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► **To cite this version:**

Jean-François Coeurjolly, Rémy Drouilhet, Jean-François Robineau. Normalized information-based divergences. 2006. hal-00022566v1

HAL Id: hal-00022566

<https://hal.science/hal-00022566v1>

Preprint submitted on 11 Apr 2006 (v1), last revised 13 Nov 2006 (v2)

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NORMALIZED INFORMATION-BASED DIVERGENCES

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April 11, 2006

Abstract

This paper is devoted to the mathematical study of some divergences based on the mutual information well-suited to categorical random vectors. These divergences are generalizations of the “entropy distance” and “information distance”. Their main characteristic is that they combine a complexity term and the mutual information. We then introduce the notion of (normalized) information-based divergence, propose several examples and discuss their mathematical properties.

Keywords and phrases. Information theory, entropy distance, information distance, triangular inequality.

1 Introduction

Shannon information theory, usually just called *information* theory was introduced in 1948, Shannon (1948). The theory aims at providing a means for measuring information. More precisely, the amount of information in an object may be measured by its *entropy* and may be interpreted as the length of the description of the object by some encoding way. In the Shannon approach, the objects to be encoded are assumed to be outcomes of a known source. Shannon theory also provides the notion of *mutual information* (related to two objects) which plays a central role in many applications, from lossy compression to machine learning methods.

Several authors noticed that it would be useful to modify the mutual information such that the resulting quantity becomes a metric in a strict sense. As a first example, Crutchfield (1990), Hillman (1998) introduced the *entropy distance* defined as the sum of the conditional entropies. Other interesting measures are the *information distance* Bennett et al. (1998) and its normalized version named *similarity metric* introduced by Li et al. (2004) in the context of the Kolmogorov complexity theory. More precisely, the information distance is defined as the maximum of the conditional Kolmogorov complexities. The similarity metric is universal in the sense defined by the authors and is not computable, since it is based on the uncomputable notion of Kolmogorov complexity.

Recent papers have demonstrated useful application of suitable version of the similarity metric in areas as diverse as genomics, virology, languages, literature, music, handwritten digits and astronomy, Cilibrasi and Vitányi (2005b). To apply the metric to real data, the authors have to replace the use of the noncomputable Kolmogorov complexity by an approximation using standard real-world compressors : GenCompress for genomics, Li et al. (2001), the *Normalized Compression Distance (NCD)* for music clustering, Cilibrasi et al. (2003), the *Normalized Google Distance (NGD)* for automatic meaning discovery, Cilibrasi and Vitányi (2005a), are examples of effective compressors. To include the information distance and the similarity metric in a framework based on information theory concepts, we make use of the principle that *expected Kolmogorov complexity equals Shannon entropy* and interested reader can refer to Grünwald and Vitányi (2004), Leung-Yan-Cheong and Cover (1978), Hammer et al. (2000) for more details. Consequently, the entropy and information distances are both expressed in terms

of conditional entropies: the first one as the sum and the second one as the maximum. Kraskov et al. (2003) gives a proof of the triangular inequality for these distances and their respective normalized versions.

In the supervised learning framework, the use of some selection method of covariables among a large number is required when it is assumed that the data size is too small with respect to the number of the available covariables in order to apply any existing discriminant analysis method. Such a problem has been widely treated, Liu and Motoda (1998). The approach undertaken by Robineau (2004) is mainly based on three kinds of methodological tools. The first one is a supervised quantization method consisting in the simplification of covariables too complex (in particular with a too large number of possible values). Indeed, our main belief is that, in order to predict the class variable generally representing a small number of categories of data, each possibly predictive covariable must not be too complex. The second one is a more usual step by step selection method combining the simplified covariables together in order to detect cluster of data of the same class. The last one is aimed at detecting redundancy among the covariables set. These three tasks may be realized using the entropy or information distances (or their normalized versions). Let us emphasize some properties allowing to understand the usefulness of these criterions in such a context. The entropy and information distances can be rewritten as the difference between a complexity term (respectively the joint entropy and the maximum of the marginal entropies) and the mutual information. Moreover, both are independence measures with the particular property to be minimal (in fact equal to 0) when random vectors share exactly the same information. Robineau (2004) proposes then to extend the definition of the entropy and information distances by introducing the notion of information-based divergence $\Delta_{\mathbf{X},\mathbf{Y}}$ between two categorical random vectors \mathbf{X} and \mathbf{Y} defined as the difference of some complexity term $C_{\mathbf{X},\mathbf{Y}}$ and the mutual information $I_{\mathbf{X},\mathbf{Y}}$ and such that $C_{\mathbf{X},\mathbf{Y}}$ is an upper bound of $I_{\mathbf{X},\mathbf{Y}}$ reached when \mathbf{X} and \mathbf{Y} share exactly the same information. The notion of normalized information-based divergence $\delta_{\mathbf{X},\mathbf{Y}}$ derives directly by dividing the associated information-based divergence $\Delta_{\mathbf{X},\mathbf{Y}}$ by the complexity term $C_{\mathbf{X},\mathbf{Y}}$. Particular examples are given by the normalized versions of the entropy and information distances.

The paper is organized as follows. In Section 2, we recall the definition and their main properties of the entropy and information distances (and their normalized version).

Similarly to Granger et al. (2004), we extract the main characteristics to define some general concept of information divergence which could be theoretically applied in a more general setting (continuous, discrete, ...). Section 3 concentrates itself on categorical data (and in particular discrete) random vectors, as it is usually the case in most of applications using entropy or information distance. We give the definition of (normalized) information-based divergence and propose several examples. We propose some sufficient condition for these divergences to verify a triangular's type inequality. Finally, we point out that the notion of information-based divergence is useful to detect redundancy.

2 Normalized entropy distance and normalized information distance

We denote by \mathbf{X} , \mathbf{Y} and \mathbf{Z} three non deterministic categorical random vectors (in particular discrete-valued random vectors).

2.1 Some notation

We denote by $H_{\mathbf{X}}$ (when it exists) the Shannon entropy of \mathbf{X} given by

$$H_{\mathbf{X}} = - \sum_{\mathbf{x} \in \Omega_{\mathbf{X}}} p_{\mathbf{X}}(\mathbf{x}) \log(p_{\mathbf{X}}(\mathbf{x})) \quad \text{with } p_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} = \mathbf{x}),$$

In the same way, one can define the joint entropy of \mathbf{X} and \mathbf{Y} denoted by $H_{\mathbf{X},\mathbf{Y}}$, the conditional entropy of \mathbf{X} (resp. \mathbf{Y}) by \mathbf{Y} (resp. \mathbf{X}) denoted by $H_{\mathbf{X}|\mathbf{Y}}$ (resp. $H_{\mathbf{Y}|\mathbf{X}}$). Finally, we denote by $I_{\mathbf{X},\mathbf{Y}}$ the mutual information between the random vectors \mathbf{X} and \mathbf{Y} . When these different quantities exist, the following relations hold (see *e.g.* Cover and Thomas (1991)):

$$H_{\mathbf{X},\mathbf{Y}} = H_{\mathbf{X}} + H_{\mathbf{Y}|\mathbf{X}} = H_{\mathbf{Y}} + H_{\mathbf{X}|\mathbf{Y}} \tag{1}$$

$$I_{\mathbf{X},\mathbf{Y}} = H_{\mathbf{X}} - H_{\mathbf{X}|\mathbf{Y}} = H_{\mathbf{Y}} - H_{\mathbf{Y}|\mathbf{X}} = H_{\mathbf{X}} + H_{\mathbf{Y}} - H_{\mathbf{X},\mathbf{Y}} \tag{2}$$

2.2 Definition and characteristics

We now shall present some measures allowing to overcome some drawbacks of the mutual information. As a first generalization, several authors noticed that it would be useful to

modify the mutual information such that the resulting quantity becomes a metric in a strict sense. Two such measures exist and are well-known in the literature. The first one called “entropy distance” is derived from the domain of information theory. The second one called “information distance” originates in works around the Kolmogorov complexity. Both measures are defined (when they exist) for two random vectors \mathbf{X} and \mathbf{Y} by:

- Entropy distance:

$$D_{\mathbf{X},\mathbf{Y}}^E = H_{\mathbf{X}|\mathbf{Y}} + H_{\mathbf{Y}|\mathbf{X}} \quad (3)$$

- Information distance:

$$D_{\mathbf{X},\mathbf{Y}}^I = \max(H_{\mathbf{X}|\mathbf{Y}}, H_{\mathbf{Y}|\mathbf{X}}). \quad (4)$$

Both measures are indeed some modifications of mutual information since from (1) and (2), we have

$$D_{\mathbf{X},\mathbf{Y}}^E = H_{\mathbf{X},\mathbf{Y}} - I_{\mathbf{X},\mathbf{Y}} \quad \text{and} \quad D_{\mathbf{X},\mathbf{Y}}^I = \max(H_{\mathbf{X}}, H_{\mathbf{Y}}) - I_{\mathbf{X},\mathbf{Y}}. \quad (5)$$

The quantities $H_{\mathbf{X},\mathbf{Y}}$ and $\max(H_{\mathbf{X}}, H_{\mathbf{Y}})$ are upper-bounds of the mutual information $I_{\mathbf{X},\mathbf{Y}}$ that are reached when \mathbf{X} and \mathbf{Y} share exactly the same information. In other words, these two measures are nonnegative and vanish if and only if $H_{\mathbf{Y}|\mathbf{X}} = H_{\mathbf{X}|\mathbf{Y}} = 0$ expressing the fact that \mathbf{X} (resp. \mathbf{Y}) predicts \mathbf{Y} (resp. \mathbf{X}) with probability 1.

These measures satisfy

$$D_{\mathbf{X},\mathbf{Y}}^E \leq H_{\mathbf{X},\mathbf{Y}} \quad \text{and} \quad D_{\mathbf{X},\mathbf{Y}}^I \leq \max(H_{\mathbf{X}}, H_{\mathbf{Y}}), \quad (6)$$

where the equality holds if the vectors \mathbf{X} and \mathbf{Y} are independent. As noticed by Kaltchenko (2004), Li and Vitányi (1997) argued that in Bioinformatics an unnormalized distance may not be a proper evolutionary distance measure. It would put two long and complex sequences that differ only by a tiny fraction of the total information as dissimilar as two short sequences that differ by the same absolute amount and are completely random with respect to one another. To overcome this problem within the algorithmic framework Li and Vitányi (1997) form two normalized versions of distances D^E and D^I . Their Shannon version have been proposed and studied by Kraskov et al. (2003)

Definition 1 *When they exist, one defines the two following measures:*

- *Normalized entropy distance:*

$$d_{\mathbf{X},\mathbf{Y}}^E = \frac{H_{\mathbf{X}|\mathbf{Y}} + H_{\mathbf{Y}|\mathbf{X}}}{H_{\mathbf{X},\mathbf{Y}}}$$

- *Normalized information distance:*

$$d_{\mathbf{X},\mathbf{Y}}^I = \frac{\max(H_{\mathbf{X}|\mathbf{Y}}, H_{\mathbf{Y}|\mathbf{X}})}{\max(H_{\mathbf{X}}, H_{\mathbf{Y}})}$$

Since $H_{\mathbf{X},\mathbf{Y}} = 0 \Leftrightarrow H_{\mathbf{X}} = H_{\mathbf{Y}} = 0 \Leftrightarrow \max(H_{\mathbf{X}}, H_{\mathbf{Y}}) = 0$, we set by convention $d_{\mathbf{X},\mathbf{Y}}^E = 0$ (resp. $d_{\mathbf{X},\mathbf{Y}}^I = 0$) when $H_{\mathbf{X}} = H_{\mathbf{Y}} = 0$.

We are encouraged to define the following class of equivalence: the vectors \mathbf{X} and \mathbf{Y} are said to be equivalent if \mathbf{X} (resp. \mathbf{Y}) predicts \mathbf{Y} (resp. \mathbf{X}) with probability 1 and one will denote

$$\mathbf{X} \sim \mathbf{Y} \Leftrightarrow H_{\mathbf{Y}|\mathbf{X}} = H_{\mathbf{X}|\mathbf{Y}} = 0 \Leftrightarrow I_{\mathbf{X},\mathbf{Y}} = H_{\mathbf{X},\mathbf{Y}} = H_{\mathbf{X}} = H_{\mathbf{Y}} \quad (7)$$

Since \mathbf{X} and \mathbf{Y} are not deterministic this leads to the following equivalences

$$d_{\mathbf{X},\mathbf{Y}}^E = 0 \Leftrightarrow d_{\mathbf{X},\mathbf{Y}}^I = 0 \Leftrightarrow \mathbf{X} \sim \mathbf{Y}.$$

From (1) and (2), one can obtain the following expressions for these two measures allowing some new interpretations.

Proposition 1 *We have the following expressions for $d_{\mathbf{X},\mathbf{Y}}^E$ and $d_{\mathbf{X},\mathbf{Y}}^I$.*

$$d_{\mathbf{X},\mathbf{Y}}^E = 1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{H_{\mathbf{X},\mathbf{Y}}} \quad (8)$$

$$d_{\mathbf{X},\mathbf{Y}}^I = 1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{\max(H_{\mathbf{X}}, H_{\mathbf{Y}})} \quad (9)$$

$$= \max\left(\frac{H_{\mathbf{X}|\mathbf{Y}}}{H_{\mathbf{X}}}, \frac{H_{\mathbf{Y}|\mathbf{X}}}{H_{\mathbf{Y}}}\right) \quad (10)$$

Proposition 2 *The measures d^E et d^I constitute two distances bounded by 1.*

To our knowledge, these results have been proved by Kraskov et al. (2003). Proofs are very similar to proofs of Li et al. (2003) who consider the algorithmic version of these distances. The proof is then omitted, but in Section 3.3, we propose a result extending this one in the sense that we give conditions on measures that can be written as (8) and (9) to constitute a metric.

2.3 Concept of information divergence

We can exhibit from the previous study related to D^I , D^E , d^I and d^E , some characteristics useful for an attempt to define the concept of information divergence denoted by $\Delta_{\mathbf{X},\mathbf{Y}}$ in a more general setting. Let us first consider a similarity measure $\mathcal{I}_{\mathbf{X},\mathbf{Y}}$ (not necessarily the mutual information) minimal (in fact equal to 0) when \mathbf{X} and \mathbf{Y} are independent, and maximal when the distributions of \mathbf{X} given $\mathbf{Y} = \mathbf{y}$ and \mathbf{Y} given $\mathbf{X} = \mathbf{x}$ are trivial. An information divergence $\Delta_{\mathbf{X},\mathbf{Y}}$ could satisfy the following properties:

[P1] symmetry: $\Delta_{\mathbf{X},\mathbf{Y}} = \Delta_{\mathbf{Y},\mathbf{X}}$.

[P2] nonnegativeness: $\Delta_{\mathbf{X},\mathbf{Y}} \geq 0$.

[P3] $\Delta_{\mathbf{X},\mathbf{Y}}$ is minimum (i.e. $\Delta_{\mathbf{X},\mathbf{Y}} = 0$) if and only if \mathbf{X} and \mathbf{Y} share exactly the same information (i.e. $\mathcal{I}_{\mathbf{X},\mathbf{Y}}$ is maximal).

[P4] $\Delta_{\mathbf{X},\mathbf{Y}}$ is maximum if and only if \mathbf{X} and \mathbf{Y} are independent (i.e. $\mathcal{I}_{\mathbf{X},\mathbf{Y}} = 0$).

Other supplementary properties could be that $\Delta_{\mathbf{X},\mathbf{Y}}$:

[P5] is normalized: $\Delta_{\mathbf{X},\mathbf{Y}} \in [0, 1]$ and $\Delta_{\mathbf{X},\mathbf{Y}} = 1$ when \mathbf{X} and \mathbf{Y} are independent.

[P6] satisfies a triangular inequality: $\Delta_{\mathbf{X},\mathbf{Y}} \leq \Delta_{\mathbf{X},\mathbf{Z}} + \Delta_{\mathbf{Z},\mathbf{Y}}$.

[P7] invariant under continuous and strictly increasing transformations $\varphi(\cdot)$, $\psi(\cdot)$ of the vectors \mathbf{X} and \mathbf{Y} , whenever they are quantitative random vectors.

There exists a large literature on the discussion of criteria satisfying the previous stated properties. We may cite Ullah (1996), or a recent work of Granger et al. (2004) who propose to detect the dependence between two possibly nonlinear processes through the Bhattacharya-Matusita-Hellinger measure of dependence given by

$$S_\rho = \frac{1}{2} \int \int \left(\sqrt{f_1(\mathbf{x}, \mathbf{y})} - \sqrt{f_2(\mathbf{x}, \mathbf{y})} \right)^2 d\mathbf{x}d\mathbf{y},$$

where f_1 (resp. f_2) is the joint density (resp. the product of marginal densities) of \mathbf{X} and \mathbf{Y} . This measure, that has the other advantage to be applicable to continuous or discrete variables, satisfies properties [P1]-[P7] (in fact let us precise that [P7] is only valid if $\varphi(\cdot) = \psi(\cdot)$).

In some framework where the purpose is to predict some reference variable, one may find interesting to work with a divergence $\Delta_{\mathbf{X},\mathbf{Y}}$ which combines the minimization of a nonnegative complexity term denoted by $\mathcal{C}_{\mathbf{X},\mathbf{Y}}$ and the maximization of a nonnegative information term $\mathcal{I}_{\mathbf{X},\mathbf{Y}}$. The quantity $\mathcal{C}_{\mathbf{X},\mathbf{Y}}$ is called a complexity term since it is assumed to be expressed as a function of $\mathcal{H}_{\mathbf{X}}$, $\mathcal{H}_{\mathbf{Y}}$ and $\mathcal{H}_{\mathbf{X},\mathbf{Y}}$ measuring in some way respectively the complexity of vectors \mathbf{X} , \mathbf{Y} and (\mathbf{X}, \mathbf{Y}) . In other words, we may expect

[P8] When \mathbf{X}_1 and \mathbf{X}_2 have the same complexity (in the sense that $\mathcal{C}_{\mathbf{Y},\mathbf{X}_1} = \mathcal{C}_{\mathbf{Y},\mathbf{X}_2}$): $\Delta_{\mathbf{Y},\mathbf{X}_1} < \Delta_{\mathbf{Y},\mathbf{X}_2}$ whenever \mathbf{X}_1 has a better knowledge about \mathbf{Y} than \mathbf{X}_2 (i.e. $\mathcal{I}_{\mathbf{Y},\mathbf{X}_1} > \mathcal{I}_{\mathbf{Y},\mathbf{X}_2}$).

[P9] When \mathbf{X}_1 and \mathbf{X}_2 have the same knowledge about \mathbf{Y} (i.e. $\mathcal{I}_{\mathbf{Y},\mathbf{X}_1} = \mathcal{I}_{\mathbf{Y},\mathbf{X}_2}$): $\Delta_{\mathbf{Y},\mathbf{X}_1} < \Delta_{\mathbf{Y},\mathbf{X}_2}$ whenever \mathbf{X}_1 is simpler than \mathbf{X}_2 in the sense that $\mathcal{C}_{\mathbf{Y},\mathbf{X}_1} < \mathcal{C}_{\mathbf{Y},\mathbf{X}_2}$. Moreover, in this particular situation the fact that

[P9bis] $\mathcal{C}_{\mathbf{Y},\mathbf{X}_1} < \mathcal{C}_{\mathbf{Y},\mathbf{X}_2}$ must be equivalent to $\mathcal{H}_{\mathbf{X}_1} < \mathcal{H}_{\mathbf{X}_2}$.

[P10] When \mathbf{X}_1 and \mathbf{X}_2 share almost exactly the same information (i.e. $\mathcal{I}_{\mathbf{X}_1,\mathbf{X}_2}$ is almost maximal and $\Delta_{\mathbf{X}_1,\mathbf{X}_2} \simeq 0$) then the difference between the divergences $\Delta_{\mathbf{Y},\mathbf{X}_1}$ and $\Delta_{\mathbf{Y},\mathbf{X}_2}$ is almost zero (i.e. $\Delta_{\mathbf{Y},\mathbf{X}_1} \simeq \Delta_{\mathbf{Y},\mathbf{X}_2}$).

A class of candidates that satisfy both of the previous statements could be of the form:

$$\Delta_{\mathbf{X},\mathbf{Y}} = \frac{\mathcal{C}_{\mathbf{X},\mathbf{Y}} - \mathcal{I}_{\mathbf{X},\mathbf{Y}}}{\mathcal{W}_{\mathbf{X},\mathbf{Y}}}, \quad (11)$$

where $\mathcal{W}_{\mathbf{X},\mathbf{Y}}$ is a positive term. When $\mathcal{W}_{\mathbf{X},\mathbf{Y}} = \mathcal{C}_{\mathbf{X},\mathbf{Y}}$ we obtain a normalized information divergence. The properties **[P2]**-**[P3]** and the form (11) implies that $\mathcal{C}_{\mathbf{X},\mathbf{Y}}$ is an upper bound of $\mathcal{I}_{\mathbf{X},\mathbf{Y}}$ reached when \mathbf{X} and \mathbf{Y} share exactly the same information.

In the rest of this paper we concentrate ourself on criteria described by (11) that are in addition well-suited to categorical random variables (and in particular discrete random variables). In such a framework, we shall only describe some entropic-based criteria (i.e. $\mathcal{H}_{\mathbf{X}} = H_{\mathbf{X}}$), and so the information term will be set to the mutual information $\mathcal{I}_{\mathbf{X},\mathbf{Y}}$.

3 Information-based divergences and their normalized versions

We denote by \mathbf{X}, \mathbf{Y} and \mathbf{Z} three non deterministic categorical random vectors (in particular discrete-valued random vectors).

3.1 Definition and examples

Definition 2 *Two criteria Δ and δ are respectively called an information-based divergence and a normalized information-based divergence (in short IB-divergence and NIB-divergence) if they can respectively be written*

$$\Delta_{\mathbf{X}, \mathbf{Y}} = C_{\mathbf{X}, \mathbf{Y}} - I_{\mathbf{X}, \mathbf{Y}} \quad (12)$$

$$\delta_{\mathbf{X}, \mathbf{Y}} = \frac{C_{\mathbf{X}, \mathbf{Y}} - I_{\mathbf{X}, \mathbf{Y}}}{C_{\mathbf{X}, \mathbf{Y}}} = 1 - \frac{I_{\mathbf{X}, \mathbf{Y}}}{C_{\mathbf{X}, \mathbf{Y}}} \quad (13)$$

where the term $C_{\mathbf{X}, \mathbf{Y}}$ constitutes a normalization term satisfying

- (i) $C_{\mathbf{X}, \mathbf{Y}} > 0$.
- (ii) $C_{\mathbf{X}, \mathbf{Y}} = C_{\mathbf{Y}, \mathbf{X}}$
- (iii) $I_{\mathbf{X}, \mathbf{Y}} \leq C_{\mathbf{X}, \mathbf{Y}}$ and this bound is achieved if the random vectors \mathbf{X} and \mathbf{Y} are equivalent, i.e. if $\mathbf{X} \sim \mathbf{Y}$.

This definition implies automatically that an IB-divergence $\Delta_{\mathbf{X}, \mathbf{Y}}$ (resp. a NIB-divergence $\delta_{\mathbf{X}, \mathbf{Y}}$) satisfies properties [P1]-[P4] (resp. [P1]-[P5]). In the rest of the paper, we impose on the term $C_{\mathbf{X}, \mathbf{Y}}$ to be expressed as

$$C_{\mathbf{X}, \mathbf{Y}} = f_C \left(H_{\mathbf{X}|\mathbf{Y}}, H_{\mathbf{Y}|\mathbf{X}}, I_{\mathbf{X}, \mathbf{Y}} \right), \quad (14)$$

where $f_C(\cdot, \cdot, \cdot)$ is a nonnegative function. Under such an expression of $C_{\mathbf{X}, \mathbf{Y}}$, the property [P7] is ensured since the conditional entropies and the mutual information depend only on the joint probability distribution of the categorical random vectors \mathbf{X} and \mathbf{Y} .

From now on, we propose a series of examples for which we adopt the following convention: an IB-divergence (resp. a NIB-divergence) satisfying the triangular inequality is denoted D (resp. d) rather than Δ (resp. δ). Moreover, each example will be

particularized by some discriminating additional letter in the same manner as D^E and D^I (resp. d^E and d^I) which clearly constitute IB-divergences (resp. NIB-divergences).

In Robineau (2004), we investigate about two new entropic criteria naturally expressed by

$$\delta_{x,y}^D = \frac{1}{2} \left(\frac{H_{x|y}}{H_x} + \frac{H_{y|x}}{H_y} \right) \quad \text{and} \quad \delta_{x,y}^S = \frac{H_{x|y} + H_{y|x}}{H_x + H_y}.$$

which can be rewritten as NIB-divergences:

$$\delta_{x,y}^D = 1 - \frac{I_{x,y}}{C_{x,y}^D} \quad \text{with} \quad C_{x,y}^D = \left(\frac{1}{2} \left(\frac{1}{H_x} + \frac{1}{H_y} \right) \right)^{-1} \quad (15)$$

$$\delta_{x,y}^S = 1 - \frac{I_{x,y}}{C_{x,y}^S} \quad \text{with} \quad C_{x,y}^S = \frac{1}{2} (H_x + H_y). \quad (16)$$

Their non normalized version are expressed as $\Delta_{x,y}^D = C_{x,y}^D - I_{x,y}$ and $D_{x,y}^S = C_{x,y}^S - I_{x,y}$. To be convinced that the measures δ^D and δ^S lie in $[0, 1]$, let us notice that these two measures are particular cases of the following family of IB-divergence or NIB-divergence with normalization terms of the form:

$$C_{x,y} = g^{-1} \left(\frac{1}{2} (g(H_x) + g(H_y)) \right) \quad (17)$$

where $g(\cdot)$ is any nonnegative monotone function on \mathbb{R}^+ . Indeed, by assuming that $g(\cdot)$ is an increasing function for example, we have

$$I_{x,y} = g^{-1} \left(\frac{1}{2} (g(I_{x,y}) + g(I_{x,y})) \right) \leq g^{-1} \left(\frac{1}{2} (g(H_x) + g(H_y)) \right).$$

By choosing $g(\cdot) = (\cdot)^\alpha$ for some $\alpha > 0$, the normalization term is given by $C_{x,y} = 2^{-1/\alpha} \|(H_x, H_y)\|_\alpha$, where $\|\mathbf{x}\|_\alpha = (\sum_{i=1}^n |x_i|^\alpha)^{1/\alpha}$ denotes the norm of some vector \mathbf{x} of length n . Note that, when $\alpha = +\infty$, we retrieve the distances D^I and d^I . Another family can be obtained by choosing the normalization term as follows:

$$C_{x,y} = g^{-1} \left(\sqrt{g(H_x) \times g(H_y)} \right) \quad (18)$$

Among both of these families, we shall consider the following IB-divergences Δ^R , Δ^P and NIB-divergences δ^R , δ^P with respective normalization terms:

$$C_{x,y}^R = \left(\frac{\sqrt{H_x} + \sqrt{H_y}}{2} \right)^2 \quad (19)$$

$$C_{x,y}^P = \sqrt{H_x \times H_y}. \quad (20)$$

Let us now propose a result to arrange the different examples considered in this paper:

Proposition 3 *For any $\Delta^{(1)}, \Delta^{(2)}$ IB-divergences or any $\delta^{(1)}, \delta^{(2)}$ NIB-divergences with respective normalization terms $C^{(1)}$ and $C^{(2)}$, the following equivalence holds:*

$$\Delta_{\mathbf{x},\mathbf{y}}^{(1)} \leq \Delta_{\mathbf{x},\mathbf{y}}^{(2)} \iff \delta_{\mathbf{x},\mathbf{y}}^{(1)} \leq \delta_{\mathbf{x},\mathbf{y}}^{(2)} \iff C_{\mathbf{x},\mathbf{y}}^{(1)} \leq C_{\mathbf{x},\mathbf{y}}^{(2)}. \quad (21)$$

Due to

$$C_{\mathbf{x},\mathbf{y}}^D \leq C_{\mathbf{x},\mathbf{y}}^P \leq C_{\mathbf{x},\mathbf{y}}^R \leq C_{\mathbf{x},\mathbf{y}}^S \leq C_{\mathbf{x},\mathbf{y}}^I \leq C_{\mathbf{x},\mathbf{y}}^E \quad (22)$$

the respective IB-divergences and NIB-divergences are then ordered according to equation (21).

Proof. Equation (21) is direct. To obtain (22) it is sufficient to see that

$$\frac{2H_{\mathbf{x}}H_{\mathbf{y}}}{H_{\mathbf{x}}+H_{\mathbf{y}}} \leq \sqrt{H_{\mathbf{x}}H_{\mathbf{y}}} \leq \left(\frac{\sqrt{H_{\mathbf{x}}} + \sqrt{H_{\mathbf{y}}}}{2} \right)^2 \leq \frac{H_{\mathbf{x}}+H_{\mathbf{y}}}{2} \leq \max(H_{\mathbf{x}}, H_{\mathbf{y}}) \leq H_{\mathbf{x},\mathbf{y}}$$

■

The following proposition gives a larger class of examples of IB-divergences and NIB-divergences.

Proposition 4 *Let $(\alpha^{(j)})_{j=1,\dots,J}$ be some vector of probability weights for some $J \geq 1$.*

(i) *Let $\Delta^{(1)}, \dots, \Delta^{(J)}$, J IB-divergences, and let $\delta^{(1)}, \dots, \delta^{(J)}$, J NIB-divergences, then the measures defined by*

$$\Delta_{\mathbf{x},\mathbf{y}} = \sum_{j=1}^J \alpha^{(j)} \Delta_{\mathbf{x},\mathbf{y}}^{(j)} \quad \text{and} \quad \delta_{\mathbf{x},\mathbf{y}} = \sum_{j=1}^J \alpha^{(j)} \delta_{\mathbf{x},\mathbf{y}}^{(j)} \quad (23)$$

are respectively an IB-divergence and a NIB-divergence with normalization term given by

$$C_{\mathbf{x},\mathbf{y}} = \left(\sum_{j=1}^J \frac{\alpha^{(j)}}{C_{\mathbf{x},\mathbf{y}}^{(j)}} \right)^{-1}. \quad (24)$$

(ii) *Let $\Delta^{(1)}, \dots, \Delta^{(j)}$, J IB-divergences and $\delta^{(1)}, \dots, \delta^{(j)}$, J NIB-divergences with normalization terms $C_{\mathbf{x},\mathbf{y}}^{(1)}, \dots, C_{\mathbf{x},\mathbf{y}}^{(j)}$ then the measures defined by*

$$\Delta_{\mathbf{x},\mathbf{y}} = C_{\mathbf{x},\mathbf{y}} - I_{\mathbf{x},\mathbf{y}} \quad \text{and} \quad \delta_{\mathbf{x},\mathbf{y}} = 1 - \frac{I_{\mathbf{x},\mathbf{y}}}{C_{\mathbf{x},\mathbf{y}}}, \quad \text{with} \quad C_{\mathbf{x},\mathbf{y}} = \sum_{j=1}^J \alpha^{(j)} C_{\mathbf{x},\mathbf{y}}^{(j)} \quad (25)$$

are also respectively an *IB-divergence* and a *NIB-divergence*.

The proof is immediate.

Given some normalization term of the form (14), the property **[P3]** may be expressed by: $f_C(x, y, z) = z \Leftrightarrow x = y = 0$. In fact, **[P3]** should be extended to the more useful assumption: $\Delta_{\mathbf{X}, \mathbf{Y}}$ is near from minimum 0 if and only if \mathbf{X} and \mathbf{Y} share almost the same information. This may be translated by:

[P3bis] The following implications hold: given any $z \geq 0$,

- for all $\gamma > 0$ there exists $\varepsilon > 0$ such that for all $x, y \geq 0$

$$|f_C(x, y, z) - z| \leq \varepsilon \implies \max(x, y) \leq \gamma. \quad (26)$$

- for all $\varepsilon > 0$ there exists $\gamma > 0$ such that for all $x, y \geq 0$

$$\max(x, y) \leq \gamma \implies |f_C(x, y, z) - z| \leq \varepsilon. \quad (27)$$

Equation (27) derives from the continuity of f_C . Moreover, we assert that any normalization term C of the form (17) or (18) including the whole of our examples satisfies (26) and so **[P3bis]** by assuming, in addition, that the function $g(\cdot)$ (and then $g^{-1}(\cdot)$) is continuous. Let us concentrate on normalization terms C of the form (17): the following implications hold

$$\begin{aligned} & g^{-1} \left(\frac{g(x+z) + g(y+z)}{2} \right) - z \rightarrow 0 \\ \implies & \frac{1}{2} (g(x+z) - g(z)) + \frac{1}{2} (g(y+z) - g(z)) \rightarrow 0 \\ \implies & \max(x, y) \rightarrow 0. \end{aligned}$$

For normalization terms C of the form (18):

$$\begin{aligned} & g^{-1} \left(\sqrt{g(x+z) \times g(y+z)} \right) - z \rightarrow 0 \\ \implies & \left(\log(g(x+z)) - \log(g(z)) \right) + \left(\log(g(y+z)) - \log(g(z)) \right) \rightarrow 0 \\ \implies & \max(x, y) \rightarrow 0. \end{aligned}$$

3.2 Prediction problem framework

We pay attention on properties related to the prediction of some fixed random vector \mathbf{Y} . Recall that our purpose is to find the random vector \mathbf{X} that minimizes $\Delta_{\mathbf{Y},\mathbf{X}}$ (resp. $\delta_{\mathbf{Y},\mathbf{X}}$) which combines a complexity term $C_{\mathbf{X},\mathbf{Y}}$ (to minimize) and an information term $I_{\mathbf{X},\mathbf{Y}}$ (to maximize). Let us imagine that we already get some \mathbf{X}_1 and its associated measure $\Delta_{\mathbf{Y},\mathbf{X}_1}$ (resp. $\delta_{\mathbf{Y},\mathbf{X}_1}$). After evaluating $\Delta_{\mathbf{Y},\mathbf{X}_2}$ (resp. $\delta_{\mathbf{Y},\mathbf{X}_2}$), we may be interested in describing the conditions under which \mathbf{X}_2 is better or worse than \mathbf{X}_1 :

Proposition 5 *Two situations may occur*

Case 1: we choose \mathbf{X}_2 instead of \mathbf{X}_1 when

$$\Delta_{\mathbf{Y},\mathbf{X}_2} < \Delta_{\mathbf{Y},\mathbf{X}_1} \iff C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} < I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} \quad (28)$$

$$\delta_{\mathbf{Y},\mathbf{X}_2} < \delta_{\mathbf{Y},\mathbf{X}_1} \iff \frac{C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} < \frac{I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}}{I_{\mathbf{Y},\mathbf{X}_1}} \quad (29)$$

Case 2: we keep \mathbf{X}_1 and reject \mathbf{X}_2 when

$$\Delta_{\mathbf{Y},\mathbf{X}_2} \geq \Delta_{\mathbf{Y},\mathbf{X}_1} \iff C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} \geq I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} \quad (30)$$

$$\delta_{\mathbf{Y},\mathbf{X}_2} \geq \delta_{\mathbf{Y},\mathbf{X}_1} \iff \frac{C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} \geq \frac{I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}}{I_{\mathbf{Y},\mathbf{X}_1}} \quad (31)$$

Tab. 1 summarizes the computations of absolute and relative differences of complexity increments for the examples presented in the previous section.

This result implies automatically that the properties [P8] and [P9] are satisfied. Let us comment more precisely the previous proposition:

- Case 1 holds when

1. \mathbf{X}_2 is simpler than \mathbf{X}_1 (i.e. $C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} < 0$) and \mathbf{X}_2 is at least as informative as \mathbf{X}_1 (i.e. $I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} \geq 0$).
2. \mathbf{X}_2 and \mathbf{X}_1 have the same complexity (i.e. $C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} = 0$) and \mathbf{X}_2 is more informative than \mathbf{X}_1 (i.e. $I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} > 0$).
3. \mathbf{X}_2 is simpler and less informative than \mathbf{X}_1 and such that the absolute (resp. relative) excess of complexity is lower than the absolute (resp. relative) gain of information that is $C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} < I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} < 0$ (resp. $\frac{C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} < \frac{I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}}{I_{\mathbf{Y},\mathbf{X}_1}} < 0$).

4. \mathbf{X}_2 is more complex and more informative than \mathbf{X}_1 and such that the absolute (resp. relative) excess of complexity is lower than the absolute (resp. relative) gain of information that is $0 < C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} < I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}$ (resp. $0 < \frac{C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} < \frac{I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}}{I_{\mathbf{Y},\mathbf{X}_1}}$).
- Case 2 holds when
 1. \mathbf{X}_2 is at least as complex as \mathbf{X}_1 (i.e. $C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} \geq 0$) and \mathbf{X}_2 is at most as informative as \mathbf{X}_1 (i.e. $I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} \leq 0$).
 2. \mathbf{X}_2 is simpler and less informative than \mathbf{X}_1 , and such that the absolute (resp. relative) excess of complexity is greater than or equal to the absolute (resp. relative) gain of information that is $0 > C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} \geq I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}$ (resp. $0 > \frac{C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} \geq \frac{I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}}{I_{\mathbf{Y},\mathbf{X}_1}}$).
 3. \mathbf{X}_2 is more complex and more informative than \mathbf{X}_1 , and such that the absolute (resp. relative) excess of complexity is greater than or equal to the absolute (resp. relative) gain of information that is $C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1} \geq I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1} > 0$ (resp. $\frac{C_{\mathbf{Y},\mathbf{X}_2} - C_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} \geq \frac{I_{\mathbf{Y},\mathbf{X}_2} - I_{\mathbf{Y},\mathbf{X}_1}}{I_{\mathbf{Y},\mathbf{X}_1}} > 0$).

More specifically, two frameworks may be of special interest:

- \mathbf{X}_2 is as informative as \mathbf{X}_1 (i.e. $I_{\mathbf{Y},\mathbf{X}_1} = I_{\mathbf{Y},\mathbf{X}_2}$): we expect to select the random variable with the smallest entropy. This is effectively what happens when **[P9bis]** is satisfied. We assert that for all the examples considered in the previous section **[P9bis]** is true. Indeed, from Tab. 1, **[P9bis]** holds for the normalization terms C^\bullet with $\bullet = I, S, R, P, D$ in the general case and for C^E in this framework since $H_{\mathbf{Y},\mathbf{X}_2} - H_{\mathbf{Y},\mathbf{X}_1} = H_{\mathbf{X}_2} - H_{\mathbf{X}_1}$.
- $\mathbf{X}_1 = g(\mathbf{X}_2)$ with g some surjective (but not injective) mapping: \mathbf{X}_2 is more complex than \mathbf{X}_1 and \mathbf{X}_2 is at least as informative as \mathbf{X}_1 . Consequently, this case is not trivial since both absolute (resp. relative) excess of complexity and absolute (resp. relative) gain of information are competing. Let us give two important examples of such a context.
 1. quantization problem: given a quantized version \mathbf{X}_1 of some (continuous) random variable with its associated partition \mathcal{A}_1 , the problem is to know

whether some new quantized version \mathbf{X}_2 with an associated partition \mathcal{A}_2 finer than \mathcal{A}_1 should be preferred to predict \mathbf{Y} .

2. variables selection problem: suppose one wants to construct an ascending selection method. The vector \mathbf{X}_1 could represent some selected set of covariables and $\mathbf{X}_2 = (\mathbf{X}_1, \mathbf{X}'_2)$ a larger set of covariables. The aim is so to know if \mathbf{X}'_2 should be integrated to the selected set or not.

Some simple algorithms of quantization and selection methods are proposed in Robineau (2004) using these results.

$C_{\mathbf{X},\mathbf{Y}}$	Expression of $C_{\mathbf{Y},\mathbf{X}_1} - C_{\mathbf{Y},\mathbf{X}_2}$	Expression of $\frac{C_{\mathbf{Y},\mathbf{X}_1} - C_{\mathbf{Y},\mathbf{X}_2}}{C_{\mathbf{Y},\mathbf{X}_1}}$
$C_{\mathbf{X},\mathbf{Y}}^E$	$H_{\mathbf{Y},\mathbf{X}_2} - H_{\mathbf{Y},\mathbf{X}_1}$	$\frac{H_{\mathbf{Y},\mathbf{X}_2} - H_{\mathbf{Y},\mathbf{X}_1}}{H_{\mathbf{Y},\mathbf{X}_1}}$
$C_{\mathbf{X},\mathbf{Y}}^I$	$\max(H_{\mathbf{Y}}, H_{\mathbf{X}_2}) - \max(H_{\mathbf{Y}}, H_{\mathbf{X}_1})$	$\frac{\max(H_{\mathbf{Y}}, H_{\mathbf{X}_2}) - \max(H_{\mathbf{Y}}, H_{\mathbf{X}_1})}{\max(H_{\mathbf{Y}}, H_{\mathbf{X}_1})}$
$C_{\mathbf{X},\mathbf{Y}}^S$	$\frac{1}{2} (H_{\mathbf{X}_2} - H_{\mathbf{X}_1})$	$\frac{H_{\mathbf{X}_1}}{H_{\mathbf{X}_1} + H_{\mathbf{Y}}} \times \frac{H_{\mathbf{X}_2} - H_{\mathbf{X}_1}}{H_{\mathbf{X}_1}}$
$C_{\mathbf{X},\mathbf{Y}}^R$	$\frac{1}{4} (H_{\mathbf{X}_2} - H_{\mathbf{X}_1}) + \frac{\sqrt{H_{\mathbf{Y}}}}{2} (\sqrt{H_{\mathbf{X}_2}} - \sqrt{H_{\mathbf{X}_1}})$	$\frac{(H_{\mathbf{X}_2} - H_{\mathbf{X}_1}) + 2\sqrt{H_{\mathbf{Y}}} (\sqrt{H_{\mathbf{X}_2}} - \sqrt{H_{\mathbf{X}_1}})}{(\sqrt{H_{\mathbf{Y}}} + \sqrt{H_{\mathbf{X}_1}})^2}$
$C_{\mathbf{X},\mathbf{Y}}^P$	$\sqrt{H_{\mathbf{Y}}} (\sqrt{H_{\mathbf{X}_2}} - \sqrt{H_{\mathbf{X}_1}})$	$\frac{\sqrt{H_{\mathbf{X}_2}} - \sqrt{H_{\mathbf{X}_1}}}{\sqrt{H_{\mathbf{X}_1}}}$
$C_{\mathbf{X},\mathbf{Y}}^D$	$\frac{2H_{\mathbf{Y}}^2}{(H_{\mathbf{Y}} + H_{\mathbf{X}_1})(H_{\mathbf{Y}} + H_{\mathbf{X}_2})} (H_{\mathbf{X}_2} - H_{\mathbf{X}_1})$	$\frac{H_{\mathbf{Y}}}{H_{\mathbf{Y}} + H_{\mathbf{X}_2}} \times \frac{H_{\mathbf{X}_2} - H_{\mathbf{X}_1}}{H_{\mathbf{X}_1}}$

Table 1: Expressions of the absolute and relative increments of normalization terms for the different examples considered.

3.3 Around the triangular inequality's property

The question arises now whether an IB-divergence or a NIB-divergence satisfies the property [P6] that is a triangular inequality. The following proposition establishes sufficient conditions for such measures to constitute a metric.

Lemma 6

$$H_{X,Y} \leq H_{X,Z} + H_{Y,Z} - H_Z \quad (32)$$

$$I_{X,Y} \geq I_{X,Z} + I_{Y,Z} - H_Z \quad (33)$$

Proof. From general properties on entropy, one can obtain

$$H_{X,Y} \leq H_{X,Y,Z} = H_{X,Z} + H_{Y|X,Z} \leq H_{X,Z} + H_{Y|Z} = H_{X,Z} + H_{Y,Z} - H_Z. \quad (34)$$

Equation (33) directly derives from (2). ■

Proposition 7 *Assume the normalization term defining an IB-divergence satisfies the following property:*

(i)

$$C_{X,Y} \leq C_{X,Z} + C_{Y,Z} - H_Z. \quad (35)$$

Then, the associated IB-divergence satisfies the triangular inequality, that is

$$\Delta_{X,Y} \leq \Delta_{X,Z} + \Delta_{Y,Z}. \quad (36)$$

In addition, if C satisfies

(ii)

$$C_{X,Z} \geq \max(H_X, H_Z), \quad (37)$$

then the associated NIB-divergence satisfies also a triangular inequality, that is

$$\delta_{X,Y} \leq \delta_{X,Z} + \delta_{Y,Z}. \quad (38)$$

Proof. Since the following quantity

$$A = -(C_{X,Y} - I_{X,Y}) + (C_{X,Z} - I_{X,Z}) + (C_{Y,Z} - I_{Y,Z}),$$

is nonnegative from (33) and (35), we have immediately (36). Moreover, the following equation is valid

$$\delta_{X,Y} \leq 1 - \frac{I_{X,Y}}{C_{X,Y} + A}, \quad (39)$$

Now, it is also easy to see from (37) that

$$A + C_{X,Y} \geq C_{X,Z} + C_{Y,Z} - H_Z \geq \max(C_{X,Z}, C_{Y,Z}).$$

From (39) it follows

$$\delta_{\mathbf{X},\mathbf{Y}} \leq \frac{C_{\mathbf{X},\mathbf{Z}} - I_{\mathbf{X},\mathbf{Z}} + C_{\mathbf{Y},\mathbf{Z}} - I_{\mathbf{Y},\mathbf{Z}}}{\max(C_{\mathbf{X},\mathbf{Z}}, C_{\mathbf{Y},\mathbf{Z}})} \leq \frac{C_{\mathbf{X},\mathbf{Z}} - I_{\mathbf{X},\mathbf{Z}}}{C_{\mathbf{X},\mathbf{Z}}} + \frac{C_{\mathbf{Y},\mathbf{Z}} - I_{\mathbf{Y},\mathbf{Z}}}{C_{\mathbf{Y},\mathbf{Z}}} = \delta_{\mathbf{X},\mathbf{Z}} + \delta_{\mathbf{Y},\mathbf{Z}}.$$

■

Remark 1 In Proposition 7, there is no implication between (35) and (37). Indeed, one may check that the NIB-divergence δ^S satisfies the first one but not the second one. Now consider a NIB-divergence with normalization term $C_{\mathbf{X},\mathbf{Y}} = \max(H_{\mathbf{X}}, H_{\mathbf{Y}}) + H_{\mathbf{X}|\mathbf{Y}} H_{\mathbf{Y}|\mathbf{X}}$. By choosing \mathbf{X}, \mathbf{Y} and \mathbf{Z} such that $H_{\mathbf{X}|\mathbf{Y}} = H_{\mathbf{Y}|\mathbf{X}} = I_{\mathbf{X},\mathbf{Y}} = H_{\mathbf{Z}}/3 = H_{\mathbf{X},\mathbf{Y}}/3 > 2$, one asserts that (37) is satisfied but not (35).

Remark 2 Let us consider a NIB-divergence δ with normalization term $C_{\mathbf{X},\mathbf{Y}} = C'_{\mathbf{X},\mathbf{Y}} + \max(H_{\mathbf{X}}, H_{\mathbf{Y}})$ such that $C'_{\mathbf{X},\mathbf{Y}} \geq 0$ (necessarily $C'_{\mathbf{X},\mathbf{Y}} = 0$ whenever $\mathbf{X} \sim \mathbf{Y}$). Then, Δ and δ satisfy a triangular inequality if C' also satisfies a triangular inequality. However, this is not a necessary condition. Indeed, the triangular inequality is not satisfied for the same example of the previous remark with $C'_{\mathbf{X},\mathbf{Y}} = H_{\mathbf{X}|\mathbf{Y}} H_{\mathbf{Y}|\mathbf{X}}$ for which $C'_{\mathbf{X},\mathbf{Z}} = C'_{\mathbf{Y},\mathbf{Z}} = 0$ whereas $C'_{\mathbf{X},\mathbf{Y}} > 0$.

Let us now propose some examples and consequences through the following corollary.

Corollary 8

- (i) The measures D^E , D^I and D^S satisfy the condition (35) and so are metrics.
- (ii) The measures d^E and d^I satisfy the conditions (35) and (37) and so are metrics.
- (iii) Let $(\alpha^{(j)})_{j=1,\dots,J}$ be some vector of probability weights for some $J \geq 1$. Let $\Delta^{(1)}, \dots, \Delta^{(J)}$, J IB-divergences (resp. $\delta^{(1)}, \dots, \delta^{(J)}$, J NIB-divergences) with normalization terms $C_{\mathbf{X},\mathbf{Y}}^{(1)}, \dots, C_{\mathbf{X},\mathbf{Y}}^{(J)}$ satisfying (35) (resp. (35) and (37)) then these measures defined by (25) satisfy a triangular inequality.

Proof. (i) and (ii) Equation (32) corresponds exactly to (35) for $C_{\mathbf{X},\mathbf{Y}}^E = H_{\mathbf{X},\mathbf{Y}}$. And since $H_{\mathbf{X},\mathbf{Z}} \geq \max(H_{\mathbf{X}}, H_{\mathbf{Z}})$, we have proved that D^E and d^E are metrics. Concerning D^I and d^I , the normalization term corresponds to $C_{\mathbf{X},\mathbf{Y}}^I = \max(H_{\mathbf{X}}, H_{\mathbf{Y}})$. Thus it is sufficient to prove (35) which is quite obvious. Indeed,

$$\max(H_{\mathbf{X}}, H_{\mathbf{Z}}) + \max(H_{\mathbf{Y}}, H_{\mathbf{Z}}) - H_{\mathbf{Z}} \geq \max(H_{\mathbf{X}}, H_{\mathbf{Y}}).$$

Concerning, D^S it is easily checked that the associated normalization term satisfies

$$C_{\mathbf{X},\mathbf{Y}}^S - C_{\mathbf{X},\mathbf{Z}}^S - C_{\mathbf{Y},\mathbf{Z}}^S + H_{\mathbf{Z}} = 0.$$

(iii) trivial. ■

We did not state in the previous corollary the obvious following result : let $\delta^{(1)}, \dots, \delta^{(J)}$, J NIB-divergences satisfying the triangular inequality (38), then the measure defined by (23), that is as a linear combination, satisfies also (38). A similar result can be written for IB-divergences that are metrics.

The measure Δ^D (and so δ^D) does not satisfy the condition (35). Indeed, let us choose \mathbf{X} , \mathbf{Y} and \mathbf{Z} such that $H_{\mathbf{X}} = H_{\mathbf{Y}} = \frac{1}{3}H_{\mathbf{Z}} > 0$, then

$$C_{\mathbf{X},\mathbf{Z}}^D + C_{\mathbf{Y},\mathbf{Z}}^D - H_{\mathbf{Z}} - C_{\mathbf{X},\mathbf{Y}}^D = -H_{\mathbf{X}} < 0,$$

which is in contradiction with (35). Moreover, if \mathbf{X} , \mathbf{Y} and \mathbf{Z} are such that $H_{\mathbf{X}} = H_{\mathbf{Y}} = \frac{1}{4}H_{\mathbf{Z}} > 0$, then

$$C_{\mathbf{X},\mathbf{Z}}^P + C_{\mathbf{Y},\mathbf{Z}}^P - H_{\mathbf{Z}} - C_{\mathbf{X},\mathbf{Y}}^P = -\frac{1}{4}H_{\mathbf{Z}} < 0.$$

And since $C_{\mathbf{X},\mathbf{Y}}^R = \frac{1}{2}C_{\mathbf{X},\mathbf{Y}}^P + \frac{1}{2}C_{\mathbf{X},\mathbf{Y}}^S$, the same choice of \mathbf{X} , \mathbf{Y} and \mathbf{Z} leads to

$$C_{\mathbf{X},\mathbf{Z}}^R + C_{\mathbf{Y},\mathbf{Z}}^R - H_{\mathbf{Z}} - C_{\mathbf{X},\mathbf{Y}}^R = \frac{1}{2} \left(C_{\mathbf{X},\mathbf{Z}}^P + C_{\mathbf{Y},\mathbf{Z}}^P - H_{\mathbf{Z}} - C_{\mathbf{X},\mathbf{Y}}^P \right) = -\frac{1}{8}H_{\mathbf{Z}} < 0.$$

Both of the previous equations imply that $\Delta_{\mathbf{X},\mathbf{Y}}^P$ and $\Delta_{\mathbf{X},\mathbf{Y}}^R$ (and so $\delta_{\mathbf{X},\mathbf{Y}}^P$ and $\delta_{\mathbf{X},\mathbf{Y}}^R$) do not satisfy the condition (35).

One can also wonder if the NIB-divergence δ^S defined by (16) satisfies (35) and (37). The associated normalization term, $C_{\mathbf{X},\mathbf{Y}} = \frac{1}{2}(H_{\mathbf{X}} + H_{\mathbf{Y}})$, clearly satisfies (35) but not (37). And so, concerning this measure we do not know if it satisfies a triangular inequality but our tool cannot be applied to prove it. However, for this measure we manage to obtain a weaker result which has nevertheless a nice interpretation.

Proposition 9 *Let us consider the following assumptions on a normalization term: there exists a constant $c \geq 1$ such that*

$$c C_{\mathbf{X},\mathbf{Z}} + c C_{\mathbf{Y},\mathbf{Z}} - H_{\mathbf{Z}} - (c-1)(I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}) \geq C_{\mathbf{X},\mathbf{Y}} \quad (40)$$

$$c C_{\mathbf{X},\mathbf{Z}} + c C_{\mathbf{Y},\mathbf{Z}} - H_{\mathbf{Z}} - (c-1)(I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}) \geq \max(C_{\mathbf{X},\mathbf{Y}}, C_{\mathbf{X},\mathbf{Z}}, C_{\mathbf{Y},\mathbf{Z}}). \quad (41)$$

If an IB-divergence satisfies (40) or a NIB-divergence satisfies (41), then

$$\Delta_{\mathbf{X},\mathbf{Y}} \leq c \times (\Delta_{\mathbf{X},\mathbf{Z}} + \Delta_{\mathbf{Y},\mathbf{Z}}) \quad \text{or} \quad \delta_{\mathbf{X},\mathbf{Y}} \leq c \times (\delta_{\mathbf{X},\mathbf{Z}} + \delta_{\mathbf{Y},\mathbf{Z}}). \quad (42)$$

Proof. In fact, by introducing

$$A = \left(C_{X,Y} - I_{X,Y} \right) + c \times (C_{X,Z} - I_{X,Z}) + c \times (C_{Y,Z} - I_{Y,Z})$$

the inequality (41) stands for the two following assumptions which are direct generalizations of assumptions of Proposition 7:

$$C_{X,Y} \leq c C_{X,Z} + c C_{Y,Z} - H_Z - (c-1)(I_{X,Z} + I_{Y,Z}) \quad (43)$$

$$c C_{X,Z} + (c-1)C_{Y,Z} - H_Z - (c-1)(I_{X,Z} + I_{Y,Z}) \geq 0 \quad (44)$$

From (43) and (33), one may assert that

$$A \geq c C_{X,Z} + c C_{Y,Z} - C_{X,Y} - H_Z - (c-1)(I_{X,Z} + I_{Y,Z}) \geq 0,$$

which implies that the result is valid for Δ . Now, from (44) one can write

$$A + C_{X,Y} \geq \max(C_{X,Z}, C_{Y,Z})$$

which leads to

$$\delta_{X,Y} \leq \frac{c \times (C_{X,Z} - I_{X,Z}) + c \times (C_{Y,Z} - I_{Y,Z})}{\max(C_{X,Z}, C_{Y,Z})} \leq c \times \delta_{X,Z} + c \times \delta_{Y,Z}.$$

■

Corollary 10

(i) The NIB-divergence δ^S satisfies (42) with $c = 2$.

(ii) The NIB-divergence with normalization term defined by

$$C_{X,Y} = \alpha \max(H_X, H_Y) + (1 - \alpha) \frac{H_X + H_Y}{2}$$

for some $\alpha > 0$, satisfies (42) with the constant $c = \frac{2}{1+\alpha}$. In particular, when $\alpha = \frac{1}{2}$ (resp. $\alpha \rightarrow 1$), we have $c = \frac{4}{3}$ (resp. $c \rightarrow 1$).

(iii) Let us give $\delta_{X,Y}^{(1)}$ and $\delta_{X,Y}^{(2)}$ two NIB-divergences with normalization term $C_{X,Y}^{(1)}$ and $C_{X,Y}^{(2)}$ such that

$$\delta_{X,Y}^{(1)} \leq \delta_{X,Y}^{(2)} \iff C_{X,Y}^{(1)} \leq C_{X,Y}^{(2)}.$$

Assume $\delta^{(1)}$ satisfies (42) for some constant κ , and assume $C^{(2)}$ satisfies (43) for some constant $c \leq \kappa$, then $\delta^{(2)}$ satisfies (42) with the constant $c = \kappa$.

(iv) The IB-divergence (resp. NIB-divergence) with normalization term $C_{X,Y}^R = \left(\frac{\sqrt{H_X} + \sqrt{H_Y}}{2} \right)^2$ satisfies (40) (resp. (41)) with $c = 2$ (resp. with $c = 4$).

As an example of (iii) consider the NIB-divergence with normalization term defined for some $\alpha > 0$ by

$$C_{\mathbf{X},\mathbf{Y}} = \alpha H_{\mathbf{X},\mathbf{Y}} + (1 - \alpha) \frac{H_{\mathbf{X}} + H_{\mathbf{Y}}}{2}.$$

Then, thanks to (ii), this NIB-divergence satisfies (42) with the constant $c = \frac{2}{1+\alpha}$.

Proof. (i) Let us denote by A the following quantity for some $c \geq 1$.

$$A = c C_{\mathbf{X},\mathbf{Z}} + (c - 1) C_{\mathbf{Y},\mathbf{Z}} - H_{\mathbf{Z}} - (c - 1) (I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}})$$

When $C_{\mathbf{X},\mathbf{Z}} = \frac{1}{2} (H_{\mathbf{X}} + H_{\mathbf{Z}})$, this leads to

$$A = \frac{c}{2} H_{\mathbf{X}} + \frac{c-1}{2} H_{\mathbf{Y}} + \left(c - \frac{3}{2} \right) H_{\mathbf{Z}} - (c - 1) (I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}).$$

In the independent case, the best constant is $c = 3/2$. So we are ensured that $c \geq 3/2$. It is impossible to prove that $c = 3/2$ implies $A \geq 0$. Since $I_{\mathbf{Y},\mathbf{Z}} \leq 1/2(H_{\mathbf{Y}} + H_{\mathbf{Z}})$, one may obtain

$$A \geq \frac{c}{2} H_{\mathbf{X}} + \left(\frac{c}{2} - 1 \right) H_{\mathbf{Z}} - (c - 1) I_{\mathbf{X},\mathbf{Z}}.$$

Again, we obtain that $c \geq 2$. And when $c = 2$, it follows that: $A \geq H_{\mathbf{X}} - I_{\mathbf{X},\mathbf{Z}} \geq 0$.

(ii) Again the two normalization terms satisfy (43) with $c = 1$. So let us concentrate on (44). We have

$$\begin{aligned} A &= c\alpha \max(H_{\mathbf{X}}, H_{\mathbf{Z}}) + c \frac{1-\alpha}{2} (H_{\mathbf{X}} + H_{\mathbf{Z}}) + (c-1)\alpha \max(H_{\mathbf{Y}}, H_{\mathbf{Z}}) \\ &\quad + (c-1) \frac{1-\alpha}{2} (H_{\mathbf{Y}} + H_{\mathbf{Z}}) - H_{\mathbf{Z}} - (c-1) (I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}). \end{aligned}$$

After some further calculation, we obtain

$$A \geq c \frac{1-\alpha}{2} H_{\mathbf{X}} + \left(c \frac{1+\alpha}{2} - 1 \right) H_{\mathbf{Z}} - (c-1) I_{\mathbf{X},\mathbf{Z}}.$$

In the independent case, the best constant is $c = 2/(1 + \alpha)$, so we are ensured that $c \geq 2/(1 + \alpha)$. And when $c = 2/(1 + \alpha)$, we have

$$A \geq \frac{1-\alpha}{1+\alpha} (H_{\mathbf{X}} - I_{\mathbf{X},\mathbf{Z}}) \geq 0,$$

which implies that (44) is satisfied.

(iii) The proof is omitted.

(iv) Let us first prove that for the penalization term defined by $C_{\mathbf{X},\mathbf{Y}}^R = \left(\frac{\sqrt{H_{\mathbf{X}}} + \sqrt{H_{\mathbf{Y}}}}{2}\right)^2$ the condition (35) is not always satisfied. For example, let us give \mathbf{X}, \mathbf{Y} and \mathbf{Z} such that $H_{\mathbf{X}} = 1, H_{\mathbf{Y}} = 4$ and $H_{\mathbf{Z}} = 2$ then

$$C_{\mathbf{X},\mathbf{Z}}^R + C_{\mathbf{Y},\mathbf{Z}}^R - H_{\mathbf{Z}} - C_{\mathbf{X},\mathbf{Y}}^R = \sqrt{\frac{5}{2}} + \sqrt{10} - 2 - \sqrt{\frac{17}{2}} < 0.$$

Now, let A denote the following quantity

$$A = 2C_{\mathbf{X},\mathbf{Z}}^R + 2C_{\mathbf{Y},\mathbf{Z}}^R - H_{\mathbf{Z}} - (I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}).$$

Since $I_{\mathbf{X},\mathbf{Z}} \leq \min(H_{\mathbf{X}}, H_{\mathbf{Z}}) \leq \sqrt{H_{\mathbf{X}}H_{\mathbf{Z}}}$, we obtain

$$\begin{aligned} A &= \frac{1}{2}H_{\mathbf{X}} + \frac{1}{2}H_{\mathbf{Y}} + \sqrt{H_{\mathbf{X}}H_{\mathbf{Z}}} + \sqrt{H_{\mathbf{Y}}H_{\mathbf{Z}}} - (I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}) \\ &\geq \frac{H_{\mathbf{X}} + H_{\mathbf{Y}}}{2} \\ &\geq \left(\frac{\sqrt{H_{\mathbf{X}}} + \sqrt{H_{\mathbf{Y}}}}{2}\right)^2 = C_{\mathbf{X},\mathbf{Y}}^R, \end{aligned}$$

which yields the result. The fact that the associated NIB-divergence satisfies (41) with $c = 4$ is left to the reader. ■

Remark 3 *The tool presented in Proposition 9 cannot be applied to the IB-divergence Δ^D and the NIB-divergence δ^D . Indeed, let us give some $c \geq 1$ and let us consider the quantity*

$$A = c C_{\mathbf{X},\mathbf{Z}}^D + c C_{\mathbf{Y},\mathbf{Z}}^D - H_{\mathbf{Z}} - (c-1)(I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}})$$

with $C_{\mathbf{X},\mathbf{Y}}^D = 2\left(\frac{1}{H_{\mathbf{X}}} + \frac{1}{H_{\mathbf{Y}}}\right)^{-1} = \frac{2H_{\mathbf{X}}H_{\mathbf{Y}}}{H_{\mathbf{X}}+H_{\mathbf{Y}}}$. In fact, one can always find $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ such that for all $c \geq 1$, the quantity A is negative. Indeed, let us choose \mathbf{Z} independent of \mathbf{X} and \mathbf{Y} and such that $H_{\mathbf{Z}} + H_{\mathbf{X}} = 6cH_{\mathbf{X}}$ and $H_{\mathbf{Z}} + H_{\mathbf{Y}} = 6cH_{\mathbf{Y}}$. Then, it is easy to see that $A = H_{\mathbf{Z}}\left(\frac{1}{3} + \frac{1}{3} - 1\right) \leq 0$. In the same manner, the tool is inapplicable to the IB-divergence Δ^P and the NIB-divergence δ^P . Indeed, let us give \mathbf{Z} independent of \mathbf{X} and \mathbf{Y} and such that $H_{\mathbf{X}} = H_{\mathbf{Y}} = \frac{1}{9c^2}H_{\mathbf{Z}}$, then

$$A = c C_{\mathbf{X},\mathbf{Z}}^P + c C_{\mathbf{Y},\mathbf{Z}}^P - H_{\mathbf{Z}} - (c-1)(I_{\mathbf{X},\mathbf{Z}} + I_{\mathbf{Y},\mathbf{Z}}) = -\frac{1}{3}H_{\mathbf{Z}} < 0.$$

The following result is an extension of Proposition 9 well-suited to be applied to δ^D .

Proposition 11 *Let us assume that there exists two positive integer I and J such that a NIB-divergence $\delta_{\mathbf{X},\mathbf{Y}}$ can be expressed as:*

$$\delta_{\mathbf{X},\mathbf{Y}} = \sum_{i=1}^I \frac{S_{\mathbf{X},\mathbf{Y}}^{(i)}}{U_{\mathbf{X},\mathbf{Y}}^{(i)}} = \sum_{j=1}^J \alpha^{(j)} \left(1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{C_{\mathbf{X},\mathbf{Y}}^{(j)}} \right)$$

where $(\alpha^{(j)})_{j=1,\dots,J}$ is some vector of probability weights. By denoting $S_{\mathbf{X},\mathbf{Y}} = \sum_{i=1}^I S_{\mathbf{X},\mathbf{Y}}^{(i)}$ and $U_{\mathbf{X},\mathbf{Y}} = \max_{i=1,\dots,I} U_{\mathbf{X},\mathbf{Y}}^{(i)}$, if there exists some real number $c \geq 1$ such that for any $j = 1, \dots, J$ the following assumptions are satisfied:

$$(i) \quad A^{(j)} = I_{\mathbf{X},\mathbf{Y}} - C_{\mathbf{X},\mathbf{Y}}^{(j)} + c(S_{\mathbf{X},\mathbf{Z}} + S_{\mathbf{Z},\mathbf{Y}}) \geq 0.$$

$$(ii) \quad A^{(j)} + C_{\mathbf{X},\mathbf{Y}}^{(j)} \geq \max(U_{\mathbf{X},\mathbf{Z}}, U_{\mathbf{Z},\mathbf{Y}}).$$

then the equation (42) is valid for $\delta_{\mathbf{X},\mathbf{Y}}$.

After noticing that

$$\delta_{\mathbf{X},\mathbf{Y}} = 1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{C_{\mathbf{X},\mathbf{Y}}} \quad \text{with} \quad C_{\mathbf{X},\mathbf{Y}} = \left(\sum_{j=1}^J \frac{\alpha^{(j)}}{C_{\mathbf{X},\mathbf{Y}}^{(j)}} \right)^{-1}$$

the different normalization terms, $C^{(j)}$, do not need to be symmetric as long as $C_{\mathbf{X},\mathbf{Y}}$ is symmetric. The measure δ^D is exactly constructed like this. Indeed, δ^D is such that $C_{\mathbf{X},\mathbf{Y}}^{(1)} = H_{\mathbf{X}}$, $C_{\mathbf{X},\mathbf{Y}}^{(2)} = H_{\mathbf{Y}}$ and $C_{\mathbf{X},\mathbf{Y}} = 2 \left(\frac{1}{H_{\mathbf{X}}} + \frac{1}{H_{\mathbf{Y}}} \right)^{-1}$, that is obviously symmetric.

Proof. Using assumptions (i) and (ii), one can prove that for all $j = 1, \dots, J$

$$\begin{aligned} 1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{C_{\mathbf{X},\mathbf{Y}}^{(j)}} &\leq 1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{C_{\mathbf{X},\mathbf{Y}}^{(j)} + A^{(j)}} \leq c \frac{S_{\mathbf{X},\mathbf{Z}} + S_{\mathbf{Y},\mathbf{Z}}}{\max(U_{\mathbf{X},\mathbf{Z}}, U_{\mathbf{Z},\mathbf{Y}})} \\ &\leq c \times (\delta_{\mathbf{X},\mathbf{Z}} + \delta_{\mathbf{Y},\mathbf{Z}}). \end{aligned}$$

It follows that

$$\delta_{\mathbf{X},\mathbf{Y}} = \sum_{j=1}^J \alpha^{(j)} \left(1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{C_{\mathbf{X},\mathbf{Y}}^{(j)}} \right) \leq \sum_{j=1}^J \alpha^{(j)} \times c \times (\delta_{\mathbf{X},\mathbf{Z}} + \delta_{\mathbf{Y},\mathbf{Z}}) = c \times (\delta_{\mathbf{X},\mathbf{Z}} + \delta_{\mathbf{Y},\mathbf{Z}}).$$

■

Corollary 12 *The measure δ^D satisfies (42) with the constant $c = 2$.*

Proof. We have

$$\delta_{\mathbf{X},\mathbf{Y}}^D = \frac{1}{2} \left(\frac{H_{\mathbf{X}|\mathbf{Y}}}{H_{\mathbf{X}}} + \frac{H_{\mathbf{Y}|\mathbf{X}}}{H_{\mathbf{Y}}} \right) = \frac{1}{2} \left(1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{H_{\mathbf{X}}} + 1 - \frac{I_{\mathbf{X},\mathbf{Y}}}{H_{\mathbf{Y}}} \right)$$

By identification with notation introduced in Proposition 11, we have $I = J = 2$, $S_{\mathbf{X},\mathbf{Y}}^{(1)} = \frac{1}{2}H_{\mathbf{X}|\mathbf{Y}}$, $S_{\mathbf{X},\mathbf{Y}}^{(2)} = \frac{1}{2}H_{\mathbf{Y}|\mathbf{X}}$, $U_{\mathbf{X},\mathbf{Y}}^{(1)} = H_{\mathbf{X}}$, $U_{\mathbf{X},\mathbf{Y}}^{(2)} = H_{\mathbf{Y}}$, $C_{\mathbf{X},\mathbf{Y}}^{(1)} = H_{\mathbf{X}}$ and $C_{\mathbf{X},\mathbf{Y}}^{(2)} = H_{\mathbf{Y}}$. Let us fix c to the value 2. We have

$$A^{(1)} = I_{\mathbf{X},\mathbf{Y}} - H_{\mathbf{X}} + \left(H_{\mathbf{X}|\mathbf{Z}} + H_{\mathbf{Z}|\mathbf{X}} + H_{\mathbf{Y}|\mathbf{Z}} + H_{\mathbf{Z}|\mathbf{Y}} \right)$$

Clearly,

$$\begin{aligned} A^{(1)} &= H_{\mathbf{Y}} - H_{\mathbf{X},\mathbf{Y}} + 2H_{\mathbf{X},\mathbf{Z}} + 2H_{\mathbf{Y},\mathbf{Z}} - H_{\mathbf{X}} - H_{\mathbf{Y}} - 2H_{\mathbf{Z}} \\ &\geq H_{\mathbf{X},\mathbf{Z}} + H_{\mathbf{Y},\mathbf{Z}} - H_{\mathbf{X}} - H_{\mathbf{Z}}, \\ &\geq 0. \end{aligned}$$

And one also has

$$H_{\mathbf{X}} + A^{(1)} \geq H_{\mathbf{X},\mathbf{Z}} + H_{\mathbf{Y},\mathbf{Z}} - H_{\mathbf{Z}} \geq \max(H_{\mathbf{X}}, H_{\mathbf{Y}}, H_{\mathbf{Z}}) = \max(U_{\mathbf{X},\mathbf{Z}}, U_{\mathbf{Y},\mathbf{Z}}).$$

It follows that $A^{(1)}$ fullfills conditions (i) and (ii) of Proposition 11 with $c = 2$. The proof is strictly similar for $A^{(2)}$. ■

We summarize the different results obtained for our examples in Tab. 2

D^E, d^E	D^I, d^I	D^S, δ^S	Δ^R, δ^R	Δ^P, δ^P	Δ^D, δ^D
$c = 1$	$c = 1$	$c = 1 \quad c \leq 2$	$c \leq 2 \quad c \leq 4$	$c = +\infty$	$c = +\infty \quad c = 2$

Table 2: Values of the parameter c ensuring an inequality of the form (42). A value $c = 1$ indicates that the considered measure constitutes a metric and a value $c = +\infty$ asserts that we proved that for any $c \geq 1$ Equation (40) is not valid.

3.4 Around the redundancy of two random vectors \mathbf{X}_1 and \mathbf{X}_2

In the future use of an IB-divergence or NIB-divergence, one would expect that if two discrete-valued random vectors \mathbf{X}_1 and \mathbf{X}_2 have the same (or almost the same) information with respect to an IB-divergence or NIB-divergence, then both have the same

effect on another vector \mathbf{Y} . This requirement, expressed by the property **[P10]**, could be used for example in a variables selection problem in the context of discrimination to detect redundant variables.

We may hope that **[P10]** holds whenever **[P3bis]** is satisfied. Let us start to write $I_{\mathbf{x}_1, \mathbf{x}_2} = H_{\mathbf{x}_1} - H_{\mathbf{x}_1 | \mathbf{x}_2} = H_{\mathbf{x}_2} - H_{\mathbf{x}_2 | \mathbf{x}_1}$, then

$$|H_{\mathbf{x}_2} - H_{\mathbf{x}_1}| = |H_{\mathbf{x}_2 | \mathbf{x}_1} - H_{\mathbf{x}_1 | \mathbf{x}_2}| \leq \max(H_{\mathbf{x}_2 | \mathbf{x}_1}, H_{\mathbf{x}_1 | \mathbf{x}_2}).$$

Otherwise, $H_{\mathbf{Y}, \mathbf{x}_1, \mathbf{x}_2} = H_{\mathbf{Y}, \mathbf{x}_1} + H_{\mathbf{x}_2 | (\mathbf{Y}, \mathbf{x}_1)} = H_{\mathbf{Y}, \mathbf{x}_2} + H_{\mathbf{x}_1 | (\mathbf{Y}, \mathbf{x}_2)}$, and so

$$\begin{aligned} |H_{\mathbf{Y}, \mathbf{x}_2} - H_{\mathbf{Y}, \mathbf{x}_1}| &= |H_{\mathbf{x}_2 | (\mathbf{Y}, \mathbf{x}_1)} - H_{\mathbf{x}_1 | (\mathbf{Y}, \mathbf{x}_2)}| \\ &\leq \max(H_{\mathbf{x}_2 | (\mathbf{Y}, \mathbf{x}_1)}, H_{\mathbf{x}_1 | (\mathbf{Y}, \mathbf{x}_2)}) \leq \max(H_{\mathbf{x}_2 | \mathbf{x}_1}, H_{\mathbf{x}_1 | \mathbf{x}_2}). \end{aligned}$$

Now,

$$\begin{aligned} |\Delta_{\mathbf{Y}, \mathbf{x}_1} - \Delta_{\mathbf{Y}, \mathbf{x}_2}| &\leq |C_{\mathbf{Y}, \mathbf{x}_1} - C_{\mathbf{Y}, \mathbf{x}_2}| + |I_{\mathbf{Y}, \mathbf{x}_1} - I_{\mathbf{Y}, \mathbf{x}_2}| \\ &\leq |C_{\mathbf{Y}, \mathbf{x}_1} - C_{\mathbf{Y}, \mathbf{x}_2}| + |H_{\mathbf{x}_2} - H_{\mathbf{x}_1}| + |H_{\mathbf{Y}, \mathbf{x}_2} - H_{\mathbf{Y}, \mathbf{x}_1}| \\ &\leq |C_{\mathbf{Y}, \mathbf{x}_1} - C_{\mathbf{Y}, \mathbf{x}_2}| + 2 \times \max(H_{\mathbf{x}_2 | \mathbf{x}_1}, H_{\mathbf{x}_1 | \mathbf{x}_2}). \end{aligned}$$

From **[P3bis]** and the continuity of $f_C(\cdot, \cdot, \cdot)$, we may state

$$\begin{aligned} \Delta_{\mathbf{x}_1, \mathbf{x}_2} \rightarrow 0 &\implies \max(H_{\mathbf{x}_2 | \mathbf{x}_1}, H_{\mathbf{x}_1 | \mathbf{x}_2}) \rightarrow 0 \\ &\implies |C_{\mathbf{Y}, \mathbf{x}_1} - C_{\mathbf{Y}, \mathbf{x}_2}| \rightarrow 0. \end{aligned}$$

Finally, we obtain

$$\Delta_{\mathbf{x}_1, \mathbf{x}_2} \rightarrow 0 \implies |\Delta_{\mathbf{Y}, \mathbf{x}_1} - \Delta_{\mathbf{Y}, \mathbf{x}_2}| \rightarrow 0.$$

In order to make the property **[P10]** applicable for practical purpose, we may find interesting to have a bound of the difference $|\Delta_{\mathbf{Y}, \mathbf{x}_1} - \Delta_{\mathbf{Y}, \mathbf{x}_2}|$ (resp. $|\delta_{\mathbf{Y}, \mathbf{x}_1} - \delta_{\mathbf{Y}, \mathbf{x}_2}|$) expressed in terms of $\Delta_{\mathbf{x}_1, \mathbf{x}_2}$ (resp. $\delta_{\mathbf{x}_1, \mathbf{x}_2}$). More precisely, the question may arise whether there exists a function $h(\cdot)$ satisfying $h(x) \rightarrow 0$ as $x \rightarrow 0$ and such that $|\Delta_{\mathbf{Y}, \mathbf{x}_1} - \Delta_{\mathbf{Y}, \mathbf{x}_2}| \leq h(\Delta_{\mathbf{x}_1, \mathbf{x}_2})$ (resp. $|\delta_{\mathbf{Y}, \mathbf{x}_1} - \delta_{\mathbf{Y}, \mathbf{x}_2}| \leq h(\delta_{\mathbf{x}_1, \mathbf{x}_2})$).

As a first answer, let us precise that if the IB-divergence or NIB-divergence satisfies a triangular inequality then this result is easily obtained with $h(x) = x$, since the triangular inequality for Δ or δ implies that

$$|\Delta_{\mathbf{Y}, \mathbf{x}_1} - \Delta_{\mathbf{Y}, \mathbf{x}_2}| \leq \Delta_{\mathbf{x}_1, \mathbf{x}_2} \quad \text{or} \quad |\delta_{\mathbf{Y}, \mathbf{x}_1} - \delta_{\mathbf{Y}, \mathbf{x}_2}| \leq \delta_{\mathbf{x}_1, \mathbf{x}_2}.$$

A priori, if an IB-divergence or NIB-divergence satisfies only an inequality like (42) with some $c > 1$, then this property does no more seem to be true: indeed, for all \mathbf{Y} , \mathbf{X}_1 and \mathbf{X}_2 , one may prove for a NIB-divergence by instance that

$$|\delta_{\mathbf{Y},\mathbf{X}_1} - \delta_{\mathbf{Y},\mathbf{X}_2}| \leq c \times \delta_{\mathbf{X}_1,\mathbf{X}_2} + (c - 1) \min(\delta_{\mathbf{Y},\mathbf{X}_1}, \delta_{\mathbf{Y},\mathbf{X}_2}).$$

The right-hand side of the previous inequality is not in general small when $\delta_{\mathbf{X}_1,\mathbf{X}_2}$ is small. Actually, this apparent disappointing result only expresses that a “redundant” property cannot (always) be derived from a triangular’s type inequality. By directly taking into account our objective, we can state a finer result for NIB-divergences.

Proposition 13 *Let us assume that there exists a real number $\varepsilon_0 \in [0, 1[$ and a positive decreasing continuous function $\alpha(\cdot)$ defined for any $\varepsilon \in [0, \varepsilon_0]$ satisfying $\alpha(\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$, such that:*

$$C_{\mathbf{X}_1,\mathbf{X}_2} \geq \alpha(\varepsilon) \max(H_{\mathbf{X}_1}, H_{\mathbf{X}_2}) \quad (45)$$

holds for all $\mathbf{X}_1, \mathbf{X}_2$ such that $\delta_{\mathbf{X}_1,\mathbf{X}_2} = \varepsilon \leq \varepsilon_0$. Then, for any \mathbf{Y}

$$|\delta_{\mathbf{Y},\mathbf{X}_1} - \delta_{\mathbf{Y},\mathbf{X}_2}| \leq \frac{1}{\alpha(\varepsilon_0)} \left(\frac{1}{(1 - \varepsilon)\alpha(\varepsilon)} - 1 \right). \quad (46)$$

Proof. First of all let us note that

$$\min(H_{\mathbf{X}_1}, H_{\mathbf{X}_2}) \geq I_{\mathbf{X}_1,\mathbf{X}_2} = (1 - \varepsilon)C_{\mathbf{X}_1,\mathbf{X}_2} \geq (1 - \varepsilon)\alpha(\varepsilon) \max(H_{\mathbf{X}_1}, H_{\mathbf{X}_2}). \quad (47)$$

We have,

$$|\delta_{\mathbf{Y},\mathbf{X}_1} - \delta_{\mathbf{Y},\mathbf{X}_2}| = \left| \frac{I_{\mathbf{Y},\mathbf{X}_1}}{C_{\mathbf{Y},\mathbf{X}_1}} - \frac{I_{\mathbf{Y},\mathbf{X}_2}}{C_{\mathbf{Y},\mathbf{X}_2}} \right| \leq \frac{|I_{\mathbf{Y},\mathbf{X}_1} - I_{\mathbf{Y},\mathbf{X}_2}|}{\min(C_{\mathbf{Y},\mathbf{X}_1}, C_{\mathbf{Y},\mathbf{X}_2})}.$$

On the one hand, we have

$$I_{\mathbf{Y},\mathbf{X}_1} \geq I_{\mathbf{Y},\mathbf{X}_2} + I_{\mathbf{X}_1,\mathbf{X}_2} - H_{\mathbf{Z}}.$$

One can deduce

$$\begin{aligned} |I_{\mathbf{Y},\mathbf{X}_1} - I_{\mathbf{Y},\mathbf{X}_2}| &\leq \max(H_{\mathbf{X}_1}, H_{\mathbf{X}_2}) - I_{\mathbf{X}_1,\mathbf{X}_2} = \max(H_{\mathbf{X}_1|\mathbf{X}_2}, H_{\mathbf{X}_2|\mathbf{X}_1}) \\ &\leq \max(H_{\mathbf{X}_1}, H_{\mathbf{X}_2}) (1 - (1 - \varepsilon)\alpha(\varepsilon)) \end{aligned} \quad (48)$$

On the other hand from assumption (45), we have

$$\min \left(C_{Y, X_1}, C_{Y, X_2} \right) \geq \alpha(\varepsilon_0) \min \left(H_{X_1}, H_{X_2} \right). \quad (49)$$

Hence, we obtain

$$\begin{aligned} \left| \delta_{Y, X_1} - \delta_{Y, X_2} \right| &\leq \frac{1 - (1 - \varepsilon)\alpha(\varepsilon)}{\alpha(\varepsilon_0)} \frac{\max \left(H_{X_1}, H_{X_2} \right)}{\min \left(H_{X_1}, H_{X_2} \right)}, \quad \text{from (48) and (49)} \\ &\leq \frac{1 - (1 - \varepsilon)\alpha(\varepsilon)}{\alpha(\varepsilon_0)(1 - \varepsilon)\alpha(\varepsilon)} \quad \text{from (47),} \end{aligned}$$

which ends the proof. ■

Remark 4 *The previous result is valid for all NIB-divergences with normalization term such that $C_{X_1, X_2} \geq \max \left(H_{X_1}, H_{X_2} \right)$. In particular, it can be applied for the NIB-divergence $C_{X_1, X_2} = \max \left(H_{X_1}, H_{X_2} \right) + H_{X_1|X_2} H_{X_2|X_1}$, for which Remark 1 stated that it was not a metric.*

Corollary 14 *The assumption (45) applied respectively to C^S , C^R , C^P and C^D is expressed by*

$$C_{X_1, X_2}^S \geq \frac{1}{1 + \varepsilon} \max \left(H_{X_1}, H_{X_2} \right), \quad \text{when } \delta_{X_1, X_2}^S = \varepsilon \leq \varepsilon_0 < 1 \quad (50)$$

$$C_{X_1, X_2}^R \geq \left(\frac{1}{2 - \sqrt{1 - \varepsilon}} \right)^2 \max \left(H_{X_1}, H_{X_2} \right), \quad \text{when } \delta_{X_1, X_2}^R = \varepsilon \leq \varepsilon_0 < 1 \quad (51)$$

$$C_{X_1, X_2}^P \geq (1 - \varepsilon) \max \left(H_{X_1}, H_{X_2} \right), \quad \text{when } \delta_{X_1, X_2}^P = \varepsilon \leq \varepsilon_0 < 1 \quad (52)$$

$$C_{X_1, X_2}^D \geq \frac{1 - 2\varepsilon}{1 - \varepsilon} \max \left(H_{X_1}, H_{X_2} \right), \quad \text{when } \delta_{X_1, X_2}^D = \varepsilon \leq \varepsilon_0 < \frac{1}{2} \quad (53)$$

which leads to

$$\left| \delta_{Y, X_1}^S - \delta_{Y, X_2}^S \right| \leq (1 + \varepsilon_0) \times \frac{2\varepsilon}{1 - \varepsilon} \quad (54)$$

$$\left| \delta_{Y, X_1}^R - \delta_{Y, X_2}^R \right| \leq 4 \left(2 - \sqrt{1 - \varepsilon_0} \right)^2 \times \frac{1 - \sqrt{1 - \varepsilon}}{1 - \varepsilon} \quad (55)$$

$$\left| \delta_{Y, X_1}^P - \delta_{Y, X_2}^P \right| \leq \frac{1}{1 - \varepsilon_0} \times \frac{\varepsilon(2 - \varepsilon)}{(1 - \varepsilon)^2} \quad (56)$$

$$\left| \delta_{Y, X_1}^D - \delta_{Y, X_2}^D \right| \leq \frac{1 - \varepsilon_0}{1 - 2\varepsilon_0} \times \frac{2\varepsilon}{1 - 2\varepsilon} \quad (57)$$

Proof. Let us first prove (50): we have $H_{X_1} \geq I_{X_1, X_2} = (1 - \varepsilon)C_{X_1, X_2}^S$ which leads to $H_{X_1} \geq \frac{1-\varepsilon}{1+\varepsilon}H_{X_2}$ and finally to

$$C_{X_1, X_2}^S = \frac{1}{2} (H_{X_1} + H_{X_2}) \geq \frac{1}{1 + \varepsilon} \max(H_{X_1}, H_{X_2}).$$

To prove (51), let us note that $H_{X_1} \geq (1 - \varepsilon)C_{X_1, X_2}^R$ leads to $\sqrt{H_{X_1}} \geq \frac{\sqrt{1-\varepsilon}}{2-\sqrt{1-\varepsilon}}\sqrt{H_{X_2}}$ and finally to

$$C_{X, Y}^R = \left(\frac{\sqrt{H_{X_1}} + \sqrt{H_{X_2}}}{2} \right)^2 \geq \left(\frac{1}{2 - \sqrt{1 - \varepsilon}} \right)^2 \max(H_{X_1}, H_{X_2}).$$

To prove (52), let us note that $H_{X_1} \geq (1 - \varepsilon)C_{X_1, X_2}^P$ leads to $\sqrt{H_{X_1}} \geq (1 - \varepsilon)\sqrt{H_{X_2}}$ and finally to

$$C_{X, Y}^P = \sqrt{H_{X_1} \times H_{X_2}} \geq (1 - \varepsilon) \max(H_{X_1}, H_{X_2}).$$

To prove (53), let us note that $H_{X_1} \geq C_{X_1, X_2}^D = 2 \left(\frac{1}{H_{X_1}} + \frac{1}{H_{X_2}} \right)^{-1}$ leads to $\frac{1}{H_{X_1}} \leq \frac{1}{1-2\varepsilon} \frac{1}{H_{X_2}}$ and finally to

$$C_{X_1, X_2}^D = 2 \left(\frac{1}{H_{X_1}} + \frac{1}{H_{X_2}} \right)^{-1} \geq \frac{1 - 2\varepsilon}{1 - \varepsilon} \max(H_{X_1}, H_{X_2}),$$

as soon as $\varepsilon \leq \varepsilon_0 < \frac{1}{2}$. ■

Remark 5 Let us give a NIB-divergence satisfying the following condition $C_{X, Y} \geq \kappa \times \max(H_X, H_Y)$ for some $\kappa > 0$ then, by taking again the proof of Proposition 13, we can prove that

$$|\delta_{Y, X_1} - \delta_{Y, X_2}| \leq \frac{1}{\kappa} \times \frac{d_{X_1, X_2}^I}{1 - d_{X_1, X_2}^I}.$$

A class of examples satisfying the previous condition is given by normalization terms defined for some $\alpha > 0$ by $C_{X, Y} = \|(H_X, H_Y)\|_\alpha$ ($\alpha = 1$ and $\alpha = 1/2$ correspond respectively to $\delta_{X, Y}^S$ and $\delta_{X, Y}^R$). Indeed, for this class we have $C_{X, Y} \geq 2^{-1/\alpha} \max(H_X, H_Y)$.

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