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Hybrid Equity-Credit Modelling

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Abstract

We propose a study of the pitfalls of the market widely used Poisson Default model in the Equity-Credit Hybrid land and show that a slight modification of the Constant Elasticity of Variance (CEV) model can, in addition to its well-known properties, capture the default event probability. Because of a need for more freedom between the volatility level, the skewness and the risk of default, we exhibit extensions of the CEV model adding stochasticity in the volatility.

Introduction

The growth of the credit derivatives market and the development of derivatives such as equity default swaps (EDSs) has led to a need for models that realistically capture stock price behaviour. The probability of default has become a crucial issue for pricing new claims. We therefore need to define "default".

The notion of default has been discussed in market financial literature for a long time and in corporate finance literature for much longer. Defaults happen when a party is unwilling or unable to pay its debt obligations. Default is usually the step before bankruptcy in corporate finance. In the US, a firm getting in trouble usually files for bankruptcy protection under Chapter 11 (Reorganization), which defines a default event. Chapter 11 allows a firm to cancel some

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or all of its debts and contracts while attempting to achieve financial stability without interruption of the operating business.

In the common structural model literature pioneered by Black & Scholes (1973) and Merton (1974), one defines default as being the event for which the asset value of a firm goes below a boundary that is a function of the firm debts. But the impossibility of knowing the barrier level leads us to consider alternatives to structural models. Reduced-form models do not model the value of the firm's assets and its capital structure, they consider the credit event to be an exogenously specified jump process. Two reduced-form model subclasses are the credit migration model family and the intensity-based model family. In the case of intensity-based models, one is interested in modelling the default event process; the traditional literature on these kinds of models (see, for instance, Jarrow and Turnbull (1995)) does not describe the behaviour of stocks just before default.

Since our aim is to build a unique model for stock prices and default events, modelling the probability of default as the consequence of the stock price falling under a certain boundary seems natural. Now, for simplicity, recent models (see, for example, Albanese and Chen (2005) and Linetsky (2005)) consider the default event as the stock price falling to zero and this is the framework we will use. We may nevertheless notice that for a firm, the fact of being under Chapter 11 doesn't imply that the stock price is equal to zero, but being under Chapter 7 (Liquidation) will imply a null stock price. As lognormal models are unable to comply with this latest feature, financial practitioners and academics have added to the diffusion model a Poisson default process. Such models were first presented in Davis and Lischka (2002), where the default probability depends on the level of the spot price. In this article, we wish to build a stock price diffusion with continuous paths, since for most companies going bankrupt the stock price behaviour doesn't default as a Poisson process does. This stock price property can be illustrated on the US stock market (see for instance WorldCom, Enron, Mirant or Kmart) and that is the reason why building continuous processes with a non-zero probability of reaching zero is a very interesting feature.

The constant elasticity of variance (CEV) model designed by Cox (1975) is a continuous path model that has the following diffusion $\frac{dS}{S} = rdt + \sigma S^{\alpha-1}dW$ and a non-zero probability of reaching zero under certain conditions on the elasticity parameter α .

First we will explain why the CEV model describes the equity market better than the Poisson default model in terms of realism of the stock price paths and pricing downside risks. By this we mean that the path continuity of the CEV model brings consistency with low-strike put options and equity default swap (EDS) market prices, for example. We will then more precisely present the stopped CEV process and price vanilla options, credit default swaps (CDSs) and equity default swaps within this model. The major drawbacks of the CEV model are the lack of independence between the skewness and the probability of default, and the high dependency between the level of volatility and the probability of default. To deal with these drawbacks, we will present some generalizations of the CEV model using stochastic volatility. Our contribution is

threefold: explaining default Poisson model mis-pricing features and illustrating the necessity of smoother stock price processes (with continuous paths) to model the default event; showing that the stopped CEV model can approximately fit vanilla options and CDSs and price EDSs more safely; introducing and presenting an extension of the stopped CEV model using Heston stochastic volatility (constant elasticity of stochastic variance (CESV)). We additionally provide closed-form pricing formulas for the Heston CESV model presented in this article.

1 Tracking a Stock Price Process that models default

To price exotic derivatives, it is first necessary to be able to reproduce existing, observable vanilla option prices with sufficient precision and a small number of adjustable parameters. The main drawback of this view is often the irrelevance of the underlying asset price behaviour and, as a consequence, a lack of accuracy for the hedging portfolio. The local probability of default model doesn't represent a typical path of a default event since the stock price process can jump to zero at any time with a probability that is a function of the underlying stock price. Our purpose is to create a model consistent with the sustainable stock price evolutions. An important feature of a stochastic model is its ability to integrate extremal events realistically. When building a model concerned with default, the choice of the diffusion may be of importance for pricing non plain-vanilla derivatives such as EDSs that are swaps where payouts occur when the stock price falls under a pre-defined level. We will now recall the Poisson default model and present a slight modification of the CEV model as an alternative to the unrealistic stock behaviour of the Poisson default process.

1.1 Poisson Default Models

The most commonly used equity-credit market models are those based on jump-diffusion processes with a jump to zero if the stock defaults. This type of model usually solves the following equation under the risk-neutral measure:

$$\frac{dS_t}{S_{t-}} = rdt + \sigma dW_t - dQ_t$$

where:

$$\tau = \inf\{t > 0; \int_0^t p(u, S_u) du \geq \Theta\}$$

$$Q_t = \mathbf{1}_{t \geq \tau} - \int_0^{t \wedge \tau} p(u, S_u) du$$

where Θ is an exponential random variable, p is a deterministic function of the time and the spot level. This model was presented for instance in Davis

and Lischka (2002) and is commonly used for the pricing of defaultable claims, especially of convertible bonds. In Andersen and Buffum (2003) and in Ayache et al. (2003) for instance, the probability function is of the following form:

$$p(S) = p_0 \left(\frac{S}{S_0} \right)^\alpha$$

where p_0 is the estimated hazard rate for the stock price level $S = S_0$. Linetsky (2005) provides closed-form formulae for vanilla option prices and corporate bonds with the specification on the local probability function presented above.

Such processes generate paths where the stock price drops down directly to zero from its level just before default. As shown in figure 1, it is not a natural hypothesis for a default modelling framework and that is why we consider alternative smoother processes.

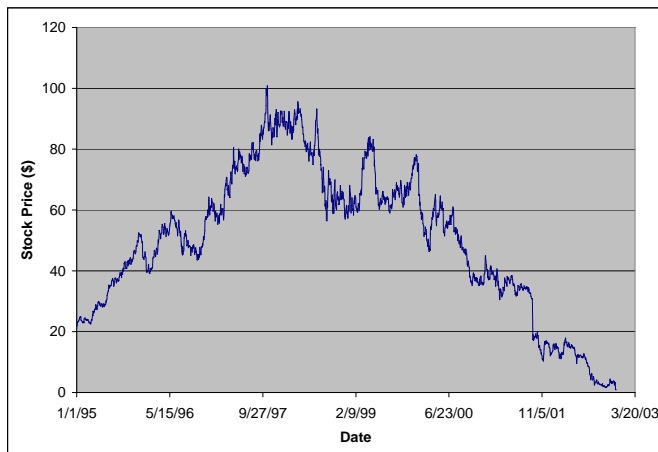


Figure 1: United Airlines Historical Stock Price prior to Default

1.2 CEV Diffusion

A positive continuous process that has a strictly positive probability of reaching zero can be found in the family of squared Bessel processes with dimensions lower than two. Among the different stock price models, the CEV model is a well-known stock price diffusion based on Bessel processes. In this article, we will consider a CEV process stopped at the first hitting time of zero in order to build a credit-coherent model under a risk-neutral pricing measure:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma S_t^{\alpha-1} dW_t \quad \text{if } t < \tau. \\ S_t &= 0 \quad \text{if } t \geq \tau. \end{aligned}$$

where $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$, σ a constant and $\alpha < 1$ in order to get a non-zero probability of default.

For a CEV process, zero is an absorbing boundary for $\frac{1}{2} < \alpha < 1$ and is a reflecting boundary for $\alpha < \frac{1}{2}$ and this is why we consider a stopped CEV process. There is much literature on CEV models. Since CEV processes are based on squared Bessel processes, they have the advantage of giving analytical formulas for many derivatives. They were introduced by Cox (1975), who only considered the case $\alpha < 1$, which takes into account the so-called leverage effect to price vanilla options. Then Emanuel and MacBeth (1982) proposed pricing formulas for $\alpha > 1$ and Schroder (1989) showed that the CEV pricing formula could be expressed in terms of noncentral chi-square distributions. More recently, Delbaen and Shirakawa (2002) proposed a call pricing formula for the stopped CEV process. We can calculate the law of this stopped process in terms of squared Bessel processes and hence in terms of non-central chi-square distributions. For a detailed study of the stopped process, we refer to Delbaen and Shirakawa (2002). But, for the purpose of self-consistency, some essential results are reproduced in the Appendix.

1.3 Poisson Default Process Problem

We aim at continuous diffusions that can reach zero. A possible inconsistency of Poisson default models comes from the pricing of EDSs. If we wish to price an EDS with a low implied volatility and a high credit grade, there won't be a significant price difference between a quarterly 20% two-year EDS and a quarterly 30% two-year EDS. For example, let us consider the US company Tyco, with a 23% one-year at-the-money implied volatility and a 250-basis point one-year credit grade with a \$36.50 spot price. For simplicity, we will consider down-and-in barrier put options whose payout is of the following form: $\mathbb{E}[e^{-r\tau} \mathbf{1}_{\tau < T}]$ with $\tau = \inf_{t < T} \{S_t < B\}$ and calculate their prices under the CEV model and under the Poisson default model. All the prices can be found in table A. The two models are fitted on the one-year at-the-money volatility and probability of default. The prices under the Poisson default model show a bad strike scaling feature. We see that reaching low barriers is equivalent to reaching zero in our jumpdiffusion framework for the pricing of down-and-in barrier put options, and that is why they all have the same price. In the CEV model, they all have different prices and they are more expensive than in the jump-diffusion model. Let us remark that to get the one-year Poisson default model price presented in table A, one would need to take a 5% barrier to get the same price under the CEV model. Nonetheless, one could argue that the default event could be chosen not to be zero but a certain small value $\hat{\epsilon}$ as presented in its generality in Ayache, Forsyth and Vetzal (2003), but that would only shift up the options prices and they would remain insensitive to strike scaling. Another explanation of this important price difference can be excerpted from a qualitative study of the hedging strategy, and this will illustrate another problem with Poisson default models. Indeed, when selling a down-and-in digital put barrier option under a Poisson default model, in the case of a jump to zero the profit will come

from the number of short stocks. But the stock price usually declines smoothly before jumping down in case of default, so the Poisson default model won't perform efficiently, whereas managing these options under the CEV model is better because the intrinsic structure of this model sees the default event as it may happen. This means that for the pricing involved, the delta for the CEV process is higher than for the default Poisson process. To summarise, these price differences come from different hedging strategies, which themselves come from different stock price behaviour modelling.

TYCO -December 2004	
DOWN-AND-IN DIGITAL BARRIER OPTION PRICES IN DOLLARS	
Poisson Default Model, $\sigma = 20.2\%$, $\alpha = 2$, $r = 2\%$ and $p_0 = 3.7$	
Strike/Maturity	1 Year
30%	0.0246
40%	0.0246
50%	0.0249
CEV Model, $\sigma = 23\%$, $\alpha = -1.6$ and $r = 2\%$	
Strike/Maturity	1 Year
30%	0.045
40%	0.062
50%	0.083

2 Consistent Pricing of Credit and Equity Derivatives within CEV

2.1 Calibration and Pricing of Vanilla Options

Since our purpose is to build a cross-asset market model for strategies that involve equity and credit assets, we calculate the European-style vanilla option prices. To ensure the absence of arbitrage, the discounted stopped CEV process has to be a true martingale. This is the case for $\alpha < 1$, as proven in Atlan and Leblanc (2004).

Let us now calculate the European-style put P_0 option price at maturity T and strike K for the stopped CEV process:

$$P_0 = e^{-rT} \mathbb{E}[(K - S_T)_+ \mathbf{1}_{T < \tau}] + K e^{-rT} \mathbb{P}(\tau \leq T)$$

We can see explicitly that the put option price incorporates the price of default and that the martingale property ensures the put-call parity relation. We can now give the option pricing formula, knowing the density of the stopped CEV process thanks to equation (11):

$$P_0 = K e^{-rT} Q\left(2\xi_T, \frac{1}{1-\alpha}, z_T\right) - S_0 \left(1 - Q\left(z_T, 2 + \frac{1}{1-\alpha}, 2\xi_T\right)\right)$$

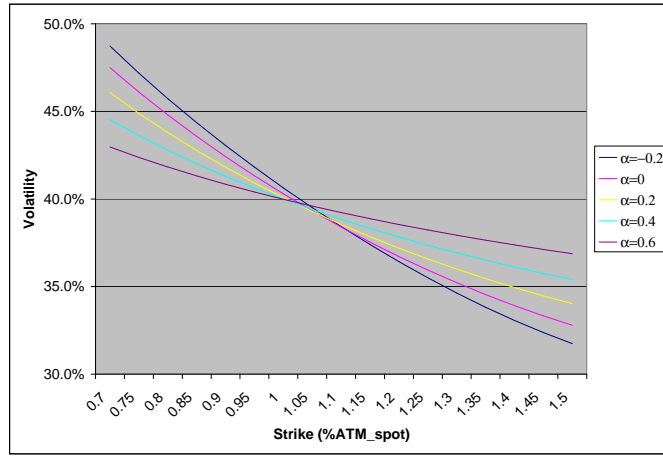


Figure 2: CEV Skewness, $\sigma = 40\%$, $r = 2\%$ and $T=1$ year

where

$$z_T = \frac{2rK^{2(1-\alpha)}}{\sigma^2(1-\alpha)(e^{2(1-\alpha)rT} - 1)}$$

$$\xi_T = \frac{rS_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1 - e^{-2(1-\alpha)rT})}$$

and Q is the complementary non-central chi-square distribution function. One can obtain the call option price thanks to the Call-Put parity relation. For homogeneity reasons, we may define σ_0 to be such as:

$$\sigma = \frac{\sigma_0}{S_0^{\alpha-1}}$$

To fit an implied volatility curve at a given maturity, take :

$$\sigma_0 \simeq \sigma_{ATM}^{BS}$$

$$\frac{\partial \sigma^{BS}}{\partial K} \simeq \frac{\sigma_0(\alpha - 1)}{S}$$

These approximations enable us to get a good idea of the parameters. Figure 2 shows several skews generated by a CEV model at a given at-the-money implied volatility for a given maturity. Figure 3 shows General Motors' implied volatility skew for the maturity January 2006 as of May 2005. The calibration of α and σ_0 was performed for a given maturity on all the call options where bids and asks were provided. Using the General Motors calibrated volatility curve, the calculated credit grade of the one year CDS with a recovery rate $R = 30$ is 326bp.

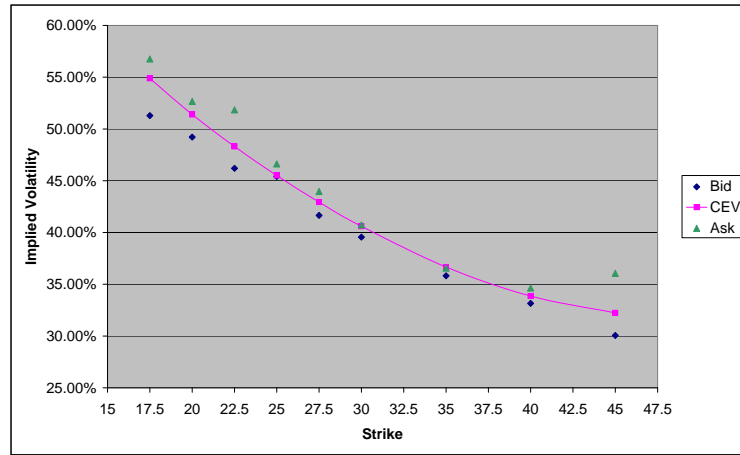


Figure 3: General Motors January 06 Implied Volatility Curve, $\sigma_0 = 43\%$, $\alpha = -0.28$, $S_0 = \$27$ and $r = 2\%$

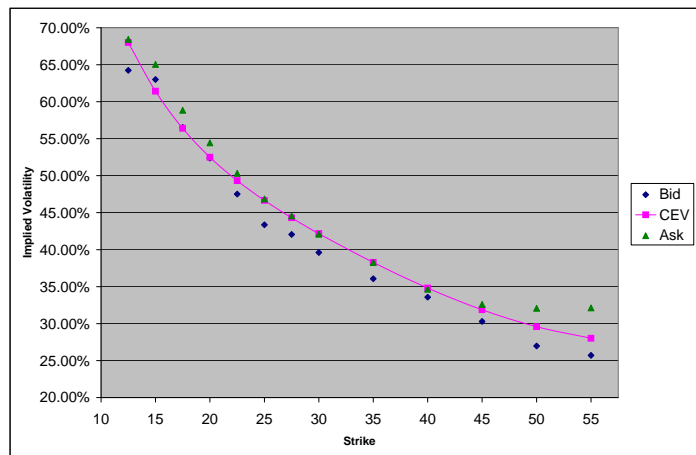


Figure 4: General Motors January 07 Implied Volatility Curve, $\sigma_0 = 41\%$, $\alpha = -0.28$, $S_0 = \$27$ and $r = 2\%$

Figure 4 displays the implied volatility curve for January 2007 call options based on the calibration of the α performed on January 2006 options and on an adjustment of σ_0 to the at-the-money volatility. It is well known that for short-term maturities, jumps are needed in the dynamic to perform a model calibration. That is the reason why adding a regular Poisson jump process to the CEV model allows a short-term maturity calibration. However, this is not the aim of this article and we leave it for further research.

2.2 Credit Derivatives Pricing

In the past few years, with the growth of the credit derivatives market, the issue of pricing CDSs with a view on the equity market has become important, especially with the recent interest in EDS pricing.

To calibrate a model to CDS market prices, we need to be able to calculate the probability of default in the CEV framework. That means we want to calculate the first hitting time of the zero cumulative distribution function. This calculation was originally done by Cox (1975). Not long afterwards, the cumulative distribution function was computed for Bessel processes by Gettoor (1979). We obtain the following simple formula for the CEV process:

$$\mathbb{P}(\tau \leq T | S_0) = G\left(\frac{1}{2(1-\alpha)}, \xi_T\right) \quad (1)$$

where G and ξ_T are defined as follow:

$$G(x, y) = \int_{z \geq y} \frac{z^{x-1} e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z > 0\}} dz$$

$$\xi_T = \frac{r S_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1 - e^{2(\alpha-1)rT})}$$

This last formula enables us to calibrate the CEV model to the CDS market. We recall the general valuation formula of a CDS initiated at time zero and evaluated at time t :

$$CDS_t(T_1, T_n; C; R) = -C \sum_{i=1}^n B(t, T_i) \mathbb{P}(\tau > T_i | S_t) + (1-R) \mathbb{E}[e^{-r(\tau-t)} \mathbf{1}_{\tau \leq T_n} | S_t]$$

where C is the coupon, T_1, \dots, T_n the payment dates, $B(t, T_i)$ the risk-free zero-coupon bonds, r the risk-free interest rate, R the recovery rate assumed to be deterministic, τ the default time and $\mathbb{P}(\tau > T_i | S_t)$ is given by formula (1). Figure 5 illustrates the different probabilities of default generated for a given level of at-the-money 1-year implied volatility within the CEV model. In the absence of arbitrage the coupon value at the inception of the contract is given by:

$$CDS_{t=0}(T_1, T_n; C; R) = 0$$

To price a CDS within the CEV Model, we just need to compute the rebate price that can be found in Davydov and Linetsky (2001).

EDSs are very similar to CDSs except that payouts occur when the stock price falls under a pre-defined level, which is often referred to as a trigger price. The trigger price is usually around 30% of the equity stock price at the beginning of the contract. Hence, these contracts provide a protection against a credit event happening on the equity market for the buyer. They were initiated by the end of 2003. At that time, it had become difficult in many countries to structure investment-grade credit portfolios with good returns because the CDS spreads

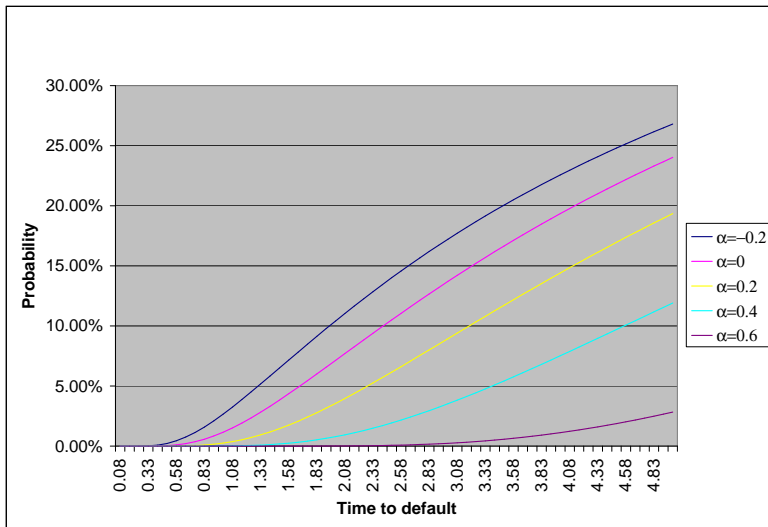


Figure 5: CEV Probabilities of default, $\sigma = 40\%$ and $r = 2\%$

were tightening, as reported by Sawyer (2003). Let us now define τ_L as the first passage of time of the stock price process under the level $L < S_0$. Formally, we write $\tau_L = \inf\{t > 0; S_t \leq L\}$. We recall the general valuation formula of an EDS:

$$EDS_t(T_1, T_n; C; R) = -C \sum_{i=1}^n B(t, T_i) \mathbb{P}(\tau_L > T_i | S_t) + \mathbb{E}[e^{-r(\tau_L - t)} \mathbf{1}_{\tau_L \leq T_n} | S_t]$$

where C is the coupon, T_1, \dots, T_n the payment dates, $B(t, T_i)$ the risk-free zero-coupon bonds and r the risk-free interest rate. Again, by absence of arbitrage, we can find the coupon price, by stating that at the initiation of the contract:

$$EDS_{t=0}(T_1, T_n; C; R) = 0$$

Analytical formulae for the EDS price are obtained using Davydov and Linetsky (2001) and can be found in Albanese and Chen (2004).

3 Heston CESV Model

Due to the limitations in the CEV model's ability to capture the main derivatives market effects - that is to say some flexibility between the level of volatility, the probability of default and the smile structure - we are led to consider a stochastic volatility instead of a constant one. More precisely, it enables us to cope with the bad time dependency of CEV credit curves for stocks with low volatilities and high probabilities of default. A well-known model of this family used in

fixed income is the SABR model introduced by Hagan et al (2002). Hence, we wish to build a Heston stochastic volatility model with a CEV diffusion for the stock price dynamics:

$$\frac{dS_t}{S_t} = rdt + \sigma\sqrt{v_t}S_t^{\alpha-1}dW_t^S \quad \text{if } t < \tau. \quad (2)$$

$$dv_t = \kappa(1 - v_t)dt + \eta\sqrt{v_t}dW_t \quad (3)$$

$$v_0 = 1 \quad (4)$$

$$S_t = 0 \quad \text{if } t \geq \tau \quad (5)$$

$$d \langle W, W^S \rangle = 0 \quad (6)$$

where W and W^S are standard Brownian motions, $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$ and $\alpha < 1$. We do not correlate the stock price return dynamics and the volatility process because the leverage effect is sufficiently well explained by the constant elasticity effect. Adding a stochastic volatility also permits the capture of a smile effect less correlated to the probability of default and changes of regimes in volatility that are shown in figure 6.

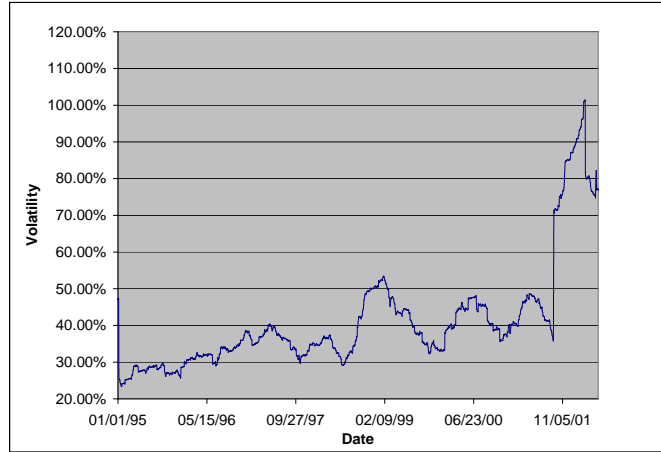


Figure 6: United Airlines 6 month Historical Volatility prior to Default

Therefore, if we consider the following process X defined as follows:

$$X_t = e^{rt} R_{\left(2 - \frac{1}{1-\alpha}, x_0\right)}^{\frac{1}{2(1-\alpha)}}(H_t) \quad \text{if } t < \tau \quad (7)$$

$$X_t = 0 \quad \text{if } \tau > t \quad (8)$$

where $H_t = \sigma^2(1-\alpha)^2 \int_0^t v_s e^{-2(1-\alpha)rs} ds$ and $R_{(\delta, x)}$ is a Squared Bessel Process, we can show that this process is a solution of Equations (2) and (5). To prove this relation, it suffices to apply Ito Formula and the change of variable formula.

A crucial point in the use of stochastic volatility is that the absence of Arbitrage is expressed by the property of the discounted stock price process being a true martingale, as mentioned above. Now, using the conditioning formula, we are able to get formulae that just depend on the law of H at terminal time. More precisely, defining the following quantity $P_0(x, K, T; S_0)$ by:

$$P_0(x, K, T; S_0) = Ke^{-rT}Q\left(\frac{S_0^{2(1-\alpha)}}{x}, \frac{1}{1-\alpha}, \frac{(Ke^{-rT})^{2(1-\alpha)}}{x}\right) - S_0\left(1 - Q\left(\frac{(Ke^{-rT})^{2(1-\alpha)}}{x}, 2 + \frac{1}{1-\alpha}, \frac{S_0^{2(1-\alpha)}}{x}\right)\right)$$

we obtain the put option price:

$$P_0 = \int_{\mathbb{R}_+} P_0(x, K, T; S_0)\mu_{H_T}(dx)$$

where μ_{H_T} is the law of H at time T . The call option price may be obtained using the call-put parity relation.

The probability of default can still be different from 0 and we have:

$$\mathbb{P}(\tau \leq T) = \int_{\mathbb{R}_+} p(x; S_0)\mu_{H_T}(dx) \quad (9)$$

where $p(x, S_0) = G\left(\frac{1}{2(1-\alpha)}, \frac{S_0^{2(1-\alpha)}}{x}\right)$.

It is well known that the law of this process can be expressed in terms of a space and time changed squared Bessel processes. A condition on v_t ensuring that 0 remains a reflecting boundary is $\frac{4\kappa}{\eta^2} > 0$ and a stability condition ensuring that the volatility process remains strictly positive is that $\frac{4\kappa}{\eta^2} > 2$. Let us for simplicity reasons consider a CEV diffusion for the forward contract F_t , it will then solve the SDE below:

$$\begin{aligned} \frac{dF_t}{F_t} &= \sigma\sqrt{v_t}S_t^{\alpha-1}dW_t^F \quad \text{if } t < \tau \\ F_t &= 0 \quad \text{if } t \geq \tau \\ dv_t &= \kappa(1 - v_t)dt + \eta\sqrt{v_t}dW_t \\ v_0 &= 1 \\ d \langle W, W^F \rangle &= 0 \end{aligned}$$

where W^F is a brownian motion and $\tau = T_0(F) = \inf\{t > 0, F_t = 0\}$.

Consequently, we are looking for the law of:

$$H_t = \sigma^2(1 - \alpha)^2 \int_0^t v_s ds \quad (10)$$

Hence, we are able to compute the Laplace transform of H_t and then get the law of H_t . It is a well-known computation for those who are for example,

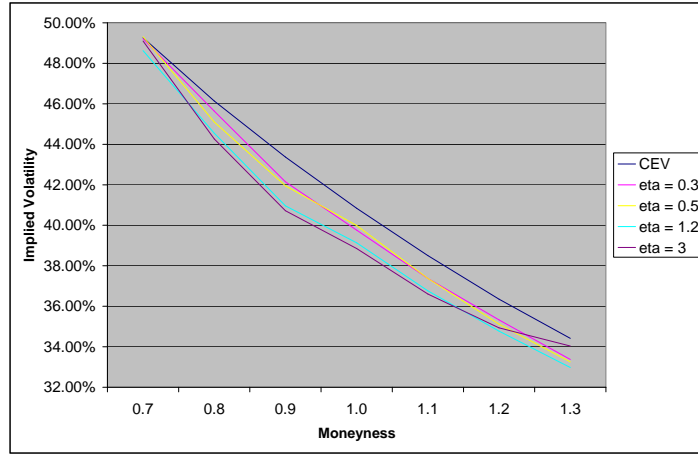


Figure 7: Heston CEV Skewness, $\sigma_0 = 40\%$, $\kappa = 2$, $\alpha = -0.3$, $r = 2\%$ and $T=1$ year

calculating the price of discount bonds within a CIR (1985) model. Let us recall its Laplace transform that one can find for instance, in Lamberton and Lapeyre (1995) $\forall \lambda \in \mathbb{R}_+$:

$$\mathbb{E}[e^{-\lambda H_t}] = e^{-\kappa \varphi_\lambda(t)} e^{-\psi_\lambda(t)}$$

where:

$$\begin{aligned} \gamma &= \sqrt{\kappa^2 + 2\eta^2 \lambda \sigma^2 (1 - \alpha)^2} \\ \varphi_\lambda(t) &= -\frac{2}{\eta^2} \ln \left(\frac{2\gamma e^{t \frac{\gamma + \kappa}{2}}}{\gamma - \kappa + e^{\gamma t} (\gamma + \kappa)} \right) \\ \psi_\lambda(t) &= \frac{2\lambda \sigma^2 (1 - \alpha)^2 (e^{\gamma t} - 1)}{\gamma - \kappa + e^{\gamma t} (\gamma + \kappa)} \end{aligned}$$

Figure 7 shows the impact of the addition of a stochastic volatility to the smile structure. One can see that a simple way to get an upward smile for upside strikes is to take κ and η such that $1 > \frac{2\kappa}{\eta^2} > 0$.

Conclusion

This article presents a study of the CEV model and an analysis of one of its possible extensions where we add a stochastic volatility (CESV model), both dedicated to the pricing of credit derivatives and equity derivatives where a downside risk is involved. We have shown that the widely used Poisson default model cannot represent the stock price behaviour of a firm defaulting, and thus a process is needed with a continuous component that by itself can "easily" reach low spot levels. This is the case of the well-known CEV model, and that

is why we considered a slight modification involving stopping the CEV process at its first-passage time by zero, to be consistent with the default event. Then, to get more freedom in the correlation structure of the skewness with the level of default, we naturally build CESV models. Moreover, for some stochastic volatility models, we are able to calculate analytical formulas. At this point, we note that we haven't performed any hedging strategies based on the CEV-type models. We leave this for future research. We also leave for future research the study of models mixing jumps and diffusions able to reach zero, such as a Poisson default CEV model that would solve the following SDE:

$$\frac{dS_t}{S_{t-}} = rdt + \sigma S_{t-}^{\alpha-1} dW_t - dQ_t$$

where :

$$Q_t = \mathbf{1}_{t \geq \tau} - \lambda t$$

This last class of models generates exogeneous default events independent of the stock price level.

We believe the CEV model and its extensions could be useful for pricing and understanding the growing equity credit-related market.

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Appendix : CEV and Bessel Processes

The law of a CEV diffusion can be thought of in terms of squared Bessel process in the following way for: $t < \tau$,

$$S_t = e^{rt} R_{\left(\frac{2\alpha-1}{\alpha-1}, x_0^{2(1-\alpha)}\right)}^{\frac{1}{2(1-\alpha)}} \left(\frac{(1-\alpha)\sigma^2}{2r} (1 - e^{-2(1-\alpha)rt}) \right)$$

where $R_{(\delta,x)}$ is a squared Bessel Process of dimension δ and starting from x solution of

$$R_{(\delta,x)}(t) = x + \delta t + 2 \int_0^t \sqrt{R_{(\delta,x)}(u)} dW_u$$

where W is a brownian motion.

Next, we are interested in the law of the stopped CEV diffusion, thanks to Girsanov theorem, we obtain for a squared Bessel process R with \mathcal{R}_t its canonical filtration

$$\mathbb{P}_x^\delta |_{\mathcal{R}_t \cap \{t < \tau\}} = \left(\frac{R_{(4-\delta,x)}(t)}{x} \right)^{\frac{\delta}{2}-1} \cdot \mathbb{P}_x^{4-\delta} |_{\mathcal{R}_t} \quad (11)$$

We can also get from Laplace transforms (see for example Delbaen and Shirakawa (2002)) the law of a squared Bessel process in terms of noncentral chi-square random variables:

$$R_{(\delta,x)}(t) \stackrel{(d)}{=} tV^{(\delta, \frac{x}{t})}$$

where $V^{(a,b)}$ is a noncentral chi-square r.v with a degrees of freedom and non-centrality parameter $b \geq 0$.