result. \square

Lemma 2.2. *For all p and $\sigma > 0$, we have*

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 > 0.$$

In particular for all $\sigma > 0$ and $y \geq 2$, we have $\phi_2(\sigma, y) > 0$.

Proof. By using the definition (2.1) of $E_p(\sigma)$, it is easy to see that

$$\begin{aligned} E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 &= \frac{4}{\pi^2} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \int_0^\pi D_p(\theta)^\sigma \sin^2 \theta \, d\theta \\ &\quad - \left(\frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta \, d\theta \right)^2 \\ &= \frac{4}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma (\log^2 D_p(\theta_1) - \log D_p(\theta_1) \log D_p(\theta_2)) \times \\ &\quad \times \sin^2 \theta_1 \sin^2 \theta_2 \, d\theta_1 \, d\theta_2. \end{aligned}$$

In view of the symmetry in θ_1 and θ_2 , the same formula holds if we exchange the roles of θ_1 and θ_2 . Thus it follows that

$$E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2 = \frac{2}{\pi^2} \int_0^\pi \int_0^\pi D_p(\theta_1)^\sigma D_p(\theta_2)^\sigma \log^2 \left(\frac{D_p(\theta_1)}{D_p(\theta_2)} \right) \sin^2 \theta_1 \sin^2 \theta_2 d\theta_1 d\theta_2.$$

This proves the first assertion and the second follows immediately. \square

Lemma 2.3. *There is an absolute constant $c \geq 2$ such that for $t \geq 4 \log c$ and $y \geq ce^t$, the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution in σ . Denoting by $\kappa(t, y)$ this solution, we have $\kappa(t, y) \asymp e^t$ uniformly for $t \geq 4 \log c$ and $y \geq ce^t$.*

Proof. According to Lemma 4.3 below with the choice of $J = 1$, we have

$$\phi_1(\sigma, y) = 2(\log_2 \sigma + \gamma) + O(1/\log \sigma)$$

for $y \geq \sigma \geq 2$. Thus

$$\begin{aligned} \phi(ce^t, y) &= 2 \log(t + \log c) + 2\gamma + O\left(\frac{1}{t + \log c}\right) \\ &> 2 \log t + 2\gamma, \\ \phi(c^{-1}e^t, y) &= 2 \log(t - \log c) + 2\gamma + O\left(\frac{1}{t - \log c}\right) \\ &< 2 \log t + 2\gamma, \end{aligned}$$

provided that c is a large constant and $t \geq 4 \log c$. On the other hand, in view of Lemma 2.2, we know that for any $y \geq 2$, $\phi_1(\sigma, y)$ is an increasing function of σ in $(0, \infty)$. Hence the equation $\phi_1(\sigma, y) = 2(\log t + \gamma)$ has a unique positive solution $\kappa(t, y)$ and $c^{-1}e^t \leq \kappa(t, y) \leq ce^t$ for $t \geq 4 \log c$ and $y \geq ce^t$. This completes the proof. \square

§ 3. Preliminary lemmas

This section is devoted to establish some preliminary lemmas, which will be useful later.

Lemma 3.1. *Let $j \geq 0$ be a fixed real number. Then we have*

$$(3.1) \quad \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta d\theta \asymp_j e^{2u} u^{-(j+3/2)} \quad (u \geq 1).$$

The implied constant depends on j only.

Proof. First we write

$$\begin{aligned} \int_0^\pi e^{2u \cos \theta} (1 - \cos \theta)^j \sin^2 \theta d\theta &= \int_0^{\pi/2} (e^{2u \cos \theta} (1 - \cos \theta)^j + e^{-2u \cos \theta} (1 + \cos \theta)^j) \sin^2 \theta d\theta \\ &= \int_0^1 (e^{2ut} (1-t)^j + e^{-2ut} (1+t)^j) (1-t^2)^{1/2} dt \\ &\asymp \int_0^1 e^{2ut} (1-t)^{j+1/2} dt + \int_0^1 e^{-2ut} (1-t)^{1/2} dt. \end{aligned}$$

By the change of variables $u(1-t) = v$, it follows that

$$\begin{aligned} \int_0^1 e^{2ut} (1-t)^{j+1/2} dt &= e^{2u} u^{-(j+3/2)} \int_0^u e^{-2v} v^{j+1/2} dv \\ &\asymp e^{2u} u^{-(j+3/2)}, \\ \int_0^1 e^{-2ut} (1-t)^{1/2} dt &\leq \int_0^1 e^{-2ut} dt \ll u^{-1}. \end{aligned}$$

We obtain the desired result by insertion of these estimates into the preceding relation. \square

Lemma 3.2. *Let $j \geq 0$ be an integer and*

$$(3.2) \quad E_{p,j}(\sigma) := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma (1 - \cos \theta)^j \sin^2 \theta \, d\theta.$$

(In particular $E_{p,0}(\sigma) = E_p(\sigma)$.) Then we have

$$E_{p,j}(\sigma) = \frac{2^{j+3}}{\pi} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du$$

and the estimate

$$(3.3) \quad E_{p,j}(\sigma)/E_p(\sigma) \ll (p/\sigma)^j$$

holds uniformly for all primes p and $\sigma > 0$. Further if $p \geq \sigma \geq 0$, we have

$$(3.4) \quad E_p(\sigma) \asymp 1.$$

The implied constant in (3.3) depends on j only and the one in (3.4) is absolute.

Proof. By the change of variables $u = \sin^2(\theta/2)$, a simple computation shows that the first assertion is true. Obviously (3.3) holds for $j = 0$.

Now assume that it is true for j . An integration by parts leads to

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1/2} (1-u)^{1/2} \, du \\ &\gg_j \left(\frac{\sigma}{p}\right)^j \int_0^1 \left\{ \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \right\} \times \\ &\quad \times \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} \, du. \end{aligned}$$

On the other hand, we have

$$0 < u < 1 \Rightarrow \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-1} \frac{4\sigma}{p} + \frac{1}{2(1-u)} \geq \left(1 + \frac{1}{p}\right)^{-2} \frac{4\sigma}{p} \geq \frac{16\sigma}{9p}.$$

Inserting it into the preceding estimate, we see that

$$\begin{aligned} E_p(\sigma) &\gg_j \left(\frac{\sigma}{p}\right)^{j+1} \int_0^1 \left[\left(1 - \frac{1}{p}\right)^2 + \frac{4u}{p} \right]^{-\sigma} u^{j+1+1/2} (1-u)^{1/2} \, du \\ &\asymp_j \left(\frac{\sigma}{p}\right)^{j+1} E_{p,j+1}(\sigma). \end{aligned}$$

Thus (3.3) holds also for $j + 1$.

Since $(1+1/p)^{-2} \leq D_p(\theta) \leq (1-1/p)^{-2}$ for all primes p and any $\theta \in \mathbb{R}$, we have $D_p(\theta)^\sigma \asymp 1$ uniformly for $p \geq \sigma \geq 0$ and $\theta \in \mathbb{R}$. This implies (3.4). \square

Introduce the function

$$(3.5) \quad g(u) := \log \left(\frac{2}{\pi} \int_0^\pi e^{2u \cos \theta} \sin^2 \theta \, d\theta \right) \quad (u \geq 0)$$

and let $h(u)$ be defined as in (1.22). Clearly we have

$$(3.6) \quad h(u) = \begin{cases} g(u) & \text{if } 0 \leq u < 1, \\ g(u) - 2u & \text{if } u \geq 1, \end{cases}$$

$$(3.7) \quad h'(u) = \begin{cases} g'(u) & \text{if } 0 \leq u < 1, \\ g'(u) - 2 & \text{if } u \geq 1, \end{cases}$$

$$(3.8) \quad h''(u) = g''(u) \quad (u \geq 0, u \neq 1).$$

Lemma 3.3. *We have*

$$(3.9) \quad h(u) \asymp \begin{cases} u^2 & \text{if } 0 \leq u < 1, \\ \log(2u) & \text{if } u \geq 1, \end{cases}$$

$$(3.10) \quad h'(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-1} & \text{if } u \geq 1, \end{cases}$$

$$(3.11) \quad h''(u) \asymp \begin{cases} 1 & \text{if } 0 \leq u < 1, \\ u^{-2} & \text{if } u \geq 1, \end{cases}$$

$$(3.12) \quad h'''(u) \asymp \begin{cases} u & \text{if } 0 \leq u < 1, \\ u^{-3} & \text{if } u \geq 1. \end{cases}$$

Proof. The last three relations are trivial. We will prove only the first four estimates.

When $0 \leq u < 1$, we have

$$e^{2u \cos \theta} = \sum_{n=0}^{\infty} \frac{(u \cos \theta)^n}{n!}.$$

From this we deduce that

$$(3.13) \quad \begin{aligned} h(u) &= \log \left(\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{u^n}{n!} \int_0^{\pi} (\cos \theta)^n \sin^2 \theta \, d\theta \right) \\ &= \log \left(1 + \sum_{\ell=1}^{\infty} \frac{2 \cdot (2\ell - 1)!!}{(2\ell)!(2\ell + 2)!!} u^{2\ell} \right), \end{aligned}$$

where we have used the following facts:

$$\int_0^{\pi} (\cos \theta)^{2\ell+1} \sin^2 \theta \, d\theta = 0$$

and

$$\frac{2}{\pi} \int_0^{\pi} (\cos \theta)^{2\ell} \sin^2 \theta \, d\theta = \begin{cases} 1 & \text{if } \ell = 0, \\ 2 \frac{(2\ell - 1)!!}{(2\ell + 2)!!} & \text{if } \ell \geq 1. \end{cases}$$

Now we easily deduce, from (3.13), the desired results in the case of $0 \leq u < 1$.

The estimates of (3.9)–(3.12) for $u > 1$ are simple consequences of (3.1), by noticing the following relations

$$\begin{aligned} h'(u) &= -2 \frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta}, \\ h''(u) &= 4 \frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta)^2 \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta} - 4 \left(\frac{\int_0^{\pi} e^{2u \cos \theta} (1 - \cos \theta) \sin^2 \theta \, d\theta}{\int_0^{\pi} e^{2u \cos \theta} \sin^2 \theta \, d\theta} \right)^2. \end{aligned}$$

This completes the proof. \square

§ 4. Estimates of $\phi_n(\sigma, y)$

The aim of this section is to prove some estimates of $\phi_n(\sigma, y)$ for $n = 0, 1, 2, 3, 4$.

Lemma 4.1. *For any fixed integer $J \geq 1$, we have*

$$(4.1) \quad \phi_0(\sigma, y) = \sigma \left\{ 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 3$, where $R_J(\sigma, y)$ is defined as in (1.20) and

$$(4.2) \quad b_{j,0} := \int_0^\infty \frac{h(u)}{u^2} (\log u)^{j-1} du.$$

Proof. By the definition (2.1) of $D_p(\theta)$ and the one of $E_p(\sigma)$, it is easy to see that for $p \geq \sigma^{1/2}$, we have

$$(4.3) \quad D_p(\theta)^\sigma = e^{2(\sigma/p) \cos \theta} \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\},$$

$$(4.4) \quad E_p(\sigma) = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \sin^2 \theta d\theta.$$

From these, we deduce that

$$(4.5) \quad \sum_{\sigma^{1/2} < p \leq y} \log E_p(\sigma) = \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}/\log \sigma)$$

where $g(u)$ is defined as in (3.5).

In order to treat the sum over $p \leq \sigma$, we write

$$E_p(\sigma) = (1 - 1/p)^{-2\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{2}{\pi} \int_0^\pi \left\{ 1 + \frac{2(1 - \cos \theta)}{p} \left(1 - \frac{1}{p}\right)^{-2} \right\}^{-\sigma} \sin^2 \theta d\theta.$$

By using the change of variables $u = \sin^2(\theta/2)$, we have

$$\begin{aligned} E_p^*(\sigma) &= \frac{8}{\pi} \int_0^\pi \left\{ 1 + \frac{4}{p} \left(1 - \frac{1}{p}\right)^{-2} \sin^2(\theta/2) \right\}^{-\sigma} \sin^2(\theta/2) \cos^2(\theta/2) d\theta \\ &\geq \frac{8}{\pi} \int_0^{p/2\sigma} \left\{ 1 + \frac{4}{p} \left(1 - \frac{1}{p}\right)^{-2} u \right\}^{-\sigma} \sqrt{u(1-u)} du \\ &\geq \frac{8}{\pi} \left(1 + \frac{8}{\sigma}\right)^{-\sigma} \int_0^{p/2\sigma} \sqrt{u(1-u)} du \\ &\geq C \left(\frac{p}{\sigma}\right)^{3/2}, \end{aligned}$$

where $C > 0$ is a constant. On the other hand, we have trivially $E_p^*(\sigma) \leq 1$ for all p and $\sigma > 0$. Thus $|\log E_p^*(\sigma)| \ll \log(\sigma/p)$ for $p \leq \sigma^{1/2}$ and

$$(4.6) \quad \sum_{p \leq \sigma^{1/2}} |\log E_p^*(\sigma)| \ll \sum_{p \leq \sigma^{1/2}} \log(\sigma/p) \ll \sigma^{1/2}.$$

Combining (4.5) and (4.6), we can write

$$\sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma^{1/2}} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} g(\sigma/p) + O(\sigma^{1/2}).$$

In view of (3.6) and the following estimate

$$\sum_{\sigma^{1/2} < p \leq \sigma} (2\sigma \log(1 - 1/p)^{-1} - 2\sigma/p) \ll \sum_{\sigma^{1/2} < p \leq \sigma} \sigma/p^2 \ll \sigma^{1/2}/\log \sigma,$$

the preceding estimate can be written as

$$(4.7) \quad \sum_{p \leq y} \log E_p(\sigma) = 2\sigma \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} + \sum_{\sigma^{1/2} < p \leq y} h(\sigma/p) + O(\sigma^{1/2}).$$

By using the prime number theorem in the form

$$(4.8) \quad \pi(t) := \sum_{p \leq t} 1 = \int_2^t \frac{dv}{\log v} + O\left(te^{-8\sqrt{\log t}}\right),$$

it follows that

$$(4.9) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt + R_0,$$

where

$$\begin{aligned} R_0 &:= h\left(\frac{\sigma}{y}\right)e^{-8\sqrt{\log y}} + h(\sigma^{1/2})e^{-4\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^y \sigma \frac{|h'(\sigma/t)|}{t^2} e^{-8\sqrt{\log t}} dt \\ &\ll \frac{\sigma^2}{y^2} e^{-8\sqrt{\log y}} + e^{-2\sqrt{\log \sigma}} + \int_{\sigma^{1/2}}^{\sigma} \frac{e^{-2\sqrt{\log t}}}{t} dt + \sigma^2 \int_{\sigma}^y \frac{e^{-8\sqrt{\log t}}}{t^3} dt \\ &\ll e^{-\sqrt{\log \sigma}} \end{aligned}$$

by use of Lemma 3.3.

In order to evaluate the integral of (4.9), we use the change of variables $u = \sigma/t$ to write

$$\begin{aligned} \int_{\sigma^{1/2}}^y \frac{h(\sigma/t)}{\log t} dt &= \sigma \int_{\sigma/y}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du \\ &= \sigma \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du + O(R'_0) \end{aligned}$$

where

$$\begin{aligned} R'_0 &:= \sigma \int_0^{\sigma/y} \frac{|h(u)|}{u^2 \log(\sigma/u)} du + \sigma \int_0^{\sigma^{-1/2}} \frac{|h(u)|}{u^2 \log(\sigma/u)} du \\ &\ll \frac{\sigma^2}{y \log y} + \frac{\sigma^{1/2}}{\log \sigma}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 \log(\sigma/u)} du &= \frac{1}{\log \sigma} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2 (1 - (\log u)/\log \sigma)} du \\ &= \sum_{j=1}^J \frac{1}{(\log \sigma)^j} \int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du + O\left(\frac{1}{(\log \sigma)^{J+1}}\right). \end{aligned}$$

Extending the interval of integration $[\sigma^{-1/2}, \sigma^{1/2}]$ to $(0, \infty)$ and bounding the contributions of $(0, \sigma^{-1/2}]$ and $[\sigma^{1/2}, \infty)$ by using (3.9) of Lemma 3.3, we have

$$\int_{\sigma^{-1/2}}^{\sigma^{1/2}} \frac{h(u)}{u^2} (\log u)^{j-1} du = b_{j,0} + O\left(\frac{(\log \sigma)^j}{\sigma^{1/2}}\right).$$

Combining these estimates, we find that

$$(4.10) \quad \sum_{\sigma^{1/2} < p \leq y} h\left(\frac{\sigma}{p}\right) = \sigma \left\{ \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}.$$

Now the desired result follows from (4.7), (4.10) and the prime number theorem in the form

$$(4.11) \quad \sum_{p \leq \sigma} \log(1 - 1/p)^{-1} = \log_2 \sigma + \gamma + O\left(e^{-2\sqrt{\log \sigma}}\right).$$

This completes the proof. \square

Remark 2. In view of (1.3), we can write (4.1) as

$$\phi_0(\sigma, y) = \sigma \left\{ \log(B_1^+ \log \sigma)^{A_1^+} + \sum_{j=1}^J \frac{b_{j,0}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 3$. In the case $\sigma < 0$, a similar asymptotic formula (with A_1^-, B_1^- and corresponding $b_{j,0}^-$ in place of A_1^+, B_1^+ and $b_{j,0}$) can be established uniformly for $y \geq -\sigma \geq 3$. As indicated in the introduction, Lemma 4.1 can be easily generalised to the general case $m \geq 1$. Thus we give an improvement and generalisation of Corollaries A and C of [13], of Theorem B of [4], and an improvement of Theorem 1.12 of [1]. It is worthy to indicate that our method seems to be simpler and more natural.

Lemma 4.2. *We have*

$$(4.12) \quad \frac{E'_p(\sigma)}{E_p(\sigma)} = \begin{cases} \log D_p(0) + O\left(\frac{1}{\sigma}\right) & \text{for all } p \text{ and } \sigma > 0, \\ \frac{1}{2} g'\left(\frac{\sigma}{p}\right) \log D_p(0) + O\left(\frac{1}{p^2} + \frac{\sigma}{p^3}\right) & \text{if } p \geq \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.13) \quad \begin{aligned} E'_p(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log D_p(\theta) \sin^2 \theta d\theta \\ &= E_p(\sigma) \log D_p(0) + R', \end{aligned}$$

where

$$(4.14) \quad R' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log \left(\frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta d\theta.$$

Since

$$\left| \log \left(\frac{D_p(\theta)}{D_p(0)} \right) \right| = \left| -\log \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) \right| \leq \frac{2p(1 - \cos \theta)}{(p-1)^2} \leq \frac{8(1 - \cos \theta)}{p},$$

it follows from (3.3) of Lemma 3.2 with $j = 1$ that

$$\frac{R'}{E_p(\sigma)} \ll \frac{E_{p,1}(\sigma)}{pE_p(\sigma)} \ll \frac{1}{\sigma}$$

for all p and $\sigma > 0$. This implies, via (4.13), the first estimate of (4.12).

We have

$$\begin{aligned} \log D_p(\theta) &= (\cos \theta)(2/p) + O(1/p^2) \\ &= (\cos \theta) \log D_p(0) + O(1/p^2). \end{aligned}$$

Inserting it and (4.3) into the first relation of (4.13) and in view of (4.4), we can write, for $p \geq \sigma^{1/2}$,

$$\begin{aligned} E_p'(\sigma) &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} \left\{ (\cos \theta) \log D_p(0) + O\left(\frac{1}{p^2}\right) \right\} \sin^2 \theta \, d\theta \\ &= \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{2}{\pi} \int_0^\pi e^{2(\sigma/p) \cos \theta} (\cos \theta) \sin^2 \theta \, d\theta \log D_p(0) + O\left(\frac{E_p(\sigma)}{p^2}\right). \end{aligned}$$

From this and (4.4), we deduce

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \frac{1}{2} g' \left(\frac{\sigma}{p} \right) \log D_p(0) + O\left(\frac{1}{p^2}\right),$$

which implies the second estimate of (4.12). This completes the proof. \square

Lemma 4.3. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_1(\sigma, y) = 2 \log_2 \sigma + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y))$$

uniformly for $y \geq \sigma \geq 3$, where the constant $b_{j,1}$ is given by

$$(4.15) \quad b_{j,1} := \int_0^\infty \frac{h'(u)}{u} (\log u)^{j-1} \, du$$

and $R_J(\sigma, y)$ is defined as in (1.20).

Proof. We have

$$\phi_1(\sigma, y) = \sum_{p \leq y} E_p'(\sigma) / E_p(\sigma).$$

Using the first relation of (4.12) for $p \leq \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leq y$, we obtain

$$\phi_1(\sigma, y) = \sum_{p \leq \sigma^{2/3}} \log D_p(0) + \frac{1}{2} \sum_{\sigma^{2/3} < p \leq y} g' \left(\frac{\sigma}{p} \right) \log D_p(0) + O\left(\frac{1}{\sigma^{1/3}}\right).$$

In view of (3.7), the preceding formula can be written as

$$(4.16) \quad \phi_1(\sigma, y) = \sum_{p \leq \sigma} \log D_p(0) + \sum_{\sigma^{2/3} < p \leq y} h' \left(\frac{\sigma}{p} \right) \log \left(1 - \frac{1}{p} \right)^{-1} + O\left(\frac{1}{\sigma^{1/3}}\right).$$

Similarly to (4.10), we can prove that

$$(4.17) \quad \sum_{\sigma^{2/3} < p \leq y} h' \left(\frac{\sigma}{p} \right) \log \left(1 - \frac{1}{p} \right)^{-1} = \sum_{j=1}^J \frac{b_{j,1}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)),$$

using (3.10), (3.11) and (4.11) instead of (3.9), (3.10) and (4.8). Now the desired result follows from (4.16), (4.10) and (4.17). \square

Lemma 4.4. *We have*

$$(4.18) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \begin{cases} O\left(\frac{1}{\sigma^2}\right) & \text{if } p \leq \sigma^{1/2}, \\ \frac{1}{p^2}g''\left(\frac{\sigma}{p}\right) + O\left(\min\left\{\frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2}\right\}\right) & \text{if } p > \sigma^{1/2}, \end{cases}$$

where $g(u)$ is defined as in (3.5).

Proof. First we write

$$(4.19) \quad \begin{aligned} E_p''(\sigma) &= \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 D_p(\theta) \sin^2 \theta \, d\theta \\ &= E_p(\sigma) \log^2 D_p(0) + R'', \end{aligned}$$

where

$$R'' := \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \left(\log^2 D_p(\theta) - \log^2 D_p(0) \right) \sin^2 \theta \, d\theta.$$

Using (4.13) and (4.19), we can deduce

$$(4.20) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} - \left(\frac{R'}{E_p(\sigma)} \right)^2,$$

where R' is defined as in (4.14).

From the definitions of R' and R'' , a simple calculation shows that

$$R'' - 2R' \log D_p(0) = \frac{2}{\pi} \int_0^\pi D_p(\theta)^\sigma \log^2 \left(\frac{D_p(\theta)}{D_p(0)} \right) \sin^2 \theta \, d\theta.$$

Since

$$\log^2 \left(\frac{D_p(\theta)}{D_p(0)} \right) = \log^2 \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = \frac{4(1 - \cos \theta)^2}{p^2} + O\left(\frac{(1 - \cos \theta)^2}{p^3} \right),$$

we have

$$R'' - 2R' \log D_p(0) = \frac{4}{p^2} E_{p,2}(\sigma) + O\left(\frac{E_{p,2}(\sigma)}{p^3} \right),$$

where $E_{p,j}(\sigma)$ is defined as in (3.2). By using (3.3) with the choice of $j = 2$ and the trivial estimate $E_{p,2}(\sigma) \leq 4E_p(\sigma)$, we deduce

$$(4.21) \quad \frac{R'' - 2R' \log D_p(0)}{E_p(\sigma)} = \frac{4}{p^2} \frac{E_{p,2}(\sigma)}{E_p(\sigma)} + O\left(\min\left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right).$$

Similarly we have

$$\log \left(\frac{D_p(\theta)}{D_p(0)} \right) = -\log \left(1 + \frac{2p(1 - \cos \theta)}{(p-1)^2} \right) = -\frac{2(1 - \cos \theta)}{p} + O\left(\frac{(1 - \cos \theta)}{p^2} \right),$$

and therefore

$$R' = -\frac{2}{p} E_{p,1}(\sigma) + O\left(\frac{E_{p,1}(\sigma)}{p^2} \right).$$

Now (3.3) with $j = 1$ and the trivial estimate $E_{p,1}(\sigma) \leq 2E_p(\sigma)$ imply

$$(4.22) \quad \begin{aligned} \left(\frac{R'}{E_p(\sigma)} \right)^2 &= \frac{4}{p^2} \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left(\frac{E_{p,1}(\sigma)^2}{p^3 E_p(\sigma)^2} \right) \\ &= \frac{4}{p^2} \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2 + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right). \end{aligned}$$

Inserting (4.21) and (4.22) into (4.20) and in view of (4.14), we deduce

$$(4.23) \quad \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{4}{p^2} h_p(\sigma) + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{p^3} \right\} \right)$$

for all p and $\sigma > 0$, where

$$h_p(\sigma) := \frac{E_{p,2}(\sigma)}{E_p(\sigma)} - \left(\frac{E_{p,1}(\sigma)}{E_p(\sigma)} \right)^2.$$

When $p \leq \sigma^{1/2}$, the inequality (3.3) of Lemma 3.2 implies that $h_p(\sigma) \ll (p/\sigma)^2$. From this and (4.23) we deduce the first estimate of (4.18).

If $p \geq \sigma^{1/2}$, we can use (4.3), (3.11) and (3.8) to write

$$\begin{aligned} 4h_p(\sigma) &= g''\left(\frac{\sigma}{p}\right) \left\{ 1 + O\left(\frac{\sigma}{p^2}\right) \right\} \\ &= g''\left(\frac{\sigma}{p}\right) + O\left(\min \left\{ \frac{\sigma}{p^2}, \frac{1}{\sigma} \right\} \right). \end{aligned}$$

Inserting it into (4.23) and in view of Lemma 3.1, we get, for $p \geq \sigma^{1/2}$,

$$\frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} = \frac{1}{p^2} g''\left(\frac{\sigma}{p}\right) + O\left(\min \left\{ \frac{1}{\sigma^2 p}, \frac{1}{\sigma p^2} \right\} \right).$$

This completes the proof. □

Lemma 4.5. *Let $J \geq 1$ be a fixed integer. Then we have*

$$\phi_2(\sigma, y) = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\}$$

uniformly for $y \geq \sigma \geq 2$, where

$$b_{j,2} := \int_0^\infty h''(u) (\log u)^{j-1} du.$$

In particular $b_{1,2} = 2$.

Proof. From Lemma 4.4 and (3.8), we deduce easily that

$$\begin{aligned} \phi_2(\sigma, y) &= \sum_{p \leq y} \frac{E_p''(\sigma)E_p(\sigma) - E_p'(\sigma)^2}{E_p(\sigma)^2} \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{g''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma} \right) \\ &= \sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} + O\left(\frac{1}{\sigma^{3/2} \log \sigma} \right). \end{aligned}$$

Similarly to (4.10), we can prove that

$$\sum_{\sigma^{1/2} < p \leq y} \frac{h''(\sigma/p)}{p^2} = \frac{1}{\sigma} \left\{ \sum_{j=1}^J \frac{b_{j,2}}{(\log \sigma)^j} + O_J(R_J(\sigma, y)) \right\},$$

by using (3.11), (3.12) and (4.8). Now the desired result follows from the preceding two estimates.

Finally

$$\begin{aligned} b_{1,2} &= \int_0^1 h''(u) du + \int_1^\infty h''(u) du \\ &= h'(1-) - h'(1+) = h'(1-) - (h'(1-) - 2) = 2. \end{aligned}$$

This completes the proof. \square

Similarly (even more easily, since we only need an upper bound instead of an asymptotic formula), we can prove the following result.

Lemma 4.6. *We have*

$$(4.24) \quad \phi_n(\sigma, y) \ll 1/(\sigma^{n-1} \log \sigma) \quad (n = 3, 4)$$

uniformly for $y \geq \sigma \geq 3$.

§ 5. Estimate of $|E(\kappa + i\tau, y)|$

Lemma 5.1. *For any $\delta \in (0, \frac{1}{4})$, there are two absolute positive constants c_1, c_2 and a positive constant $c_3 = c_3(\delta)$ such that for all $y \geq \sigma \geq 3$ we have*

$$(5.1) \quad \left| \frac{E(\sigma + i\tau, y)}{E(\sigma, y)} \right| \leq \begin{cases} 1 & \text{if } |\tau| \leq c_1 \sigma^{1/2} \log \sigma \text{ or } |\tau| \geq y^{1/\delta}, \\ e^{-c_2 \tau^2 / [\sigma (\log \sigma)^2]} & \text{if } c_1 \sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma, \\ e^{-c_3 |\tau|^\delta} & \text{if } \sigma \leq |\tau| \leq y^{1/\delta}. \end{cases}$$

Proof. First we write

$$\begin{aligned} E_p(s) &= \frac{2}{\pi} \int_0^\pi (D_p(\theta)^{-1})^{-s} \sin^2 \theta d\theta \\ &= \frac{2}{\pi} \int_0^\pi \frac{\sin^2 \theta}{(1-s)(D_p(\theta)^{-1})'} d(D_p(\theta)^{-1})^{1-s}. \end{aligned}$$

Since $(D_p(\theta)^{-1})' = 2p^{-1} \sin \theta$, after a simplification and an integration by parts it follows that

$$\begin{aligned} E_p(s) &= \frac{p}{\pi(s-1)} \int_0^\pi D_p(\theta)^{s-1} \cos \theta d\theta \\ &= \frac{p}{\pi(s-1)} \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi-\theta)^{s-1}\} \cos \theta d\theta. \end{aligned}$$

This implies that

$$(5.2) \quad \left| \frac{E_p(s)}{E_p(\sigma)} \right| = \left| \frac{\sigma-1}{s-1} \right| \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right|$$

with

$$E_p^*(s) := \int_0^{\pi/2} \{D_p(\theta)^{s-1} - D_p(\pi - \theta)^{s-1}\} \cos \theta \, d\theta.$$

1° Case of $\sigma^{1/\delta} < |\tau| \leq y^{1/\delta}$

Write

$$E_p^*(s) = \int_0^{\pi/2} D_p(\theta)^{s-1} \{1 - \Delta_p(\theta)^{s-1}\} \cos \theta \, d\theta$$

with

$$\Delta_p(\theta) := \frac{1 - 2p^{-1} \cos \theta + p^{-2}}{1 + 2p^{-1} \cos \theta + p^{-2}}.$$

It is clear that for all p , the function $\theta \mapsto \Delta_p(\theta)$ is increasing on $[0, \pi/2]$. It follows that

$$\begin{aligned} E_p^*(\sigma) &\geq \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \{1 - \Delta_p(\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &\geq \{1 - \Delta_p(\pi/4)^{\sigma-1}\} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \end{aligned}$$

for all p and $\sigma \geq 1$. This implies that

$$(5.3) \quad \left| \frac{1}{E_p^*(\sigma)} \int_0^{\pi/4} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

Similarly since the function $\theta \mapsto D_p(\theta)^{\sigma-1} \cos \theta$ is decreasing on $[0, \pi/2]$ for all p and $\sigma \geq 2$, we can deduce, via (5.3), that

$$(5.4) \quad \left| \frac{1}{E_p^*(\sigma)} \int_{\pi/4}^{\pi/2} D_p(\theta)^{\sigma-1} \cos \theta \, d\theta \right| \leq \frac{1}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

From (5.3) and (5.4), we deduce that

$$\left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \frac{2}{1 - \Delta_p(\pi/4)^{\sigma-1}}.$$

It is easy to verify that for all $p \geq \sigma \geq 2$, we have

$$\Delta_p\left(\frac{\pi}{4}\right)^{\sigma-1} \leq \left(1 - \frac{\sqrt{2}}{p} + \frac{1}{p^2}\right)^{\sigma-1} \leq 1 - \frac{\sigma-1}{4p}.$$

Combining these estimates with (5.2), we obtain

$$\left| \frac{E_p(s)}{E_p(\sigma)} \right| \leq \frac{8p}{|s-1|} \leq \frac{p^4}{|\tau|} \quad (p \geq \sigma).$$

By multiplying this inequality for $\sigma < p \leq |\tau|^\delta (\leq y)$ and the trivial inequality $|E_p(s)| \leq |E_p(\sigma)|$ for the others p , we deduce, via the prime number theorem, that

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma < p \leq |\tau|^\delta} \log |\tau| + 4 \sum_{\sigma < p \leq |\tau|^\delta} \log p \right\} \\ &\leq e^{-\{1/\delta - 4 + o(1)\}|\tau|^\delta}. \end{aligned}$$

2° Case of $c_1\sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma^{1/\delta}$

For $p \geq \sigma^{1/2} \geq 2$, we can write

$$\begin{aligned} |E_p^*(s)| &\leq \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} + D_p(\pi-\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p] \cos \theta} + e^{-2[(\sigma-1)/p] \cos \theta}) \cos \theta \, d\theta \end{aligned}$$

and

$$\begin{aligned} |E_p^*(\sigma)| &= \int_0^{\pi/2} \{D_p(\theta)^{\sigma-1} - D_p(\pi-\theta)^{\sigma-1}\} \cos \theta \, d\theta \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\} \int_0^{\pi/2} (e^{2[(\sigma-1)/p] \cos \theta} - e^{-2[(\sigma-1)/p] \cos \theta}) \cos \theta \, d\theta. \end{aligned}$$

From these, we deduce that

$$(5.5) \quad \left| \frac{E_p^*(s)}{E_p^*(\sigma)} \right| \leq \left\{1 + O\left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}}\right)\right\} \quad (2 \leq \sigma^{1/2} \leq p \leq \sigma)$$

where we have used the following facts

$$\int_0^{\pi/2} e^{2[(\sigma-1)/p] \cos \theta} \cos \theta \, d\theta \gg e^{\sigma/p} \quad \text{and} \quad \int_0^{\pi/2} e^{-2[(\sigma-1)/p] \cos \theta} \cos \theta \, d\theta \ll 1.$$

Inserting (5.5) into (5.2), for $2 \leq \sigma^{1/2} \leq p \leq \sigma$ we obtain

$$\begin{aligned} \left| \frac{E_p(s)}{E_p(\sigma)} \right| &\leq \exp \left\{ -\log \left| \frac{s-1}{\sigma-1} \right| + C \left(\frac{\sigma}{p^2} + \frac{1}{e^{\sigma/p}} \right) \right\} \\ &\leq \begin{cases} e^{-\tau^2/\sigma^2 + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } 3 \leq |\tau| \leq \sigma, \\ e^{-\frac{1}{2} \log(1+\tau^2/\sigma^2) + C\sigma/p^2 + Ce^{-\sigma/p}} & \text{if } \sigma \leq |\tau| \leq \sigma^{1/\delta}, \end{cases} \end{aligned}$$

where $C > 0$ is an absolute constant.

Now by multiplying these inequalities for $\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)$ and the trivial inequality $|E_p(s)| \leq E_p(\sigma)$ for the other p , we get

$$\begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)} \left(\frac{\tau^2}{\sigma^2} - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right) \right\} \\ &\leq \exp \left\{ - \left(\frac{\tau^2}{8\sigma(\log \sigma)^2} - 10C - \frac{10C}{\sigma \log \sigma} \right) \right\} \\ &\leq \exp \left\{ - \frac{c_2 \tau^2}{\sigma(\log \sigma)^2} \right\} \end{aligned}$$

if $c_1\sigma^{1/2} \log \sigma \leq |\tau| \leq \sigma$, and

$$(5.6) \quad \begin{aligned} \left| \frac{E(s, y)}{E(\sigma, y)} \right| &\leq \exp \left\{ - \sum_{\sigma/(4 \log \sigma) \leq p \leq \sigma/(2 \log \sigma)} \left[\frac{1}{2} \log \left(1 + \frac{\tau^2}{\sigma^2} \right) - \frac{C\sigma}{p^2} - \frac{C}{e^{\sigma/p}} \right] \right\} \\ &\leq \exp \left\{ - \left[\frac{\sigma}{8 \log \sigma} \log \left(1 + \frac{\tau^2}{\sigma^2} \right) - 10C - \frac{10C}{\sigma \log \sigma} \right] \right\} \\ &\leq \exp \left\{ - c_3 |\tau|^\delta \right\} \end{aligned}$$

if $\sigma \leq |\tau| \leq \sigma^{1/\delta}$. This completes the proof. \square

§ 6. Proof of Theorem 1

We follow the argument of Granville & Soundararajan [3] to prove Theorem 1. We shall divide the proof in several steps which are embodied in the following lemmas. The first one is an analogue for (3.6) and (3.7) of [3] (see also Lemma 3.1 of [19]).

Lemma 6.1. *Let $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Then we have*

$$(6.1) \quad \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s} \lambda s^2} ds \leq \Phi(te^{-\lambda}, y),$$

$$(6.2) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s} \lambda s^2} (e^{\lambda s} - e^{-\lambda s}) ds.$$

Proof. For any $c > 0$ and $\lambda > 0$, we have, by Perron's formula ([18], Lemma II.2.1.1),

$$(6.3) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{1}{\lambda} \int_0^\lambda \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (ye^v)^s \frac{ds}{s} \right) dv \\ = \begin{cases} 0 & \text{if } 0 < y < e^{-\lambda}, \\ 1 + (\log y)/\lambda \in [0, 1] & \text{if } e^{-\lambda} \leq y \leq 1, \\ 1 & \text{if } y > 1. \end{cases}$$

Denote by $\mathbf{1}_X(\omega)$ the characteristic function of the set $X \subset \Omega$. Then by (6.3), we have

$$\mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega), y) > (e\gamma t)^2\}}(\omega) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\natural}(\omega), y)}{(e\gamma t)^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds.$$

Integrating over Ω and interchanging the order of integrations yield

$$\Phi(t, y) \leq \int_{\Omega} \left\{ \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\natural}(\omega), y)}{(e\gamma t)^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds \right\} d\mu(\omega) \\ = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s} \lambda s^2} ds.$$

This proves the first inequality of (6.1). The second can be treated by noticing that

$$\mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega), y) > (e\gamma^{-\lambda} t)^2\}}(\omega) = \mathbf{1}_{\{\omega \in \Omega: L(1, g^{\natural}(\omega), y) > (e\gamma t)^2\}}(\omega) \\ + \mathbf{1}_{\{\omega \in \Omega: (e\gamma t)^2 \geq L(1, g^{\natural}(\omega), y) > (e\gamma^{-\lambda} t)^2\}}(\omega) \\ \geq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \left(\frac{L(1, g^{\natural}(\omega), y)}{(e\gamma t)^2} \right)^s \frac{e^{\lambda s} - 1}{\lambda s^2} ds.$$

From (6.1), we can deduce

$$\Phi(te^{-\lambda}, y) - \Phi(t, y) \leq \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma^{-\lambda} t)^{2s} \lambda s^2} ds - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s} \lambda s^2} ds \\ = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e\gamma t)^{2s} \lambda s^2} (e^{\lambda s} - e^{-\lambda s}) ds.$$

This completes the proof. \square

Lemma 6.2. *Let $t \geq 1$, $y \geq 2e^t$ and $0 < \kappa\lambda \leq 1$. Then we have*

$$\frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\kappa\lambda + \frac{\log \kappa}{\kappa}\right) \right\}.$$

Proof. First we write, for $s = \kappa + i\tau$ and $|\tau| \leq \kappa$,

$$E(s, y) = \exp \left\{ \sigma_0 + i\sigma_1\tau - \frac{\sigma_2}{2}\tau^2 - i\frac{\sigma_3}{6}\tau^3 + O(\sigma_4\tau^4) \right\},$$

$$\frac{e^{\lambda s} - 1}{\lambda s^2} = \frac{1}{\kappa} \left\{ 1 - \frac{i}{\kappa}\tau + O\left(\kappa\lambda + \frac{\tau^2}{\kappa^2}\right) \right\}.$$

Since $\sigma_1 = \log t + \gamma$, we have

$$\frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} = \frac{E(\kappa, y)}{\kappa (e^{\gamma t})^{2\kappa}} e^{-(\sigma_2/2)\tau^2} \left\{ 1 - \frac{i}{\kappa}\tau - i\frac{\sigma_3}{6}\tau^3 + O(R(\tau)) \right\}$$

with

$$R(\tau) := \kappa\lambda + \kappa^{-2}\tau^2 + \sigma_4\tau^4 + \sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau| \leq \kappa$ to obtain

$$(6.4) \quad \frac{1}{2\pi i} \int_{\kappa-i\kappa}^{\kappa+i\kappa} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds = \frac{E(\kappa, y)}{2\pi\kappa (e^{\gamma t})^{2\kappa}} \int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} \{1 + O(R(\tau))\} d\tau,$$

where we have used the fact that the integrals involving $(i/\kappa)\tau$ and $(i\sigma_3/6)\tau^3$ vanish.

On the other hand, using lemmas 4.5 and 4.6 we have

$$\int_{-\kappa}^{\kappa} e^{-(\sigma_2/2)\tau^2} d\tau = \sqrt{\frac{2\pi}{\sigma_2}} \left\{ 1 + O\left(\exp\left\{-\frac{1}{2}\kappa^2\sigma_2\right\}\right) \right\},$$

$$\int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2} R(\tau) d\tau \ll \frac{1}{\sqrt{\sigma_2}} \left(\kappa\lambda + \frac{1}{\kappa^2\sigma_2} + \frac{\sigma_3^2}{\sigma_2^2} + \frac{\sigma_4}{\sigma_2^2} \right)$$

$$\ll \frac{1}{\sqrt{\sigma_2}} \left(\kappa\lambda + \frac{\log \kappa}{\kappa} \right).$$

Inserting these into (6.4), we obtain the desired result. \square

Lemma 6.3. *Let δ and c_3 be two constants determined by Lemma 5.1. Then we have*

$$(6.5) \quad \int_{\kappa\pm i\kappa}^{\kappa\pm i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_1,$$

$$(6.6) \quad \int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y) e^{\lambda s} - 1}{(e^{\gamma t})^{2s} \lambda s^2} (e^{\lambda s} - e^{-\lambda s}) ds \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} R_2,$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda\kappa \leq 1$, where

$$R_1 := \lambda^{-1} e^{-c_3\kappa^\delta} + \lambda^{-1} (\kappa/\log \kappa)^{1/2} y^{-1/\delta},$$

$$R_2 := \lambda\kappa (\log \kappa)^{1/2} + (\kappa/\log \kappa)^{1/2} e^{-c_3\kappa^\delta} + \lambda^{-1} (\kappa/\log \kappa)^{1/2} y^{-1/\delta}.$$

Proof. We split the integral in (6.5) into two parts according to $\kappa \leq |\tau| \leq y^{1/\delta}$ or $|\tau| \geq y^{1/\delta}$. Using Lemma 5.1 with $\sigma = \kappa$ and the inequality $(e^{\lambda s} - 1)/s^2 \ll 1/\tau^2$, the integral in (6.5) is

$$\ll \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa} \lambda} \left(\frac{e^{-c_3 \kappa^\delta}}{\kappa} + \frac{1}{y^{1/\delta}} \right),$$

which implies (6.5), in view of Lemma 4.5 with $J = 1$.

Similarly we split the integral in (6.6) into four parts according to

$$|\tau| \leq c_1 \kappa^{1/2} \log \kappa, \quad c_1 \kappa^{1/2} \log \kappa < |\tau| \leq \kappa, \quad \kappa < |\tau| \leq y^{1/\delta}, \quad |\tau| \geq y^{1/\delta}.$$

By Lemma 5.1 with $\sigma = \kappa$ and the inequalities

$$(e^{\lambda s} - 1)/\lambda s \ll \min\{1, 1/(\lambda|\tau|)\}, \quad (e^{\lambda s} - e^{-\lambda s})/s \ll \min\{\lambda, 1/|\tau|\},$$

the integral in (6.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa, y)}{(e^{\gamma t})^{2\kappa}} \left(\lambda \kappa^{1/2} \log \kappa + e^{-c_3 \kappa^\delta} + \lambda^{-1} y^{-1/\delta} \right),$$

which implies (6.6), as before. \square

Now we are ready to complete the proof of Theorem 1. Lemma 6.2 and (6.5) of Lemma 6.3 give

$$(6.7) \quad \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \frac{E(s, y)}{(e^{\gamma t})^{2s}} \frac{e^{\lambda s} - 1}{\lambda s^2} ds = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \{1 + O(R')\}$$

where

$$R' := \frac{\log \kappa}{\kappa} + \kappa \lambda + \frac{e^{-c_3 \kappa^\delta} + (\kappa/\log \kappa)^{1/2} y^{-1/\delta}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geq 2e^t \asymp \kappa$ and $1/\delta > 4$, we deduce

$$(6.8) \quad R' \ll t/e^t.$$

Combining (6.7) and (6.8) with (6.1), we obtain

$$(6.9) \quad \Phi(t, y) \leq \frac{E(\kappa, y)}{\kappa \sqrt{2\pi\sigma_2} (e^{\gamma t})^{2\kappa}} \left\{ 1 + O\left(\frac{t}{e^t}\right) \right\} \leq \Phi(te^{-\lambda}, y)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$.

On the other hand, (6.2) of Lemma 6.1 and (6.6) of Lemma 6.2 imply

$$\begin{aligned} \Phi(te^{-\lambda}, y) - \Phi(t, y) &\ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} + \frac{(\kappa/\log \kappa)^{1/2}}{\lambda y^{1/\delta}} \right) \\ &\ll_\varepsilon \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} \right) \end{aligned}$$

when $y^{-1/(2\delta)} \kappa^{-1/2} (\log \kappa)^{-1} \leq \lambda \leq \kappa^{-1}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t, y)$ is a non-decreasing function of λ , we deduce

$$(6.10) \quad \Phi(te^{-\lambda}, y) - \Phi(t, y) \ll \frac{E(\kappa, y)}{\kappa \sqrt{\sigma_2} (e^{\gamma t})^{2\kappa}} \left(\lambda \kappa (\log \kappa)^{1/2} + \frac{(\kappa/\log \kappa)^{1/2}}{e^{c_3 \kappa^\delta}} + \frac{\kappa (\log \kappa)^{1/2}}{y^{1/(2\delta)}} \right)$$

uniformly for $t \geq 1$, $y \geq 2e^t$ and $0 < \lambda \leq e^{-t}$. Obviously the estimates (6.9) and (6.10) imply the desired result. This completes the proof of Theorem 1. \square

§ 7. Proof of Theorem 2

Using Lemmas 4.1 and 4.5, we can write

$$\begin{aligned} \frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e\gamma t)^{2\kappa}} &= \exp \left\{ \overline{\phi(\kappa, y)} - 2\kappa(\gamma + \log t) + O(\log \kappa) \right\} \\ &= \exp \left\{ \kappa \left(2\log_2 \kappa - 2\log t + \sum_{j=1}^J \frac{b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right) \right\}. \end{aligned}$$

On the other hand, Lemma 4.3 and (1.17) imply that

$$2\log_2 \kappa + 2\gamma + \sum_{j=1}^J \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) = 2(\log t + \gamma).$$

Combining these estimates, we can obtain

$$\frac{E(\kappa, y)}{\kappa\sqrt{2\pi\sigma_2}(e\gamma t)^{2\kappa}} = \exp \left\{ -\kappa \left[\sum_{j=1}^J \frac{b_{j,1} - b_{j,0}}{(\log \kappa)^j} + O_J(R_J(\kappa, y)) \right] \right\}.$$

In view of (1.21), (4.2) and (4.15), we have $b_{j,1} - b_{j,0} = a_j$. This completes the proof. \square

§ 8. Proof of Corollary 3

We first prove an asymptotic development of $\kappa(t, y)$ in t .

Lemma 8.1. *For each integer $J \geq 1$, there are computable constants c_1, \dots, c_J such that the asymptotic formula*

$$(8.1) \quad \kappa(t, y) = e^{t-\gamma_0} \left\{ 1 + \sum_{j=1}^J \frac{c_j}{t^j} + O_J(R_J^*(t, y)) \right\}$$

holds uniformly for $t \geq 1$ and $y \geq 2e^t$, where

$$R_N^*(t, y) := \frac{1}{t^{N+1}} + \frac{e^t}{y \log y}.$$

Further we have $c_1 = -\frac{1}{8}b_{1,1}^2 - \frac{1}{4}b_{2,1}$.

Proof. By Lemma 4.3 and (1.17), we have

$$(8.2) \quad 2\log t = 2\log_2 \kappa + \sum_{j=1}^{J+1} \frac{b_{j,1}}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)),$$

where $R_J(\kappa, y)$ is defined as in (1.20). From (8.2), we easily deduce that

$$\begin{aligned} t &= (\log \kappa) \prod_{j=1}^{J+1} \exp \left\{ \sum_{j=1}^J \frac{b_{j,1}}{2(\log \kappa)^j} \right\} \exp \{ O_J(R_{J+1}(\kappa, y)) \} \\ &= (\log \kappa) \prod_{j=1}^{J+1} \left\{ \sum_{m_j=0}^{J+1} \frac{1}{m_j!} \left(\frac{b_{j,1}}{2(\log \kappa)^j} \right)^{m_j} + O_J(R_{J+1}(\kappa, y)) \right\}. \end{aligned}$$

Developping the product, we get

$$t = (\log \kappa) \left\{ \sum_{j=0}^{J+1} \frac{b'_j}{(\log \kappa)^j} + O_J(R_{J+1}(\kappa, y)) \right\}$$

where

$$\begin{aligned} b'_j &:= \sum_{\substack{m_1 \geq 0, \dots, m_{J+1} \geq 0 \\ m_1 + 2m_2 + \dots + (J+1)m_{J+1} = j}} \frac{b_{1,1}^{m_1} \dots b_{J+1,1}^{m_{J+1}}}{(2m_1)!! \dots (2m_{J+1})!!} \\ &= \sum_{\substack{m_1 \geq 0, \dots, m_j \geq 0 \\ m_1 + 2m_2 + \dots + jm_j = j}} \frac{b_{1,1}^{m_1} \dots b_{j,1}^{m_j}}{(2m_1)!! \dots (2m_j)!!}. \end{aligned}$$

Since $b'_0 = 1$ and $b'_1 = b_{1,1}/2 = \gamma_0$, the preceding asymptotic formula can be written as

$$(8.3) \quad t = \log \kappa + \gamma_0 + \sum_{j=1}^J \frac{b'_{j+1}}{(\log \kappa)^j} + O_J(R_J^*(t, y)),$$

where we have used the fact that $\kappa(t, y) \asymp e^t$ (see Lemma 2.3) and $(\log \kappa)R_{J+1}(\kappa, y) \asymp R_J^*(t, y)$.

With the help of (8.3), a simple recurrence argument shows that there are constants γ_n such that

$$(8.4) \quad t = \log \kappa + \sum_{j=0}^J \frac{\gamma_j}{t^j} + O_J(R_J^*(t, y)).$$

In fact taking $J = 0$ in (8.3), we see that (8.4) holds for $J = 0$. Suppose that it holds for $0, \dots, J-1$, i.e.

$$t = \log \kappa + \sum_{i=0}^{J-j-1} \frac{\gamma_i}{t^i} + O(R_{J-j-1}^*(t, y)) \quad (j = 0, \dots, J-1),$$

which is equivalent to

$$(8.5) \quad \log \kappa = t \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma_{i-1}}{t^i} + O\left(\frac{R_{J-j-1}^*(t, y)}{t}\right) \right\} \quad (j = 0, \dots, J-1).$$

This holds also for $j = J$ if we use the convention:

$$\sum_{i=0}^{-1} = 0 \quad \text{and} \quad R_{-1}^*(t, y) := 1,$$

since $\log \kappa = t + O(1)$. Inserting it into (8.3), we easily see that (8.4) holds also for J . In particular we have $\gamma_1 = b'_2 = b_{1,1}^2/8 + b_{2,1}/4$.

Now (8.1) is an immediate consequence of (8.4) with

$$c_j := \sum_{\substack{m_1 \geq 0, \dots, m_J \geq 0 \\ m_1 + 2m_2 + \dots + Jm_J = j}} (-1)^{m_1 + \dots + m_J} \frac{\gamma_1^{m_1} \dots \gamma_J^{m_J}}{m_1! \dots m_J!}.$$

This completes the proof. \square

Now we are ready to prove Corollary 3.

Using (8.5), we have

$$(8.6) \quad \begin{aligned} \sum_{j=1}^J \frac{a_j}{(\log \kappa)^j} &= \sum_{j=1}^J \frac{a_j}{t^j} \left\{ 1 - \sum_{i=1}^{J-j} \frac{\gamma_{i-1}}{t^i} + O_N \left(\frac{R_{J-j-1}^*(t, y)}{t} \right) \right\}^{-j} \\ &= \sum_{j=1}^J \frac{\rho_j}{t^j} + O_J \left(\frac{R_{J-2}^*(t, y)}{t^2} \right), \end{aligned}$$

where the ρ_n are constants. In particular we have $\rho_1 = a_1 = 1$ and $\rho_2 = \gamma_0 + a_2$.

Now Theorem 2, (8.1) and (8.6) imply the result of Corollary with

$$a_1^* = \rho_1 = 1, \quad a_j^* = \rho_j + \sum_{i=1}^{j-1} c_i \rho_{j-i} \quad (j \geq 2).$$

This completes the proof of Corollary 3. □

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