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# Stochastic derivatives for fractional diffusions

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## Abstract

In this paper, we introduce some fundamental notions related to the so-called *stochastic derivatives* with respect to a given  $\sigma$ -field  $\mathcal{Q}$ . In our framework, we recall well-known results about Markov Wiener diffusions. We afterwards mainly focus on the case where  $X$  is a fractional diffusion and where  $\mathcal{Q}$  is the past, the future or the present of  $X$ . We treat some crucial examples and our main result is the existence of stochastic derivatives with respect to the present of  $X$  when  $X$  solves a stochastic differential equation driven by a fractional Brownian motion with Hurst index  $H > 1/2$ . We give explicit formulas.

*Key words:* stochastic derivatives, Nelson's derivative, fractional Brownian motion, fractional differential equation, Malliavin calculus.

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## 1 Introduction

There exist various ways to generalize the notion of differentiation on deterministic functions. We may think about fractional derivative or differentiation in the sense of the theory of distributions. In both cases, we lose a dynamical or a geometric interpretation as tangent vectors, velocities for instance. In this present work, we want to construct derivatives on stochastic processes which conserve a dynamical meaning. Our goal may be motivated by the stochastic embedding of dynamical systems introduced in [3]. This procedure aims at comprehending the following question: how to write an equation which contains the dynamical meaning of an initial ordinary differential equation and which extends this dynamical meaning on stochastic processes? We refer to [4] for more details.

Unfortunately, for most of the stochastic processes used in physical models, the limit

$$\frac{Z_{t+h} - Z_t}{h}$$

does not exist almost surely. What can we do to give a meaning to this limit? One of the main available tool is the "quantity of information" we can use to calculate it, namely a given  $\sigma$ -field  $\mathcal{Q}$ . The idea is that one can remove the divergences which appear by doing some means in the computation. This fact can be achieved by studying the behavior when  $h$  goes to zero of the conditional expectation:

$$E \left[ \frac{Z_{t+h} - Z_t}{h} \middle| \mathcal{Q} \right].$$

Such objects were introduced by Nelson in his dynamical theory of Brownian diffusion [10]. For a fixed time  $t$ , he calculates a forward (resp. backward) derivative with respect to a given  $\sigma$ -field  $\mathcal{P}_t$  which can be seen as the past of the process up to time  $t$  (resp.  $\mathcal{F}_t$ , the future of the process up to time  $t$ ). The main class with which he can work turns out to be that of Wiener diffusions.

The purpose of this paper is, on the one hand, to introduce notions to study the above mentioned quantities for general processes and, on the other hand, to treat some examples. We mainly study these notions on solutions of stochastic differential equations driven by a fractional Brownian motion with Hurst index  $H \geq \frac{1}{2}$ . In particular, we recall results on Wiener diffusions (case  $H = \frac{1}{2}$ ) in our framework. We prove that for a suitable  $\sigma$ -algebra, the stochastic derivatives of a solution of the fractional stochastic differential equation exist and we are able to give explicit formulas.

Our paper is organized as follows. In section 2, we recall some now classical facts on stochastic analysis on fractional diffusions. In section 3, we introduce the fundamental notions related to the so-called *stochastic derivatives*. In section 4, we study stochastic derivatives of Nelson's type for fractional diffusions. We show in section 5 that stochastic derivatives with respect to the present turn out to be adequate tools for fractional Brownian motion with  $H > \frac{1}{2}$ . We treat also the more difficult case of a fractional diffusion.

## 2 Basic notions for fractional Brownian motion

We briefly recall some basic facts about stochastic calculus with respect to a fractional Brownian motion. One refers to [13] for further details. Let  $B = (B_t)_{t \in [0, T]}$  be a fractional Brownian motion with Hurst parameter  $H \in (0, 1)$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We mean that  $B$  is a centered Gaussian process with the covariance function  $E(B_s B_t) = R_H(s, t)$ , where

$$R_H(s, t) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}). \quad (1)$$

If  $H = 1/2$ , then  $B$  is a Brownian motion. From (1), one can easily see that  $E|B_t - B_s|^2 = |t - s|^{2H}$ , so  $B$  has  $\alpha$ -Hölder continuous paths for any  $\alpha \in (0, H)$ .

## 2.1 Space of deterministic integrands

We denote by  $\mathcal{E}$  the set of step  $\mathbb{R}$ -valued functions on  $[0, T]$ . Let  $\mathcal{H}$  be the Hilbert space defined as the closure of  $\mathcal{E}$  with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s).$$

The mapping  $\mathbf{1}_{[0,t]} \mapsto B_t$  can be extended to an isometry between  $\mathcal{H}$  and the Gaussian space  $H_1(B)$  associated with  $B$ . We denote this isometry by  $\varphi \mapsto B(\varphi)$ .

When  $H \in (\frac{1}{2}, 1)$ , it follows from [15] that the elements of  $\mathcal{H}$  may be not functions but distributions of negative order. It will be more convenient to work with a subspace of  $\mathcal{H}$ , which contains only functions. Such a space is the set  $|\mathcal{H}|$  of measurable functions on  $[0, T]$  such that

$$\int_0^T \int_0^T |f(u)||f(v)||u-v|^{2H-2} dudv < \infty$$

endowed with the scalar product

$$\langle f, g \rangle = H(2H-1) \int_0^T \int_0^T f(u)g(v)|u-v|^{2H-2} dudv. \quad (2)$$

Actually, we have the inclusions

$$L^2([0, T]) \subset L^{\frac{1}{H}}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}. \quad (3)$$

## 2.2 Fractional operators

The covariance kernel  $R_H(t, s)$  introduced in (1) can be written as

$$R_H(t, s) = \int_0^{s \wedge t} K_H(s, u)K_H(t, u)du,$$

where  $K_H(t, s)$  is the square integrable kernel defined by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad 0 < s < t \quad (4)$$

where  $c_H^2 = H(2H-1)\beta(2-2H, H-1/2)^{-1}$  and  $\beta$  denotes the Beta function. By convention, we set  $K_H(t, s) = 0$  if  $s \geq t$ .

Let  $\mathcal{K}_H^* : \mathcal{E} \rightarrow L^2([0, T])$  be the linear operator defined by:

$$\mathcal{K}_H^* (\mathbf{1}_{[0,t]}) = K_H(t, \cdot).$$

The following equality holds for any  $\phi, \psi \in \mathcal{E}$

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \langle \mathcal{K}_H^* \phi, \mathcal{K}_H^* \psi \rangle_{L^2([0, T])} = \mathbb{E} (B(\phi)B(\psi))$$

and then  $\mathcal{K}_H^*$  provides an isometry between the Hilbert spaces  $\mathcal{H}$  and  $L^2([0, T])$ .

The process  $W = (W_t)_{t \in [0, T]}$  defined by

$$W_t = B((\mathcal{K}_H^*)^{-1}(\mathbf{1}_{[0, t]}))$$

is a Wiener process, and the process  $B$  has an integral representation of the form

$$B_t = \int_0^t K_H(t, s) dW_s.$$

Hence, for any  $\phi \in \mathcal{H}$ ;

$$B(\phi) = W(\mathcal{K}_H^* \phi).$$

Let  $a, b \in \mathbb{R}$ ,  $a < b$ . For any  $p \geq 1$ , we denote by  $L^p = L^p([a, b])$  the usual Lebesgue spaces of functions on  $[a, b]$ . Let  $f \in L^1$  and  $a > 0$ . The left-sided and right-sided fractional Riemann-Liouville integrals of  $f$  of order  $\alpha$  are defined for almost all  $x \in (a, b)$  by

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy,$$

and

$$I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy,$$

respectively, where  $\Gamma$  denotes the usual Euler function. These integrals extend the classical integral of  $f$  when  $\alpha = 1$ .

If  $f \in I_{a+}^\alpha(L^p)$  (resp.  $f \in I_{b-}^\alpha(L^p)$ ) and  $\alpha \in (0, 1)$ , then for almost all  $x \in (a, b)$ , the left-sided and right-sided Riemann-Liouville derivative of  $f$  of order  $\alpha$  are defined by

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right)$$

and

$$D_{b-}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^{\alpha+1}} dy \right)$$

respectively, where  $a \leq x \leq b$ .

We define the operator  $\mathcal{K}_H$  on  $L^2([0, T])$  by

$$(\mathcal{K}_H h)(t) = \int_0^t K_H(t, s) h(s) ds.$$

It is an isomorphism from  $L^2([0, T])$  onto  $I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$  and it can be expressed as follows when  $H > \frac{1}{2}$ :

$$\mathcal{K}_H h = I_{0+}^1 s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} s^{\frac{1}{2}-H} h$$

where  $h \in L^2([0, T])$ . The crucial point is that any absolutely continuous function belongs to the space  $I_{0+}^{H+\frac{1}{2}}(L^2([0, T]))$  when  $H > \frac{1}{2}$ . For these functions  $\phi$ , the inverse operator  $\mathcal{K}_H^{-1}$  is given by

$$\mathcal{K}_H^{-1}\phi = s^{H-\frac{1}{2}}D_{0+}^{H-\frac{1}{2}}s^{\frac{1}{2}-H}\phi'.$$

When  $H > \frac{1}{2}$ , we introduce the operator  $\mathcal{O}_H$  on  $L^2([0, T])$  defined by

$$(\mathcal{O}_H\varphi)(s) := \left(\frac{d}{dt}\mathcal{K}_H\right)(\varphi)(s) = s^{H-\frac{1}{2}}I_{0+}^{H-\frac{1}{2}}s^{\frac{1}{2}-H}\varphi(s). \quad (5)$$

Let  $f : [a, b] \rightarrow \mathbb{R}$  be  $\alpha$ -Hölder continuous and  $g : [a, b] \rightarrow \mathbb{R}$  be  $\beta$ -Hölder continuous with  $\alpha + \beta > 1$ . Then, for any  $s, t \in [a, b]$ , the Young integral [20]  $\int_s^t f dg$  exists and we can express it in terms of fractional derivatives (see [21]): for any  $\gamma \in (1 - \beta, \alpha)$ , we have

$$\int_s^t f dg = (-1)^\gamma \int_s^t D_{s+}^\gamma f(x) D_{t-}^{1-\gamma} g_{t-}(x) dx, \quad (6)$$

where  $g_{t-}(x) = g(x) - g(t)$ . In particular, we deduce that:

$$\forall s < t \in [a, b], \quad \left| \int_s^t (f(r) - f(s)) dg(r) \right| \leq \kappa \|f\|_\alpha \|g\|_\beta |t - s|^{\alpha+\beta}, \quad (7)$$

where  $\kappa$  is a constant depending only on  $a, b, \alpha$  and  $\beta$ , and if  $h : [a, b] \rightarrow \mathbb{R}$  and  $\mu \in (0, 1]$ ,

$$\|h\|_\mu = \sup_{a \leq s < t \leq b} \frac{|h(t) - h(s)|}{|t - s|^\mu}.$$

### 2.3 Malliavin calculus

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function with compact support and consider the random variable  $F = f(B_{t_1}, \dots, B_{t_n})$  (we then say that  $F$  is a smooth random variable). The derivative process of  $F$  (w.r.t.  $B$ ) is the element of  $L^2(\Omega, \mathcal{H})$  defined by

$$D_s^B F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_n}) \mathbf{1}_{[0, t_i]}(s), \quad s \in [0, T].$$

In particular  $D_s^B B_t = \mathbf{1}_{[0, t]}(s)$ . As usual,  $\mathbb{D}^{1,2} = \mathbb{D}^{1,2}(\mathcal{H})$  is the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 = \mathbb{E}[F^2] + \mathbb{E}[\|D^B F\|_{\mathcal{H}}^2].$$

The Malliavin derivative  $D^B$  verifies the chain rule: if  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C_b^1$  and if  $(F_i)_{i=1, \dots, n}$  is a sequence of elements of  $\mathbb{D}^{1,2}$  then  $\varphi(F_1, \dots, F_n) \in \mathbb{D}^{1,2}$  and we have, for any  $s \in [0, T]$ :

$$D_s^B \varphi(F_1, \dots, F_n) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_n) D_s^B F_i.$$

The divergence operator  $\delta^B$  is the adjoint of the derivative operator  $D^B$ . If a random variable  $u \in L^2(\Omega, \mathcal{H})$  belongs to the domain of the divergence operator, then  $\delta^B(u)$  is defined by the duality relationship

$$\mathbb{E}(F\delta^B(u)) = \mathbb{E}\langle D^B F, u \rangle_{\mathcal{H}},$$

for every  $F \in \mathbb{D}^{1,2}$ .

## 2.4 Pathwise integration with respect to $B$

If  $X = (X_t)_{t \in [0, T]}$  and  $Z = (Z_t)_{t \in [0, T]}$  are two continuous processes, we define the forward integral of  $Z$  w.r.t.  $X$ , in the sense of Russo-Vallois, by

$$\int_0^\bullet Z_s dX_s = \lim_{\varepsilon \rightarrow 0} \text{ucp} \quad \varepsilon^{-1} \int_0^\bullet Z_s (X_{s+\varepsilon} - X_s) ds, \quad t \in [0, T], \quad (8)$$

provided the limit exists. Here "ucp" means "uniform convergence in probability". If  $X$  (resp.  $Z$ ) has a.s. Hölder continuous paths of order  $\alpha$  (resp.  $\beta$ ) with  $\alpha + \beta > 1$  then  $\int_0^\bullet Z_s dX_s$  exists and coincides with the usual Young integral (see [16], Proposition 2.12). When  $H > 1/2$ , if  $u = (u_t)_{t \in [0, T]}$  is in the space  $\mathbb{D}^{1,2}(|\mathcal{H}|)$  and verifies a.s.

$$\int_0^T \int_0^T |D_s^B u_t| |t - s|^{2H-2} ds dt < +\infty,$$

then  $\int_0^\bullet u_s dB_s$  exists and we have (see [1])

$$\int_0^t u_s dB_s = \delta^B(u \mathbf{1}_{[0, t]}) + H(2H - 1) \int_0^t \int_0^t D_s^B u_\tau |\tau - s|^{2H-2} ds d\tau. \quad (9)$$

## 2.5 Stochastic differential equation driven by $B$

Here we assume that  $H > 1/2$ . If  $\sigma \in C_b^2$  and if  $b \in C_b^1$ , then the equation

$$X_t = x_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \quad t \in [0, T] \quad (10)$$

admits a unique solution  $X$  in the set of processes whose paths are Hölder continuous of order  $\alpha > 1 - H$ . Here, the integral w.r.t.  $B$  is in the sense of Russo-Vallois, see (8). Moreover, we have a Doss-Sussmann [6, 18] type representation:

$$X_t = \phi(A_t, B_t), \quad t \in [0, T],$$

where  $\phi$  and  $A$  are given respectively by

$$\frac{\partial \phi}{\partial x_2}(x_1, x_2) = \sigma(\phi(x_1, x_2)), \quad \phi(x_1, 0) = x_1, \quad x_1, x_2 \in \mathbb{R}$$

and

$$A_t = \exp\left(-\int_0^{B_t} \sigma'(\phi(A_t, s)) ds\right) b(\phi(A_t, B_t)), \quad A_0 = x_0, \quad t \in [0, T].$$

Using this representation, we can show that  $X$  belongs to  $\mathbb{D}^{1,2}$  and that

$$D_s^B X_t = \sigma(X_s) \exp\left(\int_s^t b'(X_u) du + \int_s^t \sigma'(X_u) dB_u\right) \mathbf{1}_{[0,t]}(s), \quad s, t \in [0, T].$$

(see [11], proof of Theorem B).

### 3 Notions related to stochastic derivatives

Let  $(Z_t)_{t \in [0, T]}$  be a stochastic process defined on  $(\Omega, \mathcal{F}, \mathbf{P})$ . For all  $t \in (0, T)$  and  $h \neq 0$  such that  $t + h \in (0, T)$ , we set:

$$\Delta_h Z_t = \frac{Z_{t+h} - Z_t}{h}.$$

#### 3.1 Stochastic derivatives in a strong sense

**Definition 1** *Set  $t \in (0, T)$ . We say that  $\mathcal{A}^t$  (resp.  $\mathcal{B}^t$ ) is a good forward  $\sigma$ -field (resp. good backward  $\sigma$ -field) for  $Z$  if  $E[\Delta_h Z_t | \mathcal{A}^t]$  (resp.  $E[\Delta_{-h} Z_t | \mathcal{B}^t]$ ) converges in probability when  $h \downarrow 0$ . In these cases, we define the forward and backward derivatives*

$$\begin{aligned} D_+^{\mathcal{A}^t} Z_t &= \lim_{h \downarrow 0} E[\Delta_h Z_t | \mathcal{A}^t], \\ D_-^{\mathcal{B}^t} Z_t &= -\lim_{h \downarrow 0} E[\Delta_{-h} Z_t | \mathcal{B}^t]. \end{aligned}$$

The set of all good forward  $\sigma$ -fields (resp. good backward  $\sigma$ -fields) for  $Z$  at time  $t$  is denoted by  $\mathcal{M}_Z^{+(t)}$  (resp.  $\mathcal{M}_Z^{-(t)}$ ). The more  $\mathcal{M}^{\pm(t)}$  is high, the more  $Z$  is regular at time  $t$ . For instance, one has obviously that  $\{\emptyset, \Omega\} \in \mathcal{M}_Z^{+(t)}$  (resp.  $\in \mathcal{M}_Z^{-(t)}$ ) if and only if  $s \mapsto E(Z_s)$  is right differentiable (resp. left differentiable) in  $t$ . At the opposite, one has that  $\mathcal{F} \in \mathcal{M}_Z^{+(t)}$  (resp.  $\in \mathcal{M}_Z^{-(t)}$ ) if and only if  $s \mapsto Z_s$  is a.s. right differentiable (resp. left differentiable) at time  $t$ .

**Definition 2** *We say that  $(\mathcal{A}^t, \mathcal{B}^t)_{t \in (0, T)}$  is a good collection of  $\sigma$ -fields for  $Z$  if, for any  $t \in (0, T)$ ,  $\mathcal{A}^t$  is a good forward  $\sigma$ -field and  $\mathcal{B}^t$  is a good backward  $\sigma$ -field. If  $\mathcal{A}^t = \mathcal{B}^t$ , we write  $(\mathcal{A}^t, \mathcal{B}^t) = \mathcal{A}^t$  to simplify.*

An other interesting notion is the following definition:

**Definition 3** *We say that  $(\mathcal{A}^t, \mathcal{B}^t)_{t \in (0, T)}$  is a very good collection of  $\sigma$ -fields for  $Z$  if  $(\mathcal{A}^t, \mathcal{B}^t)_{t \in (0, T)}$  is a good collection of  $\sigma$ -fields for  $Z$  and if it satisfies the following property:*

$$\left(\forall t \in (0, T), D_+^{\mathcal{A}^t} Z_t = D_-^{\mathcal{B}^t} Z_t = 0\right) \implies Z \text{ is a.s. a constant process on } [0, T].$$



An obvious example of a very good collection of  $\sigma$ -fields for a process with differentiable paths is  $\{\mathcal{A}^t = \mathcal{F}, t \in (0, T)\}$ . If  $Z$  is a process such that  $s \mapsto E(Z_s)$  is differentiable on  $(0, T)$  then the collection  $\{\mathcal{A}^t = \{\emptyset, \Omega\}, t \in [0, T]\}$  is good but, in general, not very good.

Let us now consider a more advanced example. Let  $B = (B_t)_{t \in [0, T]}$  be a fractional Brownian motion with Hurst index  $H \in (1/2, 1)$ . Let us denote by  $\mathcal{P}_t$  the  $\sigma$ -field generated by  $B_s$  for  $0 \leq s \leq t$  and, if  $g : \mathbb{R} \rightarrow \mathbb{R}$ , by  $\mathcal{T}_t^g$  the  $\sigma$ -field generated by  $g(B_t)$ .

**Example 1** For any  $t \in (0, T)$ ,  $\mathcal{P}_t$  is not a good forward  $\sigma$ -field for  $B$ .

We refer to Proposition 10 in [5] for a proof. This result is extended to the case of Volterra processes in this paper, see Proposition 2.

**Example 2** For any even function  $g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $(\mathcal{T}_t^g)_{t \in (0, T)}$  is a good, but not very good, collection of  $\sigma$ -fields for  $B$ .

*Proof:* Since  $B$  and  $-B$  have the same law, we have that  $E[\Delta_h B_t | g(B_t)] = 0$  for any  $t \in (0, T)$  and  $h \neq 0$  such that  $t + h \in (0, T)$ . The conclusion follows easily.  $\square$

**Example 3**  $(\mathcal{T}_t^{\text{id}})_{t \in (0, T)}$  is a very good collection of  $\sigma$ -fields for  $B$ .

*Proof:* Using a linear Gaussian regression, we can write

$$E[\Delta_h B_t | B_t] = \frac{(1 + h/t)^{2H} - 1 - (h/t)^{2H}}{2} B_t \xrightarrow{h \downarrow 0} H \frac{B_t}{t} \text{ in probability.}$$

Since  $H t^{-1} B_t \neq 0$  a.s. then  $(\mathcal{T}_t^{\text{id}})_{t \in (0, T)}$  is a very good collection of  $\sigma$ -fields for  $B$ .  $\square$

Thus, for the fractional Brownian motion, stochastic derivatives w.r.t. the present (that is w.r.t.  $\mathcal{T}_t^{\text{id}}$ ) turns out to be an adequate tool (see section 5 below, for a more precise study).

### 3.2 Stochastic derivatives in a weak sense

A way to weaken Definition 1 is to consider stochastic derivatives as follows:

**Definition 4** Set  $t \in (0, T)$  and let  $\mathcal{A}^t$  be a  $\sigma$ -field. We say that  $Z$  admits a weak forward stochastic derivative w.r.t.  $\mathcal{A}^t$  if

$$\lim_{h \downarrow 0} E[A_t \Delta_h Z_t] \text{ exists}$$

for any bounded  $\mathcal{A}^t$ -measurable random variable  $A_t$  belonging to a dense subspace of  $L^2(\Omega, \mathcal{A}^t, \mathbb{P})$ . We similarly define the notion of weak backward stochastic derivative w.r.t. a  $\sigma$ -field  $\mathcal{B}^t$ .

Of course, if  $\mathcal{A}^t$  (resp.  $\mathcal{B}^t$ ) is a good forward (resp. backward)  $\sigma$ -field for  $Z$  then  $Z$  admits a weak forward (resp. backward) stochastic derivative w.r.t.  $\mathcal{A}^t$  (resp.  $\mathcal{B}^t$ ). But the converse is not true in general.

Let  $\Upsilon$  be the set of the so-called fractional diffusions  $X = (X_t)_{t \in [0, T]}$  defined by

$$X_t = x_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds, \quad t \in [0, T] \quad (11)$$

where  $\sigma \in \mathbb{D}^{1,2} \cap C^\alpha$  ( $\alpha > 1 - H$ ) and  $b \in L_{\text{loc}}^1$  are processes which are adapted w.r.t. the natural filtration associated to  $X$ .

**Lemma 1** *The decomposition (11) is unique: if*

$$x_0 + \int_0^t \sigma_s dB_s + \int_0^t b_s ds = \tilde{x}_0 + \int_0^t \tilde{\sigma}_s dB_s + \int_0^t \tilde{b}_s ds, \quad t \in [0, T] \quad (12)$$

then  $x_0 = \tilde{x}_0$ ,  $\sigma = \tilde{\sigma}$  and  $b = \tilde{b}$ .

*Proof:* The equality  $x_0 = \tilde{x}_0$  is obvious and (12) is then equivalent to

$$\int_0^t (\sigma_s - \tilde{\sigma}_s) dB_s = \int_0^t (\tilde{b}_s - b_s) ds, \quad t \in [0, T]$$

which implies, by setting  $t_k = \frac{kT}{n}$ :

$$\begin{aligned} & (|\sigma_{t_k} - \tilde{\sigma}_{t_k}| |B_{t_{k+1}} - B_{t_k}|)^{1/H} \\ &= \left| \int_{t_k}^{t_{k+1}} (b_s - \tilde{b}_s) ds + \int_{t_k}^{t_{k+1}} (\sigma_s - \tilde{\sigma}_{t_k}) dB_s + \int_{t_k}^{t_{k+1}} (\tilde{\sigma}_s - \tilde{\sigma}_{t_k}) dB_s \right|^{1/H} \\ &\leq C \left[ \left| \int_{t_k}^{t_{k+1}} (b_s - \tilde{b}_s) ds \right|^{1/H} + \left| \int_{t_k}^{t_{k+1}} (\sigma_s - \tilde{\sigma}_{t_k}) dB_s \right|^{1/H} + \left| \int_{t_k}^{t_{k+1}} (\tilde{\sigma}_s - \tilde{\sigma}_{t_k}) dB_s \right|^{1/H} \right] \end{aligned}$$

We easily deduce, using (7), that

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} |\sigma_{t_k} - \tilde{\sigma}_{t_k}|^{1/H} |B_{t_{k+1}} - B_{t_k}|^{1/H} = 0 \text{ in probability.}$$

But, on the other hand, it is easy to obtain (see, for instance, Theorem 4.4 in [8]) that:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} |\sigma_{t_k} - \tilde{\sigma}_{t_k}|^{1/H} |B_{t_{k+1}} - B_{t_k}|^{1/H} = \int_0^T |\sigma_s - \tilde{\sigma}_s|^{1/H} ds \text{ in probability.}$$

We deduce that  $\sigma = \tilde{\sigma}$  and then  $b = \tilde{b}$ . □

In section 4, we will see that the past of  $X \in \Upsilon$  before time  $t$  is not, in general, a good forward  $\sigma$ -field. At the opposite, we will see in section 5 that the present of  $X \in \Upsilon$  is, in general, a good forward  $\sigma$ -field.

However,  $X$  admits a weak forward stochastic derivative w.r.t. any  $\sigma$ -field  $\mathcal{A}^t$ .

**Proposition 1** *Let  $X \in \Upsilon$  given by (11),  $t \in (0, T)$  and  $(\mathcal{A}^t, \mathcal{B}^t)$  be a couple of  $\sigma$ -fields. Then  $X$  admits a weak forward (resp. backward) stochastic derivative w.r.t.  $\mathcal{A}^t$  (resp.  $\mathcal{B}^t$ ).*

*Proof:* For simplicity, we only prove the forward case, the backward case being similar. Proposition 1 is in fact a direct consequence of the two following lemmas.

**Lemma 2** *Let  $X \in \Upsilon$  given by (11),  $t \in (0, T)$  and  $\mathcal{A}^t$  be a  $\sigma$ -field. We note by  $\mathcal{D}_t$  the set of bounded  $\mathcal{A}^t$ -measurable random variables belonging to  $\mathbb{D}^{1,2}$ . If*

$$\lim_{h \downarrow 0} \mathbb{E} [\sigma_t A_t \Delta_h B_t]$$

*exists for any  $A_t \in \mathcal{D}_t$  then  $X$  admits a weak forward stochastic derivative w.r.t.  $\mathcal{A}^t$ .*

*Proof:* We have, by Fubini and Lebesgue theorems:

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[ \int_t^{t+h} b_s ds \mid \mathcal{A}_t \right] = \mathbb{E}[b_t \mid \mathcal{A}_t] \text{ in probability.} \quad (13)$$

Moreover, using the inequality (7), the following limit holds:

$$\lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} \left[ \int_t^{t+h} (\sigma_s - \sigma_t) dB_s \mid \mathcal{A}_t \right] = 0 \text{ in probability.} \quad (14)$$

Thus, since  $\mathcal{D}_t$  is dense in  $L^2(\Omega, \mathcal{A}_t, \mathbb{P})$  and

$$X_{t+h} - X_t = \sigma_t (B_{t+h} - B_t) + \int_t^{t+h} b_s ds + \int_t^{t+h} (\sigma_s - \sigma_t) dB_s, \quad (15)$$

we can easily conclude. □

**Lemma 3** *Let  $X \in \Upsilon$  given by (11) and  $t \in (0, T)$ . For any bounded  $\mathbb{D}^{1,2}$ -random variable  $A_t$ , the limit*

$$\lim_{h \downarrow 0} \mathbb{E} [\sigma_t A_t \Delta_h B_t]$$

*exists.*

*Proof:* We write

$$\begin{aligned}
\mathbb{E}[\sigma_t(B_{t+h} - B_t)A_t] &= \mathbb{E}[\delta^B(\mathbf{1}_{[t,t+h]})\sigma_t A_t] \\
&= \mathbb{E}[\sigma_t \langle \mathbf{1}_{[t,t+h]}, D^B A_t \rangle_{\mathcal{H}}] + \mathbb{E}[A_t \langle \mathbf{1}_{[t,t+h]}, D^B \sigma_t \rangle_{\mathcal{H}}] \\
&= H(2H - 1) \mathbb{E} \left[ \sigma_t \int_0^T D_s^B A_t \int_t^{t+h} |v - s|^{2H-2} dv ds \right] \\
&\quad + H(2H - 1) \mathbb{E} \left[ A_t \int_0^T D_s^B \sigma_t \int_t^{t+h} |v - s|^{2H-2} dv ds \right].
\end{aligned}$$

We conclude, by Fubini and Lebesgue theorems, that

$$\lim_{h \rightarrow 0} h^{-1} \mathbb{E}[\sigma_t(B_{t+h} - B_t)A_t]$$

exists and equals

$$H(2H - 1) \mathbb{E} \left[ \sigma_t \int_0^T D_s^B A_t |t - s|^{2H-2} ds + A_t \int_0^T D_s^B \sigma_t |t - s|^{2H-2} ds \right].$$

□

We straightforwardly deduce Proposition 1.

□

## 4 Stochastic derivatives of Nelson's type

Let  $Z$  be a stochastic process defined on  $(\Omega, \mathcal{F}, \mathbf{P})$ . We define the past of  $Z$  before time  $t$ :

$$\mathcal{P}_t^Z := \varsigma(Z_s, 0 \leq s \leq t)$$

and the future of  $Z$  after time  $t$ :

$$\mathcal{F}_t^Z := \varsigma(Z_s, t \leq s \leq T).$$

If  $\mathcal{P}_t^Z$  and  $\mathcal{F}_t^Z$  are respectively good forward and backward  $\sigma$ -fields for  $Z$ , we call  $D_+^{\mathcal{P}_t^Z} Z_t$  and  $D_-^{\mathcal{F}_t^Z} Z_t$  respectively the forward and backward stochastic derivatives of Nelson's type in reference of Nelson's work [10]. In the sequel, we denote them by  $D_+^{\mathcal{P}} Z_t$  and  $D_+^{\mathcal{F}} Z_t$  for simplicity.

## 4.1 The case of Wiener diffusions

We denote by  $\Lambda_d$  the space of diffusion processes  $X$  satisfying the following conditions:

1.  $X$  solves the stochastic differential equation :

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0, \quad (16)$$

where  $x_0 \in \mathbb{R}^d$ ,  $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$  are Borel measurable functions satisfying the hypothesis : there exists a constant  $K > 0$  such that for every  $x, y \in \mathbb{R}^d$  we have

$$\begin{aligned} \sup_t (|\sigma(t, x) - \sigma(t, y)| + |b(t, x) - b(t, y)|) &\leq K|x - y|, \\ \sup_t (|\sigma(t, x)| + |b(t, x)|) &\leq K(1 + |x|). \end{aligned}$$

2. For any  $t \in (0, T)$ ,  $X_t$  has a density  $p_t$ .
3. Setting  $a_{ij} = (\sigma\sigma^*)_{ij}$ , for any  $i, j \in \{1, \dots, n\}$ , for any  $t_0 \in (0, T)$ , for any bounded open set  $O \subset \mathbb{R}^d$ ,

$$\int_{t_0}^T \int_O |\partial_j(a_{ij}(t, x)p_t(x))| dxdt < +\infty.$$

4. The functions  $b$  and  $\delta(t, x) \mapsto \frac{1}{p_t(x)}\partial_j(a_{ij}(t, x)p_t(x))$  are bounded, belong to  $C^{1,2}([0, T] \times \mathbb{R}^d)$ , and have all its first and second order derivatives bounded (we use the usual convention that the term involving  $\frac{1}{p_t(x)}$  is 0 if  $p_t(x) = 0$ ).

These conditions are introduced in [9] and ensure the existence of a drift and a diffusion coefficient for the time reversed process  $\bar{X}_t := X_{T-t}$ . Föllmer focuses in [7] Proposition 2.5 on the important relation between drifts and derivatives of Nelson's type. It allows him to compute the drift of the time reversal of a Brownian diffusion with a constant diffusion coefficient both in the Markov and non Markov case (see Theorem 3.10 and 4.7 in [7]).

For a Markov diffusion with a rather general diffusion coefficient, we have the following

**Theorem 1** *Let  $X \in \Lambda_d$  given by (16). Then  $X$  is a Markov diffusion with respect to  $\mathcal{P}^X$  and  $\mathcal{F}^X$ . Moreover,  $(\mathcal{P}^X, \mathcal{F}^X)$  is a good collection and :*

$$\begin{aligned} D_+^{\mathcal{P}} X_t &= b(t, X_t), \\ (D_-^{\mathcal{F}} X_t)_i &= b_i(t, X_t) - \frac{1}{p_t(X_t)} \sum_j \partial_j(a_{ij}(t, X_t)p_t(X_t)) \end{aligned}$$

where the convention that the term involving  $\frac{1}{p_t(x)}$  is 0 if  $p_t(x) = 0$ .

We refer to [4] for a proof: it is based on the proof of Proposition 4.1 in [19] and Theorem 2.3 in [9].

Moreover, Theorem 11.11 in [10] shows that  $(\mathcal{P}^X, \mathcal{F}^X)$  is a very good collection for  $X$  when for all  $t \in (0, T)$ ,  $X_t \in L^2(\Omega)$ .

## 4.2 The case of fBm and of Volterra processes

Let  $K$  be an  $L^2$ -kernel, that is a function  $K : [0, T] \times [0, T] \rightarrow \mathbb{R}$  verifying  $\int_{[0, T]^2} K(t, s)^2 dt ds < +\infty$ . We assume moreover that  $K$  is Volterra: that is it vanishes on  $\{(t, s) \in [0, T]^2 : s > t\}$ , and non degenerate: that is the family  $\{K(t, \cdot), t \in [0, T]\}$  is free and span a vector space dense in  $L^2([0, T])$ . For such a kernel  $K$ , we associate the so-called Volterra process

$$G_t = \int_0^t K(t, s) dW_s, \quad 0 \leq t \leq T \quad (17)$$

where  $W$  denotes a standard Brownian motion. Assumptions made on  $K$  imply in particular that the natural filtrations associated to  $W$  and  $G$  are the same (see for instance [2], Remark 3).

**Proposition 2** *Let  $G$  be a Volterra process associated to a non degenerate Volterra kernel  $K$ . We denote by  $\frac{\partial^+ K}{\partial t}$  the right derivative of  $K$  with respect to  $t$  (with the convention that it equals to  $+\infty$  if it does not exist). Let  $t \in (0, T)$ . The forward Nelson derivative  $D_+^{\mathcal{P}} G_t$  exists if and only if  $\int_0^t \frac{\partial^+ K}{\partial t}(t, s)^2 ds < +\infty$ . In this case, we have  $D_+^{\mathcal{P}} G_t = \int_0^t \frac{\partial^+ K}{\partial t}(t, s) dW_s$ .*

*Proof:* We adapt the proof of [5], Proposition 10. Using the representation (17), we deduce that

$$\begin{aligned} \mathbb{E} [\Delta_h G_t | \mathcal{P}_t^G] &= \mathbb{E} [\Delta_h G_t | \mathcal{P}_t^W] \\ &= \frac{1}{h} \int_0^t [K(t+h, s) - K(t, s)] dW_s =: Z_h. \end{aligned}$$

Remark that  $Z = (Z_h)_{h>0}$  is a centered Gaussian process. First assume that  $\int_0^t \frac{\partial^+ K}{\partial t}(t, s)^2 ds = +\infty$ . It is classical that, if  $Z_h$  converges in probability as  $h \downarrow 0$ , then  $\text{Var}(Z_h)$  converges as  $h \downarrow 0$ . But, from Fatou's lemma, we deduce

$$\liminf_{h \downarrow 0} \text{Var}(Z_h) \geq \int_0^t \frac{\partial^+ K}{\partial t}(t, s)^2 ds = +\infty.$$

Thus,  $Z_h$  does not converge in probability as  $h \downarrow 0$ . Conversely, assume that  $\int_0^t \frac{\partial^+ K}{\partial t}(t, s)^2 ds < +\infty$ . In this case, we can apply the stochastic version of bounded Lebesgue convergence to obtain that  $Z_h \rightarrow \int_0^t \frac{\partial^+ K}{\partial t}(t, s) dW_s$  in probability, as  $h \downarrow 0$ . In other words,  $D_+^{\mathcal{P}} G_t$  exists and equals  $\int_0^t \frac{\partial^+ K}{\partial t}(t, s) dW_s$ .  $\square$

The result of Proposition 10 in [5] is then a particular case. If  $B$  denotes a fractional Brownian motion with Hurst index  $H \in (0, 1/2) \cup (1/2, 1)$  and if  $t \in (0, T)$ , then  $D_+^{\mathcal{P}} B_t$  does not exist. Indeed, we have  $B_t = \int_0^t K_H(t, s) dW_s$  where  $K_H$  is the non-degenerate Volterra kernel given by (4) and verifying

$$\frac{\partial K_H}{\partial t}(t, s) = c_H \left( \frac{t}{s} \right)^{H-1/2} (t-s)^{H-3/2}.$$

**Remark 1** For a stochastic process  $Z$ , let us define

$$\xi(Z) = \text{Leb}\{t \in [0, T], D_+^{\mathcal{P}}Z_t \text{ exists}\}. \quad (18)$$

For instance, if  $B$  is a fractional Brownian motion with Hurst index  $H \in (0, 1)$ , then  $\xi(B) = T$  if  $H = 1/2$  and  $\xi(B) = 0$  otherwise. A real  $c \in [0, T]$  being fixed, it is in fact not difficult, by using Proposition 2, to construct a continuous process  $Z$  such that  $\xi(Z) = c$ . For instance, we can consider the Volterra process associated to the Volterra kernel

$$K(t, s) = \begin{cases} (t-s)^{H(t)} & \text{if } s \leq t \\ 0 & \text{otherwise} \end{cases} \quad \text{with } H(t) = \begin{cases} 0 & \text{if } t \leq c \\ (t-c) \wedge 1/4 & \text{if } t > c \end{cases}.$$

The study of backward derivatives seems to be much difficult. Among these difficulties, we mention the fact that it is not easy to obtain backward representation of fractional diffusion (see [5]). However, for a fBm, we are able to prove the following proposition:

**Proposition 3** Set  $H > 1/2$ . The limit

$$\lim_{h \downarrow 0} \mathbb{E} \left[ \frac{B_t - B_{t-h}}{h} \middle| \mathcal{F}_t^B \right]$$

exists neither as an element in  $L^p(\Omega)$  for any  $p \in [1, \infty)$  nor as an almost sure limit.

*Proof:* We fix  $t \in (0, T)$ . We set

$$G_h := \mathbb{E} \left[ \frac{B_t - B_{t-h}}{h} \middle| \mathcal{F}_t^B \right] \quad \text{and} \quad Z_h := \mathbb{E} \left[ \frac{B_t - B_{t-h}}{h} \middle| B_t, B_{t+h} \right].$$

Since  $(G_h)_{h>0}$  is a family of gaussian random variables, it only suffices to prove that  $\text{Var}(G_h)$  diverges when  $h$  goes to 0.

We have :  $Z_h = \mathbb{E}[G_h | B_t, B_{t+h}]$ . So, by Jensen inequality,  $Z_h^2 \leq \mathbb{E}[G_h^2 | B_t, B_{t+h}]$  and  $\text{Var}(Z_h) \leq \text{Var}(G_h)$ . Let us show that  $\lim_{h \downarrow 0} \text{Var}(Z_h) = +\infty$ .

The covariance matrix of the Gaussian vector  $(B_{t-h} - B_t, B_t, B_{t+h})$  reads

$$\begin{pmatrix} a & v \\ v^* & M \end{pmatrix}$$

where  $a = \text{Var}(B_{t-h} - B_t)$ ,  $v = (R(t-h, t) - R(t, t); R(t-h, t+h) - R(t, t+h))$  and

$$M = \begin{pmatrix} R(t, t) & R(t, t+h) \\ R(t, t+h) & R(t+h, t+h) \end{pmatrix}.$$

Since  $d_h := R(t, t)R(t+h, t+h) - R(t+h, t)^2 \neq 0$ ,  $M$  is invertible. Therefore  $hZ_h = vM^{-1}Q^*$  where  $Q = (B_t, B_{t+h})$ . Since  $M = \mathbb{E}[Q^*Q]$ , we deduce that

$$\text{Var}(hZ_h) = \mathbb{E}[vM^{-1}Q^*(vM^{-1}Q^*)^*] = vM^{-1}v^*.$$

Hence

$$\text{Var}(hZ_h) = \frac{1}{d_h} (R(t+h, t+h)v_1^2 - 2R(t+h, t)v_1v_2 + R(t, t)v_2^2).$$

This expression is homogeneous in  $t^{2H}$ , so we henceforth work with  $t = 1$ . Tedious computations give  $d_h \sim h^{2H}$  as  $h \downarrow 0$ . Moreover we note that  $v_2 = v_1 + c h^{2H}$  where  $c$  is a constant depending only on  $H$ . Thus

$$d_h \text{Var}(hZ_h) = v_1 c h^{2H} (1 - (1+h)^{2H} + h^{2H}) + h^{2H} v_1^2 + c^2 h^{4H}.$$

Since  $2H > 1$  and the function  $x \mapsto x^{2H}$  is derivable, the quantities  $\frac{v_1}{h}$  and  $\frac{1-(1+h)^{2H}+h^{2H}}{h}$  converge as  $h \downarrow 0$ . But  $2H < 2$  and  $\frac{h^{4H}}{h^2 h^{2H}} = h^{2H-2} \rightarrow +\infty$  as  $h \downarrow 0$ . Thus

$$\lim_{h \downarrow 0} \text{Var}(Z_h) = +\infty$$

which concludes the proof. □

### 4.3 The case of fractional diffusions

**Proposition 4** *Let  $X \in \Upsilon$  given by (11) and  $t \in (0, T)$ . If  $u_t = 0$  a.s. then  $\mathcal{P}_t^X$  is a good forward  $\sigma$ -field for  $X$ . If  $u_t \neq 0$  a.s. then  $\mathcal{P}_t^X$  is not a good forward  $\sigma$ -field for  $X$ .*

**Remark 2** *When  $0 < P(u_t \neq 0) < 1$ , it seems to us that we can not conclude in all generality.*

*Proof:* First assume that  $u_t = 0$  a.s. Then, using (13), (14) and (15), we have that  $\mathcal{P}_t^X$  is a good forward  $\sigma$ -field for  $X$ .

Assume now that  $u_t \neq 0$  a.s. We can write

$$B_t = \int_0^t \frac{1}{u_s} dX_s - \int_0^t \frac{b_s}{u_s} ds.$$

In particular,  $\mathcal{P}_t^B \subset \mathcal{P}_t^X$ . Assume, for a moment, that

$$\text{E}[\Delta_h B_t | \mathcal{P}_t^X]$$

converges in probability as  $h \downarrow 0$ . At this level, we need the following obvious lemma:

**Lemma 4** *Let  $(X_n)$  be a sequence of random variables such that  $X_n \rightarrow X$  as  $n \rightarrow \infty$ , in probability. For any  $\sigma$ -field  $\mathcal{A}$ , we have that  $\text{E}[X_n | \mathcal{A}] \rightarrow \text{E}[X | \mathcal{A}]$  as  $n \rightarrow \infty$ , in probability.*



*Proof:* It is well-known that  $Y_n \rightarrow Y$  in probability if and only if  $E(|Y_n - Y| \wedge 1) \rightarrow 0$ . Let  $\varepsilon > 0$  and set  $\Omega_{\varepsilon,n} = \{|X_n - X| > \varepsilon/2\}$ . We have

$$E(|E[X_n | \mathcal{A}] - E[X | \mathcal{A}]| \wedge 1) \leq P(\Omega_{\varepsilon,n}) + \varepsilon/2.$$

Since  $X_n \rightarrow X$  in probability, there exists  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow P(\Omega_{\varepsilon,n}) \leq \varepsilon/2$ . The conclusion follows easily.  $\square$

Thanks to Lemma 4, we deduce that

$$E[\Delta_h B_t | \mathcal{P}_t^B] = E[E[\Delta_h B_t | \mathcal{P}_t^X] | \mathcal{P}_t^B]$$

converges in probability as  $h \downarrow 0$ . It is a contradiction with Proposition 10 in [5] or Proposition 2 of this paper. Thus, using again (13), (14) and (15), we deduce this time that  $\mathcal{P}_t^X$  is not a good forward  $\sigma$ -field for  $X$ .  $\square$

#### 4.4 The case of fractional differential equations with analytic volatility

**Proposition 5** *Let  $X \in \Xi$  given by (10) and  $t \in (0, T)$ . We assume moreover that  $\sigma$  is a real analytic function. Then  $\mathcal{P}_t^X$  is a good forward  $\sigma$ -field for  $X$  if and only if  $\sigma \equiv 0$ .*

*Proof:* If  $\sigma \equiv 0$  then  $X$  is deterministic, and differentiable in  $t$ . Consequently,  $\mathcal{P}_t^X$  is a good forward  $\sigma$ -field. Assume now that  $\sigma \not\equiv 0$ . According to the Bouleau-Hirsch optimal criterium for fractional differential equations (see [11], Theorem B), we have that the law of  $X_t$  is absolutely continuous w.r.t. the Lebesgue measure (we have indeed  $\text{int } \sigma^{-1}(\{0\}) = \emptyset$ ). We deduce that  $P(\sigma(X_t) = 0) = 0$  since  $\text{Leb}(\sigma^{-1}(\{0\})) = 0$  ( $\sigma$  has only isolated zeros). Proposition 4 allows to conclude that  $\mathcal{P}_t^X$  is not a good forward  $\sigma$ -field.  $\square$

**Remark 3** *The case where  $\sigma$  is not assumed analytical seems more difficult to reach. We conjecture however that, in this case,  $\mathcal{P}_t^X$  is a good forward  $\sigma$ -field for  $X$  if and only if  $t < t_x$  where  $t_x$  is the deterministic time defined by*

$$t_x = \inf\{t \geq 0 : x_t \notin \text{int } \sigma^{-1}(\{0\})\}$$

*with  $(x_t)_{t \in [0, T]}$  the solution to  $x_t = x_0 + \int_0^t b(x_s) ds$ . If this conjecture is true, we have that  $\xi(X) = t_x$ , see (18).*

## 5 Stochastic derivatives with respect to the present

### 5.1 Definition

A consequence of Proposition 2 is that the  $\sigma$ -field  $\mathcal{P}_t^X$  generated by  $X_s$ ,  $0 \leq s \leq t$  (the past of  $X$ ) is not an adequate tool when we work with the fractional Brownian motion. Moreover, we can stress on the following important fact: the Markov property of a Wiener diffusion  $X \in \Lambda_d$  implies that to take expectations w.r.t.  $\mathcal{P}_t^X$  produces the same effect as to take expectations only w.r.t.  $X_t$ . The following definition is then natural.

**Definition 5** Let  $Z = (Z_t)_{t \in [0, T]}$  be a stochastic process defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and, for any  $t \in (0, T)$ ,  $\mathcal{T}_t$  be the  $\sigma$ -field generated by  $Z_t$ . We say that  $Z$  admits a forward (resp. backward) stochastic derivative w.r.t. the present  $t \in (0, T)$  if  $\mathcal{T}_t$  is a good forward (resp. backward)  $\sigma$ -field for  $Z$ . In this case, we set  $\mathbf{D}^+ Z_t := \mathbf{D}_+^{\mathcal{T}_t} Z_t$  (resp.  $\mathbf{D}^- Z_t := \mathbf{D}_-^{\mathcal{T}_t} Z_t$ ).

**Example 4** Let  $B$  be a fractional Brownian motion with Hurst index  $H \in (0, 1)$  and  $t \in (0, T)$ . Then

$$\mathbf{D}^+ B_t = \begin{cases} H t^{-1} B_t & \text{if } H > 1/2 \\ 0 & \text{if } H = 1/2 \\ \text{does not exist} & \text{if } H < 1/2 \end{cases} \quad \text{and}$$

$$\mathbf{D}^- B_t = \begin{cases} H t^{-1} B_t & \text{if } H > 1/2 \\ t^{-1} B_t & \text{if } H = 1/2 \\ \text{does not exist} & \text{if } H < 1/2 \end{cases}$$

(see also Example 3). In particular, we would say that the fractional Brownian motion with Hurst index  $H > 1/2$  is more regular than the Brownian motion ( $H = 1/2$ ), because of the equality between the forward and backward derivatives in the case  $H > 1/2$  contrary to the case  $H = 1/2$ . We can identify the cause of these different regularities: the covariance function  $R_H$  is differentiable along the diagonal  $(t, t)$  in the case  $H > 1/2$  while it is not when  $H = 1/2$ .

### 5.2 Case of fractional differential equations

We denote by  $\Xi$  the set of fractional differential equations, that is the subset of  $\Upsilon$  whose elements are processes  $X = (X_t)_{t \in [0, T]}$  solution of (10) with  $\sigma \in \mathcal{C}_b^2$  and  $b \in \mathcal{C}_b^1$ .

In the sequel, we compute  $\mathbf{D}^\pm X_t$  for  $X \in \Xi$  and  $t \in (0, T)$ . Let us begin by a simple case.

**Proposition 6** Let  $X \in \Xi$  given by (10),  $t \in (0, T)$  and  $\mathcal{T}_t$  be the  $\sigma$ -field generated by  $X_t$ . Assume moreover that  $\sigma$  and  $b$  are proportional. Then  $X$  admits a forward and a backward stochastic derivative w.r.t. the present  $t$ , given by

$$\mathbf{D}^+ X_t = \mathbf{D}^- X_t = H t^{-1} \sigma(X_t) B_t + b(X_t). \quad (19)$$

Moreover, the collection of  $\sigma$ -fields  $(\mathcal{T}_t)_{t \in (0, T)}$  is very good for  $X$ .

*Proof:* We make only the proof for  $\mathbf{D}^+X_t$ , the computation for  $\mathbf{D}^-X_t$  being similar. Assume that  $b(x) = r\sigma(x)$  with  $r \in \mathbb{R}$ . Then  $X_t = f(B_t + rt)$  with  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(0) = x_0$  and  $f' = \sigma(f)$ . If  $\sigma(x_0) = 0$  then  $X_t \equiv x_0$  and  $\mathbf{D}_t^+X = \sigma(X_t)Ht^{-1}B_t + b(X_t)$ . If  $\sigma(x_0) \neq 0$  then it is classical that  $f$  is strictly monotonous. We can then write  $B_t = f^{-1}(X_t) - rt$ . In particular, the random variables which are measurable with respect to  $X_t$ , are measurable with respect to  $B_t$ , and vice-versa. On the other hand, by using a linear Gaussian regression, it is easy to show that  $\mathbf{D}^+B_t = Ht^{-1}B_t$  (see also Example 3). Finally, convergences (13) and (14) and equality (15) allow to conclude that we have (19). Moreover, if  $Ht^{-1}\sigma(X_t)B_t + b(X_t) = \sigma(X_t)(Ht^{-1}B_t + r) = 0$  a.s. then  $\sigma(X_t) = 0$  a.s. and  $X_t \equiv x_0$  a.s., see (10). In other words, the collection of  $\sigma$ -fields  $(\mathcal{T}_t)_{t \in (0, T)}$  is very good.  $\square$

Let us now describe a more general case.

**Theorem 2** *Let  $X \in \Xi$  given by (10) and  $t \in (0, T)$ . Assume moreover that  $\sigma$  does not vanish. Then  $X$  admits a forward  $\mathbf{D}^+X_t$  and a backward  $\mathbf{D}^-X_t$  stochastic derivative w.r.t. the present  $t$ , both given by*

$$b(X_t) + H \frac{\sigma(X_t)}{t} \left\{ \int_0^{X_t} \frac{dy}{\sigma(y)} - \mathbb{E} \left[ \int_0^t \frac{b}{\sigma}(X_s) ds + \int_0^t \int_0^t \beta_r^H(s) \delta B_r ds - t \int_0^t \beta_r^H(t) \delta B_r \middle| X_t \right] \right\} \quad (20)$$

where

$$\beta_r^H(t) = \left( \mathcal{O}_H \int_0^r \frac{b'\sigma - b\sigma'}{\sigma}(X_s) \mathbf{1}_{s \geq \cdot} ds \right) (t).$$

Recall that  $\mathcal{O}_H$  is defined by (5).

*Proof:* We only make the proof for  $\mathbf{D}^+X_t$ , the computation for  $\mathbf{D}^-X_t$  being similar.

**First step.** Assume that  $\sigma \equiv 1$ . Using the transfer principle and the isometry  $\mathcal{K}_H$ , it holds that

$$X_t = \int_0^t K_H(t, s) dY_s$$

where

$$Y_t = W_t + \int_0^t a_r dr.$$

Here, we set

$$a_r = \left( \mathcal{K}_H^{-1} \int_0^\cdot b(X_s) ds \right) (r).$$

We know (see [14], Theorem 2) that the process  $X = (X_t)_{t \in [0, T]}$  is a fBm under the new probability measure  $\mathbf{Q} = G \cdot \mathbf{P}$  where

$$\delta G = \exp \left( - \int_0^T a_s dW_s - \frac{1}{2} \int_0^T a_s^2 ds \right).$$

Using the integration by part of Malliavin calculus, we can write, for  $g : \mathbb{R} \rightarrow \mathbb{R} \in C_b^1$ :

$$\begin{aligned} \mathbb{E}[(X_{t+h} - X_t)g(X_t)] &= \mathbb{E}_{\mathbf{Q}} [G^{-1}g(X_t)\delta^X(\mathbf{1}_{[t,t+h]})] \\ &= \mathbb{E}_{\mathbf{Q}} [G^{-1}\langle \mathbf{1}_{[t,t+h]}, D^X g(X_t) \rangle_{\mathcal{H}}] + \mathbb{E}_{\mathbf{Q}} [g(X_t)\langle \mathbf{1}_{[t,t+h]}, D^X G^{-1} \rangle_{\mathcal{H}}] \\ &= \mathbb{E}[g'(X_t)]\langle \mathbf{1}_{[t,t+h]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} + \mathbb{E} [Gg(X_t)\langle \mathcal{K}_H^* \mathbf{1}_{[t,t+h]}, \mathcal{K}_H^* D^X G^{-1} \rangle_{L^2}]. \end{aligned} \quad (21)$$

But  $\mathcal{K}_H^* D^X G^{-1} = D^Y G^{-1}$  (transfer principle). Since

$$\delta G^{-1} = \exp \left( \int_0^T a_s dY_s - \frac{1}{2} \int_0^T a_s^2 ds \right),$$

we have

$$\begin{aligned} G \times D_t^Y (G^{-1}) &= a_t + \int_0^T D_t^Y a_s dY_s - \int_0^T a_s D_t^Y a_s ds \\ &= a_t + \int_0^T D_t^Y a_s dW_s. \end{aligned}$$

Moreover

$$\int_0^T D_s^Y a_r dW_r = \int_0^T (\mathcal{K}_H^* D_s^X a)(r) dW_r = \int_0^T D_s^X a_r \delta B_r := \Phi(s),$$

and

$$(\mathcal{K}_H^* \mathbf{1}_{[0,t]})(s) = K_H(t, s) \mathbf{1}_{[0,t]}(s).$$

Therefore

$$\begin{aligned} \langle \mathcal{K}_H^* \mathbf{1}_{[t,t+h]}, G \mathcal{K}_H^* D^X G^{-1} \rangle_{L^2} &= (\mathcal{K}_H a)(t+h) - (\mathcal{K}_H a)(t) + (\mathcal{K}_H \Phi)(t+h) - (\mathcal{K}_H \Phi)(t) \\ &= \int_t^{t+h} b(X_u) du + (\mathcal{K}_H \Phi)(t+h) - (\mathcal{K}_H \Phi)(t). \end{aligned}$$

By the stochastic Fubini theorem, we have  $(\mathcal{O}_H \Phi)(t) = \int_0^T (\mathcal{O}_H D^X a_r)(t) \delta B_r$ . We set

$$\beta_r^H(t) = (\mathcal{O}_H D^X a_r)(t) = \left( \mathcal{O}_H \int_0^T b'(X_s) \mathbf{1}_{s \geq \cdot} ds \right)(t).$$

We then deduce

$$\begin{aligned} \mathbb{E}[(X_{t+h} - X_t)g(X_t)] &= \mathbb{E}[g'(X_t)]\langle \mathbf{1}_{[t,t+h]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} \\ &+ \mathbb{E} \left[ g(X_t) \left( \int_t^{t+h} b(X_s) ds + \int_t^{t+h} \int_0^T \beta_r^H(s) \delta B_r ds \right) \right]. \end{aligned} \quad (22)$$

By developing  $\mathbb{E}[X_t g(X_t)]$  as in (21), we obtain

$$t^{2H} \mathbb{E}[g'(X_t)] = \mathbb{E} \left[ g(X_t) \left( X_t - \int_0^t b(X_s) ds - \int_0^t \int_0^T \beta_r^H(s) \delta B_r ds \right) \right].$$

Then:

$$\begin{aligned} \mathbb{E}[\Delta_h X_t | X_t] &= h^{-1} \langle \mathbf{1}_{[t, t+h]}, \mathbf{1}_{[0, t]} \rangle_{\mathcal{H}} \left( X_t - \mathbb{E} \left[ \int_0^t b(X_s) ds + \int_0^t \int_0^T \beta_r^H(s) \delta B_r ds \middle| X_t \right] \right) \\ &\quad + h^{-1} \mathbb{E} \left[ \int_t^{t+h} b(X_s) ds + \int_t^{t+h} \int_0^T \beta_r^H(s) \delta B_r ds \middle| X_t \right]. \end{aligned}$$

We deduce that  $\mathbb{E}[\Delta_h X_t | X_t]$  converges in probability, as  $h \downarrow 0$ , to

$$b(X_t) + \frac{H}{t} X_t - \frac{H}{t} \mathbb{E} \left[ \int_0^t b(X_s) ds + \int_0^t \int_0^T \beta_r^H(s) \delta B_r ds - \int_0^T \beta_r^H(t) \delta B_r \middle| X_t \right].$$

Since  $\lim_{h \downarrow 0} \mathbb{E}[\Delta_h X_t | X_t]$  does not depend on  $T$ , we finally obtain (20) in the particular case where  $\sigma \equiv 1$ , by letting  $T \downarrow t$ .

**Second step.** Assume that  $\sigma$  does not vanish. Set  $Y_t = h(X_t)$  where  $h(x) = \int_0^x \frac{dy}{\sigma(y)}$ . Using the change of variable formula, we obtain that  $Y$  verifies

$$Y_t = y_0 + B_t + \int_0^t \frac{b}{\sigma} \circ h^{-1}(Y_s) ds, \quad t \in [0, T].$$

Since, on the one hand, the  $\sigma$ -fields generated by  $X_t$  and  $Y_t$  are the same and, on the other hand,  $X$  has  $\alpha$ -Hölder continuous paths with  $\alpha > 1/2$ , we have

$$\mathbf{D}_t^+ X = \sigma(X_t) \mathbf{D}_t^+ Y.$$

The expression (20) is then a consequence of the first step of the proof.  $\square$

**Remark 4** When  $\sigma$  does not vanish and  $b \equiv r \sigma$  with  $r \in \mathbb{R}$ , we can apply either Proposition 6 or Theorem 2 to compute  $\mathbf{D}^\pm X_t$ . Of course, the conclusions are the same. Indeed, since we have, in this case,  $b' \sigma - b \sigma' \equiv 0$  and  $\int_0^{X_t} \frac{dy}{\sigma(y)} = B_t + r t$  (since  $X_t = f(B_t + r t)$  with  $f$  verifying  $f' = \sigma \circ f$ ), formula (20) can be simplified in (19).

Compared to the case where  $\sigma$  and  $b$  are proportional, it is here more difficult to decide if the collection of  $\sigma$ -fields  $(\mathcal{T}_t)_{t \in (0, T)}$  (with  $\mathcal{T}_t$  the  $\sigma$ -field generated by  $X_t$ ) is very good or not.

In the framework of the stochastic embedding of dynamical systems introduced in [3], the set of processes, called set of Nelson differentiable processes, which satisfy the equality between a stochastic forward and stochastic backward derivatives plays a fundamental role (see [4], Chapters 3 and 7). We stress on the fact that solutions of stochastic differentiable equations driven by a fBm with  $H > 1/2$  are the first example of non absolutely continuous Nelson differentiable process.

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## References

- [1] E. Alòs and D. Nualart (2003). *Stochastic Integration with respect to the fractional Brownian motion*. Stochastics and Stochastics Reports **75**, 129-152.
- [2] F. Baudoin and D. Nualart (2003). *Equivalence of Volterra processes*. Stochastic Proc. Appl. **107**, 327-350.
- [3] J. Cresson and S. Darses (2006): *Plongement stochastique des systèmes lagrangiens*. C.R. Acad. Sci. Paris Ser. I **342** (5), 333-336.
- [4] J. Cresson and S. Darses (2005): *Stochastic embedding of dynamical systems*. Available online at <http://arxiv.org/pdf/math.PR/0509713>.
- [5] S. Darses and B. Saussereau (2006). *Time reversal for drifted fractional Brownian motion with Hurst index  $H > 1/2$* . Preprint Besancon.
- [6] H. Doss (1977). *Liens entre équations différentielles stochastiques et ordinaires*. Ann. Inst. H. Poincaré Probab. Statist **13**, 99-125.
- [7] H. Föllmer (1984). *Time reversal on Wiener space*. Stochastic processes - mathematics and physics (Bielefeld). Lecture Notes in Math. **1158**, 119-129.
- [8] J. Guerra and D. Nualart (2005). *The  $1/H$ -variation of the divergence integral with respect to the fractional Brownian motion for  $H > 1/2$  and fractional Bessel processes*. Stochastic Proc. Appl. **115**, 91-115.
- [9] A. Millet, D. Nualart and M. Sanz (1989): *Integration by parts and time reversal for diffusion processes*. Annals of Probab. **17**, No. 1, 208-238.
- [10] E. Nelson (2001). *Dynamical theory of Brownian motion*. Princeton University Press. Second edition. Available online at <http://www.math.princeton.edu/~nelson/books/bmotion.pdf>.
- [11] I. Nourdin and T. Simon (2006). *On the absolute continuity of one-dimensional SDEs driven by a fractional Brownian motion*. Statistics and Probability Letters **76** (9), 907-912.
- [12] D. Nualart (1996). *The Malliavin Calculus and Related Topics*. Springer Verlag.
- [13] D. Nualart (2003). *Stochastic calculus with respect to the fractional Brownian motion and applications*. Contemporary Mathematics **336**, 3-39.
- [14] D. Nualart and Y. Ouknine (2002). *Regularization of differential equations by fractional noise*. Stochastic Proc. Appl. **102**, 103-116.
- [15] V. Pipiras and M. Taqqu (2001). *Are classes of deterministic integrands for fractional Brownian motion on an interval complete?* Bernoulli **7** (6), 873-897.

- [16] F. Russo and P. Vallois (2005). *Elements of stochastic calculus via regularisation*. To appear in Séminaires de Probabilités. Available online at <http://arxiv.org/pdf/math.PR/0603224>.
- [17] S.G. Samko, A.A. Kilbas and O.I. Marichev (1993). *Fractional integrals and derivatives, Theory and applications*. Gordon and Breach Science Publishers, Yvendon.
- [18] H. J. Sussmann (1977). *An interpretation of stochastic differential equations as ordinary differential equations which depend on a sample point*. Bull. Amer. Math. Soc. **83**, 296-298.
- [19] Thieullen M. (1993). *Second order stochastic differential equations and non-Gaussian reciprocal diffusions*. Probab. Theory and Rel. Fields **97**, 231-257.
- [20] Young, L.C. (1936): *An inequality of the Hölder type connected with Stieltjes integration*. Acta Math. **67**, 251-282.
- [21] Zähle, M. (1998): *Integration with respect to fractal functions and stochastic calculus I*. Probab. Theory Relat. Fields **111**, 333-374.