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Totally umbilic surfaces in homogeneous 3-manifolds

Rabah Souam & ***Eric Toubiana***

Abstract. We discuss existence and classification of totally umbilic surfaces in the model geometries of Thurston and the Berger spheres. We classify such surfaces in $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$ and the *Sol* group. We prove nonexistence in the Berger spheres and in the remaining model geometries other than the space forms.

1. Introduction

During the recent years, there has been a rapidly growing interest in the geometry of surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ focusing on minimal and constant mean curvature surfaces. This was initiated by H. Rosenberg, [16]. More generally many works are devoted to studying the geometry of surfaces in homogeneous 3-manifolds. See for example [14], [6], [7], [17], [15], [12], [13], [11], [9], [4], [2], [10], [5] and [8].

In the space forms the classification of totally umbilic surfaces is well known and very useful, see [21]. In $\mathbb{R}^3$ they are planes and round spheres and in $\mathbb{S}^3$ they are round spheres. In $\mathbb{H}^3$ they are totally geodesic planes and their equidistants, horospheres and round spheres. In particular they all have constant mean curvature.

A natural question is to understand the totally umbilic surfaces in the remaining homogeneous 3-manifolds. Untill now the only known result in this direction was the non-existence of totally umbilic surfaces in the Heisenberg space due to A. Sanini, see [17]. In this paper we study totally umbilic surfaces in simply connected and homogeneous 3-manifolds. More precisely we first consider the manifolds having a 4-dimensional isometry group, denoted by $\mathbb{M}^3(\kappa, \tau)$ (see section 2). Namely these manifolds are $\mathbb{S}^2(\kappa) \times \mathbb{R}$ ($\kappa > 0,\ \tau = 0$), $\mathbb{H}^2(\kappa) \times \mathbb{R}$ ($\kappa < 0,\ \tau = 0$), the Berger spheres ($\kappa > 0,\ \tau \neq 0$) and the manifolds having the isometry group of either the Heisenberg space ($\kappa = 0,\ \tau \neq 0$) or $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ ($\kappa < 0,\ \tau \neq 0$), see [3], [19] or [22]. Except for the Berger spheres, these manifolds are four of the eight model geometries of Thurston [22]. The remaining model geometries are the three space forms and the *Sol* geometry which has a 3-dimensional isometry group. As a matter of fact we also consider the *Sol* geometry.

In section 2 we prove (Theorem 1) the non-existence of totally umbilic surfaces, in particular the non-existence of totally geodesic ones, in the homogeneous manifolds $\mathbb{M}^3(\kappa, \tau)$ for $\tau \neq 0$, that is those which are not Riemannian products. This extends the result of Sanini, [17].

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In section 3 we construct and classify all rotational and totally umbilic surfaces in $\mathbb{S}^2(\kappa)\times\mathbb{R}$. In section 4 we construct and classify all totally umbilic surfaces in $\mathbb{H}^2(\kappa)\times\mathbb{R}$ which are invariant under a one-parameter group of ambient isometries. Except for the totally geodesic ones, these surfaces do not have constant mean curvature. In section 5, we prove that the surfaces obtained in sections 3 and 4 are the only totally umbilic surfaces in respectively $\mathbb{S}^2(\kappa)\times\mathbb{R}$ and $\mathbb{H}^2(\kappa)\times\mathbb{R}$.

In section 6 we show that there exist, up to ambient isometries, only two totally umbilic surfaces in *Sol*, one of them being totally geodesic.

Finally, in section 7 we apply our results to prove that any conformal diffeomorphism of $\mathbb{H}^2\times\mathbb{R}$, $\mathbb{S}^2\times\mathbb{R}$ and *Sol* is an isometry.

Throughout this paper all the surfaces are assumed of class C^3, see however the remark 20.

We are grateful to H. Rosenberg for valuable comments and to the referee for his observations which improved the paper.

2. Non-existence of totally umbilic surfaces in some homogeneous 3-manifolds

In this section we consider the connected and simply connected homogeneous Riemannian 3-manifolds, whose isometry group has dimension 4 and which are not Riemannian products. We recall that such a manifold is a fibration over some complete and simply connected surface, $\mathbb{M}^2(\kappa)$, of constant curvature $\kappa\in\mathbb{R}$, with geodesic fibers. Actually, for each κ, there is a one-parameter family $\mathbb{M}^3(\kappa,\tau)$ of such fibrations, parametrized by the bundle curvature $\tau\in\mathbb{R}^*$. The unit vector field ξ tangent to the fibers is a Killing field and satisfies :

$$\overline{\nabla}_X\xi=\tau(X\wedge\xi), \tag{1}$$

for any tangent vector X in $T\mathbb{M}^3(\kappa,\tau)$, where $\overline{\nabla}$ is the connection on $\mathbb{M}^3(\kappa,\tau)$. The field ξ defines the *vertical* direction of the Riemannian submersion $\mathbb{M}^3(\kappa,\tau)\to\mathbb{M}^2(\kappa)$. As a matter of fact, the bundle curvature τ can be zero, but in this case $\mathbb{M}^3(\kappa,0)$ is just a Riemannian product $\mathbb{M}^2(\kappa)\times\mathbb{R}$. These product manifolds will be considered in the following sections. Moreover we assume $\kappa-4\tau^2\neq 0$, otherwise the manifold is a space form and its isometry group has dimension 6. These manifolds are of three types : when $\kappa>0$ they are the Berger spheres, for $\kappa=0$ they have the isometry group of the Heisenberg space, Nil_3, and for $\kappa<0$ they have the isometry group of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$.

For more details we refer to [3], [19] and [22].

We can now state our result.

Theorem 1. *There exist no totally umbilic surfaces (even non complete) in the 3-manifolds $\mathbb{M}^3(\kappa,\tau)$, with $\tau\neq 0$ and $\kappa-4\tau^2\neq 0$. In particular, there are no totally geodesic surfaces.*

For the special case of the Heisenberg space ($\kappa=0,\tau=1/2$), we recover the result proved by A. Sanini, see [17].

Proof. Let S be an immersed totally umbilic surface in $\mathbb{M}^3(\kappa,\tau)$. Locally S is the image of an embedding $X:\Omega\to\mathbb{M}^3(\kappa,\tau)$, where Ω is an open disk in $\mathbb{R}^2$. Call (u,v) the coordinates on Ω and consider a unit normal field N on $X(\Omega)$. As X is totally umbilic, there exists a function $\lambda:\Omega\to\mathbb{R}$ such that

$$\begin{cases}\overline{\nabla}_{X_u}N=\lambda X_u\\ \overline{\nabla}_{X_v}N=\lambda X_v\end{cases}$$

Therefore

$$\begin{cases}\overline{\nabla}_{X_v}(\overline{\nabla}_{X_u}N)=\lambda_v X_u+\lambda\overline{\nabla}_{X_v}X_u\\ \overline{\nabla}_{X_u}(\overline{\nabla}_{X_v}N)=\lambda_u X_v+\lambda\overline{\nabla}_{X_u}X_v\end{cases}$$

Substracting the second equation from the first one we get

$$\overline{\nabla}_{X_v}(\overline{\nabla}_{X_u}N)-\overline{\nabla}_{X_u}(\overline{\nabla}_{X_v}N)=\lambda_v X_u-\lambda_u X_v.$$

That is

$$R(X_u,X_v)N=\lambda_v X_u-\lambda_u X_v, \tag{2}$$

where R denotes the curvature tensor of $\mathbb{M}^3(\kappa,\tau)$.

We define the function ν on Ω setting $\nu:=\langle N,\xi\rangle$. We denote by T the projection of ξ on S, that is $T=\xi-\nu N$.

As the projection $\mathbb{M}^3(\kappa,\tau)\to\mathbb{M}^2(\kappa)$ is a Riemannian submersion, we have the following formula derived by Daniel, see [7]:

$$R(X_u,X_v)N=(\kappa-4\tau^2)\nu\left(\langle X_v,T\rangle X_u-\langle X_u,T\rangle X_v\right).$$

Taking into account the relation (2) we get

$$\nabla\lambda=(\kappa-4\tau^2)\nu T, \tag{3}$$

where ∇ denotes the gradient on S.

Observe that if $T=0$ on a nonempty open set, then we can take $N=\xi$ on this set and we deduce from (1) that this surface cannot be umbilic. We can thus assume that T does not vanish on Ω.

Set $JT=N\wedge T$, thus JT is tangent to S and horizontal.

Claim: We have $[T,JT]\equiv 0$.

We need to show that: $\overline{\nabla}_T JT=\overline{\nabla}_{JT}T$. Since $JT=N\wedge T=N\wedge(\xi-\nu N)=N\wedge\xi$, we have using (1)

$$\overline{\nabla}_T JT=\overline{\nabla}_T(N\wedge\xi)=\overline{\nabla}_T N\wedge\xi+N\wedge\overline{\nabla}_T\xi=\lambda(T\wedge\xi)+\tau(N\wedge(T\wedge\xi)).$$

As $T\wedge\xi=T\wedge(T+\nu N)=\nu(T\wedge N)=-\nu JT$, we deduce that

$$\overline{\nabla}_T JT=-\lambda\nu JT+\tau\nu T.$$

On the other hand

$$\overline{\nabla}_{JT}T=\overline{\nabla}_{JT}(\xi-\nu N)=\tau(JT\wedge\xi)-JT.(\nu)N-\lambda\nu JT. \tag{4}$$

We have

$$JT \wedge \xi = JT \wedge (T + \nu N) = JT \wedge T + \nu JT \wedge N = -|T|^2 N + \nu T, \tag{5}$$

and

$$\begin{aligned} JT.(\nu) = JT.\langle N, \xi\rangle &= \langle \overline{\nabla}_{JT} N, \xi\rangle + \langle N, \overline{\nabla}_{JT}\xi\rangle \\ &= \lambda\langle JT, \xi\rangle + \tau\langle N, JT \wedge \xi\rangle = \tau\langle N, JT \wedge \xi\rangle. \end{aligned}$$

Using (5) we obtain

$$JT.(\nu) = -\tau|T|^2. \tag{6}$$

Inserting (5) and (6) in (4) we end with:

$$\overline{\nabla}_{JT} T = \tau\nu T - \lambda\nu JT = \overline{\nabla}_T JT,$$

which proves the claim.

Now (3) implies

$$\begin{cases} JT.(\lambda) = 0 \\ T.(\lambda) = (\kappa - 4\tau^2)\nu|T|^2 = (\kappa - 4\tau^2)\nu(1 - \nu^2) \end{cases}$$

Since $[T, JT] = 0$, we get $(\kappa - 4\tau^2)JT.(\nu - \nu^3) = 0$. As $\kappa - 4\tau^2 \neq 0$ we infer that $(1 - 3\nu^2)JT.(\nu) = 0$. This implies easily

$$JT.(\nu) = 0.$$

As $JT.(\nu) = -\tau|T|^2$, see (6), and $\tau \neq 0$, we deduce that $T \equiv 0$, which is a contradiction. This concludes the proof. □

3. Symmetric totally umbilic surfaces in $\mathbb{S}^2(\kappa) \times \mathbb{R}$

In this section we classify totally umbilic surfaces which are rotationally invariant in $\mathbb{S}^2 \times \mathbb{R}$. The classification in $\mathbb{M}^2(\kappa) \times \mathbb{R}$, for any $\kappa > 0$ is completely analogous. We will see that besides the obvious totally geodesic ones, up to isometries of $\mathbb{S}^2 \times \mathbb{R}$, there are two one-parameter families of complete totally umbilic rotationally invariant surfaces homeomorphic to the two-sphere and a unique complete surface which has the topology of $\mathbb{R}^2$. The surfaces of the first family are homologous to zero and those of the second family are not. Moreover these surfaces are embedded, analytic and any totally umbilic rotationally invariant surface is a part of one of these complete surfaces.

A rotational surface in $\mathbb{S}^2 \times \mathbb{R}$ is by definition a surface obtained by rotating a curve in a totally geodesic cylinder $\mathcal{C} := \Gamma \times \mathbb{R}$, where $\Gamma \subset \mathbb{S}^2$ is a geodesic, around an axis $R = \{p\} \times \mathbb{R}$ where p is a fixed point of Γ.

In the coordinates (x, y, t) given by the stereographic projection with respect to the north pole, the metric on $\mathbb{S}^2 \times \mathbb{R}$ reads as follows:

$$\mathrm{d}\tilde{s}^2 = \left(\frac{2}{1 + x^2 + y^2}\right)^2 (\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2,$$

where $x, y, t \in \mathbb{R}$.

Up to an ambient isometry we can assume that $\Gamma \subset \mathbb{S}^2$ corresponds to the complete geodesic defined by $y = 0$ and that $p = (0, 0, 0) \in \Gamma$ is the south pole of $\mathbb{S}^2$. Therefore the axis is $R = \{(0, 0, t),\ t \in \mathbb{R}\}$.

Remark 2. *Let us remark that for any given curve α in the cylinder $\mathcal{C}$, the surface generated by rotating α around the axis through the south pole, that is R, is the same as the one generated by rotating α around the axis through the north pole.*

We consider the vertical (noncomplete) geodesic plane $P = \{y = 0\} \subset \mathcal{C}$, that is $\mathcal{C} = P \cup (\{N\} \times \mathbb{R})$, where $N \in \mathbb{S}^2$ is the north pole.

Let $\rho \in]-\pi, \pi[$ denote the signed distance to the origin $(0, 0)$ on $\Gamma \cap P$. Thus we have $x = \tan(\rho/2)$. In the coordinates (ρ, t) the metric on the plane P writes

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}t^2.$$

Consider now a smooth curve $\alpha(s) = (\rho(s), t(s))$ parametrized by arclength in P. Let $\theta(s)$ be the oriented angle between the ρ-axis and $\alpha'(s)$. Therefore, we have:

$$\begin{cases} \rho'(s) = \cos\theta(s) \\ t'(s) = \sin\theta(s) \end{cases} \tag{7}$$

In the plane P we consider the unit normal N to the curve α so that the basis $(\alpha'(s), N(s))$ is positively oriented for each s. We orient by N the symmetric surface generated by α. The principal curvatures computed with respect to this orientation are as follows:

$$\begin{cases} \lambda_1(s) = \theta'(s) \\ \lambda_2(s) = \dfrac{\sin\theta(s)}{\tan\rho(s)}. \end{cases}$$

Thus, the umbilicity condition is:

$$\theta'(s) = \frac{\sin\theta(s)}{\tan\rho(s)}. \tag{8}$$

A priori the equation (8) makes sense only for $\rho \neq 0$, but as we will see later the surfaces we obtain are regular even at such points.

Differentiating the first equation in (7) and using equation (8) we get

$$\rho'' = \frac{(\rho'^2 - 1)}{\tan\rho}. \tag{9}$$

Assume that $\rho'(s_0) = 1$ for some s_0 where $\rho(s_0) \neq 0$. Note that the function $f(s) = \rho(s_0) + s - s_0$ is a solution of (9) with the same initial conditions at s_0 than ρ. Therefore $\rho \equiv f$ and $t' \equiv 0$ and the surface is part of a slice $\mathbb{S}^2 \times \{t_0\}$. The same happens in case where $\rho'(s_0) = -1$. Henceforth we will assume that $\rho'^2(s) \neq 1$ for all

s and (9) is equivalent to

$$\frac{\rho''}{(\rho'^2-1)}=\frac{\cos\rho}{\sin\rho}.$$

Multiplying both sides by $2\rho'$ and integrating we get

$$\rho'^2-1=\lambda\sin^2\rho,$$

for some nonzero real constant λ. Since the curve α is parametrized by arclength we must have $\rho'^2<1$. Thus $\lambda=-a^2$ for some $a>0$.

Conversely, any solution ρ of (9) satisfying $\rho'^2\leq 1$ defines a function $\theta(s)$ setting $\cos\theta(s)=\rho'(s)$. Consider the function t defined by setting $t'(s)=a\sin\rho(s)$ and $t(s_0)=t_0$ for some s_0 in the domain of ρ and some real number t_0. Then t satisfies the second equation of (7) and therefore the curve $\alpha(s)=(\rho(s),t(s))\in\mathcal{C}$ generates a rotational totally umbilic surface in $\mathbb{S}^2\times\mathbb{R}$.

Let ρ be a solution of (9) satisfying $\rho'^2<1$. Observe that equations (7) and (8) show that ρ' cannot be identically zero on an open interval unless the generated surface is part of the totally geodesic cylinder $\rho\equiv\pi/2$. Henceforth we assume that ρ is not this trivial solution and so, up to restricting the domain of ρ, we can suppose that ρ takes its values in $]0,\pi/2[$ or $]\pi/2,\pi[$.

So, we can consider an interval on which ρ' never vanishes. Changing s into $-s$ if needed we can suppose that $\rho'>0$. Therefore we get

$$\rho'=\sqrt{1-a^2\sin^2\rho}. \tag{10}$$

It is interesting to note that, when $a\neq 1$, the function ρ is the *Jacobi amplitude* function : $\rho(s)=\mathrm{am}(s,a^2)$ and, up to the sign, we have $t'(s)=a\ \mathrm{sn}(s,a^2)$ and $\theta'(s)=a\ \mathrm{cn}(s,a^2)$, where $\mathrm{sn}(s,a^2)=\sin\mathrm{am}(s,a^2)$ and $\mathrm{cn}(s,a^2)=\cos\mathrm{am}(s,a^2)$ are respectively the *sinus and cosinus amplitudinis* elliptic Jacobi functions, see for instance [1, Chapter 16] and [18, pp 286-307]. However, for reader's convenience and to be self-contained, we will treat in a direct and elementary way this ODE.

Now observe that the transformation $(\rho,t)\mapsto(\pi-\rho,t)$ is an isometry which changes rotations around the axis through the south pole into rotations around the axis through the north pole. Therefore, taking into account Remark 2, we can assume that, up to an ambient isometry, ρ takes its values in $]0,\pi/2[$.

Let us call ρ_a the maximal solution of (10) extending ρ without restrictions on its values, that is for the moment we do not require that ρ_a takes its values in $]-\pi,\pi[$.

Lemma 3. *Up to a reparametrization of the form $s\to s+c$ for some real constant c, the maximal solution ρ_a is defined on an interval $]-\delta,\delta[$, where $\delta\in]0,+\infty]$. Furthermore ρ_a is odd and so satisfies $\rho_a(0)=0$ and $\rho_a'(0)=1$.*

Proof. Let us call $]u,v[$ the domain of ρ_a where $-\infty\leq u<v\leq+\infty$.

We first show that ρ_a vanishes at some point. Since $\rho_a'\leq 1$ such a point clearly exists if $u=-\infty$. In case u is finite, ρ_a has a limit $l\in[-\infty,+\infty[$ at u as it is nondecreasing.

If $l < 0$ then ρ_a vanishes at some point since ρ is positive.

Consider now the case where $l \geq 0$ and call $I \subset]u, v[$ the domain of ρ. Suppose that, as s decreases starting from I, ρ_a never vanishes. Then the function ρ_a' increases (since $\rho(I) \subset]0, \pi/2[$) and thus has a positive limit at u. But then we could extend the solution ρ_a of equation (10) beyond u, which contradicts the maximality of ρ_a.

Therefore $\rho_a(s_0) = 0$ for some point s_0. Changing s into $s - s_0$ we can assume that $\rho_a(0) = 0$. The function $f(s) := -\rho_a(-s)$ is then also a solution of (10) satisfying $f(0) = 0$. We conclude that $f \equiv \rho_a$, which means ρ_a is odd. □

Lemma 4. *Suppose $a \in]0, 1[$. Then ρ_a is defined on the whole of $\mathbb{R}$ and it gives rise to a unique, up to an ambient isometry, curve α_a generating a rotational totally umbilic surface. This curve is an analytic Jordan curve in the cylinder $\mathcal{C}$, it is nonhomologous to zero and symmetric with respect to the axis of rotation R. The rotational totally umbilic surface, $S_1(a)$, generated by α_a is analytic, embedded and homeomorphic to the sphere. Moreover $S_1(a)$ is nonhomologous to zero in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. With the notations of the lemma 3, if $\delta < +\infty$ then, since ρ_a' takes its values in $]0, 1]$, ρ_a would have a finite limit l at δ. Since $a \in]0, 1[$, we have $1 - a^2 \sin^2 l > 0$ which allows to extend ρ_a beyond δ, contradicting the maximality of ρ_a. This shows that $\delta = +\infty$.

Since $\rho_a' \geq \sqrt{1 - a^2} > 0$ and $\rho_a(0) = 0$, there is a smallest $s_1 > 0$ such that $\rho_a(s_1) = \pi$.

Now let us consider the function $f(s) := 2\pi - \rho_a(2s_1 - s)$, $s \in \mathbb{R}$. We observe that f is also a solution of equation (10) and satisfies $f(s_1) = 2\pi - \rho_a(s_1) = \pi = \rho_a(s_1)$. Consequently $f \equiv \rho_a$, that is:

$$\rho_a(2s_1 - s) = 2\pi - \rho_a(s), \ \forall s \in \mathbb{R}. \tag{11}$$

As we are interested in curves generating rotational totally umbilic surfaces, we look for a function t satisfying $t'^2 = 1 - \rho_a'^2 = a^2 \sin^2 \rho_a$. Let t_a be the function defined on $\mathbb{R}$ by setting $t_a'(s) = a \sin \rho_a(s)$ and $t_a(0) = 0$.

Since ρ_a is an odd function and $t_a(0) = 0$ we deduce that t_a is an even function. Observe that the function $g(s) := t_a(2s_1 - s)$ satisfies $g'(s) = t_a'(s)$ (using equation (11)) and $g(s_1) = t_a(s_1)$. Thus $g \equiv t_a$, that is $t_a(2s_1 - s) = t_a(s)$ for any $s \in \mathbb{R}$. Using the evenness of t_a we get

$$t_a(s + 2s_1) = t_a(s), \ \forall s \in \mathbb{R}. \tag{12}$$

Using equation (11) and the oddness of ρ_a we obtain for any $s \in \mathbb{R}$

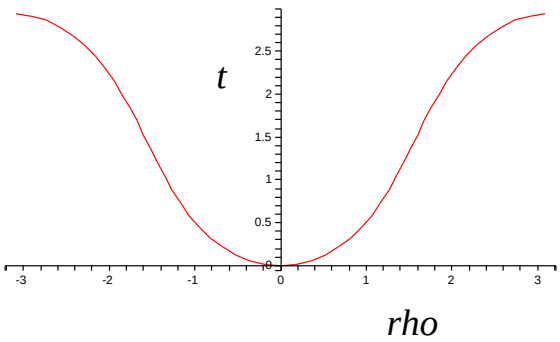
$$\rho_a(s) = -\rho_a(-s) = -(2\pi - \rho_a(2s_1 + s)),$$

and so

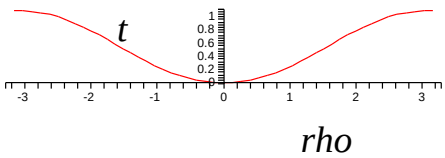
$$\rho_a(s + 2s_1) = 2\pi + \rho_a(s), \ \forall s \in \mathbb{R}. \tag{13}$$

Now the curve $\widetilde{\alpha}_a(s) = (\rho_a(s), t_a(s))$, $s \in \mathbb{R}$, is a curve in the Riemannian universal cover $\widetilde{\mathcal{C}}$ of $\mathcal{C}$. Observe that the equations (12) and (13) show that restricting s to

a=0.9



a=0.5



$[-s_1, s_1]$, $\widetilde{\alpha}_a$ gives rise to an analytic closed curve α_a in $\mathcal{C}$. Since ρ_a is increasing on $[-s_1, s_1]$ and $\rho_a(-s_1) = -\rho_a(s_1) = -\pi$ we deduce that α_a is embedded and nonhomologous to zero.

As ρ_a is odd and t_a is even, the curve α_a has the desired symmetry.

It is clear that the other choice $t_a'(s) = -a \sin \rho_a(s)$ leads to the curve deduced from α_a by the isometry $(\rho, t) \mapsto (\rho, -t)$. $\square$

Remark 5. *We observe that the curve α_a is globally invariant under the isometry $(\rho, t) \mapsto (\pi - \rho \bmod(2\pi), t_a(2s_0) - t)$.*

Lemma 6. *Assume $a = 1$. Then $\rho_1(s) = \pi/2 - 2\arctan e^{-s}$, $s \in \mathbb{R}$. This gives rise to a unique, up to an ambient isometry, curve α_1 in $P \subset \mathcal{C}$ generating a rotational totally umbilic surface. The curve α_1 is complete, open, embedded and symmetric with respect to the axis of rotation R. The rotational totally umbilic surface, S_1, generated by α_1 in $\mathbb{S}^2 \times \mathbb{R}$ is complete, properly embedded, analytic and homeomorphic to $\mathbb{R}^2$.*

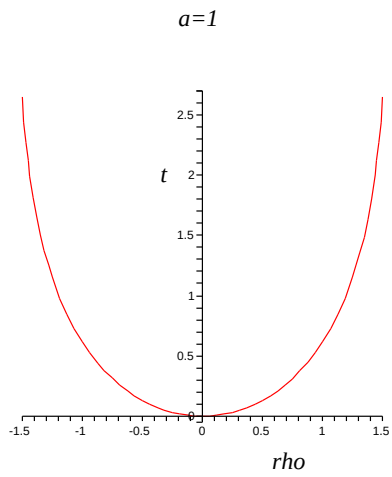
Proof. Since ρ_1 is the maximal solution of $\rho' = \sqrt{1 - \sin^2 \rho}$ satisfying $\rho(0) = 0$, we deduce that ρ_1 is solution of $\rho' = \cos \rho$. A straightforward computation shows that the maximal solution of this last equation with the initial condition $\rho(0) = 0$ is

$$\rho_1(s) = \frac{\pi}{2} - 2\arctan e^{-s}, \ \forall s \in \mathbb{R}.$$

Note that ρ_1 takes its values in $]-\pi/2, \pi/2[$.

As in the proof of the lemma 4, we can assume that t satisfies $t'(s) = \sin \rho_1(s)$, up to an ambient isometry. We consider the function t_1 defined by setting $t_1'(s) = \sin \rho_1(s)$ and $t_1(0) = 0$. It is straightforward to check that $\sin \rho_1(s) = \tanh(s)$ and then $t_1(s) = \log \cosh(s)$.

As ρ_1 is odd and t_1 is even, the curve α_1 has the desired symmetry. As a matter of fact, α_1 is the graph of the function $t(\rho) = -\log \cos \rho$, $\rho \in]-\pi/2, \pi/2[$. This concludes the proof. □



Lemma 7. *Assume $a > 1$. Then the maximal solution ρ_a is defined on a bounded interval $]-\delta_a, \delta_a[$ where δ_a is a positive number. It gives rise to a unique, up to an ambient isometry, Jordan curve α_a in $P \subset \mathcal{C}$ generating a rotational totally umbilic surface. The curve α_a is analytic and symmetric with respect to the axis of rotation R. The rotational totally umbilic surface, $S_2(a)$, generated by α_a is analytic, embedded and homeomorphic to the sphere. Furthermore, $S_2(a)$ is homologous to zero in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. Since $a > 1$ we deduce that $\rho_a(s) \in]-\arcsin 1/a, \arcsin 1/a[$. Recall that ρ_a is defined on an open interval $]-\delta_a, \delta_a[$, see the lemma 3. We first show that δ_a is finite. Assume by contradiction $\delta_a = +\infty$. Since ρ_a is nondecreasing it admits a limit $l \in]0, \arcsin 1/a]$ as $s \to +\infty$. Necessarily $l = \arcsin 1/a$ since otherwise it would follow from the equation (10) that $\rho_a' > \sqrt{1 - a^2 \sin^2 l} > 0$ for all $s > 0$. Therefore ρ_a would not be bounded.

Using the equation (9) we see that for s big enough : $\rho_a''(s) \leq -1/(2\tan l) < 0$. Consequently ρ_a' would be negative for s big enough which is a contradiction. This proves that δ_a is finite.

Let us call again l the limit of ρ_a as $s \to \delta_a$. If $l < \arcsin 1/a$ then we could extend the solution ρ_a, which is maximal, beyond δ_a, which is absurd. So $l = \arcsin 1/a$ and $\rho_a' \to 0$ as $s \to \delta_a$.

Observe that, since the function ρ_a satisfies equations (9) and (10), it satisfies also the following equation

$$\rho'' = -a^2 \sin\rho \cos\rho. \tag{14}$$

As the second member of (14) is bounded, its maximal solutions are defined on the whole of $\mathbb{R}$. Call $\widetilde{\rho}_a$ the maximal solution of (14) extending ρ_a. Set $f(s) := \widetilde{\rho}_a(2\delta_a - s)$. It is clear that f and $\widetilde{\rho}_a$ satisfy equation (14) with the same initial conditions at δ_a. Thus we have

$$\widetilde{\rho}_a(2\delta_a - s) = \widetilde{\rho}_a(s), \ \forall s \in \mathbb{R}. \tag{15}$$

As we are interested in curves generating rotational totally umbilic surfaces, we look for a function t satisfying $t'^2 = 1 - \widetilde{\rho}_a'^2 = a^2 \sin^2 \widetilde{\rho}_a$. Let t_a be the function defined by $t_a' = a \sin \widetilde{\rho}_a$ and $t_a(0) = 0$. As $\widetilde{\rho}_a$ is an odd function we deduce that t_a is even. Observe that the function $g(s) := 2t_a(\delta_a) - t_a(2\delta_a - s)$ satisfies $g'(s) = t_a'(s)$ (using equation (15)) and $g(\delta_a) = t_a(\delta_a)$. Thus $g \equiv t_a$, that is

$$t_a(2\delta_a - s) = 2t_a(\delta_a) - t_a(s), \ \forall s \in \mathbb{R}. \tag{16}$$

It follows from (15) and the oddness of $\widetilde{\rho}_a$ that

$$\widetilde{\rho}_a(s + 4\delta_a) = \widetilde{\rho}_a(s), \ \forall s \in \mathbb{R}.$$

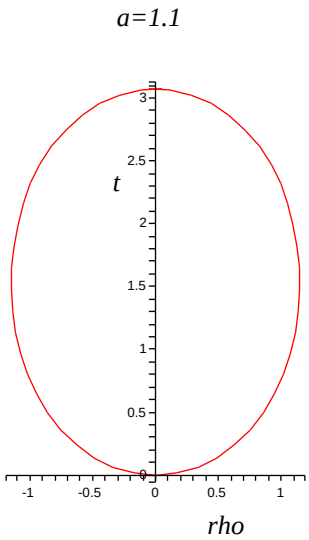
In the same way, using (16) and the evenness of t_a, we get

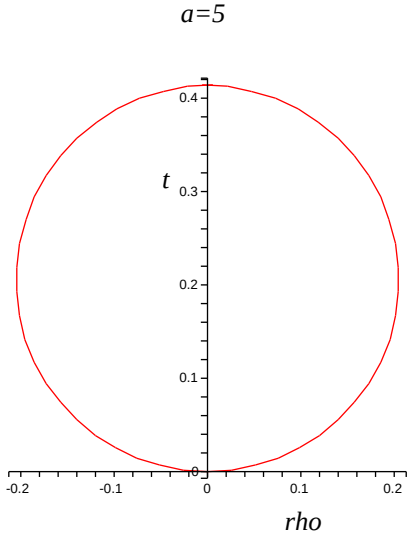
$$t_a(s + 4\delta_a) = t_a(s), \ \forall s \in \mathbb{R}.$$

Now the curve $\alpha_a(s) = (\widetilde{\rho}_a(s), t_a(s))$, $s \in \mathbb{R}$, parametrizes a closed analytic curve in $P \subset \mathcal{C}$. Taking into account (15), the oddnes of $\widetilde{\rho}_a$, (16) and the evenness of t_a, we deduce that the curve α_a is symmetric with respect to the axis R.

Considering the fact that t_a is increasing on $[0, 2\delta_a]$ and the symmetry of α_a, we infer that α_a defines a Jordan curve in P.

To conclude the proof observe that the other choice for t_a, that is $t_a' = -a \sin \widetilde{\rho}_a$, leads to the curve deduced from α_a by the isometry $(\rho, t) \mapsto (\rho, -t)$. $\square$





Remark 8. *The relations (15) and (16) show that the curve α_a in the lemma 7 is symmetric with respect to the horizontal reflection $(\rho, t) \mapsto (\rho, 2t_a(\delta_a) - t)$ in P. Consequently, the surface $S_3(a)$ is symmetric with respect to the slice $\mathbb{S}^2 \times \{t_a(\delta_a)\}$.*

Furthermore, we observe that each surface $S_3(a)$ is contained in $\mathbb{S}^2_- \times \mathbb{R}$ where $\mathbb{S}^2_-$ is the south hemispshere.

Summarizing we can state the following result.

Theorem 9. *Besides the totally geodesic slices $\mathbb{S}^2 \times \{t_0\}$ and the vertical cylinder $\Gamma \times \mathbb{R}$ where $\Gamma \subset \mathbb{S}^2$ is a geodesic, the surfaces introduced in the lemmas 4, 6 and 7 are, up to ambient isometries, the only complete totally umbilic rotational surfaces in $\mathbb{S}^2 \times \mathbb{R}$. In particular they are all embedded and homeomorphic either to $\mathbb{R}^2$ or to $\mathbb{S}^2$. Among the surfaces homeomorphic to $\mathbb{S}^2$ some are homologous to zero and some are not.*

Remark 10. *It is interesting to observe that unlike in the case of space forms, the totally umbilic surfaces we obtained do not have constant mean curvature, except for the totally geodesic ones.*

4. Symmetric totally umbilic surfaces in $\mathbb{H}^2(\kappa) \times \mathbb{R}$

In this section, we classify the totally umbilic surfaces in $\mathbb{H}^2 \times \mathbb{R}$ which are invariant under a one-parameter group of isometries. The case of $\mathbb{M}^2(\kappa) \times \mathbb{R}$, $\kappa < 0$, is completely similar.

We recall that in $\mathbb{H}^2$ there are three kinds of one-parameter families of positive isometries: the rotations around a fixed point (*elliptic isometries*), the translations along a fixed geodesic (*hyperbolic isometries*) and the "translations" along the horocycles sharing the same point at infinity (*parabolic isometries*). An isometry of $\mathbb{H}^2$ obviously induces an isometry of $\mathbb{H}^2 \times \mathbb{R}$ fixing the factor $\mathbb{R}$ pointwise. Such an isometry of $\mathbb{H}^2 \times \mathbb{R}$ obtained from an elliptic (resp. parabolic, hyperbolic) isometry of $\mathbb{H}^2$ will thus be called elliptic (resp. parabolic, hyperbolic).

We will see that for each of the associated families of isometries of $\mathbb{H}^2 \times \mathbb{R}$ there exist complete and globally invariant totally umbilic surfaces. In fact, we are going to classify all of them. More precisely, we prove they are all embedded, those which are invariant under elliptic isometries are either totally geodesic slices $\mathbb{H}^2 \times \{t_0\}$ or homeomorphic to the sphere and the remaining ones are all homeomorphic to the plane. In particular the only totally geodesic ones are the slices and the products $\Gamma \times \mathbb{R}$ where $\Gamma \subset \mathbb{H}^2$ is a geodesic.

We will work with the disk model for $\mathbb{H}^2$, so that

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 < 1\},$$

and the metric is

$$\mathrm{d}s^2_{\mathbb{H}} = \left(\frac{2}{1 - (x^2 + y^2)}\right)^2 (\mathrm{d}x^2 + \mathrm{d}y^2).$$

Therefore the product metric on $\mathbb{H}^2 \times \mathbb{R}$ reads as follows:

$$\mathrm{d}\tilde{s}^2 = \left(\frac{2}{1 - (x^2 + y^2)}\right)^2 (\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}t^2,$$

where $(x, y) \in \mathbb{H}^2$ and $t \in \mathbb{R}$. We consider the following particular geodesics of $\mathbb{H}^2$:

$$\Gamma = \{(x, 0),\ x \in]-1, 1[\,\} \subset \mathbb{H}^2$$
$$L = \{(0, y),\ y \in]-1, 1[\,\} \subset \mathbb{H}^2$$

Up to ambient isometries, we can assume that the symmetric surfaces are generated by curves in the geodesic plane $P := \Gamma \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$.

On the geodesic Γ we denote by $\rho \in \mathbb{R}$ the signed distance to the origin $(0,0)$, thus $x = \tanh \rho/2$. Therefore the metric on P is

$$\mathrm{d}s^2 = \left(\frac{2}{1-x^2}\right)^2 \mathrm{d}x^2 + \mathrm{d}t^2 = \mathrm{d}\rho^2 + \mathrm{d}t^2.$$

Given a curve $\alpha(s) = (\rho(s), t(s))$ parametrized by arclength in P, we let $\theta(s)$ be the oriented angle between the ρ-axis and $\alpha'(s)$. Therefore, we have:

$$\begin{cases} \rho'(s) = \cos\theta(s) \\ t'(s) = \sin\theta(s) \end{cases} \tag{17}$$

In the elliptic case, the isometries of $\mathbb{H}^2 \times \mathbb{R}$ under consideration are the rotations around the vertical axis $R := \{(0,0)\} \times \mathbb{R}$. In the parabolic case, the isometries are the ones corresponding to the point at infinity $(-1,0) \in \partial_\infty \mathbb{H}^2$. Finally, the hyperbolic isometries correspond to translations along L in $\mathbb{H}^2$.

In the plane P we consider the unit normal N to the curve α so that the basis $(\alpha'(s), N(s))$ is positively oriented for each s. In the three cases we orient by N the symmetric surface generated by α. The principal curvatures computed with respect to this orientation are as follows:

$$\lambda_1(s) = \theta'(s)$$

and

$$\lambda_2(s) = \begin{cases} \dfrac{\sin\theta(s)}{\tanh\rho(s)} & \text{(elliptic case)} \\ \sin\theta(s) & \text{(parabolic case)} \\ \sin\theta(s)\tanh\rho(s) & \text{(hyperbolic case)} \end{cases}$$

4.1. **Elliptic case.**

The umbilicity condition is

$$\theta'(s) = \frac{\sin\theta(s)}{\tanh\rho(s)}.$$

This case is similar to the case $a > 1$ in $\mathbb{S}^2 \times \mathbb{R}$, so we will omit the details.

Differentiating the first equation in (17) and using the umbilicity condition we get

$$\rho'' = \frac{(\rho'^2 - 1)}{\tanh\rho}. \tag{18}$$

Discarding the trivial totally geodesic surfaces $\mathbb{H}^2 \times \{t_0\}$, we can show as in the case of $\mathbb{S}^2 \times \mathbb{R}$ that $\rho'^2(s) \neq 1$ for any s such that $\rho(s) \neq 0$. Therefore we may assume that $\rho'^2 < 1$.

We can state the following.

Proposition 11. *Any local solution of (18) satisfying $\rho'^2 < 1$ gives rise to a unique, up to ambient isometries, complete rotational totally umbilic and nongeodesic surface*

in $\mathbb{H}^2 \times \mathbb{R}$. Moreover, there exists a one-parameter family of such surfaces and all of them are analytic, embedded and homeomorphic to the sphere.

Besides the totally geodesic slices $\mathbb{H}^2 \times \{t\}$, these surfaces are the only complete rotational and totally umbilic surfaces in $\mathbb{H}^2 \times \mathbb{R}$.

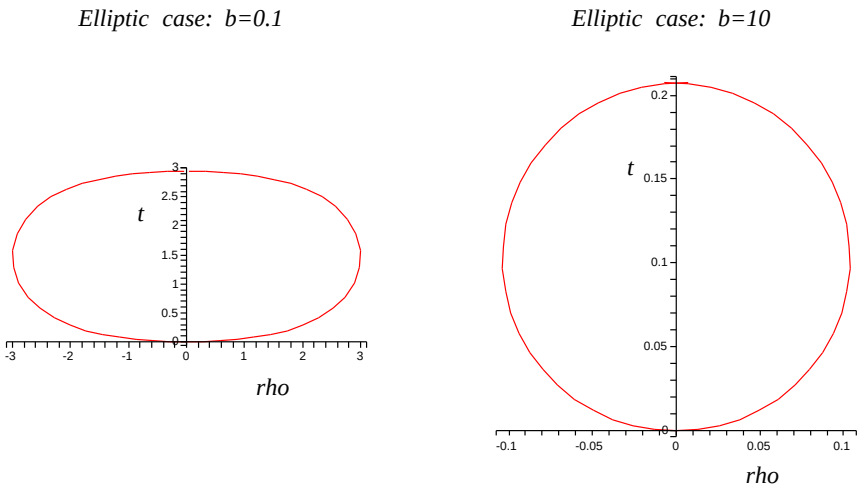
Furthermore, any rotational umbilic (including geodesic) surface in $\mathbb{H}^2 \times \mathbb{R}$ is, up to an ambient isometry, part of one of the above surfaces.

Proof. Let ρ be a local solution of (18). Proceeding as in the case of $\mathbb{S}^2 \times \mathbb{R}$, we can suppose that $\rho' > 0$ and so

$$\rho' = \sqrt{1 - b^2 \sinh^2 \rho}, \tag{19}$$

for some real number $b > 0$. As in the case of $\mathbb{S}^2 \times \mathbb{R}$, the functions $\rho(s)$ and $t(s)$ are related to the Jacobi elliptic functions as follows: $\rho(s) = -i\mathrm{am}(is, -b^2)$ and, up to the sign, $t'(s) = ib\ \mathrm{sn}(is, -b^2)$, see [1, Chapter 16]. Again, we prefer to give direct and elementary arguments.

Let ρ_b be the maximal solution of (19) extending ρ. As in the proof of the lemma 3, we can prove that ρ_b vanishes at some point. Thus, up to a reparametrization we can assume that $\rho_b(0) = 0$, consequently we prove as in lemma 3 that ρ_b is an odd function. Therefore ρ_b is defined on an interval $]-\delta_b, \delta_b[$. As in lemma 7, it can be shown that δ_b is a finite positive number, that ρ_b has a finite limit $l = \mathrm{argsinh}\, 1/b$ at δ_b and that $\rho_b'(s) \to 0$ as $s \to \delta_b$.



From equations (18) and (19) we deduce that ρ_b satisfies

$$\rho'' = -b^2 \cosh \rho \sinh \rho \tag{20}$$

with the initial conditions $\rho(0) = 0$ and $\rho'(0) = 1$. Therefore we can extend the solution ρ_b of (20) beyond δ_b. Let $\widetilde{\rho}_b$ be the maximal solution of (20) extending ρ_b. Observe that for any s_0 where $\widetilde{\rho}_b{}'(s_0) = 0$ we have the symmetry $\widetilde{\rho}_b(2s_0 - s) = \widetilde{\rho}_b(s)$.

As $\widetilde{\rho}_b{}'(\delta_b) = 0$ and $\widetilde{\rho}_b$ is odd, we deduce that $\widetilde{\rho}_b$ is defined on all of $\mathbb{R}$ and that it is $4\delta_b$-periodic.

As we are interested in curves generating rotational totally umbilic surfaces, we look for a function t satisfying $t'^2 = 1 - \widetilde{\rho}_b{}'^2 = b^2 \sinh^2 \widetilde{\rho}_b$. Let t_b be the function defined by $t_b' = b \sinh \widetilde{\rho}_b$ and $t_b(0) = 0$, thus t_b is an even function. As in the proof of lemma 7, we can show that t_b satisfies $t_b(2\delta_b - s) = 2t_b(\delta_b) - t_b(s)$ for any $s \in \mathbb{R}$ and that it is also $4\delta_b$-periodic.

Taking into account that t_b is increasing on $[0, 2\delta_b]$, we deduce that the curve $\alpha_b(s) = (\widetilde{\rho}_b(s), t(s))$, $s \in \mathbb{R}$, parametrizes an analytic Jordan curve in P, symmetric with respect to the axis R.

To conclude the proof we just observe that the other choice for t_b, that is $t_b' = -b \sinh \widetilde{\rho}_b$, leads to the curve deduced from α_b by the isometry $(\rho, t) \mapsto (\rho, -t)$. $\square$

4.2. **Parabolic case.**

The umbilicity condition is

$$\theta'(s) = \sin \theta(s)$$

Integrating this equation we get

$$\theta(s) = 2 \arctan \lambda e^s, \ \forall s \in \mathbb{R},$$

for some real constant λ.

First observe that $\lambda = 0$ leads to the curve Γ which generates a slice $\mathbb{H}^2 \times \{t_0\}$.

Now if $\lambda < 0$ then θ is a negative function. Note that the symmetry $(\rho, t) \mapsto (\rho, -t)$ changes θ into $-\theta$. Therefore, up to an ambient isometry, we can assume that θ is positive and then $\lambda > 0$. Finally observe that up to the reparametrization $s \mapsto s - \log \lambda$ we can assume $\lambda = 1$ and then:

$$\theta(s) = 2 \arctan e^s, \ \forall s \in \mathbb{R}, \tag{21}$$

Taking into account the first equation in (17) and (21) we obtain

$$\rho'(s) = \cos 2 \arctan e^s = -\tanh s, \ \forall s \in \mathbb{R}.$$

Thus

$$\rho(s) = -\log \cosh s + \mu, \ \forall s \in \mathbb{R},$$

for some real constant μ. Note that the isometries of $\mathbb{H}^2 \times \mathbb{R}$ obtained from the hyperbolic translations along the geodesic Γ in $\mathbb{H}^2$ send any surface invariant under the parabolic isometries fixing the point at infinity $(-1, 0) \in \partial_\infty \mathbb{H}^2$ to a surface of the same type. Consequently, up to an ambient isometry, we can assume $\mu = 0$. Thus

$$\rho(s) = -\log \cosh s, \ \forall s \in \mathbb{R},$$

As for the function t, taking into account the second equation in (17) and (21) we obtain

$$t'(s) = \sin 2 \arctan e^s = \frac{2e^s}{1 + e^{2s}}, \ \forall s \in \mathbb{R}.$$

Integrating we get

$$t(s) = 2\arctan e^s + \beta,\ \forall s \in \mathbb{R},$$

for some real constant β. Up to a vertical translation we can take $\beta = -\pi/2$ so that $t(0) = 0$ and

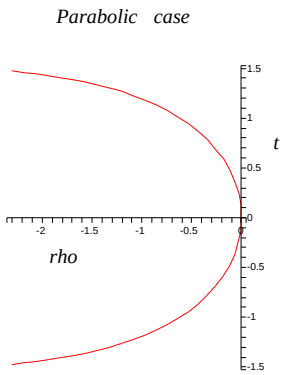
$$t(s) = 2\arctan e^s - \frac{\pi}{2},\ \forall s \in \mathbb{R}.$$

Note that $t(-s) = -t(s)$ and $\rho(-s) = \rho(s)$ for any $s \in \mathbb{R}$ so that the curve L_{par} parametrized by $(\rho(s), t(s)),\ s \in \mathbb{R}$, is symmetric with respect to Γ.

Summarizing we state the following proposition.

Proposition 12. *Besides the slices $\mathbb{H}^2 \times \{t\}$, up to ambient isometries, there exists a unique complete totally umbilic surface S_P in $\mathbb{H}^2 \times \mathbb{R}$ invariant under parabolic isometries. This surface is analytic, properly embedded, homeomorphic to a plane and is invariant under reflection with respect to a horizontal slice.*

Moreover, any totally umbilic surface invariant under parabolic isometries is either part of a slice or, up to an ambient isometry, part of this surface.



Remark 13. *Consider the surface M in $\mathbb{H}^2 \times \mathbb{R}$ generated by the same curve $L_{par} \subset P$ under parabolic isometries fixing, now, the point at infinity $(1,0) \in \partial_\infty \mathbb{H}^2$ (and not $(-1,0)$ as before). Observe that at each point of M the principal curvatures $\widetilde{\lambda}_1,\ \widetilde{\lambda}_2$ of M are given by $\widetilde{\lambda}_1 = \lambda_1$ and $\widetilde{\lambda}_2 = -\lambda_2$. Therefore we get*

$$\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda_1 - \lambda_2 \equiv 0.$$

We deduce that M is a complete minimal surface of $\mathbb{H}^2 \times \mathbb{R}$, embedded and invariant under parabolic isometries. Consequently, M is foliated by horocycles. This minimal surface was considered by B. Daniel [6], *R. Sa Earp* [8] *and L. Hauswirth* [12].

4.3. **Hyperbolic case.**

The umbilicity condition is

$$\theta'(s) = \sin\theta(s)\tanh\rho(s)$$

Proceeding as in the elliptic case, we discard the totally geodesic surfaces $\mathbb{H}^2 \times \{t\}$ and therefore we can we assume that ρ satisfies $\rho'^2 < 1$ and:

$$\rho'' = (\rho'^2 - 1)\tanh\rho, \quad \rho'^2 - 1 = -c^2\cosh^2\rho, \tag{22}$$

for some real constant $c \in \,]0,1[$. Again, invoking the Jacobi elliptic functions, it can be shown that, up to the sign, we have $\rho(s) = i\mathrm{am}(is + K, c^2) - i\pi/2$ and $t'(s) = c\,\mathrm{sn}(is + K, c^2)$ where $K = \int_0^{\pi/2} \frac{dt}{\sqrt{1-c^2\sin^2 t}}$, see [1, Chapter 16]. Nevertheless, as in the previuous cases, we prefer to give direct and elementary arguments.

We deduce from (22) that

$$\rho'' = -c^2\cosh\rho\sinh\rho. \tag{23}$$

If $\rho \equiv 0$, then the generated surface is the vertical totally geodesic plane $\{(0, y, t),\ -1 < y < 1,\ t \in \mathbb{R}\}$. Discarding this case, we consider only the nontrivial solutions of (23).

Again, as in the elliptic case, it can be shown that any maximal solution of the last equation is defined on the whole of $\mathbb{R}$, is periodic, vanishes at some point and, up to a reparametrization, is odd. We can therefore assume that there exists a unique maximal solution ρ_c satisfying $\rho(0) = 0$ and $\rho'(0) = \sqrt{1-c^2}$.

As we are interested in curves generating totally umbilic surfaces, we look for a function t satisfying $t'^2 = 1 - \rho_c'^2 = c^2\cosh^2\rho_c$. Let t_c be the function defined by $t_c' = c\cosh\rho_c$ and $t_c(0) = 0$. The function t_c is odd. Moreover consider any $s_0 \in \mathbb{R}$ such that $\rho_c'(s_0) = 0$, then it can be shown that $\rho_c(2s_0 - s) = \rho_c(s)$. Set $T(s) := -t_c(2s_0 - s) + 2t_c(s_0)$, $s \in \mathbb{R}$. We have $T' \equiv t_c'$ and $T(s_0) = t_c(s_0)$, therefore $T = t_c$. As the function t_c is odd, we have $t_c(s + 4s_0) = 4t_c(s_0) + t_c(s)$ for every $s \in \mathbb{R}$. We can deduce that t_c is an increasing and nonbounded function on $\mathbb{R}$. This shows that the curve $\alpha_c = (\rho_c, t_c)$ is properly embedded.

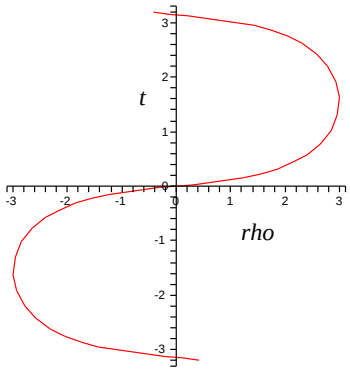
Observe that the other choice $t' = -c\cosh\rho_c$ changes the curve $\alpha_c = (\rho_c, t_c)$ into the symmetric curve $(\rho_c, -t_c)$ with respect to Γ.

We summarize stating the following.

Proposition 14. *Any non identically zero local solution of (23) satisfying $\rho'^2 < 1$ gives rise to a unique, up to ambient isometries, complete totally umbilic and non-geodesic surface S_c, $c > 0$, in $\mathbb{H}^2 \times \mathbb{R}$ invariant under hyperbolic isometries. Moreover, there exists a one-parameter family of such surfaces and all of them are analytic, properly embedded and homeomorphic to the plane. These surfaces are periodic in the vertical direction and symmetric with respect to a discrete set of horizontal slices.*

Hyperbolic case: c=0.7

Hyperbolic case: c=0.1



Furthermore, any umbilic surface in $\mathbb{H}^2 \times \mathbb{R}$ invariant under hyperbolic isometries is either part of a vertical totally geodesic plane, a slice $\mathbb{H}^2 \times \{t\}$ or, up to an ambient isometry, part of one of the surfaces S_c.

5. Unicity of totally umbilic surfaces in $\mathbb{H}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{R}$

In this section $\mathbb{M}^2$ stands for $\mathbb{H}^2$ or $\mathbb{S}^2$. The cases $\mathbb{M}^2 = \mathbb{M}^2(\kappa)$ for $\kappa < 0$ and $\kappa > 0$ are completely analogous.

We will need the following result which is of independent interest.

Proposition 15. *Let $S \subset \mathbb{M}^2 \times \mathbb{R}$ be an orientable surface transversal to each slice $\mathbb{M}^2 \times \{t\}$. We suppose the following:*

(1) *The geodesic curvature of each horizontal curve $S_t := S \cap (\mathbb{M}^2 \times \{t\})$ in $\mathbb{M}^2$ is constant (depending on t).*
(2) *The angle between S and $\mathbb{M}^2 \times \{t\}$ is constant along S_t for each t.*

Then:

In case $\mathbb{M}^2 = \mathbb{S}^2$ the surface S is part of rotational surface.

In case $\mathbb{M}^2 = \mathbb{H}^2$ the surface S is part of either

(1) *a rotational surface,*
(2) *or a surface invariant by a family of parabolic isometries having the same fixed point at infinity,*
(3) *or a surface invariant by a family of hyperbolic isometries along the same fixed geodesic of $\mathbb{H}^2$.*

Proof. Let N be a unit normal field along S. We define the function ν on S setting $\nu := \langle N, \frac{\partial}{\partial t} \rangle$. We denote by T the projection of $\frac{\partial}{\partial t}$ on S, that is $T = \frac{\partial}{\partial t} - \nu N$.

As the angle between S and $\mathbb{M}^2 \times \{t\}$ is constant along S_t, we deduce that S_t is a line of curvature. Indeed let $\gamma : s \in I \subset \mathbb{R} \to \gamma(s) \in S_t$ be a regular parametrization

of S_t, then taking into account that $\frac{\partial}{\partial t}$ is a parallel field and the definition of T:

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\langle N, \frac{\partial}{\partial t}\rangle = \langle \overline{\nabla}_{\gamma'(s)} N, \frac{\partial}{\partial t}\rangle = \langle \overline{\nabla}_{\gamma'(s)} N, T\rangle,$$

where $\overline{\nabla}$ is the connection on $\mathbb{M}^2 \times \mathbb{R}$. It follows that T is a principal direction on S.

Let $c : u \in I \subset \mathbb{R} \to c(u) \in S$ be a line of curvature associated to the field T. We are going to show that $c(I)$ is contained in a vertical totally geodesic plane. This is equivalent to showing that the horizontal projection $c_h : I \subset \mathbb{R} \to \mathbb{M}^2$ of c is a geodesic.

Assume first that c is never vertical, that is $\nu \neq 0$ along c. Thus c_h' does not vanish.

Let ∇ be the connection on $\mathbb{M}^2$. It is sufficient to show that $\nabla_{c_h'} c_h'$ is always parallel to c_h'. As $c' = c_h' + (1-\nu^2)\frac{\partial}{\partial t}$ we have

$$\begin{aligned}\overline{\nabla}_T T = \overline{\nabla}_{c'} c' &= \nabla_{c_h'} c_h' + \frac{\mathrm{d}}{\mathrm{d}u}(1-\nu^2)\frac{\partial}{\partial t} \\ &= \nabla_{c_h'} c_h' - 2\nu\nu'\frac{\partial}{\partial t}\end{aligned}$$

As T is a principal direction there exists a function λ such that $\overline{\nabla}_T N = \lambda T$. Therefore

$$\nu' = \frac{\mathrm{d}}{\mathrm{d}u}\langle N, \frac{\partial}{\partial t}\rangle = \langle \overline{\nabla}_T N, \frac{\partial}{\partial t}\rangle = \lambda(1-\nu^2).$$

Thus we obtain:

$$\overline{\nabla}_T T = \nabla_{c_h'} c_h' - 2\lambda\nu(1-\nu^2)\frac{\partial}{\partial t}.$$

Moreover we have

$$\begin{aligned}\overline{\nabla}_T T = \overline{\nabla}_T(\frac{\partial}{\partial t} - \nu N) &= -\overline{\nabla}_T \nu N \\ &= -\nu' N - \lambda\nu T \\ &= (\frac{\nu\prime}{\nu} - \lambda\nu)c_h' + (-\nu'\nu - \lambda\nu(1-\nu^2))\frac{\partial}{\partial t} \\ &= (\frac{\nu\prime}{\nu} - \lambda\nu)c_h' - 2\lambda\nu(1-\nu^2)\frac{\partial}{\partial t}.\end{aligned}$$

Thus we get

$$\nabla_{c_h'} c_h' = (\frac{\nu\prime}{\nu} - \lambda\nu)c_h',$$

which shows that $c_h(I)$ is a geodesic in $\mathbb{M}^2$.

We denote by w a unit horizontal field along c tangent to S and for each $u \in I$ we let $P(u)$ be the vertical totally geodesic plane containing $c(u)$ and orthogonal at $c(u)$ to $w(u)$.

Suppose now that that ν vanishes on an open interval $J \subset I$. Let $u_0 \in J$. Observe that along the horizontal curve of S through $c(u_0)$ the vector field N is horizontal. This means that an open set of S, including $c(J)$, is part of a cylinder $\gamma \times \mathbb{R}$ where

$\gamma \subset \mathbb{M}^2$ is some horizontal curve. Clearly this implies that w is constant along J, and thus so is P.

Combining those two arguments we see that P is locally constant on an open and dense subset of I. As $P(u)$ depends in a differentiable way on u, we conclude that P is constant.

Let us now consider a horizontal curve $\gamma : I \to S_t$ parametrized by arclength. Let $s_1, s_2 \in I$ and call $c :]-\varepsilon, \varepsilon[\to S$ the integral curve of T such that $c(0) = \gamma(s_1)$ and $\widetilde{c} :]-\varepsilon, \varepsilon[\to S$ the integral curve of T such that $\widetilde{c}(0) = \gamma(s_2)$.

Let us call c_3 (resp. $\widetilde{c}_3$) the vertical coordinate of c (resp. $\widetilde{c}$). Calling again u the parameter in $]-\varepsilon, \varepsilon[$, we have

$$c_3'(u) = \langle c'(u), \frac{\partial}{\partial t} \rangle = \langle T(c(u)), \frac{\partial}{\partial t} \rangle = 1 - \nu^2(c(u)) = 1 - \nu^2(c_3(u)).$$

Thus c_3 and $\widetilde{c}_3$ verify the same first order differential equation with the same initial condition at $u = 0$. We conclude that $c_3 \equiv \widetilde{c}_3$.

Recall that c and $\widetilde{c}$ are contained in vertical totally geodesic planes P and $\widetilde{P}$.

Let us call $\Gamma \subset \mathbb{M}^2$ the complete constant geodesic curvature line defined by γ, that is $\gamma \subset \Gamma$.

Observe that there is a unique positive isometry φ of $\mathbb{M}^2$ such that $\varphi(\Gamma) = \Gamma$, $\varphi(c(0)) = \widetilde{c}(0)$ and preserving the orientation of Γ. Therefore the isometry $\Phi(z, t) = (\varphi(z), t)$ of $\mathbb{M}^2 \times \mathbb{R}$ sends P to $\widetilde{P}$. Note that the curves $\widetilde{c}$ and $\Phi \circ c$ in the vertical plane $\widetilde{P}$ have the same vertical component and make the same angle with the horizontal for each $u \in]-\varepsilon, \varepsilon[$. We deduce that these curves coincide: $\Phi \circ c = \widetilde{c}$. This concludes the proof. □

We now state the main result of this section.

Theorem 16. *Let $S \subset \mathbb{M}^2 \times \mathbb{R}$ be an immersed totally umbilic surface. Then S is part of a complete and embedded totally umbilic surface $\widetilde{S}$ which is invariant by a one-parameter group of isometries of $\mathbb{M}^2 \times \mathbb{R}$. More precisely, up to an ambient isometry, in case $\mathbb{M}^2 = \mathbb{S}^2$, then $\widetilde{S}$ is one of the examples described in the section 3, and in case $\mathbb{M}^2 = \mathbb{H}^2$ then $\widetilde{S}$ is one of the examples described in the section 4.*

In particular, any totally geodesic surface is part of a slice $\mathbb{M}^2 \times \mathbb{R}$ or part of a product $\Gamma \times \mathbb{R}$ where $\Gamma \subset \mathbb{M}^2$ is a geodesic.

Proof. Locally S is the image of an embedding $X : \Omega \to \mathbb{M}^2 \times \mathbb{R}$, where Ω is an open disk in $\mathbb{R}^2$. As X is totally umbilic, there exists a function $\lambda : \Omega \to \mathbb{R}$ such that

$$\overline{\nabla}_w N = \lambda w,$$

for any vector w tangent to S.

Proceeding as in the proof of Theorem 1, as $\tau = 0$ we obtain:

$$\nabla \lambda = \kappa \nu T, \tag{24}$$

where κ is the Gaussian curvature of $\mathbb{M}^2$, that is $\kappa = 1$ if $\mathbb{M}^2 = \mathbb{S}^2$ and $\kappa = -1$ if $\mathbb{M}^2 = \mathbb{H}^2$.

Assume for the moment that λ has no critical point. In particular each level curve of λ is orthogonal to T and is therefore horizontal, that is belongs to some $\mathbb{M}^2 \times \{t_0\}$. Let $\gamma : I \subset \mathbb{M}^2 \times \{t_0\}$ be such a curve parametrized by arclength. We have

$$\frac{\mathrm{d}\nu}{\mathrm{d}s} = \langle \overline{\nabla}_{\gamma'(s)} N, \frac{\partial}{\partial t} \rangle + \langle N, \overline{\nabla}_{\gamma'(s)} \frac{\partial}{\partial t} \rangle = \lambda(\gamma(s)) \langle \gamma'(s), \frac{\partial}{\partial t} \rangle = 0,$$

therefore ν is constant along γ.

We now call n the unit normal field in $T(\mathbb{M}^2 \times \{t_0\})$ along γ with the orientation induced by N. Let θ be the oriented angle between n and N, hence $N(\gamma(s)) = \cos\theta(s) n(\gamma(s)) + \sin\theta(s) \frac{\partial}{\partial t}(\gamma(s))$, we deduce that θ is constant along γ.

On the other hand:

$$\begin{aligned}
\lambda(\gamma(s)) &= \langle \overline{\nabla}_{\gamma'(s)} N, \gamma'(s) \rangle \\
&= \langle \overline{\nabla}_{\gamma'(s)} (\cos\theta \; n + \sin\theta \frac{\partial}{\partial t}), \gamma'(s) \rangle \\
&= \cos\theta \langle \overline{\nabla}_{\gamma'(s)} n, \gamma'(s) \rangle + \sin\theta \langle \overline{\nabla}_{\gamma'(s)} \frac{\partial}{\partial t}, \gamma'(s) \rangle \\
&= \cos\theta \langle \overline{\nabla}_{\gamma'(s)} n, \gamma'(s) \rangle.
\end{aligned}$$

Now observe that $\langle \overline{\nabla}_{\gamma'(s)} n, \gamma'(s) \rangle$ is the geodesic curvature of γ in $\mathbb{M}^2 \times \{t_0\}$. Since λ and θ are constant along γ we deduce that γ has constant geodesic curvature. We conclude using the proposition 15 and results in the sections 3 and 4 that S is as stated.

Suppose now that λ has some critical points.

Let $U \subset S$ be a connected component, if any, of the interior of the set of critical points of λ. The formula (24) shows that N is either always vertical or always horizontal in U. In the former case U is part of a slice $\mathbb{M}^2 \times \{t_0\}$ and in the latter case U is part of a cylinder, that is part of a product $\Gamma \times \mathbb{R}$ where Γ is some curve in $\mathbb{M}^2$. As S is totally umbilic, Γ has to be a geodesic and so U is totally geodesic.

Let now $V \subset S$ be a connected component, if any, of the set of regular points of λ. From the first part of the proof, we know that V is part of one the symmetric examples given in the sections 3 and 4.

Therefore S is obtained by gluing pieces of totally geodesic surfaces and pieces of the symmetric examples constructed in the sections 3 and 4. A closer look at these different types of surfaces shows that the whole of S is either totally geodesic or part of one of the complete symmetric examples, which concludes the proof. □

Remark 17. *The local existence of totally umbilic surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ can be seen in an alternate way. Indeed, it is known that umbilicity is preserved by conformal diffeomorphisms, see* [21] *(Vol. 4). It can be shown that $\mathbb{S}^2 \times \mathbb{R}$ is conformally diffeomorphic to $\mathbb{R}^3 \setminus \{(0,0,0)\}$, see Section 7. This implies the umbilic*

surfaces in $\mathbb{S}^2 \times \mathbb{R}$ *correspond through this conformal diffeomorphism to those of* $\mathbb{R}^3 \setminus \{(0,0,0)\}$. *However to classify them in* $\mathbb{S}^2 \times \mathbb{R}$ *up to congruences, and to understand their geometry requires a nontrivial work. Regarding* $\mathbb{H}^2 \times \mathbb{R}$, *it can be shown that* $\mathbb{H}^2 \times]0, \pi[$ *is conformally diffeomorphic to* $\mathbb{H}^3$, *see Section 7. Nonetheless this is not enough to understand the global geometry and topology of the umbilic surfaces in* $\mathbb{H}^2 \times \mathbb{R}$.

6. Totally umbilic surfaces in *Sol*

The *Sol* geometry is the eighth model geometry of Thurston, see [22]. It is a Lie group endowed with a left-invariant metric, it is a homogeneous simply connected 3-manifold with a 3-dimensional isometry group, see [3]. It is isometric to $\mathbb{R}^3$ equipped with the metric:

$$ds^2 = e^{2z}\,\mathrm{d}x^2 + e^{-2z}\,\mathrm{d}y^2 + \mathrm{d}z^2.$$

The group structure of *Sol* is given by

$$(x', y', z') \star (x, y, z) = (e^{-z'}x + x', e^{z'}y + y', z + z').$$

The isometries are:

$$(x, y, z) \mapsto (\pm e^{-c}x + a, \pm e^{c}y + b, z + c) \quad \text{and} \quad (x, y, z) \mapsto (\pm e^{-c}y + a, \pm e^{c}x + b, -z + c),$$

where a, b and c are any real numbers. We set $E_1 = e^{-z}\frac{\partial}{\partial x}$, $E_2 = e^{z}\frac{\partial}{\partial y}$ and $E_3 = \frac{\partial}{\partial z}$. Thus (E_1, E_2, E_3) is a global orthonormal frame. A straightforward computation gives:

$$\begin{array}{lll} \overline{\nabla}_{E_1}E_1 = -E_3 & \overline{\nabla}_{E_2}E_1 = 0 & \overline{\nabla}_{E_3}E_1 = 0 \\ \overline{\nabla}_{E_1}E_2 = 0 & \overline{\nabla}_{E_2}E_2 = E_3 & \overline{\nabla}_{E_3}E_2 = 0 \\ \overline{\nabla}_{E_1}E_3 = E_1 & \overline{\nabla}_{E_2}E_3 = -E_2 & \overline{\nabla}_{E_3}E_3 = 0 \end{array} \tag{25}$$

We deduce from (25) that the vertical planes $\{x = x_0\}$ and $\{y = y_0\}$ are totally geodesic complete surfaces and that the horizontal planes $\{z = z_0\}$ are not totally umbilic surfaces (in fact they are minimal surfaces).

We now look for totally umbilic surfaces which are invariant under the one parameter group of isometries $(x, y, z) \mapsto (x + c, y, z)$. Clearly, such a surface is generated by a curve γ in the totally geodesic plane $\{x = 0\}$. Discarding the trivial case of a vertical plane $\{y = y_0\}$, we can assume that γ locally is a graph over the y-axis. Thus γ is given by $\gamma(y) = (0, y, z(y))$. Therefore the generated surface is parametrized by:

$$X(t, y) := (t, y, z(y)).$$

We have $X_t = (1, 0, 0) = e^{z}E_1$ and $X_y = (0, 1, z') = e^{-z}E_2 + z'E_3$. As a unit normal field we can take

$$N = \frac{e^{z}z'}{\sqrt{1 + e^{2z}z'^2}}E_2 - \frac{1}{\sqrt{1 + e^{2z}z'^2}}E_3.$$

We have:

$$\overline{\nabla}_{X_t} N = -\frac{1}{\sqrt{1+e^{2z}z'^2}} X_t$$

$$\overline{\nabla}_{X_y} N = \frac{e^{-z}}{(1+e^{2z}z'^2)^{3/2}}(1+2e^{2z}z'^2+e^{2z}z'')E_2 + \frac{z'}{(1+e^{2z}z'^2)^{3/2}}(1+2e^{2z}z'^2+e^{2z}z'')E_3$$

So that X is a totally umbilic immersion if and only if

$$\overline{\nabla}_{X_y} N = -\frac{1}{\sqrt{1+e^{2z}z'^2}} X_y,$$

that is if and only if

$$z'' + 3z'^2 + 2e^{-2z} = 0. \tag{26}$$

A first integral of (26) is

$$z'^2 = ae^{-6z} - e^{-2z},$$

where a is any positive real number.

Assume $z'(y_0) = 0$ for some y_0. Considering the function $f(y) = z(2y_0 - y)$, we can see that the curve γ is symmetric with respect to the vertical geodesic $\{y = y_0\}$. Therefore, up to the isometry $(x, y, z) \mapsto (x, -y, z)$ and restricting the domain of z if needed, we can assume $z' > 0$. Therefore z satisfies

$$z' = e^{-z}\sqrt{ae^{-4z} - 1}. \tag{27}$$

We consider the maximal solution of (27) defined by z and we call it again z, it is defined on an open interval $]y_1, y_2[$, $-\infty \leq y_1 < y_2 \leq +\infty$. By (27) the function z is bounded above and is increasing we deduce using (26) that z'' has a negative limit at y_2. Taking into account the fact that z' is positive, we deduce that y_2 is finite, $y_2 < +\infty$. Moreover, since z' is a positive and decreasing function, it has a nonnegative limit at y_2. If this limit were positive, we could extend the solution z of (27) beyond y_2 which contradicts the maximality of z. Thus we have $\lim_{y\to y_2} z'(y) = 0$ and consequently $\lim_{y\to y_2} z(y) = \frac{1}{4}\log a$.

Consider now the maximal solution of (26) defined by z and call it z_a. As $z_a'(y_2) = 0$ we have $z_a(2y_2 - y) = z_a(y)$. Up to the horizontal translation $(x, y, z) \mapsto (x, y - y_2, z)$, which is an ambient isometry, we can assume that $y_2 = 0$ and therefore z_a is an even fonction and is defined on an interval $]-y_a, y_a[$ where $0 < y_a \leq +\infty$. Observe that there exist $A > 0$ and $y_3 > 0$ such that $z_a''(y) < -A < 0$ for any $y > y_3$. Therefore, if $y_a = +\infty$ we have $\lim_{y\to+\infty} z_a(y) = -\infty$. Suppose now that y_a is finite. If z_a had a finite limit at y_a then z_a' would have also a finite limite but then we could extend the solution z_a beyond y_a, which is absurd. We deduce that in both cases, that is $y_a < +\infty$ and $y_a = +\infty$, we have $\lim_{y\to y_a} z_a(y) = -\infty$.

Now we show that $y_a < +\infty$. Indeed, as z_a is a solution of (26) satisfying $z_a(0) = 0$ we have

$$z'(y) = -\sqrt{a}e^{-3z}\sqrt{1 - \frac{e^{4z}}{a}},$$

for any $y > 0$. Since $\lim_{y \to y_a} z_a(y) = -\infty$, we deduce that for some $y_4 > 0$ we have

$$\frac{1}{\sqrt{a}} z' e^{3z} < -\frac{1}{2}$$

for any $y > y_4$. Therefore we have

$$\frac{1}{3\sqrt{a}} e^{3z} < -\frac{y}{2} + c$$

for some real constant c and for any $y > y_4$. This implies that $y_a < +\infty$.

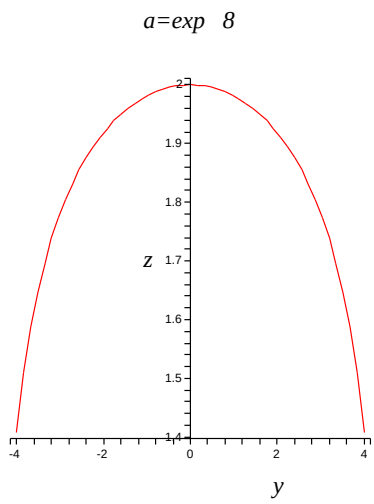
Call γ_a the graph of the function $z_a : \gamma_a := \{(0, y, z_a(y)),\ -y_a < y < \ y_a\}$ and call F_a the totally umbilic complete surface generated by γ_a:

$$F_a := \{(x, y, z_a(y)),\ x \in \mathbb{R},\ -y_a < y < y_a\}$$

Let z_a and z_b be two maximal solutions of (26) where a and b are any real numbers. Set $c = \frac{1}{4} \log \frac{b}{a}$ and consider the ambient isometry $(x, y, z) \mapsto (e^{-c}x, e^c y, z + c)$. Observe that this isometry maps the planar curve γ_a onto the planar curve γ_b and maps any Euclidean line parallel to the x-axis onto a line of the same type. Therefore this isometry maps the totally umbilic surface F_a onto the totally umbilic surface F_b.

Summarizing, we state the following result.

Proposition 18. *Up to ambient isometries, there exist only two complete totally umbilic surfaces in the Sol group invariant under the horizontal translations* $(x, y, z) \mapsto (x + t, y, z),\ t \in \mathbb{R}$. *The first one is the totally geodesic plane* $\{y = 0\}$. *The second one is nongeodesic, is contained in a slab delimited by two totally geodesic planes* $\{y = \pm y_0\},\ y_0 > 0$, *and is asymptotic to these planes. Moreover it is symmetric with respect to the totally geodesic plane* $\{y = 0\}$.



As a matter of fact we have the following.

Theorem 19. *Up to ambient isometries, any totally umbilic surface in the Sol group is part of one of the two complete totally umbilic surfaces given in the proposition 18. In particular, up to ambient isometries, there exists a unique complete totally geodesic surface in the Sol group.*

Proof. Let S be an immersed totally umbilic surface in the *Sol* group. Locally S is the image of an embedding $X : \Omega \to Sol$, where Ω is an open disk in $\mathbb{R}^2$. Call (u, v) the coordinates on Ω and consider a unit normal field N on $X(\Omega)$. As X is totally umbilic, there exists a function $\lambda : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \overline{\nabla}_{X_u} N = \lambda X_u \\ \overline{\nabla}_{X_v} N = \lambda X_v \end{cases}$$

where $\overline{\nabla}$ is the Riemannian connection of the *Sol* group. As in the proof of the theorem 1 we find

$$R(X_u, X_v)N = \lambda_v X_u - \lambda_u X_v, \tag{28}$$

where R denotes the curvature tensor of the *Sol* group. Let us express the later. let X, Y, Z and W be any vector fields. Proceeding as in [7], Proposition 2.1, after some computations and using (25) we obtain the following:

$$\begin{aligned} \langle R(X,Y)Z, W\rangle = {} & (\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle) \\ & + 2\Big(\langle X, W\rangle\langle Y, \frac{\partial}{\partial z}\rangle\langle Z, \frac{\partial}{\partial z}\rangle + \langle Y, Z\rangle\langle X, \frac{\partial}{\partial z}\rangle\langle W, \frac{\partial}{\partial z}\rangle \\ & - \langle X, Z\rangle\langle Y, \frac{\partial}{\partial z}\rangle\langle W, \frac{\partial}{\partial z}\rangle - \langle Y, W\rangle\langle X, \frac{\partial}{\partial z}\rangle\langle Z, \frac{\partial}{\partial z}\rangle\Big). \end{aligned}$$

We define the function ν on Ω setting $\nu := \langle N, \frac{\partial}{\partial z}\rangle$. We denote by T the projection of $\frac{\partial}{\partial z}$ on S, that is $T = \frac{\partial}{\partial z} - \nu N$. We then have:

$$\begin{aligned} R(X_u, X_v)N &= 2\nu(\langle X_v, \frac{\partial}{\partial z}\rangle X_u - \langle X_u, \frac{\partial}{\partial z}\rangle X_v) \\ &= 2\nu(\langle X_v, T\rangle X_u - \langle X_u, T\rangle X_v) \end{aligned}$$

From what we deduce using (28):

$$\nabla\lambda = 2\nu T. \tag{29}$$

Assume first that ν and T do not vanish on Ω. Thus T is of the form

$$T = \alpha E_1 + \beta E_2 + \gamma E_3,$$

where α and β do not vanish simultaneously. Since $|T|^2 = 1 - \nu^2$ we have $\alpha^2 + \beta^2 = \nu^2(1 - \nu^2)$. We thus have

$$N = -\frac{\alpha}{\nu}E_1 - \frac{\beta}{\nu}E_2 + \nu E_3.$$

We set

$$JT = -\frac{\beta}{\nu}E_1 + \frac{\alpha}{\nu}E_2$$

therefore JT is tangent to S, orthogonal to T and E_3 and satisfies $|JT|^2 = |T|^2$. Furthermore we have

$$N \wedge T = JT,\ T \wedge JT = (1-\nu^2)N,\ JT \wedge N = T.$$

We now compute the derivative $[T, JT](\lambda)$ in two different ways.

We first compute $[T, JT] = \overline{\nabla}_T JT - \overline{\nabla}_{JT} T$. We have

$$\begin{aligned}
\overline{\nabla}_T JT &= \overline{\nabla}_T N \wedge T + N \wedge \overline{\nabla}_T T \\
&= N \wedge \overline{\nabla}_T T \quad (\text{since } \overline{\nabla}_T N = \lambda N) \\
&= N \wedge \overline{\nabla}_T (E_3 - \nu N) \\
&= N \wedge \overline{\nabla}_T E_3 - \lambda\nu JT
\end{aligned}$$

Furthermore, using (25) we obtain

$$\begin{aligned}
\overline{\nabla}_T E_3 &= \alpha\overline{\nabla}_{E_1} E_3 + \beta\overline{\nabla}_{E_2} E_3 + (1-\nu^2)\overline{\nabla}_{E_3} E_3 \\
&= \alpha E_1 - \beta E_2
\end{aligned}$$

from what we deduce after some straightforward computations

$$\overline{\nabla}_T E_3 = \frac{\alpha^2-\beta^2}{1-\nu^2}T - 2\frac{\alpha\beta}{\nu(1-\nu^2)}JT - \nu(1-\nu^2)N.$$

Consequently:

$$\overline{\nabla}_T JT = 2\frac{\alpha\beta}{\nu(1-\nu^2)}T + \frac{\alpha^2-\beta^2}{1-\nu^2}JT - \lambda\nu JT.$$

In the same way we obtain:

$$\begin{aligned}
\overline{\nabla}_{JT} T &= \overline{\nabla}_{JT} E_3 - \overline{\nabla}_{JT}\nu N \\
&= -2\frac{\alpha\beta}{\nu(1-\nu^2)}T + \nu JT - \lambda\nu JT
\end{aligned}$$

We deduce that

$$[T, JT] = 4\frac{\alpha\beta}{\nu(1-\nu^2)}T + (\frac{\alpha^2-\beta^2}{1-\nu^2} - \nu)JT.$$

Using this last expression and (29), we find

$$[T, JT](\lambda) = 8\alpha\beta. \tag{30}$$

On the other hand, using again (29) we have

$$\begin{aligned}[T, JT](\lambda) &= T(JT(\lambda)) - JT(T(\lambda)) \\ &= -JT(\langle \nabla\lambda, T\rangle) \\ &= 2(-1+3\nu^2)JT(\nu) \\ &= 4\frac{\alpha\beta}{\nu}(-1+3\nu^2)\end{aligned} \tag{31}$$

since an easy computation gives $JT(\nu) = 2\alpha\beta/\nu$. From (30) and (31) we deduce

$$\alpha\beta(3\nu^2 - 2\nu - 1) = 0.$$

Observe that if ν is constant on an open subset then $JT(\nu) \equiv 0$ which implies $\alpha\beta \equiv 0$. So in all cases we have $\alpha\beta \equiv 0$.

Recall that α and β do not vanish simultaneously since by our assumption $\nu \neq 0$. Therefore we have either $\alpha \equiv 0$ or $\beta \equiv 0$.

Considering the isometry $(x, y, z) \mapsto (y, x, -z)$ we can assume that $\alpha \equiv 0$. Then the surface is part of a product $\mathbb{R}\times\Gamma$ where Γ is a curve in the geodesic plane $\{x = 0\}$. This case is considered in the proposition (18).

Let us suppose now that $T \equiv 0$ on an open set. Then this open set is part of a horizontal plane $\{z = z_0\}$, but this contradicts the assumption of umbilicity.

To finish the proof we consider the case where $\nu \equiv 0$ on an open subset. Therefore $T \equiv E_3$ and so this piece of the surface is part of a product $L \times \mathbb{R}$ where L is a curve in the horizontal plane $\{z = 0\}$. If L is contained in a line parallel to the y-axis, then the surface is contained in a totally geodesic plane $\{x = x_0\}$. Discarding this trivial case, we can assume that L is a graph over the x-axis. Consequently, the embedding X is given by

$$X(x, t) = (x, y(x), t).$$

As a unit normal we take

$$\begin{aligned}N &= \frac{1}{\sqrt{e^{-2t}y'^2 + e^{2t}}}(e^{-2t}y', -e^{2t}, 0) \\ &= \frac{y'}{\sqrt{y'^2 + e^{4t}}}E_1 - \frac{1}{\sqrt{e^{-4t}y'^2 + 1}}E_2\end{aligned}$$

As $X_t = E_3$, using (25) we obtain:

$$\overline{\nabla}_{X_t}N = -2\frac{y'e^{4t}}{(y'^2 + e^{4t})^{3/2}}E_1 - 2\frac{e^{-4t}y'^2}{(e^{-4t}y'^2 + 1)^{3/2}}E_2.$$

The condition $\overline{\nabla}_{X_t}N = \lambda X_t$ is therefore equivalent to $\lambda \equiv 0$ and $y' \equiv 0$. So L is part of a line parallel to the x-axis and the surface is part of a geodesic plane $\{y = y_0\}$. This concludes the proof. □

Remark 20. *It can be proved that any twice differentiable totally umbilic surface in a space form, in* $\mathbb{S}^2(\kappa) \times \mathbb{R}$ *or in* $\mathbb{H}^2(\kappa) \times \mathbb{R}$*, is in fact* C^3 *and then analytic by the previous discussions, see* [20].

7. An application

As an application of the classification of totally umbilic surfaces obtained in the previous sections, we can prove the following result:

Theorem 21. *Any conformal diffeomorphism of* $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$ *and Sol is an isometry.*

Proof. The result for $\mathbb{S}^2 \times \mathbb{R}$ is a consequence of the fact that the mapping:

$$\mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}^3 - \{(0,0,0)\}$$
$$(p,t) \mapsto e^t p$$

is a conformal diffeomorphism -here $\mathbb{S}^2$ is viewed as the unit sphere of $\mathbb{R}^3$ centered at the origin. Indeed the conformal diffeomorphisms of $\mathbb{R}^3 - \{(0,0,0)\}$ are the Möbius transformations fixing $(0,0,0)$ or sending $(0,0,0)$ to the point at infinity and these transformations correspond through the above conformal diffeomorphism to isometries of $\mathbb{S}^2 \times \mathbb{R}$. We leave the details to the reader.

We now prove the result for the space $\mathbb{H}^2 \times \mathbb{R}$. We claim that, except for the slices $\mathbb{H}^2 \times \{t_0\}$, all the non-compact maximal (for the inclusion) totally umbilic surfaces in $\mathbb{H}^2 \times \mathbb{R}$ are conformal to $\mathbb{C}$. This is clear for the products $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{H}^2$ is a geodesic. As for the surfaces invariant under a one parameter group of parabolic transformations and which are all congruent to the surface S_P described in 4.2, this is seen as follows. Consider in $\mathbb{H}^3$ a totally geodesic plane which we call $\mathbb{H}^2$ and denote by N a unit normal along it. We let exp denote the exponential map in $\mathbb{H}^3$. Then the map:

$$\mathbb{H}^2 \times]0,\pi[\to \mathbb{H}^3$$
$$(p,t) \mapsto \exp_p(\ln(\tan(\frac{t}{2}))N(p))$$

is a conformal diffeomorphism (cf. [20] for the details) which sends S_P onto a totally umbilic surface of $\mathbb{H}^3$ with one point at infinity, that is a horosphere. It remains to consider the case of the surfaces invariant under a one parameter group of hyperbolic transformations. Consider such a surface Σ. We know that Σ is invariant under a set of reflections of $\mathbb{H}^2 \times \mathbb{R}$ through horizontal slices $\mathbb{H}^2 \times \{t = a + nb\}$, for all $n \in \mathbb{Z}$, and $a, b \in \mathbb{R}$ depending on Σ. Suppose by contradiction that Σ is conformal to $\mathbb{H}^2$. The isometries of Σ induced by those reflections correspond then to conformal diffeomorphisms of $\mathbb{H}^2$ and so to isometries of $\mathbb{H}^2$. In particular all the horizontal curves $\Sigma \cap \{t = an + b\}$ correspond to geodesics of $\mathbb{H}^2$. Now observe that all these curves are invariant by the hyperbolic isometries leaving Σ invariant. We thus get

isometries of $\mathbb{H}^2$ which leave (globally) invariant more than one geodesic. This is a contradiction as only the identity of $\mathbb{H}^2$ has this property.

Take now a conformal diffeomorphism ϕ of $\mathbb{H}^2 \times \mathbb{R}$. Then ϕ sends any horizontal slice $\mathbb{H}^2 \times \{t_0\}$ to a maximal totally umbilic non-compact surface which is conformal to $\mathbb{H}^2$. From what preceeds it follows that ϕ sends $\mathbb{H}^2 \times \{t_0\}$ conformally to some horizontal slice $\mathbb{H}^2 \times \{t_1\}$ and so isometrically. As ϕ is conformal this implies that for any $x \in \mathbb{H}^2 \times \{t_0\}$, the tangent map $D_x\phi$ is an isometry. So ϕ is an isometry of $\mathbb{H}^2 \times \mathbb{R}$.

The case of the *Sol* group is treated analogously. There are two maximal totally umbilic surfaces up to congruences. With the notations of Section 6, the first one is the totally geodesic plane $\{x = x_0\}$ and is easily seen to be isometric to $\mathbb{H}^2$. The second one is the surface parametrized by: $X(t, y) = (t, y, z(y))$, where $t \in \mathbb{R}, \quad y \in]-y_1, y_1[$ and z is the maximal solution to the equation: $z' = e^{-z}\sqrt{e^{-4z} - 1}$ (we have chosen $a = 1$ with the notations of Section 6). The metric on this surface writes:

$$ds^2 = e^{2z}dt^2 + e^{-z}dy^2.$$

Making the change of coordinate $\xi = \int e^{-4z(y)}dy$, the metric writes:

$$ds^2 = e^{2z}(dt^2 + d\xi^2).$$

As the function z is bounded from above (cf. Section 6), the flat metric $dt^2 + d\xi^2$ is complete. It follows that the surface is conformal to $\mathbb{C}$. As the totally geodesic planes $\{x = const.\}$ fill the whole space *Sol*, we conclude as before that any conformal diffeomorphism of *Sol* is an isometry. $\square$

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