HAL
open science

The maximum of the local time of a diffusion process in a drifted Brownian potential

Alexis Devulder

To cite this version:

Alexis Devulder. The maximum of the local time of a diffusion process in a drifted Brownian potential. 2015. hal-00022214v2

HAL Id: hal-00022214

https://hal.science/hal-00022214v2

Preprint submitted on 17 Nov 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THE MAXIMUM OF THE LOCAL TIME OF A DIFFUSION PROCESS IN A DRIFTED BROWNIAN POTENTIAL

ALEXIS DEVULDER

Abstract. We consider a one-dimensional diffusion process X in a $(-\kappa/2)$-drifted Brownian potential for $\kappa \neq 0$. We are interested in the maximum of its local time, and study its almost sure asymptotic behaviour, which is proved to be different from the behaviour of the maximum local time of the transient random walk in random environment. We also obtain the convergence in law of the maximum local time of X under the annealed law after suitable renormalization when $\kappa \geq 1$. Moreover, we characterize all the upper and lower classes for the hitting times of X, in the sense of Paul Lévy, and provide laws of the iterated logarithm for the diffusion X itself. To this aim, we use annealed technics.

1. Introduction

1.1. **Presentation of the model.** We consider a diffusion process in random environment, defined as follows. For $\kappa \in \mathbb{R}$, we introduce the random potential

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x, \qquad x \in \mathbb{R}, \tag{1.1}$$

where $(W(x),\ x \in \mathbb{R})$ is a standard two-sided Brownian motion. Informally, a diffusion process $(X(t),\ t \geq 0)$ in the random potential W_κ is defined by

$$\begin{cases} \mathrm{d}X(t) = \mathrm{d}\beta(t) - \frac{1}{2}W_\kappa'(X(t))\mathrm{d}t, \\ X(0) = 0, \end{cases}$$

where $(\beta(t),\ t \geq 0)$ is a Brownian motion independent of W. More rigorously, $(X(t),\ t \geq 0)$ is a diffusion process such that $X(0) = 0$, and whose conditional generator given W_κ is

$$\frac{1}{2}e^{W_\kappa(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-W_\kappa(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Let P be the probability measure associated to W_κ. We denote by P_{W_κ} the law of X conditionally on the environment W_κ, and call it the *quenched law*. We also define the *annealed law* $\mathbb{P}$ as follows:

$$\mathbb{P}(\cdot) := \int P_{W_\kappa}(\cdot)P(W_\kappa \in \mathrm{d}\omega).$$

Notice in particular that X is a Markov process under P_{W_κ}, but not under $\mathbb{P}$. Such a diffusion can also be constructed from a Brownian motion through (random) changes of time and scale (see (6.1) below). This diffusion X, introduced by Schumacher [43] and Brox [11], is generally considered as the continuous time analogue of random walks in random environment (RWRE), which have many applications in physics and biology (see e.g. Le Doussal et al. [38]); for an account of general properties of RWRE, we refer to Révész [40] and Zeitouni [56]. This diffusion has been studied for example by Kawazu and Tanaka [36], see Theorem 1.1 below, later improved

Date: November 17, 2015.

2010 *Mathematics Subject Classification.* 60K37, 60J60, 60J55, 60F15.

Key words and phrases. Random environment, diffusion in a random potential, maximum local time, Lévy class, law of the iterated logarithm.

This research was partially supported by the french ANR project MEMEMO2 2010 BLAN 0125.

by Hu, Shi and Yor [33]. Large deviations results are proved in Taleb [50] and Talet [51] (see also Devulder [20] for some properties of the rate function), and moderate deviations are given by Hu and Shi [32] in the recurrent case, and by Faraud [25] in the transient case. A localization result and an aging theorem are provided by Andreoletti and Devulder [3] in the case $0 < \kappa < 1$. For a relation between RWRE and the diffusion X, see e.g. Shi [45]. See also Carmona [12], Cheliotis [13], Mathieu [39], Singh [47], [48] and Tanaka [52] for diffusions in other potentials.

In this paper, we are interested in the transient case, that is, we suppose $\kappa \neq 0$. If X is a diffusion in the random potential W_κ, then $-X$ is a diffusion in the random potential $(W_\kappa(-x),\ x \in \mathbb{R})$ which has the same law as $(W_{-\kappa}(x),\ x \in \mathbb{R})$. Hence we may assume without loss of generality that $\kappa > 0$. In this case, $X(t) \to_{t\to+\infty} +\infty$ $\mathbb{P}$–almost surely.

Our goal is to study the asymptotics of the maximum of the local time of X. Corresponding problems for RWRE have attracted much attention, and have been studied, for example, in Révész ([40], Chapter 29), Shi [44], Gantert et al. [26], [27], Hu et al. [29], Dembo et al. [17] and Andreoletti ([1], see also [2]). Moreover the local time of such processes in random environment plays an important role in estimation problems (see e.g. Comets et al. [15]), in persistence (see Devulder [21]) and in the study of processes in random scenery (see Zindy [57]).

1.2. **Maximum local time.** We denote by $(L_X(t,x),\ t \geq 0,\ x \in \mathbb{R})$ the local time of X, which is the jointly continuous process satisfying, for any positive measurable function f,

$$\int_0^t f(X(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x) L_X(t,x)\mathrm{d}x, \qquad t \geq 0. \tag{1.2}$$

The existence of such a process was proved by Hu and Shi ([29], eq. (2.6)); see (6.2) below for an expression of L_X. We are interested in the *maximum local time* of X at time t, defined as

$$L_X^*(t) := \sup_{x\in\mathbb{R}} L_X(t,x), \qquad t \geq 0.$$

In the recurrent case $\kappa = 0$, Hu and Shi [29] first proved that for any $x \in \mathbb{R}$,

$$\frac{\log L_X(t,x)}{\log t} \xrightarrow{\mathcal{L}} U \wedge \hat{U},$$

where U and $\hat{U}$ are two independent random variables uniformly distributed in $[0,1]$, and " $\xrightarrow{\mathcal{L}}$ " denotes convergence in law under the annealed law $\mathbb{P}$. Moreover, thoughout the paper, log denotes the natural logarithm. The limit law of $L_X^*(t)$, suitably renormalized, is determined by Andreoletti and Diel [5] when $\kappa = 0$:

$$\frac{L_X^*(t)}{t} \xrightarrow{\mathcal{L}} \Big(\int_{-\infty}^{\infty} e^{-\widetilde{W}(x)}\mathrm{d}x\Big)^{-1}, \tag{1.3}$$

where $\big(\widetilde{W}(x),\ x \in \mathbb{R}\big)$ is a two-sided Brownian motion conditioned to stay positive. Furthermore, Shi [44] proved the following surprising result: $\mathbb{P}$-almost surely when $\kappa = 0$,

$$\limsup_{t\to+\infty} L_X^*(t)/(t \log\log\log t) \geq 1/32. \tag{1.4}$$

The question whether this is the good renormalization remained open during 13 years, until Diel [22] gave a positive answer to this question. He proved indeed that in this recurrent case $\kappa = 0$,

$$\limsup_{t\to+\infty} L_X^*(t)/(t \log\log\log t) \leq e^2/2, \qquad j_0^2/64 \leq \liminf_{t\to+\infty} L_X^*(t)/[t/(\log\log\log t)] \leq e^2\pi^2/4$$

$\mathbb{P}$-almost surely, where j_0 is the smallest strictly positive root of the Bessel function J_0. Moreover, the convergence in law (1.3) is extended to the case of stable Lévy environment by Diel and Voisin [23]. Finally, related questions about favorite sites, that is, locations in which the local time is maximum at time t, are considered by Hu and Shi [31], Cheliotis [14], and Andreoletti et al. [4].

1.3. **Results.** We define the first hitting time of r by X as follows:

$$H(r) := \inf\{t \geq 0, \quad X(t) > r\}, \qquad r \geq 0. \tag{1.5}$$

We recall that there are three different regimes for H in the transient case $\kappa > 0$:

Theorem 1.1. *(Kawazu and Tanaka,* [36]*) When r tends to infinity,*

$$\frac{H(r)}{r^{1/\kappa}} \xrightarrow{\mathcal{L}} c_0 S_\kappa^{ca}, \qquad 0 < \kappa < 1, \tag{1.6}$$

$$\frac{H(r)}{r \log r} \xrightarrow{P.} 4, \qquad \kappa = 1, \tag{1.7}$$

$$\frac{H(r)}{r} \xrightarrow{a.s.} \frac{4}{\kappa - 1}, \qquad \kappa > 1, \tag{1.8}$$

where $c_0 = c_0(\kappa) > 0$ is a finite constant, the symbols " $\xrightarrow{\mathcal{L}}$ ", " $\xrightarrow{P.}$ " and " $\xrightarrow{a.s.}$ " denote respectively convergence in law, in probability and almost sure convergence, with respect to the annealed probability $\mathbb{P}$. Moreover, for $0 < \kappa < 1$, S_κ^{ca} is a completely asymmetric stable variable of index κ, and is a positive variable (see (2.1) *for its characteristic function).*

The asymptotics of the maximum local time $L_X^*(t)$ heavily depend on the value of κ. We start with the upper asymptotics of $L_X^*(t)$:

Theorem 1.2. *If $0 < \kappa < 1$, then*

$$\limsup_{t \to +\infty} \frac{L_X^*(t)}{t} = +\infty \qquad \mathbb{P}\text{–}a.s.$$

Theorem 1.2 tells us that in the case $0 < \kappa < 1$, the maximum local time of X has a completely different behaviour from the maximum local time of RWRE (the latter is trivially bounded by $t/2$ for any positive integer t, for example). Such a peculiar phenomenon has already been observed (see (1.4)) by Shi [44] in the recurrent case, and is even more surprising here since X is transient.

Theorem 1.3 gives, in the case $\kappa > 1$, an integral test which completely characterizes the upper functions of $L_X^*(t)$, in the sense of Paul Lévy.

Theorem 1.3. *Let $a(\cdot)$ be a positive nondecreasing function. If $\kappa > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \iff \limsup_{t \to \infty} \frac{L_X^*(t)}{[ta(t)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s.$$

This is in agreement with a result of Gantert and Shi [27] for RWRE. We notice in particular that $\limsup_{t \to +\infty} L_X^*(t)/t$ is almost surely $+\infty$ when $0 < \kappa < 1$ by Theorem 1.2, whereas it is 0 when $\kappa > 1$ by Theorem 1.3. We have not been able to prove whether $\limsup_{t \to +\infty} L_X^*(t)/t$ is infinite in the very delicate case $\kappa = 1$, since a proof similar to that of Theorem 1.2 just shows that it is greater than a positive deterministic constant (see Remark page 25 for more details).

We now turn to the lower asymptotics of $L_X^*(t)$.

Theorem 1.4. *We have*

$$\begin{aligned}
\liminf_{t \to \infty} \frac{L_X^*(t)}{t/\log\log t} &\leq \kappa^2 c_1(\kappa) \qquad \mathbb{P}\text{–}a.s. \quad \text{if } 0 < \kappa < 1, \\
\liminf_{t \to \infty} \frac{L_X^*(t)}{t/[(\log t)\log\log t]} &\leq \frac{1}{2} \qquad \mathbb{P}\text{–}a.s. \quad \text{if } \kappa = 1, \\
\liminf_{t \to \infty} \frac{L_X^*(t)}{(t/\log\log t)^{1/\kappa}} &= 4\left(\frac{(\kappa-1)\kappa^2}{8}\right)^{1/\kappa} \qquad \mathbb{P}\text{–}a.s. \quad \text{if } \kappa > 1,
\end{aligned}$$

where $c_1(\kappa)$ *is defined in* (5.13).

Theorem 1.5. *We have, for any* $\varepsilon > 0$,

$$\liminf_{t\to\infty} \frac{L_X^*(t)}{t/[(\log t)^{1/\kappa}(\log\log t)^{(2/\kappa)+\varepsilon}]} = +\infty \qquad \mathbb{P}\text{–}a.s. \quad \text{if } 0<\kappa\le 1.$$

In the case $0 < \kappa \le 1$, Theorems 1.4 and 1.5 give different bounds, for technical reasons.

We also get the convergence in law under the annealed law $\mathbb{P}$ of $L_X^*(t)$, suitably renormalized, when $\kappa \ge 1$:

Theorem 1.6. *We have as* $t \to +\infty$, *under the annealed law* $\mathbb{P}$,

$$\frac{L_X^*(t)}{t/\log t} \xrightarrow{\mathcal{L}} \frac{1}{2\mathcal{E}} \qquad \text{if } \kappa = 1,$$

$$\frac{L_X^*(t)}{t^{1/\kappa}} \xrightarrow{\mathcal{L}} 4\big[\kappa^2(\kappa-1)/8\big]^{1/\kappa}\mathcal{E}^{-1/\kappa} \qquad \text{if } \kappa > 1,$$

where $\mathcal{E}$ *denotes an exponential variable with mean* 1.

We notice that in the previous theorem, the case $0 < \kappa < 1$ is lacking. Indeed, we did not succeed in obtaining it with the annealed technics of the present paper, because due to (1.6), $H(r)$ suitably renormalized converges in law but does not converge in probability to a positive constant in this case. This is why we used quenched technics in Andreoletti et al. [4] to prove that $L_X^*(t)/t$ converges in law under $\mathbb{P}$ as $t \to +\infty$ when $0 < \kappa < 1$. To this aim, we used and extended to local time the quenched tools developed in Andreoletti et al. [3] to get the localization of X in this case $0 < \kappa < 1$, combined with some additional tools such as two dimensional Lévy processes and convergence in Skorokhod topology.

So, Theorem 1.6 completes the results of [4] and [5] (see our (1.3)), that is, these 3 results give the convergence in law of $L_X^*(t)$ suitably renormalized for any value of $\kappa \in \mathbb{R}$.

In the proof of Theorems 1.2, 1.4 and 1.5, we will frequently need to use the almost sure asymptotics of the first hitting times $H(\cdot)$. In view of the last part (1.8) of Theorem 1.1, we only need to study the case $\kappa \in (0, 1]$.

Theorem 1.7. *Let* $a(\cdot)$ *be a positive nondecreasing function. If* $0 < \kappa < 1$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \iff \limsup_{r\to\infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s.$$

If $\kappa = 1$, *the statement holds under the additional assumption that* $\limsup_{r\to+\infty}(\log r)/a(r) < \infty$.

Theorem 1.8. *We have (*Γ *denotes the usual gamma function)*

$$\liminf_{r\to+\infty} \frac{H(r)}{r^{1/\kappa}/(\log\log r)^{(1/\kappa)-1}} = \frac{8\kappa[\pi\kappa]^{1/\kappa}(1-\kappa)^{\frac{1-\kappa}{\kappa}}}{\big[2\Gamma^2(\kappa)\sin(\pi\kappa)\big]^{1/\kappa}} =: c_2(\kappa) \quad \mathbb{P}\text{–}a.s. \text{ if } 0<\kappa<1, \tag{1.9}$$

$$\liminf_{r\to+\infty} \frac{H(r)}{r\log r} = 4 \qquad \mathbb{P}\text{–}a.s. \text{ if } \kappa = 1. \tag{1.10}$$

The following corollary follows immediately from Theorem 1.7 and gives a negative answer to a question raised in Hu, Shi and Yor ([33], Remark 1.3 p. 3917):

Corollary 1.9. *The convergence in probability $H(r)/(r\log r) \to 4$ in Theorem 1.1 in the case $\kappa = 1$ cannot be strengthened into an almost sure convergence.*

We observe that in the case $0 < \kappa < 1$, the process $H(\cdot)$ has the same almost sure asymptotics as κ–stable subordinators (see Bertoin [7] p. 92).

Finally, define $\log_1 := \log$ and $\log_k := \log_{k-1} \circ \log$ for $k > 1$. Theorems 1.7 and 1.8, and the fact that $X(t)$ is not very far from $\sup_{0\le s\le t} X(s)$ (see Lemma 4.1 below) lead to

Corollary 1.10. *Recall that $c_2(\kappa)$ is defined in* (1.9)*. We have for $k \in \mathbb{N}^*$,*

$$\limsup_{t\to\infty} \frac{X(t)}{t^\kappa(\log\log t)^{1-\kappa}} = \frac{2\Gamma^2(\kappa)\sin(\pi\kappa)}{\pi 8^\kappa \kappa^{\kappa+1}(1-\kappa)^{1-\kappa}} = \frac{1}{[c_2(\kappa)]^\kappa} \qquad \mathbb{P}\text{–}a.s. \text{ if } 0<\kappa<1, \tag{1.11}$$

$$\limsup_{t\to\infty} \frac{X(t)}{t/\log t} = \frac{1}{4} \qquad \mathbb{P}\text{–}a.s. \text{ if } \kappa = 1, \tag{1.12}$$

$$\begin{cases} \alpha \le 1 \\ \alpha > 1 \end{cases} \Longleftrightarrow \liminf_{t\to+\infty} \frac{X(t)}{t^\kappa/[(\log t)\dots(\log_{k-1} t)(\log_k t)^\alpha]} \begin{cases} = 0 \\ = +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s. \text{ if } 0<\kappa\le 1, \tag{1.13}$$

where for $k = 1$, $(\log t)\dots(\log_{k-1} t) = 1$ by convention. These results remain true if we replace $X(t)$ by $\sup_{0\le s\le t} X(s)$.

Corresponding results in the recurrent case $\kappa = 0$ are proved by Hu et al. [30], extended later by Singh [47] to some asymptotically stable potentials and following results of Deheuvels et al. [16] for Sinai's walk.

Our proof hinges upon stochastic calculus. In particular, one key ingredient of the proofs of Theorems 1.2–1.8 is an approximation of the joint law of the hitting time $H[F(r)]$ of $F(r) \approx r$ by X and the maximum local time $L^*_X[H(F(r))]$ of X at this time, stated in Lemma 2.7, and proved in Section 6. Another important tool is a modification of the Borel-Cantelli lemma, stated in Lemma 2.8, which, loosely speaking, says that one can chop the real half line $[0,\infty)$ into regions in which the diffusion X behaves in an "independent" way.

The rest of the paper is organized as follows. In Section 2.1, we give some preliminaries on local time and Bessel processes. We present in Section 2.2 some estimates which will be needed later on; the proof of one key estimate (Lemma 2.7) is postponed until Section 6. Section 3 is devoted to the study of the almost sure asymptotics of $L^*_X[H(r)]$, stated in Theorems 3.1 and 3.2. In Section 4, we study the Lévy classes for the hitting times $H(r)$ and prove Theorems 1.7 and 1.8 and Corollary 1.10. In Section 5, we study $L^*_X[H(r)]/H(r)$ and prove Theorems 1.2–1.6. Section 6 is devoted to the proof of Lemma 2.7. Finally, we prove in Section 7 some lemmas dealing with Bessel processes, Jacobi processes and Brownian motion.

Throughout the paper, the letter c with a subscript denotes constants that are finite and positive.

2. Some preliminaries

2.1. **Preliminaries on local time and Bessel processes.** We first define, for any Brownian motion $(B(t),\ t\ge 0)$ and $r > 0$, the hitting time

$$\sigma_B(r) := \inf\{t > 0,\ B(t) = r\}.$$

Moreover, we denote by $(L_B(t,x),\ t\ge 0,\ x\in\mathbb{R})$ the local time of B, i.e., the jointly continuous process satisfying $\int_0^t f(B(s))\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)L_B(t,x)\mathrm{d}x$ for any positive measurable function f.

We define the inverse local time of B at 0 as

$$\tau_B(a) := \inf\{t \geq 0,\ L_B(t,0) > a\}, \qquad a > 0.$$

Furthermore, for any $\delta \in [0,\infty)$ and $x \in [0,\infty)$, the unique strong solution of the stochastic differential equation

$$Z(t) = x + 2\int_0^t \sqrt{Z(s)}\mathrm{d}\beta(s) + \delta t,$$

where $(\beta(s), s \geq 0)$ is a (one dimensional) Brownian motion, is called a δ–dimensional squared Bessel process starting from x. A Bessel process with dimension δ (or equivalently with order $\delta/2-1$) starting from $x \geq 0$ is defined as the (nonnegative) square root of a δ–dimensional squared Bessel process starting from x^2 (see e.g. Borodin et al. [10], 39 p. 73 for a more general definition as a linear diffusion with generator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{\delta-1}{2x}\frac{d}{dx}$ for every $\delta \in \mathbb{R}$; see also Göing-Jaeschke et al. [28] definition 3 p. 329). We recall some important results.

Fact 2.1. *(first Ray–Knight theorem) Consider $r > 0$ and a Brownian motion $(B(t),\ t \geq 0)$. The process $(L_B(\sigma_B(r), r-x),\ x \geq 0)$ is a continuous inhomogeneous Markov process, starting from 0. It is a 2–dimensional squared Bessel process for $x \in [0,r]$ and a 0–dimensional squared Bessel process for $x \geq r$.*

Fact 2.2. *(second Ray–Knight theorem) Fix $r > 0$, and let $(B(t),\ t \geq 0)$ be a Brownian motion. The process $(L_B(\tau_B(r), x),\ x \geq 0)$ is a 0–dimensional squared Bessel process starting from r.*

See e.g. Revuz and Yor ([41], chap. XI) for more details about Ray–Knight theorems and Bessel processes. Following the method used by Hu et al. ([33], see eq. (3.8)), we also need the following well known result:

Fact 2.3. *(Lamperti representation theorem, see Yor [54] eq. (2.e)) Consider $W_\kappa(x) = W(x) - \kappa x/2$ as in (1.1) with $\kappa > 0$, where $(W(x),\ x \geq 0)$ is a Brownian motion. There exists a $(2-2\kappa)$–dimensional Bessel process $(\rho(t),\ t \geq 0)$, starting from $\rho(0) = 2$, such that $\exp[W_\kappa(t)/2] = \rho(A(t))/2$ for all $t \geq 0$, where $A(r) := \int_0^r e^{W_\kappa(s)}\mathrm{d}s$, $r \geq 0$.*

We also recall the following extension to Bessel processes of Williams' time reversal theorem (see Yor [55], p. 80; see also Göing-Jaeschke et al. [28] eq. (34)).

Fact 2.4. *One has, for $\delta < 2$,*

$$(R_\delta(T_0 - s),\ s \leq T_0) \stackrel{\mathcal{L}}{=} (R_{4-\delta}(s),\ s \leq \gamma_a),$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law, $(R_\delta(s),\ s \geq 0)$ denotes a δ–dimensional Bessel process starting from $a > 0$, $T_0 := \inf\{s \geq 0,\ R_\delta(s) = 0\}$, $(R_{4-\delta}(s),\ s \geq 0)$ is a $(4-\delta)$–dimensional Bessel process starting from 0, and $\gamma_a := \sup\{s \geq 0,\ R_{4-\delta}(s) = a\}$.

Let S_κ^{ca} be a (positive) completely asymmetric stable variable of index κ for $0 < \kappa < 1$, and C_8^{ca} a (positive) completely asymmetric Cauchy variable of parameter 8. Their characteristic functions are given by:

$$\mathbb{E}e^{itS_\kappa^{ca}} = \exp\left[-|t|^\kappa\left(1 - i\,\mathrm{sgn}(t)\tan\left(\frac{\pi\kappa}{2}\right)\right)\right], \qquad \mathbb{E}e^{itC_8^{ca}} = \exp\left[-8\Big(|t| + it\frac{2}{\pi}\log|t|\Big)\right]. \tag{2.1}$$

Throughout the paper, we set $\lambda := 4(1+\kappa)$. If $(B(t),\ t \geq 0)$ denotes, as before, a Brownian motion, we introduce

$$K_\beta(\kappa) := \int_0^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x)\mathrm{d}x, \qquad 0 < \kappa < 1, \tag{2.2}$$

$$C_\beta := \int_0^1 \frac{L_\beta(\tau_\beta(8), x) - 8}{x}\mathrm{d}x + \int_1^{+\infty} \frac{L_\beta(\tau_\beta(8), x)}{x}\mathrm{d}x. \tag{2.3}$$

We have the following equalities in law:

Fact 2.5. *(Biane and Yor [8]) For $0 < \kappa < 1$,*

$$C_\beta \stackrel{\mathcal{L}}{=} 8c_3 + (\pi/2)C_8^{ca}, \qquad K_\beta(\kappa) \stackrel{\mathcal{L}}{=} \big(\kappa^{2-1/\kappa} c_4(\kappa)/4\big) S_\kappa^{ca},$$

where $c_3 > 0$ denotes an unimportant constant, and

$$\psi(\kappa) := \left(\frac{\pi\kappa}{4\Gamma^2(\kappa)\sin(\pi\kappa/2)}\right)^{1/\kappa}, \qquad c_4(\kappa) := 8\psi(\kappa)\lambda^{1/\kappa}\kappa^{-1/\kappa}. \tag{2.4}$$

This fact is proved in (Biane and Yor [8]); the identity in law related to C_β is given in its paragraph (4.3.2) pp 64-66 and the one related to $K_\beta(\kappa)$ follows from its (1.a) p. 24.

Finally, the first Ray-Knight theorem leads to the following formula. For $v > 0$ and $y > 0$,

$$\mathbb{P}\Big(\sup_{0\le s\le \tau_\beta(v)} \beta(s) < y\Big) = \mathbb{P}\big[L_\beta(\sigma_\beta(y), 0) > v\big] = \mathbb{P}\big(R_2^2(y) > v\big) = \exp\left(-\frac{v}{2y}\right), \tag{2.5}$$

where $(R_2(s),\ s \ge 0)$ is a 2–dimensional Bessel process starting from 0.

2.2. **Some preliminaries on the diffusion.** We assume in the rest of the paper that $\kappa > 0$, and so X is a.s. transient to the right. We start by introducing

$$A(x) := \int_0^x e^{W_\kappa(y)}\mathrm{d}y, \quad x \in \mathbb{R}, \qquad A_\infty := \int_0^\infty e^{W_\kappa(y)}\mathrm{d}y < \infty \text{ a.s.}$$

We recall that A is a scale function of X under the quenched law P_{W_κ} (see e.g. Shi [45] eq. (2.2)). That is, if $P^y_{W_\kappa}$ denotes the law of the diffusion X in the potential W_κ, starting from y instead of 0, we have conditionally on the potential W_κ,

$$P^y_{W_\kappa}\big[H(z) < H(x)\big] = \big[A(y) - A(x)\big]/\big[A(z) - A(x)\big], \qquad x < y < z. \tag{2.6}$$

We observe that, since $\kappa > 0$, $A(x) \to A_\infty < \infty$ a.s. when $x \to +\infty$.

For technical reasons, we have to introduce the random function F as follows. Fix $r > 0$. Since the function $x \mapsto A_\infty - A(x) =: D(x)$ is almost surely continuous and (strictly) decreasing and has limits $+\infty$ and 0 respectively on $-\infty$ and $+\infty$, there exists a unique $F(r) \in \mathbb{R}$, depending only on the process W_κ, such that

$$A_\infty - A(F(r)) = \exp(-\kappa r/2) =: \delta(r). \tag{2.7}$$

Our first estimate describes how close $F(r)$ is to r, for large r.

Lemma 2.6. *Let $\kappa > 0$ and $0 < \delta_0 < 1/2$. Define for $r > 0$,*

$$E_1(r) := \big\{\big(1 - 5r^{-\delta_0}/\kappa\big)r \le F(r) \le \big(1 + 5r^{-\delta_0}/\kappa\big)r\big\}. \tag{2.8}$$

Then for all large r,

$$\mathbb{P}\big[E_1(r)^c\big] \le \exp\big(-r^{1-2\delta_0}\big). \tag{2.9}$$

As a consequence, for any $\varepsilon > 0$, we have, almost surely, for all large r,

$$(1-\varepsilon)r \le F(r) \le (1+\varepsilon)r. \tag{2.10}$$

Proof of Lemma 2.6. Let $0 < \delta_0 < 1/2$, and fix $r > 0$. We have

$$\mathbb{P}\big[E_1(r)^c\big] \le \mathbb{P}\big[F(r) < (1 - 5r^{-\delta_0}/\kappa)r\big] + \mathbb{P}\big[F(r) > (1 + 5r^{-\delta_0}/\kappa)r\big]. \tag{2.11}$$

Define $s_\pm := (1 \pm 5r^{-\delta_0}/\kappa)r$, and $A_\infty^{(s)} := \int_s^\infty \exp(W_\kappa(u) - W_\kappa(s))\mathrm{d}u$ for $s \geq 0$. Observe that D is stricly decreasing, $D(F(r)) = e^{-\kappa r/2}$ and that $D(s_\pm) = A_\infty^{(s_\pm)} \exp(W_\kappa(s_\pm))$. Consequently,

$$\mathbb{P}\big[F(r) < (1 - 5r^{-\delta_0}/\kappa)r\big] \leq \mathbb{P}\big[D(F(r)) > D(s_-)\big] = \mathbb{P}\big[-\kappa r/2 > \log\big(A_\infty^{(s_-)}\big) + W_\kappa(s_-)\big].$$

Moreover, $A_\infty^{(s_\pm)} \overset{\mathcal{L}}{=} A_\infty \overset{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter $(\kappa, 1)$ (see Dufresne [24] or Borodin et al. [10] IV.48 p. 78), i.e., γ_κ has density $\frac{1}{\Gamma(\kappa)} e^{-x} x^{\kappa-1} \mathbf{1}_{\mathbb{R}_+}(x)$. Hence

$$\begin{aligned}\mathbb{P}\big[F(r) < (1 - 5r^{-\delta_0}/\kappa)r\big] &\leq \mathbb{P}\big[\log(2/\gamma_k) < -r^{1-\delta_0}\big] + \mathbb{P}\big[W(s_-) < -3r^{1-\delta_0}/2\big] \\ &\leq 2\exp\big(-9r^{1-2\delta_0}/8\big),\end{aligned}$$

for large r, since $\mathbb{P}[W(1) < -x] \leq e^{-x^2/2}$ for $x \geq 1$. Similarly, we have for large r,

$$\mathbb{P}\big[F(r) > s_+\big] \leq \mathbb{P}\big[\log(2/\gamma_k) > r^{1-\delta_0}/2\big] + \mathbb{P}\big[W(s_+) > 2r^{1-\delta_0}\big] \leq \exp\big(-9r^{1-2\delta_0}/8\big).$$

This yields (2.9) in view of (2.11).

Then $\sum_{n\geq 1} \mathbb{P}\big[E_1(n)^c\big] < \infty$, so (2.10) follows from the Borel–Cantelli lemma and the monotonicity of $F(\cdot)$. □

In the rest of the paper, we define, for $\delta_1 > 0$ and any $r > 0$,

$$c_5 := 2(\lambda/\kappa)^{\delta_1}, \qquad \psi_\pm(r) := 1 \pm \frac{c_5}{r^{\delta_1}}, \qquad t_\pm(r) := \frac{\kappa\psi_\pm(r)r}{\lambda}. \tag{2.12}$$

Taking $\psi_\pm(r)$ as defined above instead of simply $1 \pm \varepsilon$ is necessary e.g. in Lemma 5.1 below. Moreover, if $(\beta(s),\ s \geq 0)$ is a Brownian motion and $v > 0$, we define the Brownian motion $(\beta_v(s),\ s \geq 0)$ by $\beta_v(s) := (1/v)\beta(v^2 s)$, $s \geq 0$.

We prove in Section 6 the following approximation of the joint law of $\big(L_X^*[H(F(r))], H(F(r))\big)$.

Lemma 2.7. *Let $\kappa > 0$ and $\varepsilon \in (0,1)$. For $\delta_1 > 0$ small enough, there exists $c_6 > 0$ and $\alpha > 0$ such that for r large enough, there exist a Brownian motion $(\beta(t),\ t \geq 0)$ such that the following holds:*

(i) *Whenever $\kappa > 0$, we have*

$$\mathbb{P}[E_2(r)] \geq 1 - r^{-\alpha},$$

where

$$E_2(r) := \Big\{(1-\varepsilon)\widehat{L}_-(r) \leq L_X^*[H(F(r))] \leq (1+\varepsilon)\widehat{L}_+(r)\Big\}, \tag{2.13}$$

$$\widehat{L}_\pm(r) := 4[\kappa t_\pm(r)]^{1/\kappa}\Big[\sup_{0\leq u\leq \tau_{\beta_{t_\pm(r)}}(\lambda)} \beta_{t_\pm(r)}(u)\Big]^{1/\kappa} = 4\Big[\sup_{0\leq u\leq \tau_\beta(\lambda t_\pm(r))} \kappa\beta(u)\Big]^{1/\kappa}. \tag{2.14}$$

(ii) *If $0 < \kappa \leq 1$, we have*

$$\mathbb{P}[E_3(r)] \geq 1 - r^{-\alpha},$$

where, using the notation introduced in (2.2) and (2.3),

$$E_3(r) := \Big\{(1-\varepsilon)\widehat{I}_-(r) \leq H(F(r)) \leq (1+\varepsilon)\widehat{I}_+(r)\Big\}, \tag{2.15}$$

$$\widehat{I}_\pm(r) := \begin{cases} 4\kappa^{1/\kappa-2} t_\pm(r)^{1/\kappa}\big[K_{\beta_{t_\pm(r)}}(\kappa) \pm c_6 t_\pm(r)^{1-1/\kappa}\big], & 0 < \kappa < 1, \\ 4t_\pm(r)\big[C_{\beta_{t_\pm(r)}} + 8\log t_\pm(r)\big], & \kappa = 1. \end{cases} \tag{2.16}$$

Notice in particular that the Brownian motion β is the same in **(i)** and **(ii)**; this allows to approximate the law of quantities depending on both $L_X^*[H(F(r))]$ and $H(F(r))$, such as $L_X^*[H(F(r))]/H(F(r))$, which is useful in Section 5. This is possible because we kept the random function $F(r)$ in the expressions $L_X^*[H(F(r))]$ and $H(F(r))$, in order to have the same Brownian

motion β in the left hand side and the right hand side of the inequalities defining $E_2(r)$ and $E_3(r)$.

The proof of Lemma 2.7 is postponed to Section 6.

With an abuse of notation, for $z \geq 0$, we denote by $X \circ \Theta_{H(z)}$ the process $(X(H(z)+t) - z,\ t \geq 0)$. Notice that due to the strong Markov property applied at stopping time $H(z)$ under the quenched law P_{W_κ} , $X \circ \Theta_{H(z)}$ is, conditionally on W_κ, a diffusion in the $(-\kappa/2)$-drifted Brownian potential $W_\kappa \circ \Theta_z := (W_\kappa(x+z) - W_\kappa(z),\ x \in \mathbb{R})$, starting from 0. Define $H_{X\circ\Theta_{H(z)}}(s) = H(z+s) - H(z)$, $s \geq 0$, which is the hitting time of s by $X \circ \Theta_{H(z)}$. In view of (2.7), we also define $F_{W_\kappa\circ\Theta_z}$ by $\int_{F_{W_\kappa\circ\Theta_z}(r)}^{\infty} e^{W_\kappa\circ\Theta_z(u)}\mathrm{d}u = \delta(r)$, $r > 0$. That is, $F_{W_\kappa\circ\Theta_z}$ plays the same role for $W_\kappa \circ \Theta_z$ (resp. for $X \circ \Theta_{H(z)}$) as F does for W_κ (resp. for X). Similarly, $L^*_{X\circ\Theta_{H(z)}}$ and $(L^* \circ H)_{X\circ\Theta_{H(z)}}$ denote respectively the processes L^* and $L^* \circ H$ for the diffusion $X \circ \Theta_{H(z)}$, with $(L^*)_X := L^*_X$. The following lemma is a modification of the Borel–Cantelli lemma.

Lemma 2.8. *Let $\kappa > 0$, $\alpha > 0$, $r_n := \exp(n^\alpha)$ and $Z_n := \sum_{k=1}^n r_k$ for $n \geq 1$. Assume f is a continuous function $(0,+\infty)^2 \to \mathbb{R}$ and $(\Delta_n)_{n\geq 1}$ is a sequence of open sets in $\mathbb{R}$ such that*

$$\sum_{n\geq 1} \mathbb{P}\big\{f[(H\circ F)(r_{2n}), (L^*_X \circ H \circ F)(r_{2n})] \in \Delta_n\big\} = +\infty. \tag{2.17}$$

Then for any $0 < \varepsilon < 1/2$, $\mathbb{P}$ almost surely, there exist infinitely many n such that for some $t_n \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$,

$$f\big[H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n), (L^* \circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)\big] \in \Delta_n.$$

The results remain true if $r_n = n^n$ for every $n \geq 1$.

Proof of Lemma 2.8. We divide $\mathbb{R}_+$ into some regions in which the diffusion X will behave "independently", in order to apply the Borel–Cantelli lemma.

To this aim, let $n \geq 1$ and

$$E_4(n) := \left\{ \inf_{\{t:\ H(Z_{2n-1})\leq t\leq H(Z_{2n}+r_{2n+1}/2)\}} X(t) > Z_{2n-2} + \frac{1}{2}r_{2n-1} \right\}.$$

Define $x_n := r_{2n-1}/2$. For any environment, i.e., for any realization of W_κ, X is a Markov process under P_{W_κ}, and $H(Z_{2n-1})$ is a stopping time. Hence, $P_{W_\kappa}(E_4(n)^c)$ is the probability that the diffusion in the potential W_κ started at Z_{2n-1} hits level $Z_{2n-2} + x_n$ before $Z_{2n} + x_{n+1}$, that is

$$P_{W_\kappa}\big[E_4(n)^c\big] = \left(1 + \frac{\int_{Z_{2n-2}+x_n}^{Z_{2n-1}} e^{W_\kappa(u)}\mathrm{d}u}{\int_{Z_{2n-1}}^{Z_{2n}+x_{n+1}} e^{W_\kappa(u)}\mathrm{d}u}\right)^{-1} \leq \frac{\int_{Z_{2n-1}}^{Z_{2n}+x_{n+1}} e^{W_\kappa(u)}\mathrm{d}u}{\int_{Z_{2n-2}+x_n}^{Z_{2n-1}} e^{W_\kappa(u)}\mathrm{d}u}, \tag{2.18}$$

where we used (2.6). Observe that $r_{2n-1} - x_n = x_n$ and define for some $0 < \varepsilon_0 < \kappa/4$,

$$E_5(n) := \left\{ \sup_{0\leq u\leq r_{2n-1}-x_n} \left| W_\kappa(u + Z_{2n-2} + x_n) - W_\kappa(Z_{2n-2} + x_n) + \frac{\kappa}{2}u \right| \leq \varepsilon_0(r_{2n-1} - x_n) \right\}$$

and $E_6(n) := \big\{\sup_{u\geq 0}[W_\kappa(u + Z_{2n-1}) - W_\kappa(Z_{2n-1})] \leq v_n\big\}$, where $v_n := 2(\log n)/\kappa$. Since $\sup_{0\leq u\leq x_n} W(u) \overset{\mathcal{L}}{=} |W(x_n)|$ and $\sup_{x\geq 0} W_\kappa(x)$ has an exponential law of parameter κ (see e.g. Borodin et al. [10] 1.1.4 (1) p. 251), we have for large n,

$$\mathbb{P}\big[E_5(n)^c\big] = \mathbb{P}\Big(\sup_{0\leq u\leq x_n} |W(u)| > \varepsilon_0 x_n\Big) \leq 4\exp\Big[-\frac{\varepsilon_0^2 x_n}{2}\Big] \quad \text{and} \quad \mathbb{P}\big[E_6(n)^c\big] = \exp(-\kappa v_n) = n^{-2}. \tag{2.19}$$

Moreover by (2.18), we have for n large enough, on $E_5(n) \cap E_6(n)$,

$$\begin{aligned} P_{W_\kappa}\big[E_4(n)^c\big] &\leq \kappa \frac{(r_{2n}+x_{n+1})\exp[v_n + W_\kappa(Z_{2n-1})]}{\exp[W_\kappa(Z_{2n-2}+x_n) - \varepsilon_0(r_{2n-1}-x_n)]} \\ &\leq \kappa(r_{2n}+x_{n+1})\exp[v_n + (2\varepsilon_0 - \kappa/2)(r_{2n-1}-x_n)]. \end{aligned} \tag{2.20}$$

Now, integrate (2.20) over $E_5(n) \cap E_6(n)$. Since $\mathbb{P}[E_5(n)^c]$ and $\mathbb{P}[E_6(n)^c]$ are summable, this yields since $\varepsilon_0 < \kappa/4$,

$$\sum_{n=1}^{+\infty} \mathbb{P}\big[E_4(n)^c\big] < \infty. \tag{2.21}$$

To complete the proof of Lemma 2.8, let $0 < \varepsilon < 1/2$, and define

$$\begin{aligned} \mathcal{D}_n &:= \Big\{\exists t_n \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}], \\ &\qquad f\big[H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n), (L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)\big] \in \Delta_n\Big\}, \\ \mathcal{E}_n &:= \Big\{\Big(1 - 5r_{2n}^{-\delta_0}/\kappa\Big)r_{2n} \leq F_{W_\kappa\circ\Theta_{Z_{2n-1}}}(r_{2n}) \leq \Big(1 + 5r_{2n}^{-\delta_0}/\kappa\Big)r_{2n}\Big\}. \end{aligned}$$

Let $\widetilde{t}_n := F_{W_\kappa\circ\Theta_{Z_{2n-1}}}(r_{2n})$. We have uniformly for large n,

$$\mathcal{D}_n \cap E_4(n) \supset \Big\{f\big[H_{X\circ\Theta_{H(Z_{2n-1})}}(\widetilde{t}_n), (L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(\widetilde{t}_n)\big] \in \Delta_n\Big\} \cap E_4(n) \cap \mathcal{E}_n. \tag{2.22}$$

Due to our assumption (2.17), $\sum_{n\geq 1}\mathbb{P}\{f[H_{X\circ\Theta_{H(Z_{2n-1})}}(\widetilde{t}_n), (L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(\widetilde{t}_n)] \in \Delta_n\} = \infty$, since $X\circ\Theta_{H(Z_{2n-1})}$ is a diffusion process in the $(-\kappa/2)$-drifted Brownian potential $W_\kappa\circ\Theta_{Z_{2n-1}}$, which also gives $\mathbb{P}(\mathcal{E}_n) = \mathbb{P}(E_1(r_{2n}))$. In view of (2.21), (2.22) and Lemma 2.6, this yields $\sum_{n\in\mathbb{N}}\mathbb{P}(\mathcal{D}_n \cap E_4(n)) = +\infty$.

Define $x \wedge y := \inf\{x, y\}$, $(x, y) \in \mathbb{R}^2$. Since $\varepsilon r_{2n} \leq r_{2n+1}/2$ for large n, the event $\mathcal{D}_n \cap E_4(n)$ is measurable with respect to the σ-field generated by $(W_\kappa(x+Z_{2n-1}) - W_\kappa(Z_{2n-1}), -r_{2n-1}/2 \leq x \leq Z_{2n} + r_{2n+1}/2 - Z_{2n-1})$ and $\big(X\circ\Theta_{H(Z_{2n-1})}(t),\ 0 \leq t \leq H_{X\circ\Theta_{H(Z_{2n-1})}}(-r_{2n-1}/2) \wedge H_{X\circ\Theta_{H(Z_{2n-1})}}(Z_{2n} + r_{2n+1}/2 - Z_{2n-1}\big)$. So, the events $\mathcal{D}_n \cap E_4(n)$, $n \geq 1$, are independent by the strong Markov Property, because the intervals $\big[Z_{2n-1} - r_{2n-1}/2, Z_{2n} + r_{2n+1}/2\big)$, $n \geq 1$ are disjoint. Hence, Lemma 2.8 follows by an application of the Borel–Cantelli lemma. □

3. Almost sure asymptotics of $L_X^*[H(r)]$

As a warm up, we first prove the following results, which are useful in Section 5.

Theorem 3.1. *Let $\kappa > 0$. For any positive nondecreasing function $a(\cdot)$, we have*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < \infty \\ = +\infty \end{cases} \Longleftrightarrow \limsup_{r\to\infty} \frac{L_X^*[H(r)]}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \qquad \mathbb{P}\text{–}a.s.$$

Theorem 3.2. *For $\kappa > 0$,*

$$\liminf_{r\to+\infty} \frac{L_X^*[H(r)]}{(r/\log\log r)^{1/\kappa}} = 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–}a.s.$$

3.1. **Proof of Theorem 3.1.** Let $r_n := e^n$ and $Z_n := \sum_{k=1}^n r_k$. Denote by $a(\cdot)$ be a positive nondecreasing function. We begin with the upper bound in Theorem 3.1.

First, notice that for $\widehat{L}_\pm$ which is defined in (2.14), and any positive y and r, we have

$$\mathbb{P}\left(\widehat{L}_\pm(r) < (yr)^{1/\kappa}\right) = \mathbb{P}\left[\sup_{0\le u\le \tau_\beta(\lambda t_\pm(r))} \beta(u) < \frac{yr}{4^\kappa \kappa}\right] = \exp\left(-\frac{\kappa^2 4^\kappa \psi_\pm(r)}{2y}\right), \tag{3.1}$$

by (2.5) and (2.12). This together with Lemma 2.7 gives, for some $\alpha > 0$, $\varepsilon > 0$ and all large r,

$$\mathbb{P}\left\{L_X^*[H(F(r))] > \left(ra(e^{-2}r)\right)^{1/\kappa}\right\} \le 1 - \exp\left(-\frac{(1+\varepsilon)^\kappa \kappa^2 4^\kappa \psi_+(r)}{2a(e^{-2}r)}\right) + r^{-\alpha} \le \frac{c_7}{a(e^{-2}r)} + r^{-\alpha}, \tag{3.2}$$

since $1-e^{-x} \le x$ for all $x \in \mathbb{R}$. Assume $\sum_{n=1}^{+\infty} \frac{1}{na(n)} < \infty$, which is equivalent to $\sum_{n=1}^{+\infty} \frac{1}{a(r_n)} < \infty$. Then it follows from (3.2) that $\sum_{n=1}^{+\infty} \mathbb{P}\{L_X^*[H(F(r_n))] > [r_n a(r_{n-2})]^{1/\kappa}\} < \infty$.

So by the Borel–Cantelli lemma, almost surely for all large n, $L_X^*[H(F(r_n))] \le [r_n a(r_{n-2})]^{1/\kappa}$. On the other hand, $r_{n-1} \le F(r_n)$ almost surely for all large n (see (2.10)). As a consequence, almost surely for all large n, $L_X^*[H(r_{n-1})] \le [r_n a(r_{n-2})]^{1/\kappa}$. Let $r \in [r_{n-2}, r_{n-1}]$, for such large n. Then

$$L_X^*[H(r)] \le L_X^*[H(r_{n-1})] \le [r_n a(r_{n-2})]^{1/\kappa} \le e^{2/\kappa}[ra(r)]^{1/\kappa}.$$

Consequently,

$$\limsup_{r\to+\infty} \frac{L_X^*[H(r)]}{[ra(r)]^{1/\kappa}} \le e^{2/\kappa} \qquad \mathbb{P}\text{–a.s.} \tag{3.3}$$

Since $\sum_{n=1}^{+\infty} \frac{1}{n\varepsilon a(n)}$ is also finite, (3.3) holds for $a(\cdot)$ replaced by $\varepsilon a(\cdot)$, $\varepsilon > 0$. Letting $\varepsilon \to 0$ yields the "zero" part of Theorem 3.1.

Now we turn to the proof of the "infinity" part. Assume $\sum_{n=1}^{+\infty} \frac{1}{na(n)} = +\infty$, that is, $\sum_{n=1}^{+\infty} \frac{1}{a(r_n)} = +\infty$. Observe that we may restrict ourselves to the case $a(x) \to +\infty$ when $x \to +\infty$, since the result in this case yields the result when a is bounded.

By an argument similar to that leading to (3.2), we have, for some $\alpha > 0$ and all large r,

$$\mathbb{P}\left\{L_X^*[H(F(r))] > \left(ra(e^2 r)\right)^{1/\kappa}\right\} \ge \frac{c_8}{a(e^2 r)} - r^{-\alpha},$$

which implies $\sum_{n=1}^{+\infty} \mathbb{P}\left\{(L_X^* \circ H \circ F)(r_{2n}) > [r_{2n} a(r_{2n+2})]^{1/\kappa}\right\} = +\infty$. Let $0 < \varepsilon < 1/2$ and recall that $Z_n = \sum_{k=1}^n r_k$; by Lemma 2.8, almost surely, there exist infinitely many n such that

$$\sup_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} (L^* \circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(s) > [r_{2n} a(r_{2n+2})]^{1/\kappa}.$$

For such n, we have $\sup_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} L_X^*[H(Z_{2n-1}+s)] > [r_{2n} a(r_{2n+2})]^{1/\kappa}$. Consequently,

$$\sup_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{L_X^*(H(Z_{2n-1}+s))}{[(Z_{2n-1}+s)a(Z_{2n-1}+s)]^{1/\kappa}} \ge c_9,$$

almost surely for infinitely many n. This gives

$$\limsup_{r\to+\infty} \frac{L_X^*[H(r)]}{[ra(r)]^{1/\kappa}} \ge c_9 \qquad \mathbb{P}\text{–a.s.}$$

Replace $a(\cdot)$ by $a(\cdot)/\varepsilon$, and let $\varepsilon \to 0$. This yields the "infinity" part of Theorem 3.1. □

3.2. **Proof of Theorem 3.2.** We fix $\varepsilon \in (0,1)$. By Lemma 2.7 and (3.1), we get for some $\alpha > 0$, for every positive function g and all large r,

$$\mathbb{P}\Big[L_X^*[H(F(r))] < [r/g(r)]^{1/\kappa}\Big] \le \exp\Big[-\kappa^2 4^\kappa (1-\varepsilon)^\kappa \psi_-(r) g(r)/2\Big] + r^{-\alpha}. \tag{3.4}$$

We choose $g(r) := \frac{2(1+\varepsilon)}{\kappa^2 4^\kappa (1-\varepsilon)^{\kappa+1}\psi_-(r)} \log\log r$. Let $s_n := \exp(n^{1-\varepsilon})$. It follows from (3.4) that $\sum_{n=1}^\infty \mathbb{P}\big\{L_X^*[H(F(s_n))] < [s_n/g(s_n)]^{1/\kappa}\big\} < \infty$. Hence by the Borel–Cantelli lemma, almost surely for all large n,

$$L_X^*[H(F(s_n))] \ge [s_n/g(s_n)]^{1/\kappa}.$$

On the other hand, by (2.9) and the Borel-Cantelli lemma, $s_n \ge F(s_{n-1})$ almost surely for all large n, which implies that, for $r \in [s_n, s_{n+1}]$,

$$L_X^*[H(r)] \ge L_X^*[H(F(s_{n-1}))] \ge [s_{n-1}/g(s_{n-1})]^{1/\kappa} \ge (1-\varepsilon)[r/g(r)]^{1/\kappa},$$

since $s_{n-1}/s_{n+1} \to 1$ as $n \to +\infty$. Consequently,

$$\liminf_{r\to\infty} \frac{L_X^*[H(r)]}{(r/\log\log r)^{1/\kappa}} \ge 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–a.s.}$$

Now we prove the inequality "$\le$". Let $\varepsilon \in (0,1/2)$, $r_n := \exp(n^{1+\varepsilon})$, $Z_n := \sum_{k=1}^n r_k$, $n \ge 1$, and $\widetilde{g}(r) := \frac{2(1-\varepsilon)}{\kappa^2 4^\kappa (1+\varepsilon)^{\kappa+1}\psi_+(r)} \log\log r$. By Lemma 2.7 and (3.1), for some $\alpha > 0$ and all large r,

$$\mathbb{P}\Big[L_X^*[H(F(r))] < [r/\widetilde{g}(r)]^{1/\kappa}\Big] \ge \exp\Big[-\kappa^2 4^\kappa (1+\varepsilon)^\kappa \psi_+(r)\widetilde{g}(r)/2\Big] - r^{-\alpha}.$$

Therefore,

$$\sum_{n\ge1} \mathbb{P}\Big[L_X^*[H(F(r_{2n}))] < \big[r_{2n}/\widetilde{g}(r_{2n})\big]^{1/\kappa}\Big] = +\infty.$$

It follows from Lemma 2.8 that, almost surely, there are infinitely many n such that

$$\inf_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} (L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(s) < \big[r_{2n}/\widetilde{g}(r_{2n})\big]^{1/\kappa}. \tag{3.5}$$

On the other hand, an application of Theorem 3.1 with $a(x) \sim_{x\to+\infty} (\log x)^2$ gives that almost surely for large n, $L_X^*[H(Z_{2n-1})] \le [Z_{2n-1}\log^2 Z_{2n-1}]^{1/\kappa} \le \varepsilon\big[r_{2n}/\widetilde{g}(r_{2n})\big]^{1/\kappa}$, since $Z_p \le p r_p \le p\exp(-p^\varepsilon) r_{p+1}$ for p large enough. Therefore, $\inf_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} L_X^*[H(Z_{2n-1}+s)] \le (1+\varepsilon)\big[r_{2n}/\widetilde{g}(r_{2n})\big]^{1/\kappa}$ almost surely, for infinitely many n, where we used $L_X^*[H(r+s)] \le L_X^*[H(r)] + (L^*\circ H)_{X\circ\theta_{H(r)}}(s)$, $r \ge 0$, $s \ge 0$. Hence, for such n,

$$\inf_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{L_X^*[H(Z_{2n-1}+s)]}{[(Z_{2n-1}+s)/\log\log(Z_{2n-1}+s)]^{1/\kappa}} \le (1+c_{10}\varepsilon)\left(\frac{\kappa^2 4^\kappa \psi_+(r_{2n})}{2}\right)^{1/\kappa}.$$

This yields

$$\liminf_{r\to+\infty} \frac{L_X^*(H(r))}{(r/\log\log r)^{1/\kappa}} \le 4\left(\frac{\kappa^2}{2}\right)^{1/\kappa} \qquad \mathbb{P}\text{–a.s.},$$

proving Theorem 3.2. □

4. Proof of Theorems 1.7 and 1.8 and Corollary 1.10

Recall $\widehat{I}_\pm$ from (2.16) and $c_4(\kappa)$ from (2.4). By Fact 2.5,

$$\widehat{I}_\pm(r) \stackrel{\mathcal{L}}{=} t_\pm(r)^{1/\kappa}\big\{c_4(\kappa)\, S_\kappa^{ca} \pm c_{11}\, t_\pm(r)^{1-1/\kappa}\big\}, \qquad 0<\kappa<1, \tag{4.1}$$

$$\widehat{I}_\pm(r) \stackrel{\mathcal{L}}{=} 4t_\pm(r)[8c_3 + (\pi/2)C_8^{ca} + 8\log t_\pm(r)] \qquad \kappa = 1, \tag{4.2}$$

where $c_{11}>0$ and $c_3>0$ are unimportant constants. We have now all the ingredients to prove Theorems 1.7 and 1.8.

4.1. **Proof of Theorem 1.7.**

4.1.1. *Case* $0<\kappa<1$. We assume $0<\kappa<1$. Let $a(\cdot)$ be a positive nondecreasing function. Without loss of generality, we suppose that $a(r)\to\infty$ (as $r\to\infty$).

It is known (see e.g. Samorodnitsky and Taqqu [42], (1.2.8) p. 16) that

$$\mathbb{P}\big(S_\kappa^{ca} > x\big) \underset{x\to+\infty}{\sim} c_{12}x^{-\kappa},$$

where $f(x) \underset{x\to+\infty}{\sim} g(x)$ means $\lim\limits_{x\to+\infty} f(x)/g(x) = 1$, and $c_{12}>0$ is a constant depending on κ.

Recall $t_\pm(\cdot)$ from (2.12). By Lemma 2.7 and (4.1), for some $\alpha>0$, we have for large r,

$$\mathbb{P}\big[H(F(r)) > \big(a(e^{-2}r)t_+(r)\big)^{1/\kappa}\big] \;\le\; \frac{c_{13}}{a(e^{-2}r)} + r^{-\alpha}. \tag{4.3}$$

As in Section 3.1, we define $r_n := e^n$ and $Z_n := \sum_{k=1}^n r_k$. Assume $\sum_{n\ge1}\frac{1}{a(r_n)}<\infty$, which is equivalent to $\sum_{n\ge1}\frac{1}{na(n)}<\infty$. By the Borel–Cantelli lemma, almost surely for n large enough,

$$H[F(r_n)] \le [a(r_{n-2})t_+(r_n)]^{1/\kappa}. \tag{4.4}$$

On the other hand, by Lemma 2.6, almost surely for all large n, we have $r_{n+1}\le F(r_{n+2})$, which together with (4.4) implies that for $r\in[r_n, r_{n+1}]$,

$$H(r) \le H[F(r_{n+2})] \le [\psi_+(r_{n+2})\kappa r_{n+2}a(r_n)/\lambda]^{1/\kappa} \le c_{14}[ra(r)]^{1/\kappa}.$$

Therefore, $\limsup_{r\to+\infty}\frac{H(r)}{[ra(r)]^{1/\kappa}} \le c_{14}$ $\mathbb{P}$–a.s., implying the "zero" part of Theorem 1.7, since we can replace $a(\cdot)$ by any constant multiple of $a(\cdot)$.

To prove the "infinity" part, we assume $\sum_{n\ge1}\frac{1}{na(n)} = +\infty$, and observe that, by an argument similar to that leading to (4.3), we have, for some $\alpha>0$ and all r large enough,

$$\mathbb{P}\big[H(F(r)) > \big(a(e^2r)t_-(r)\big)^{1/\kappa}\big] \ge \frac{c_{15}}{a(e^2r)} - r^{-\alpha}. \tag{4.5}$$

It follows from Lemma 2.8 that $\sup_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H_{X\circ\Theta_{H(Z_{2n-1})}}(s) > [a(r_{2n+2})t_-(r_{2n})]^{1/\kappa}$, almost surely for infinitely many n. Since $H(Z_{2n-1}+s) \ge H_{X\circ\Theta_{H(Z_{2n-1})}}(s)$ for all $s>0$, this implies, for these n,

$$\sup_{s\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H(Z_{2n-1}+s)/[a(Z_{2n-1}+s)(Z_{2n-1}+s)]^{1/\kappa} \ge c_{16}. \tag{4.6}$$

This gives $\limsup_{r\to+\infty}\frac{H(r)}{[ra(r)]^{1/\kappa}} \ge c_{16}$ $\mathbb{P}$-a.s., proving the "infinity" part in Theorem 1.7, in the case $0<\kappa<1$ by replacing $a(.)$ by any constant multiple of $a(.)$. □

4.1.2. *Case* $\kappa = 1$. Let $r_n := e^n$ and $Z_n := \sum_{k=1}^n r_k$. We recall that there exists a constant $c_{17} := \frac{16}{\pi}$ such that $\mathbb{P}(C_8^{ca} > x) \underset{x\to+\infty}{\sim} \frac{c_{17}}{x}$ (see e.g. Samorodnitsky et al. [42], prop. 1.2.15 p. 16). Hence, by Lemma 2.7 and (4.2), for some $\alpha > 0$ and all large r,

$$\mathbb{P}\left\{H(F(r)) > 4t_+(r)(1+\varepsilon)\big[8c_3 + a(e^{-2}r) + 8\log t_+(r)\big]\right\} \le c_{18}/a(e^{-2}r) + r^{-\alpha}. \tag{4.7}$$

Assume $\sum_{n\ge1} \frac{1}{na(n)} < \infty$. Then by the Borel–Cantelli lemma, almost surely, for all large n,

$$H[F(r_n)] \le 4(1+\varepsilon)t_+(r_n)[8c_3 + a(r_{n-2}) + 8\log(\psi_+(r_n)\kappa r_n/8)].$$

Under the additional assumption $\limsup_{r\to+\infty}(\log r)/a(r) < \infty$, we have, almost surely, for all large n and $r \in [r_n, r_{n+1}]$ (thus $r \le F(r_{n+2})$ by Lemma 2.6),

$$H(r) \le H[F(r_{n+2})] \le c_{19} r_{n+2}[a(r_n) + \log r_{n+2}] \le c_{20} r a(r).$$

As in the case $0 < \kappa < 1$, this yields the "zero" part of Theorem 1.7 in the case $\kappa = 1$.

For the "infinity" part, we assume $\sum_{n\ge1} \frac{1}{na(n)} = +\infty$. As in (4.7), we have, for some $\alpha > 0$

$$\mathbb{P}\left\{H(F(r)) > 4t_-(r)(1-\varepsilon)a(e^2 r)\right\} \ge c_{21}/a(e^2 r) - r^{-\alpha},$$

for large r. As in the displays between (4.5) and (4.6), this yields the "infinity part" of Theorem 1.7 in the case $\kappa = 1$. □

4.2. **Proof of Theorem 1.8.**

4.2.1. *Case* $0 < \kappa < 1$. We have $\mathbb{E}\big(e^{-tS_\kappa^{ca}}\big) = \exp[-t^\kappa/\cos(\pi\kappa/2)]$, $t \ge 0$, e.g. by Samorodnitsky et al. ([42], Proposition 1.2.12, in the notation of [42], S_κ^{ca} is distributed as $S_\kappa(1,1,0)$). So by Bingham et al. ([9] Example p. 349),

$$\log \mathbb{P}\big(S_\kappa^{ca} < x\big) \underset{x\to0,\ x>0}{\sim} -c_{22} x^{-\kappa/(1-\kappa)}, \tag{4.8}$$

where $c_{22} := (1-\kappa)\kappa^{\kappa/(1-\kappa)}[\cos(\pi\kappa/2)]^{-1/(1-\kappa)}$. By Lemma 2.7, (4.1) and (4.8), for any (strictly) positive function f such that $\lim_{x\to+\infty} f(x) = 0$ and $\varepsilon > 0$ small enough, we have for large r,

$$\mathbb{P}\big[H(F(r)) < t_-(r)^{1/\kappa} f(r)\big] \le \exp\left[-(c_{22}-\varepsilon)\left(\frac{(1-\varepsilon)c_4(\kappa)}{f(r) + (1-\varepsilon)c_{11}t_-(r)^{1-1/\kappa}}\right)^{\kappa/(1-\kappa)}\right] + r^{-\alpha}. \tag{4.9}$$

We define for $\varepsilon > 0$ and $r > 1$,

$$f_\varepsilon^\pm(r) := (1\pm\varepsilon)c_4(\kappa)\left(\frac{(1\pm\varepsilon)(c_{22}\pm\varepsilon)}{(1\mp\varepsilon)\log\log r}\right)^{(1-\kappa)/\kappa} \pm c_{11}(1\pm\varepsilon)t_\pm(r)^{1-1/\kappa}.$$

So, (4.9) gives

$$\mathbb{P}\big[H(F(r)) < t_-(r)^{1/\kappa} f_\varepsilon^-(r)\big] \le (\log r)^{-(1+\varepsilon)/(1-\varepsilon)} + r^{-\alpha}.$$

With $s_n := \exp(n^{1-\varepsilon})$, this gives $\sum_n^{+\infty} \mathbb{P}\big[H(F(s_n)) < t_-(s_n)^{1/\kappa} f_\varepsilon^-(s_n)\big] < \infty$, which, by the Borel–Cantelli lemma, implies that, almost surely, for all large n, $H[F(s_n)] \ge t_-(s_n)^{1/\kappa} f_\varepsilon^-(s_n)$.

Recall from Lemma 2.6 that, almost surely, for all large n, we have $F(s_n) \le (1+\varepsilon)s_n$. Let r be large. There exists n (large) such that $(1+\varepsilon)s_n \le r \le (1+2\varepsilon)s_n$. Then if r is large,

$$H(r) \ge H[F(s_n)] \ge t_-(s_n)^{1/\kappa} f_\varepsilon^-(s_n) \ge t_-^{1/\kappa}\left(\frac{r}{1+2\varepsilon}\right) f_\varepsilon^-\left(\frac{r}{1+\varepsilon}\right).$$

Plugging the value of $t_-(\frac{r}{1+2\varepsilon})$ (defined in (2.12)), this yields inequality "$\ge$" of (1.9) with

$$c_2(\kappa) := 8\psi(\kappa)c_{22}^{(1-\kappa)/\kappa} = 8[\pi\kappa]^{1/\kappa}(1-\kappa)^{\frac{1-\kappa}{\kappa}}\kappa/\big[2\Gamma^2(\kappa)\sin(\pi\kappa)\big]^{1/\kappa} \tag{4.10}$$

where $c_{22} = c_{22}(\kappa)$ is defined after (4.8) and ψ and $c_4(\kappa)$ in (2.4).

To prove the upper bound, let $r_n := \exp(n^{1+\varepsilon})$ and $Z_n := \sum_{k=1}^n r_k$. By means of an argument similar to that leading to (4.9), we have $\sum_{n\geq 1} \mathbb{P}\big[H(F(r_{2n})) < t_+(r_{2n})^{1/\kappa} f_\varepsilon^+(r_{2n})\big] = +\infty$. So by Lemma 2.8, for $0 < \varepsilon < 1/2$, there exist almost surely infinitely many n such that

$$\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} H_{X\circ\Theta_{H(Z_{2n-1})}}(u) < [t_+(r_{2n})]^{1/\kappa} f_\varepsilon^+(r_{2n}).$$

In addition, by Theorem 1.7, $H(Z_{2n-1}) < \big[Z_{2n-1}\log^2 Z_{2n-1}\big]^{1/\kappa} \leq \varepsilon[t_+(r_{2n})]^{1/\kappa} f_\varepsilon^+(r_{2n})$ almost surely for all large n, since $\sum_{n\geq 1} 1/(n\log^2 n) < \infty$ and $Z_p \leq p\exp(-p^\varepsilon) r_{p+1}$ for all large p as before. This yields almost surely for large n,

$$\inf_{v\in[Z_{2n-1}+(1-\varepsilon)r_{2n},\, Z_{2n-1}+(1+\varepsilon)r_{2n}]} H(v) < (1+\varepsilon)[t_+(r_{2n})]^{1/\kappa} f_\varepsilon^+(r_{2n}).$$

Consequently,

$$\liminf_{r\to+\infty} \frac{H(r)}{r^{1/\kappa}(\log\log r)^{(\kappa-1)/\kappa}} \leq 8\psi(\kappa) c_{22}^{(1-\kappa)/\kappa} = c_2(\kappa) \qquad \mathbb{P}\text{–a.s.}$$

This gives inequality "$\leq$" of (1.9) and thus yields Theorem 1.8 in the case $0 < \kappa < 1$.

4.2.2. *Case* $\kappa = 1$. Assume $\kappa = 1$ (thus $\lambda = 8$). By Samorodnitsky et al. ([42], Proposition 1.2.12), $\mathbb{E}[\exp(-C_8^{ca})] = 1$ (in the notation of [42], C_8^{ca} is distributed as $S_1(8,1,0)$). Hence,

$$\mathbb{P}[C_8^{ca} \leq -\varepsilon\log r] \leq r^{-\varepsilon}\,\mathbb{E}[\exp(-C_8^{ca})] = r^{-\varepsilon}, \qquad r > 0, \tag{4.11}$$

for $\varepsilon > 0$. By Lemma 2.7 and (4.2), we have if $\varepsilon > 0$ is small enough, for all large r,

$$\mathbb{P}\big\{H[F(r)] \leq 32t_-(r)(1-2\varepsilon)[c_3 + \log t_-(r)]\big\} \leq \mathbb{P}\left(C_8^{ca} \leq -\varepsilon\log r\right) + \mathbb{P}[E_3(r)^c] \leq 2r^{-\varepsilon}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. Thus, by the Borel–Cantelli lemma, almost surely, for all large n,

$$H[F(s_n)] > 32t_-(s_n)(1-2\varepsilon)[c_3 + \log t_-(s_n)] \geq 4(1-3\varepsilon)s_n\log s_n.$$

In view of the last part of Lemma 2.6, this yields inequality "$\geq$" in (1.10) similarly as before (4.10). The inequality "$\leq$", on the other hand, follows immediately from Theorem 1.1 (that $H(r)/(r\log r) \to 4$ in probability). Theorem 1.8 is proved. □

4.3. **Proof of Corollary 1.10.** First, we need the following lemma, which says that X does not go back too far on the left, and so $X(t)$ is very close from $\sup_{0\leq s\leq t} X(s)$:

Lemma 4.1. *For every $\kappa > 0$, there exists a constant $c_{23}(\kappa)$ such that $\mathbb{P}$ a.s. for large t,*

$$0 \leq \sup_{0\leq s\leq t} X(s) - X(t) \leq c_{23}(\kappa)\log t. \tag{4.12}$$

Notice that this is not true in the recurrent case $\kappa = 0$. An heuristic explanation for $0 \leq \kappa < 1$ would be that the valleys of height approximatively $\log t$ have a length of order $(\log t)^2$ in the case $\kappa = 0$, whereas they have a height of order at most $\log t$ in the case $0 < \kappa < 1$, see e.g. Andreoletti et al. ([3], Lem. 2.7).

Proof of Lemma 4.1: Let $\kappa > 0$. By Kawazu et al. ([35], Theorem p. 79 applied with $c = \kappa/2$ to our $-X$), there exists a constant $c_{24}(\kappa) > 0$ such that $\mathbb{P}\big[\inf_{u\geq 0} X(u) < -c_{24}(\kappa)\log n\big] \leq 1/n^2$ for large n. Since $\inf_{u\geq 0} X(H(n)+u) - n$ has the same law under $\mathbb{P}$ as $\inf_{u\geq 0} X(u)$ due to the strong Markov property as explained before Lemma 2.8, this gives $\sum_n \mathbb{P}\big[\inf_{u\geq 0} X(H(n)+u) - n < -c_{24}(\kappa)\log n\big] < \infty$. So by the Borel-Cantelli lemma, almost surely for large n,

$$\inf_{u\geq 0} X(H(n)+u) - n \geq -c_{24}(\kappa)\log n. \tag{4.13}$$

For $t > 0$, there exists $n \in \mathbb{N}$ such that $H(n) \le t < H(n+1)$. We have by (4.13), almost surely if t is large,

$$\sup_{0\le s\le t} X(s) - X(t) \le \sup_{0\le s\le H(n+1)} X(s) - X(t) = n+1 - X[H(n) + (t - H(n))] \le 1 + c_{24}(\kappa)\log n.$$

Moreover, we have $\log v \le 2\log H(v)$ $\mathbb{P}$ a.s. for large v, by Theorem 1.1 if $\kappa > 1$ and by Theorem 1.8 if $0 < \kappa \le 1$. Hence almost surely for large t, with the same notation as before,

$$\sup_{0\le s\le t} X(s) - X(t) \le 1 + c_{24}(\kappa)\log n \le 1 + 2c_{24}(\kappa)\log H(n) \le 1 + 2c_{24}(\kappa)\log t.$$

This proves the second inequality of (4.12). The first one is clear. □

Proof of Corollary 1.10: By Lemma 4.1,

$$\limsup_{t\to\infty} \big[X(t)/(t/\log t)\big] = \limsup_{t\to\infty} \Big[\Big(\sup_{0\le s\le t} X(s)\Big)/(t/\log t)\Big].$$

So, (1.12) is equivalent to (1.12) with $X(t)$ replaced by $\sup_{0\le s\le t} X(s)$. The same remark also applies to (1.11) and (1.13).

Now, we have $\sup_{0\le s\le y} X(s) \ge r \iff H(r) \le y$, $r > 0$, $y > 0$. Consequently (1.11), (1.12) and (1.13) with $X(t)$ replaced by $\sup_{0\le s\le t} X(s)$ follow respectively from (1.9), (1.10) and Theorem 1.7 applied to $a(r) = (\log r)\dots(\log_{k-1} r)(\log_k r)^\alpha$. Indeed for (1.13) when $\kappa = 1$, cases $k = 1$, $\alpha \le 1$ and $k = 2$, $\alpha \le 0$ follow from the case $k = 3$, $\alpha = 1$. This proves Corollary 1.10. □

5. Proof of Theorems 1.2 to 1.6

Proof of Theorem 1.4: case $\kappa > 1$. Follows from Theorems 3.2 and 1.1. □

Proof of Theorem 1.3. Follows from Theorems 3.1 and 1.1. □

Proof of Theorem 1.6. We first notice that for every $\kappa > 0$, thanks to Lemma 2.7 **(i)**,

$$L_X^*[H(F(r))]/r^{1/\kappa} \xrightarrow{\mathcal{L}} 4\big[\kappa^2/\lambda\big]^{1/\kappa}\big(\sup_{0\le u\le \tau_\beta(\lambda)} \beta(u)\big)^{1/\kappa}, \tag{5.1}$$

where $\xrightarrow{\mathcal{L}}$ denotes convergence in law under $\mathbb{P}$ as $r \to +\infty$.

We now assume $\kappa > 1$. In this case, $H(F(r))/r \to_{r\to+\infty} 4/(\kappa-1)$ $\mathbb{P}$–a.s. by Lemma 2.6 eq. (2.10) and Theorem 1.1 eq. (1.8). This, combined with (5.1) leads to the convergence in law under $\mathbb{P}$ of $L_X^*(t)/t^{1/\kappa}$ to $4[\kappa^2(\kappa-1)/(4\lambda)]^{1/\kappa}\big(\sup_{0\le u\le\tau_\beta(\lambda)} \beta(u)\big)^{1/\kappa}$. Since $\sup_{0\le u\le\tau_\beta(\lambda)} \beta(u)$ has by (2.5) the same law as $\lambda/(2\mathcal{E})$, where $\mathcal{E}$ is an exponential variable with mean 1, this proves Theorem 1.6 when $\kappa > 1$.

We finally assume $\kappa = 1$. In this case, $H(F[t/(4\log t)])/t \to_{t\to+\infty} 1$ in probability under $\mathbb{P}$ by Lemma 2.6 and Theorem 1.1 eq. (1.7). This, combined with (5.1) leads to the convergence in law of $L_X^*(t)/(t/\log t)$ to $\lambda^{-1}\sup_{0\le u\le\tau_\beta(\lambda)} \beta(u)$, which proves Theorem 1.6 when $\kappa = 1$. □

We now assume $0 < \kappa \le 1$, and need to prove Theorems 1.2, 1.4 and 1.5. Unfortunately, it follows immediately from Theorems 1.7 and 1.8 that there is no almost sure convergence result for $H(r)$ in this case due to strong fluctuations; hence a joint study of $L_X^*[H(r)]$ and $H(r)$ is useful. In Section 5.1, we prove a lemma which will be needed later on. Section 5.2 is devoted to the proof of Theorems 1.2, 1.4 and 1.5 in the case $0 < \kappa < 1$, whereas Section 5.3 to the proof of Theorems 1.4 and 1.5 in the case $\kappa = 1$.

5.1. **A lemma.** In this section we assume $0 < \kappa \le 1$. Let $\delta_1 > 0$ and recall the definitions of $t_\pm(r)$ from (2.12) and $\widehat{L}_\pm(r)$ from (2.14).

Lemma 5.1. *Define $E_7(r) := \big\{\widehat{L}_-(r) = \widehat{L}_+(r)\big\}$. For all $\delta_2 \in (0, \delta_1)$ and all large r, we have*

$$\mathbb{P}\big[E_7(r)^c\big] \le r^{-\delta_2}.$$

Proof. Let $\delta_2 \in (0, \delta_1)$. Observe that

$$1 \le \left(\frac{\widehat{L}_+(r)}{\widehat{L}_-(r)}\right)^\kappa \le \max\left(1, \frac{\sup_{0\le u\le \tau_{\widetilde{\beta}}\{[\psi_+(r)-\psi_-(r)]\kappa r\}} \widetilde{\beta}(u)}{\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)}\right), \tag{5.2}$$

where $\widetilde{\beta}(u) := \beta[u + \tau_\beta(\psi_-(r)\kappa r)]$, $u \ge 0$, is a Brownian motion independent of the random variable $\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)$. By (2.5) and the usual inequality $1 - e^{-x} \le x$ (for $x \ge 0$),

$$\begin{aligned}
\mathbb{P}\Bigg(\sup_{0\le u\le \tau_{\widetilde{\beta}}\{[\psi_+(r)-\psi_-(r)]\kappa r\}} \widetilde{\beta}(u) > [\psi_+(r) - \psi_-(r)]\kappa r^{1+\delta_2}\Bigg) &\le \frac{1}{2r^{\delta_2}},\\
\mathbb{P}\Bigg(\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u) < \frac{\psi_-(r)\kappa r}{4\delta_2 \log r}\Bigg) &= \frac{1}{r^{2\delta_2}} \le \frac{1}{2r^{\delta_2}},
\end{aligned}$$

for large r. By definition, $\psi_\pm(r) = 1 \pm c_5 r^{-\delta_1}$ (see (2.12)). Therefore, we have for large r, with probability greater than $1 - r^{-\delta_2}$,

$$\frac{\sup_{0\le u\le \tau_{\widetilde{\beta}}\{[\psi_+(r)-\psi_-(r)]\kappa r\}} \widetilde{\beta}(u)}{\sup_{0\le u\le \tau_\beta(\psi_-(r)\kappa r)} \beta(u)} \le \frac{[\psi_+(r) - \psi_-(r)]\kappa r^{1+\delta_2}}{\psi_-(r)\kappa r/(4\delta_2 \log r)} = \frac{8c_5\delta_2 r^{-(\delta_1-\delta_2)} \log r}{1 - c_5 r^{-\delta_1}} < 1.$$

This, combined with (5.2), yields the lemma. □

5.2. **Case $0 < \kappa < 1$.** This section is devoted to the proof of Theorems 1.2, 1.4 and 1.5 in the case $0 < \kappa < 1$.

For any Brownian motion $(\beta(u),\ u \ge 0)$, let

$$N_\beta := \frac{\int_0^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x)\mathrm{d}x}{\big[\sup_{0\le u\le \tau_\beta(\lambda)} \beta(u)\big]^{1/\kappa}}.$$

So, in the notation of (2.2), (2.12) and (2.14), $N_{\beta_{t_\pm(r)}} = 4[\kappa t_\pm(r)]^{1/\kappa} K_{\beta_{t_\pm(r)}}(\kappa)/\widehat{L}_\pm(r)$, $r > 0$.

On $E_2(r) \cap E_3(r) \cap E_7(r)$ (the events $E_2(r)$ and $E_3(r)$ are defined in Lemma 2.7, whereas $E_7(r)$ in Lemma 5.1), we have, for some constant c_{25}, $\varepsilon > 0$ small enough and all large r,

$$\begin{aligned}
\frac{H(F(r))}{L_X^*[H(F(r))]} &\ge \frac{4(1-\varepsilon)\kappa^{1/\kappa-2} t_-(r)^{1/\kappa}\{K_{\beta_{t_-(r)}}(\kappa) - c_6 t_-(r)^{1-1/\kappa}\}}{(1+\varepsilon)\widehat{L}_-(r)}\\
&\ge (1-3\varepsilon)\kappa^{-2} N_{\beta_{t_-(r)}} - c_{25} t_-(r)/\widehat{L}_-(r).
\end{aligned} \tag{5.3}$$

Similarly, on $E_2(r) \cap E_3(r) \cap E_7(r)$, for some constant c_{26} and all large r,

$$\frac{H(F(r))}{L_X^*[H(F(r))]} \le (1+3\varepsilon)\kappa^{-2} N_{\beta_{t_+(r)}} + c_{26}\frac{t_+(r)}{\widehat{L}_+(r)}. \tag{5.4}$$

Define $E_8(r) := \big\{c_{25} t_-(r)/\widehat{L}_-(r) \le \varepsilon,\ c_{26} t_+(r)/\widehat{L}_+(r) \le \varepsilon\big\}$. By (3.1), $\mathbb{P}[E_8(r)^c] \le 1/r^2$ for large r. Thus $\mathbb{P}[E_2(r) \cap E_3(r) \cap E_7(r) \cap E_8(r)] \ge 1 - r^{-\alpha_1}$ for some $\alpha_1 > 0$ and all large r by

Lemmas 2.7 and 5.1. In view of (5.3) and (5.4), we have, for some $\alpha_1 > 0$ and all large r,

$$\mathbb{P}\bigg((1-3\varepsilon)\kappa^{-2}N_{\beta_{t_-(r)}} - \varepsilon \le \frac{H(F(r))}{L_X^*[H(F(r))]} \le (1+3\varepsilon)\kappa^{-2}N_{\beta_{t_+(r)}} + \varepsilon\bigg) \ge 1 - \frac{1}{r^{\alpha_1}}. \tag{5.5}$$

We now proceed to the study of the law of N_β. By the second Ray–Knight theorem (Fact 2.2), there exists a 0–dimensional Bessel process $(U(x),\ x \ge 0)$, starting from $\sqrt{\lambda}$, such that

$$\begin{aligned}
(L_\beta(\tau_\beta(\lambda), x), x \ge 0) &= \big(U^2(x), x \ge 0\big), && (5.6)\\
\sup_{0\le u\le \tau_\beta(\lambda)} \beta(u) &= \inf\{x \ge 0,\ U(x) = 0\} =: \zeta_U, && (5.7)\\
N_\beta &= \zeta_U^{-1/\kappa}\int_0^{\zeta_U} x^{1/\kappa-2}U^2(x)\mathrm{d}x. && (5.8)
\end{aligned}$$

By Williams' time reversal theorem (Fact 2.4), there exists a 4–dimensional Bessel process $(R(s),\ s \ge 0)$, starting from 0, such that

$$(U(\zeta_U - s),\ s \le \zeta_U) \overset{\mathcal{L}}{=} (R(s),\ s \le \gamma_a), \qquad a := \sqrt{\lambda}, \qquad \gamma_a := \sup\{s \ge 0,\ R(s) = \sqrt{\lambda}\}. \tag{5.9}$$

Therefore,

$$N_\beta \overset{\mathcal{L}}{=} \gamma_a^{-1/\kappa}\int_0^{\gamma_a} x^{1/\kappa-2}R^2(\gamma_a - x)\mathrm{d}x = \int_0^1 (1-v)^{1/\kappa-2}\left(\frac{R(\gamma_a v)}{\sqrt{\gamma_a}}\right)^2 \mathrm{d}v.$$

Recall (Yor [55], p. 52) that for any bounded measurable functional G,

$$\mathbb{E}\left[G\left(\frac{R(\gamma_a u)}{\sqrt{\gamma_a}}, u \le 1\right)\right] = \mathbb{E}\left(\frac{2}{R^2(1)}G\big(R(u), u \le 1\big)\right). \tag{5.10}$$

In particular, for $x > 0$,

$$\mathbb{P}\big(N_\beta > x\big) = \mathbb{E}\left(\frac{2}{R^2(1)}\mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > x\}}\right). \tag{5.11}$$

5.2.1. *Proof of Theorem 1.5 (case $0 < \kappa < 1$).* Fix $y > 0$. By (5.11), for $r > 1$,

$$\begin{aligned}
\mathbb{P}(N_\beta > y\log\log r) &\le \mathbb{E}\left(\frac{2}{R^2(1)}\mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > y\log\log r,\ R^2(1)\le 1\}}\right)\\
&\quad + 2\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > y\log\log r\right)\\
&:= \Pi_1(r) + \Pi_2(r) && (5.12)
\end{aligned}$$

with obvious notation.

We first consider $\Pi_2(r)$. Let $\mathcal{H} := \big\{\big(t \in [0,1] \mapsto \int_0^t f(s)\mathrm{d}s\big),\ f \in L^2\big([0,1], \mathbb{R}^4\big)\big\}$. As R is the Euclidean norm of a 4–dimensional Brownian motion $(\gamma(t),\ t \ge 0)$, we have by Schilder's theorem (see e.g. Dembo and Zeitouni [18], Thm. 5.2.3),

$$\begin{aligned}
&\lim_{r\to+\infty}\frac{1}{y\log\log r}\log\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > y\log\log r\right)\\
&= -\inf\left\{\frac{1}{2}\int_0^1 \|\phi'(v)\|^2\mathrm{d}v :\quad \phi \in \mathcal{H},\ \int_0^1 (1-v)^{1/\kappa-2}\|\phi(v)\|^2\mathrm{d}v \ge 1\right\} =: -c_1(\kappa), && (5.13)
\end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm. For $\phi \in \mathcal{H}$, $\|\phi(v)\|^2 = \big\|\int_0^v \phi'(u)\mathrm{d}u\big\|^2 \le v\int_0^1 \|\phi'(u)\|^2\mathrm{d}u$, where we applied Cauchy-Schwarz to each coordinate; thus $\int_0^1 (1-v)^{1/\kappa-2}\|\phi(v)\|^2\mathrm{d}v \le \big[\int_0^1 (1-v)^{1/\kappa-2}v\mathrm{d}v\big]\int_0^1 \|\phi'(v)\|^2\mathrm{d}v$. So, $c_1(\kappa) \in (0,\infty)$.

By (5.13), for $0<\varepsilon<1$ and large r,

$$\Pi_2(r) \le (\log r)^{-(1-\varepsilon)yc_1(\kappa)}. \tag{5.14}$$

Now, we consider $\Pi_1(r)$. As R is the Euclidean norm of a 4–dimensional Brownian motion $(\gamma(t),\ t\ge 0)$, we have

$$\Pi_1(r) = \mathbb{E}\left(\frac{2}{\|\gamma(1)\|^2}\mathbf{1}_{\{\|\gamma(1)\|\le 1\}}\mathbf{1}_{\{\int_0^1(1-v)^{1/\kappa-2}\|\gamma(v)\|^2\mathrm{d}v>y\log\log r\}}\right).$$

By the triangular inequality, for any finite positive measure μ on $[0,1]$,

$$\sqrt{\int_0^1\|\gamma(v)\|^2\mathrm{d}\mu(v)} \le \sqrt{\int_0^1\|\gamma(v)-v\gamma(1)\|^2\mathrm{d}\mu(v)} + \sqrt{\int_0^1 v^2\mathrm{d}\mu(v)}\,\|\gamma(1)\|.$$

Therefore, applying this to $\mathrm{d}\mu(v)=(1-v)^{1/\kappa-2}\mathrm{d}v$, we have for large r,

$$\Pi_1(r) \le \mathbb{E}\left(\frac{2}{\|\gamma(1)\|^2}\mathbf{1}_{\{\int_0^1(1-v)^{1/\kappa-2}\|\gamma(v)-v\gamma(1)\|^2\mathrm{d}v>(\sqrt{y\log\log r}-c_{27})^2\}}\right) := \mathbb{E}\left(\frac{2}{\|\gamma(1)\|^2}\mathbf{1}_E\right),$$

where $c_{27}:=\sqrt{\int_0^1 v^2(1-v)^{1/\kappa-2}\mathrm{d}v}$. By the independence of $\gamma(1)$ and $(\gamma(v)-v\gamma(1),\ v\in[0,1])$, the expectation on the right hand side is $=\mathbb{E}\big(\frac{2}{\|\gamma(1)\|^2}\big)\mathbb{P}(E)=\mathbb{P}(E)$ (the last identity being a consequence of (5.10) by taking $G=1$ there). Therefore, $\Pi_1(r)\le\mathbb{P}(E)$.

Again, by the independence of $\gamma(1)$ and $(\gamma(v)-v\gamma(1),\ v\in[0,1])$, we see that, by writing $c_{28}:=1/\mathbb{P}(\|\gamma(1)\|\le 1)$, $\Pi_1(r)\le c_{28}\,\mathbb{P}(E,\ \|\gamma(1)\|\le 1)$. By another application of the triangular inequality, this leads to, for large r:

$$\Pi_1(r) \le c_{28}\,\mathbb{P}\left(\int_0^1(1-v)^{1/\kappa-2}\|\gamma(v)\|^2\mathrm{d}v > \left(\sqrt{y\log\log r}-2c_{27}\right)^2\right).$$

In view of (5.13), we have, for all large r, $\Pi_1(r)\le(\log r)^{-(1-\varepsilon)yc_1(\kappa)}$. Plugging this into (5.12) and (5.14) yields that, for any $y>0$, $\varepsilon>0$ and all large r,

$$\mathbb{P}(N_\beta > y\log\log r) \le 2(\log r)^{-(1-\varepsilon)yc_1(\kappa)}. \tag{5.15}$$

Let $0<\varepsilon<1/2$, and $s_n:=\exp(n^{1-\varepsilon})$. We get $\sum_{n=1}^{+\infty}\mathbb{P}\big\{\frac{H(F(s_n))}{L_X^*[H(F(s_n))]}>\frac{(1+4\varepsilon)\log\log s_n}{(1-\varepsilon)^3\kappa^2c_1(\kappa)}\big\}<\infty$ due to (5.5) and (5.15). By the Borel–Cantelli lemma, almost surely, for all large n,

$$\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} \le \frac{1+4\varepsilon}{(1-\varepsilon)^3\kappa^2c_1(\kappa)}\log\log s_n. \tag{5.16}$$

We now bound $\frac{H(F(s_{n+1}))}{H(F(s_n))}$. Observe that for large n, $s_{n+1}-s_n\le n^{-\varepsilon}s_n$. By Lemma 2.6, almost surely for all large n,

$$\begin{aligned} H[F(s_{n+1})]-H[F(s_n)] &\le H\big[\big(1+5s_{n+1}^{-\delta_0}/\kappa\big)s_{n+1}\big]-H\big[\big(1-5s_n^{-\delta_0}/\kappa\big)s_n\big] \\ &\le H\big[\big(1-5s_n^{-\delta_0}/\kappa\big)s_n+(2-\varepsilon)n^{-\varepsilon}s_n\big]-H\big[\big(1-5s_n^{-\delta_0}/\kappa\big)s_n\big] \\ &= \inf\left\{u\ge 0:\ \widehat{X}_n(u)>(2-\varepsilon)n^{-\varepsilon}s_n\right\}, \end{aligned} \tag{5.17}$$

where $\big(\widehat{X}_n(u),\ u\ge 0\big)$ is, conditionally on W_κ, a diffusion process in the random potential $\widehat{W}_\kappa(x):=W_\kappa\big[x+\big(1-\frac{5}{\kappa}s_n^{-\delta_0}\big)s_n\big]-W_\kappa\big[\big(1-\frac{5}{\kappa}s_n^{-\delta_0}\big)s_n\big]$, $x\in\mathbb{R}$, starting from 0. We denote by $\widehat{H}_n(r)$ the hitting time of $r\ge 0$ by $\widehat{X}_n$, so that

$$\inf\left\{u\ge 0:\ \widehat{X}_n(u)>(2-\varepsilon)n^{-\varepsilon}s_n\right\} = \widehat{H}_n\big[(2-\varepsilon)n^{-\varepsilon}s_n\big]. \tag{5.18}$$

Note that for any $r > 0$, under $\mathbb{P}$, $\widehat{H}_n(r)$ is distributed as $H(r)$. Therefore, applying (4.3) and Lemma 2.6 to $r = 2n^{-\varepsilon}s_n$ yields that, for any $0 < \delta_0 < \frac{1}{2}$,

$$\sum_n \mathbb{P}\left[\widehat{H}_n\left(\left[1 - 5(2n^{-\varepsilon}s_n)^{-\delta_0}/\kappa\right]2n^{-\varepsilon}s_n\right) > \left[n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)\right]^{1/\kappa}\right] < \infty.$$

Since $\left[1 - \frac{5}{\kappa}(2n^{-\varepsilon}s_n)^{-\delta_0}\right]2n^{-\varepsilon}s_n \geq (2-\varepsilon)n^{-\varepsilon}s_n$ (for large n), it follows from the Borel–Cantelli lemma that, almost surely for all large n, $\widehat{H}_n\left[(2-\varepsilon)n^{-\varepsilon}s_n\right] \leq \left[n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)\right]^{1/\kappa}$. This, together with (5.17) and (5.18), yields that, almost surely for all large n,

$$H[F(s_{n+1})] - H[F(s_n)] \leq \left[n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)\right]^{1/\kappa} \leq c_{29}\left[n^{1-\varepsilon}(\log n)^{1+\varepsilon}s_n\right]^{1/\kappa}.$$

Recall from Lemma 2.6 and Theorem 1.8 that, almost surely, for all large n, $H[F(s_n)] \geq H[(1-\varepsilon)s_n] \geq \frac{c_{30}s_n^{1/\kappa}}{(\log\log s_n)^{1/\kappa-1}}$, which yields

$$\frac{H[F(s_{n+1})]}{H[F(s_n)]} \leq 1 + \frac{c_{29}[n^{1-\varepsilon}(\log n)^{1+\varepsilon}s_n]^{1/\kappa}}{c_{30}s_n^{1/\kappa}/(\log\log s_n)^{1/\kappa-1}} \leq c_{31}(\log s_n)^{1/\kappa}(\log\log s_n)^{(2+\varepsilon)/\kappa-1}.$$

In view of (5.16), this yields that, almost surely, for large n and $t \in [H(F(s_n)),\, H(F(s_{n+1}))]$,

$$\frac{t}{L_X^*(t)} \leq \frac{H[F(s_n)]}{L_X^*[H(F(s_n))]}\frac{H[F(s_{n+1})]}{H[F(s_n)]} < c_{32}(\log s_n)^{1/\kappa}(\log\log s_n)^{(2+\varepsilon)/\kappa}.$$

Since, almost surely for all large n, $\log H[F(s_n)] \geq \log H[(1-\varepsilon)s_n] \geq \frac{1-\varepsilon}{\kappa}\log s_n$ (this is seen first by Lemma 2.6, and then by Theorem 1.8), we have proved that

$$\liminf_{t\to+\infty} \frac{L_X^*(t)}{t(\log t)^{-1/\kappa}(\log\log t)^{-(2+\varepsilon)/\kappa}} \geq c_{33} \qquad \mathbb{P}\text{–a.s.}$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, this proves Theorem 1.5 in the case $0 < \kappa < 1$. □

5.2.2. *Proof of Theorem 1.4 (case $0 < \kappa < 1$).* By (5.11), for any $s > 0$ and $u > 0$,

$$\begin{aligned}\mathbb{P}(N_\beta > s) &\geq \frac{2}{u}\,\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > s,\ R^2(1) \leq u\right)\\ &\geq \frac{2}{u}\,\mathbb{P}\left(\int_0^1 (1-v)^{1/\kappa-2}R^2(v)\mathrm{d}v > s\right) - \frac{2}{u}\,\mathbb{P}\left(R^2(1) > u\right).\end{aligned}$$

The first probability term on the right hand side is taken care of by (5.13), whereas for the second, we have $\frac{1}{u}\log\mathbb{P}(R^2(1) > u) \to -\frac{1}{2}$, for $u \to \infty$, since $R^2(1)$ has a chi-squared distribution with 4 degrees of freedom. Taking $u := \exp(\sqrt{\log\log r}\,)$ leads to: for any $y > 0$,

$$\liminf_{r\to\infty} \frac{\log\mathbb{P}(N_\beta > y\log\log r)}{\log\log r} \geq -yc_1(\kappa).$$

Plugging this into (5.5) yields that, for $r_n := \exp(n^{1+\varepsilon})$,

$$\sum_{n\geq 1}\mathbb{P}\left(\frac{(H\circ F)(r_{2n})}{(L_X^*\circ H\circ F)(r_{2n})} > \frac{(1-3\varepsilon)\log\log r_{2n}}{\kappa^2c_1(\kappa)(1+\varepsilon)^3} - \varepsilon\right) = +\infty.$$

Let $Z_n := \sum_{k=1}^n r_k$. By Lemma 2.8 (in its notation), almost surely, for infinitely many n,

$$\sup_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} \frac{H_{X\circ\Theta_{H(Z_{2n-1})}}(u)}{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u)} > \frac{(1-8\varepsilon)\log\log r_{2n}}{\kappa^2c_1(\kappa)}, \tag{5.19}$$

if $\varepsilon > 0$ is small enough. Observe that

$$(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u) = \sup_{x\in\mathbb{R}} L_{\widetilde{X}_n}\big(\widetilde{H}_n(u), x\big) =: L^*_{\widetilde{X}_n}\big(\widetilde{H}_n(u)\big), \tag{5.20}$$

where $\big(\widetilde{X}_n(v),\ v\ge 0\big)$ is a diffusion process in the random potential $W_\kappa(x+Z_{2n-1})-W_\kappa(Z_{2n-1})$, $x\in\mathbb{R}$, $\big(L_{\widetilde{X}_n}(t,x),\ t\ge 0,\ x\in\mathbb{R}\big)$ is its local time and $\widetilde{H}_n(r):=\inf\big\{t>0,\ \widetilde{X}_n(t)>r\big\}$, $r>0$. Hence, for any $u>0$, under $\mathbb{P}$, the left hand side of (5.20) is distributed as $L^*_X(H(u))$. Applying (3.4) and Lemma 2.6 to $\widetilde{r}_{2n}:=(1-\varepsilon)^2 r_{2n}$, there exists $c_{34}>0$ such that

$$\sum_n \mathbb{P}\left[L^*_{\widetilde{X}_n}\left(\widetilde{H}_n\big[\big(1+5(\widetilde{r}_{2n})^{-\delta_0}/\kappa\big)\widetilde{r}_{2n}\big]\right)<c_{34}[r_{2n}/\log\log r_{2n}]^{1/\kappa}\right]<\infty.$$

Since $\big(1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0}\big)\widetilde{r}_{2n}\le(1-\varepsilon)r_{2n}$ for large n, the Borel–Cantelli lemma gives that, almost surely, for all large n,

$$c_{34}\big[r_{2n}/\log\log r_{2n}\big]^{1/\kappa}\le L^*_{\widetilde{X}_n}\left(\widetilde{H}_n[(1-\varepsilon)r_{2n}]\right)\le L^*_{\widetilde{X}_n}\left(\widetilde{H}_n(u)\right) \tag{5.21}$$

for any $u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]$. Applying Theorem 3.1, we have almost surely for large n,

$$L^*_X[H(Z_{2n-1})]\le[Z_{2n-1}\log^2 Z_{2n-1}]^{1/\kappa}\le\varepsilon[r_{2n}/\log\log r_{2n}]^{1/\kappa}\le(\varepsilon/c_{34})L^*_{\widetilde{X}_n}\left(\widetilde{H}_n(u)\right)$$

for $u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]$, since $Z_k\le k\exp(-k^\varepsilon)r_{k+1}$ for large k. Hence,

$$L^*_X[H(Z_{2n-1}+u)]\le(1+\varepsilon/c_{34})L^*_{\widetilde{X}_n}\left(\widetilde{H}_n(u)\right). \tag{5.22}$$

On the other hand, we have by Theorem 1.7, almost surely, for all large n,

$$\log\log r_{2n}\ge(1-\varepsilon)\log\log H(Z_{2n-1}+u),\qquad u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}].$$

Consequently, almost surely for infinitely many n, by (5.22) and (5.19),

$$\begin{aligned}&\inf_{v\in[Z_{2n-1}+(1-\varepsilon)r_{2n},Z_{2n-1}+(1+\varepsilon)r_{2n}]}\frac{L^*_X[H(v)]}{H(v)/\log\log H(v)}\\ \le\ &(1+c_{35}\varepsilon)\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}\frac{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u)}{H_{X\circ\Theta_{H(Z_{2n-1})}}(u)/\log\log r_{2n}}\le(1+c_{36}\varepsilon)\kappa^2c_1(\kappa),\end{aligned}$$

proving Theorem 1.4 in the case $0<\kappa<1$. □

5.2.3. *Proof of Theorem 1.2.* Assume $0<\kappa<1$. Fix $x>0$, and let $r_n:=\exp(n^{1+\varepsilon})$. Since $\mathbb{P}(N_\beta<x)>0$, (5.5) implies $\sum_{n\in\mathbb{N}}\mathbb{P}\left(\frac{(H\circ F)(r_{2n})}{(L^*_X\circ H\circ F)(r_{2n})}<\frac{(1+3\varepsilon)x}{\kappa^2}+\varepsilon\right)=+\infty$. By Lemma 2.8, for small $\varepsilon>0$, almost surely for infinitely many n,

$$\inf_{u\in[(1-\varepsilon)r_{2n},\ (1+\varepsilon)r_{2n}]}\frac{H_{X\circ\Theta_{H(Z_{2n-1})}}(u)}{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u)}<\frac{(1+3\varepsilon)x}{\kappa^2}+\varepsilon. \tag{5.23}$$

With the same notation as in (5.20), $H_{X\circ\Theta_{H(Z_{2n-1})}}(u)=H(Z_{2n-1}+u)-H(Z_{2n-1})$ is the hitting time $\widetilde{H}_n(u)$ of u by the diffusion $\widetilde{X}_n$. For any u, under $\mathbb{P}$, it has the same distribution as $H(u)$. Hence, applying (4.9) and Lemma 2.6 to $\widetilde{r}_{2n}=(1-\varepsilon)^2r_{2n}$ leads to (for $0<\delta_0<1/2$)

$$\sum_n\mathbb{P}\left[\widetilde{H}_n\left(\left(1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0}\right)\widetilde{r}_{2n}\right)<r_{2n}^{1/\kappa}/\log r_{2n}\right]<\infty.$$

Since $\big(1+\frac{5}{\kappa}(\widetilde{r}_{2n})^{-\delta_0}\big)\widetilde{r}_{2n}<(1-\varepsilon)r_{2n}$ for large n, it follows from the Borel–Cantelli lemma that, almost surely, for all large n,

$$\frac{r_{2n}^{1/\kappa}}{\log r_{2n}}\le\widetilde{H}_n[(1-\varepsilon)r_{2n}]\le\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}H_{X\circ\Theta_{H(Z_{2n-1})}}(u). \tag{5.24}$$

On the other hand, by Theorem 1.7, $H(Z_{2n-1}) \le [Z_{2n-1}\log^2 Z_{2n-1}]^{1/\kappa} \le \varepsilon \frac{r_{2n}^{1/\kappa}}{\log r_{2n}}$ almost surely, for all large n. This and (5.24) give, for $u \in [(1-\varepsilon)r_{2n},\ (1+\varepsilon)r_{2n}]$, $H(Z_{2n-1}+u) \le (1+\varepsilon)H_{X\circ\Theta_{H(Z_{2n-1})}}(u)$. Plugging this into (5.23) yields that, almost surely, for infinitely many n,

$$\inf_{u\in[(1-\varepsilon)r_{2n},\ (1+\varepsilon)r_{2n}]} \frac{H(Z_{2n-1}+u)}{L_X^*(H(Z_{2n-1}+u))} < \frac{(1+\varepsilon)(1+3\varepsilon)x}{\kappa^2} + \varepsilon(1+\varepsilon).$$

Hence $\limsup_{t\to+\infty} \frac{L_X^*(t)}{t} \ge \frac{\kappa^2}{x}$, a.s. Sending $x \to 0$ completes the proof of Theorem 1.2. □

5.3. **Case $\kappa = 1$.** This section is devoted to the proofs of Theorems 1.4 and 1.5 in the case $\kappa = 1$ (thus $\lambda = 8$; since $\lambda = 4(1+\kappa)$). Let

$$N_\beta(t) := \frac{1}{\sup_{0\le u\le \tau_\beta(8)} \beta(u)} \left[\int_0^1 \frac{L_\beta(\tau_\beta(8),x)-8}{x}\mathrm{d}x + \int_1^{+\infty} \frac{L_\beta(\tau_\beta(8),x)}{x}\mathrm{d}x + 8\log t \right].$$

Exactly as in (5.5), we have, for some $\alpha_1 > 0$, any $\varepsilon \in (0, 1/3)$, and all large r,

$$\mathbb{P}\left((1-3\varepsilon)N_{\beta_{t_-(r)}}[t_-(r)] \le \frac{H(F(r))}{L_X^*[H(F(r))]} \le (1+3\varepsilon)N_{\beta_{t_+(r)}}[t_+(r)] \right) \ge 1 - \frac{1}{r^{\alpha_1}}, \tag{5.25}$$

where $t_\pm(\cdot)$ are defined in (2.12), and C_β in (2.3). (Compared to (5.5), we no longer have the extra "$\pm\varepsilon$" terms, since they are already taken care of by the presence of $8\log t$ in the definition of $N_\beta(t)$).

With the same notation as in (5.6) and (5.7), the second Ray–Knight theorem (Fact 2.2) gives

$$N_\beta(t) = \frac{1}{\zeta_U}\left[\int_0^1 \frac{U^2(x)-8}{x}\mathrm{d}x + \int_1^{+\infty} \frac{U^2(x)}{x}\mathrm{d}x + 8\log t \right] \tag{5.26}$$

$$= \frac{1}{\zeta_U}\left[\int_0^{\zeta_U} \frac{U^2(x)-8}{x}\mathrm{d}x + 8\log\zeta_U + 8\log t \right], \tag{5.27}$$

since $U(x) = 0$ for every $x \ge \zeta_U$.

5.3.1. *Proof of Theorem 1.5 (case $\kappa = 1$).* We have $\lambda = 8$ in the case $\kappa = 1$. Since $\sup_{x>0} \frac{\log x}{x} < \infty$, we have

$$N_\beta(t) \le c_{37} + \frac{1}{\zeta_U}\int_0^{\zeta_U} \frac{|U^2(x)-8|}{x}\mathrm{d}x + \frac{8\log t}{\zeta_U}. \tag{5.28}$$

We claim that for some constant $c_{38} > 0$,

$$\limsup_{y\to+\infty} \frac{1}{y}\log\mathbb{P}\left(\frac{1}{\zeta_U}\int_0^{\zeta_U} \frac{|U^2(x)-8|}{x}\mathrm{d}x > y \right) \le -c_{38}. \tag{5.29}$$

Indeed, $\zeta_U = \sup_{0\le u\le\tau_\beta(8)} \beta(u)$ by definition (see (5.7)), which, in view of (2.5), implies that $\mathbb{P}(\zeta_U > z) = 1 - e^{-4/z} \le 4/z$ for $z > 0$. Therefore, if we write $p(y)$ for the probability expression at (5.29), we have, for any $z > 0$,

$$p(y) \le \frac{4}{z} + \mathbb{P}\left(\frac{1}{\zeta_U}\int_0^{\zeta_U} \frac{|U^2(x)-8|}{x}\mathrm{d}x > y,\ \zeta_U \le z \right).$$

In the notation of (5.9)–(5.10), this yields

$$\begin{aligned} p(y) &\le \frac{4}{z} + \mathbb{P}\left(\frac{1}{\gamma_a}\int_0^1 \frac{|R^2(\gamma_a v)-8|}{1-v}\mathrm{d}v > y,\ \gamma_a \le z\right) \\ &= \frac{4}{z} + \mathbb{E}\left(\frac{2}{R^2(1)}\mathbf{1}_{\{\int_0^1 \frac{|R^2(v)-R^2(1)|}{1-v}\mathrm{d}v>y,\ R^2(1)\ge 8/z\}}\right) \\ &\le \frac{4}{z} + \frac{z}{4}\,\mathbb{P}\left(\int_0^1 \frac{|R^2(v)-R^2(1)|}{1-v}\mathrm{d}v > y\right). \end{aligned} \tag{5.30}$$

In order to apply Schilder's theorem as in (5.13), let $\phi \in \mathcal{H}$. As before between (5.13) and (5.14), we have $\|\phi(t)\| \le \sqrt{t}\,\big[\int_0^1 \|\phi'(s)\|^2\mathrm{d}s\big]^{1/2}$. Similarly, $\big|\|\phi(u)\| - \|\phi(1)\|\big| \le \|\phi(u)-\phi(1)\| \le \sqrt{1-u}\,\big[\int_0^1 \|\phi'(s)\|^2\mathrm{d}s\big]^{1/2}$. Hence,

$$\begin{aligned} \int_0^1 \frac{\big|\|\phi(u)\|^2 - \|\phi(1)\|^2\big|}{1-u}\mathrm{d}u &= \int_0^1 \frac{\big|\|\phi(u)\| - \|\phi(1)\|\big|}{1-u}\big[\|\phi(u)\| + \|\phi(1)\|\big]\mathrm{d}u \\ &\le 2\left(\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u}}\right)\int_0^1 \|\phi'(s)\|^2\mathrm{d}s. \end{aligned}$$

Consequently,

$$c_{39} := \inf\left\{\frac{1}{2}\int_0^1 \big\|\phi'(u)\big\|^2\mathrm{d}u : \quad \phi \in \mathcal{H},\ \int_0^1 \frac{\big|\|\phi(u)\|^2 - \|\phi(1)\|^2\big|}{1-u}\mathrm{d}u > 1\right\} \in (0,\infty).$$

Applying Schilder's theorem gives that $\limsup_{y\to+\infty} \frac{1}{y}\log \mathbb{P}\left(\int_0^1 \frac{|R^2(v)-R^2(1)|}{1-v}\mathrm{d}v > y\right) \le -c_{39}$. Plugging this into (5.30), and taking $z = \exp(\frac{c_{39}}{2}y)$ there, we obtain the claimed inequality in (5.29), with $c_{38} := c_{39}/2$.

On the other hand, by (2.5) and (5.7),

$$\mathbb{P}\left(\frac{8\log t}{\zeta_U} > 2(1+2\varepsilon)(\log t)\log\log t\right) = \frac{1}{(\log t)^{1+2\varepsilon}}.$$

This, combined with (5.28) and (5.29) gives, for all large t,

$$\mathbb{P}\left\{N_\beta(t) > 2(1+3\varepsilon)(\log t)\log\log t\right\} \le \frac{2}{(\log t)^{1+2\varepsilon}}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. By (5.25), $\sum_{n=1}^{+\infty}\mathbb{P}\left(\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} > 2(1+3\varepsilon)^2(\log s_n)\log\log s_n\right) < \infty$, which, by means of the Borel–Cantelli lemma, implies that, almost surely, for all large n,

$$\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} \le 2(1+3\varepsilon)^2(\log s_n)\log\log s_n. \tag{5.31}$$

Now we give an upper bound for $\frac{H(F(s_{n+1}))}{H(F(s_n))}$. By Lemma 2.6, almost surely for n large enough, $F(s_n) \ge (1-\varepsilon)s_n$. An application of Theorem 1.8 yields that, almost surely, for large n,

$$H[F(s_n)] \ge H[(1-\varepsilon)s_n] \ge 4(1-2\varepsilon)s_n\log s_n. \tag{5.32}$$

With the same notation and the same arguments as in (5.17) and (5.18), almost surely for all large n, $H[F(s_{n+1})] - H[F(s_n)] \le \widehat{H}_n[(2-\varepsilon)n^{-\varepsilon}s_n]$. Moreover, $\widehat{H}_n(r)$ is distributed as $H(r)$ under $\mathbb{P}$ for any $r > 0$. Hence, applying Lemma 2.6 and (4.7) to $r = \widetilde{s}_n := 2n^{-\varepsilon}s_n$ and $a(e^{-2}\widetilde{s}_n) = 8n(\log n)^{1+\varepsilon}$ for $0 < \delta_0 < \frac{1}{2}$, we get

$$\sum_n \mathbb{P}\left[\widehat{H}_n\left((1-5(\widetilde{s}_n)^{-\delta_0}/\kappa)\widetilde{s}_n\right) > 32(1+\varepsilon)t_+(\widetilde{s}_n)\big[c_3 + n(\log n)^{1+\varepsilon} + \log t_+(\widetilde{s}_n)\big]\right] < \infty.$$

Since $[1-\frac{5}{\kappa}(\widetilde{s}_n)^{-\delta_0}]\widetilde{s}_n \geq (2-\varepsilon)n^{-\varepsilon}s_n$ (for large n), the Borel–Cantelli lemma yields that

$$\widehat{H}_n\big((2-\varepsilon)n^{-\varepsilon}s_n\big) \leq 32(1+\varepsilon)t_+(2n^{-\varepsilon}s_n)[c_3 + n(\log n)^{1+\varepsilon} + \log t_+(2n^{-\varepsilon}s_n)],$$

almost surely for large n. Hence, $H[F(s_{n+1})] - H[F(s_n)] \leq c_{39}s_n(\log s_n)(\log n)^{1+\varepsilon}$. Hence, by (5.32), we have, almost surely, for all large n,

$$H[F(s_{n+1})]/H[F(s_n)] \leq c_{40}(\log\log s_n)^{1+\varepsilon}.$$

Let $t \in [H(F(s_n)),\, H(F(s_{n+1}))]$. By (5.31), if t is large enough,

$$\frac{t}{L_X^*(t)} \leq \frac{H[F(s_n)]}{L_X^*[H(F(s_n))]}\frac{H[F(s_{n+1})]}{H[F(s_n)]} < 3c_{40}(\log s_n)(\log\log s_n)^{2+\varepsilon}.$$

Since almost surely for large n, $\log H[F(s_n)] \geq \log H[(1-\varepsilon)s_n] \geq \log s_n$ (by Lemma 2.6 and Theorem 1.8), this yields

$$\liminf_{t\to+\infty} \frac{L_X^*(t)}{t/[(\log t)(\log\log t)^{2+\varepsilon}]} \geq \frac{1}{3c_{40}} \qquad \mathbb{P}\text{–a.s.}$$

Theorem 1.5 is proved in the case $\kappa = 1$. □

5.3.2. *Proof of Theorem 1.4 (case $\kappa = 1$).* Again, $\lambda = 8$. Let $0 < \varepsilon < 1/2$. Recall that $\zeta_U = \sup_{0\leq u\leq \tau_\beta(8)} \beta(u)$, and that $N_\beta(t) = \frac{1}{\zeta_U}\left[\int_0^{\zeta_U} \frac{U^2(x)-8}{x}\mathrm{d}x + 8\log\zeta_U + 8\log t\right]$ (see (5.7) and (5.27)). This time, we need to bound $N_\beta(t)$ from below. By (2.5) for large z,

$$\mathbb{P}\left(8\frac{\log\zeta_U}{\zeta_U} < -z\right) \leq \mathbb{P}\left(-\frac{1}{\zeta_U^2} < -z\right) = \mathbb{P}\left(\zeta_U < \frac{1}{\sqrt{z}}\right) = \exp\left(-4\sqrt{z}\right).$$

By (2.5) again,

$$\mathbb{P}\left(\frac{8\log t}{\zeta_U} > 2(1-\varepsilon)(\log t)\log\log t\right) = \frac{1}{(\log t)^{1-\varepsilon}}.$$

On the other hand, for all large y, $\mathbb{P}\big(\frac{1}{\zeta_U}\int_0^{\zeta_U}\frac{|U^2(x)-8|}{x}\mathrm{d}x > y\big) \leq e^{-c_{41}y}$ (see (5.29)). Assembling these pieces yields that, for all large t,

$$\mathbb{P}[N_\beta(t) > 2(1-2\varepsilon)(\log t)\log\log t] \geq \frac{1}{2(\log t)^{1-\varepsilon}}.$$

Let $r_n := \exp(n^{1+\varepsilon})$. In view of (5.25) and Lemma 2.8, we get almost surely for infinitely many n,

$$\sup_{u\in[(1-\varepsilon)r_{2n},\,(1+\varepsilon)r_{2n}]} \frac{H_{X\circ\Theta_{H(Z_{2n-1})}}(u)}{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u)} > 2(1-2\varepsilon)(1-3\varepsilon)(\log r_{2n})\log\log r_{2n}. \tag{5.33}$$

The expression on the left hand side of (5.33) is "close to" $H(r_{2n})/L_X^*[H(r_{2n})]$, but we need to prove this rigorously. With the same argument as in the displays between (5.20) and (5.21), we get that there exists $c_{42} > 0$ such that, almost surely for large n,

$$\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} (L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u) \geq c_{42}r_{2n}/\log\log r_{2n}.$$

Observe that $Z_k \leq k\exp(-k^\varepsilon)r_{k+1}$ for large k, as in the paragraph after (3.5). Exactly as in the case $0 < \kappa < 1$, we apply Theorem 3.1, to see that almost surely for large n,

$$L_X^*[H(Z_{2n-1})] \leq \varepsilon r_{2n}/\log\log r_{2n} \leq (\varepsilon/c_{42}) \inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]} (L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u),$$

which implies, for all $u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$,

$$L_X^*[H(Z_{2n-1}+u)] \leq (1+\varepsilon/c_{42})(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u). \tag{5.34}$$

By Theorem 1.7, almost surely for all large n, $\sup_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}\log H(Z_{2n-1}+u)\le(1+\varepsilon)\log r_{2n}$. In view of (5.34) and then (5.33), there are almost surely infinitely many n such that

$$\begin{aligned}&\inf_{v\in[Z_{2n-1}+(1-\varepsilon)r_{2n},Z_{2n-1}+(1+\varepsilon)r_{2n}]}\frac{L_X^*[H(v)]}{H(v)/[(\log H(v))\log\log H(v)]}\\ \le\ &(1+c_{43}\varepsilon)\inf_{u\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]}\frac{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(u)}{H_{X\circ\Theta_{H(Z_{2n-1})}}(u)[(\log r_{2n})\log\log r_{2n}]^{-1}}\le(1+c_{44}\varepsilon)/2.\end{aligned}$$

This proves Theorem 1.4 in the case $\kappa=1$. □

Remark: Assume $\kappa=1$. We also prove that in this case, $\mathbb{P}$ almost surely,

$$\limsup_{t\to+\infty}L_X^*(t)/t\ \ge 8/[c_{17}\pi]=1/2. \tag{5.35}$$

This is in agreement with Theorem 1.1 of Gantert and Shi [27] for RWRE. However, we could not prove whether this lim sup is finite or not, contrarily to the cases $\kappa\in(0,1)$ and $\kappa>1$, and to the case of RWRE, for which the maximum local time at time t is clearly less than $t/2$.

We now prove (5.35). With the same notation as in (5.6) and (5.7), let $\widehat{C}_U:=\int_0^1\frac{U^2(x)-8}{x}\mathrm{d}x+\int_1^{+\infty}\frac{U^2(x)}{x}\mathrm{d}x$, $\varepsilon\in(0,1/3)$ and $y:=(1+\varepsilon)^2c_{17}\pi/[8(1-\varepsilon)]$. We have for $z>0$, by (5.26),

$$\mathbb{P}[N_\beta(t)<y]=\mathbb{P}\big[\widehat{C}_U+8\log t<y\zeta_U\big]\ge\mathbb{P}\big[(z+8)\log t<y\zeta_U,\widehat{C}_U\le z\log t\big].$$

Notice that $\widehat{C}_U\overset{\mathcal{L}}{=}C_\beta\overset{\mathcal{L}}{=}8c_3+(\pi/2)C_8^{ca}$ first by (5.6) and (2.3), then by Fact 2.5. So, $\mathbb{P}\big[\widehat{C}_U>z\log t\big]=\mathbb{P}\big[C_8^{ca}>(2z/\pi)\log t-16c_3/\pi\big]\sim_{t\to+\infty}\pi c_{17}/(2z\log t)$ (see before (4.7)). Moreover, $\mathbb{P}[(z+8)\log t<y\zeta_U]\sim_{t\to+\infty}4y/[(z+8)\log t]$ by (5.7) and (2.5). Thus,

$$\begin{aligned}\mathbb{P}[N_\beta(t)<y]\ &\ge\ \mathbb{P}\big[(z+8)\log t<y\zeta_U\big]-\mathbb{P}\big[\widehat{C}_U>z\log t\big]\\ &\ge\ [4(1-\varepsilon)y/(z+8)-(1+\varepsilon)\pi c_{17}/(2z)]/\log t\\ &=\ (1+\varepsilon)c_{17}\pi[(1+\varepsilon)/(z+8)-1/z]/(2\log t)\end{aligned}$$

So we can choose z so that $\mathbb{P}[N_\beta(t)<y]\ge c_{45}/\log t$ for some constant $c_{45}>0$. We now set $r_k:=k^k$, $k\in\mathbb{N}^*$. This and (5.25) give for some $\alpha_1>0$,

$$\mathbb{P}\left[\frac{L_X^*[H(F(r_{2n}))]}{H(F(r_{2n}))}>[(1+3\varepsilon)y]^{-1}\right]\ \ge\ \mathbb{P}\Big[N_\beta[t_+(r_{2n})]<y\Big]-\frac{1}{r_{2n}^{\alpha_1}}\ge\frac{c_{45}}{2\log r_{2n}}\ge\frac{c_{45}}{5n\log n}$$

for large n. Hence by Lemma 2.8 in its notation, $\mathbb{P}$ almost surely, there exist infinitely many n such that for some $t_n\in[(1-\varepsilon)r_{2n},(1+\varepsilon)r_{2n}]$,

$$\frac{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)}{H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)}>[(1+3\varepsilon)y]^{-1}. \tag{5.36}$$

Notice that $Z_{2n-1}=\sum_{k=1}^{2n-1}k^k\le(2n-1)^{2n-1}+\sum_{k=1}^{2n-2}(2n-2)^{2n-2}\le2(2n)^{2n-1}=r_{2n}/n$. We have by Theorem 1.7, almost surely for all large n,

$$H(Z_{2n-1})\le Z_{2n-1}(\log Z_{2n-1})(\log\log Z_{2n-1})^2\le\varepsilon r_{2n}\log r_{2n} \tag{5.37}$$

On the other hand, first by Lemma 2.6 and Lemma 2.7, then by (4.2) and since $\mathbb{E}[\exp(-C_8^{ca})]=1$ as before (4.11), for every $\varepsilon>0$ small enough,

$$\begin{aligned}&\mathbb{P}\big[H_{X\circ\Theta_{H(Z_{2n-1})}}((1-\varepsilon)r_{2n})<(1-10\varepsilon)32t_-(r_{2n})[\log t_-(r_{2n})+c_3]\big]\\ \le\ &\mathbb{P}\big[(1-\varepsilon)\widehat{I}_-((1-2\varepsilon)r_{2n})<(1-10\varepsilon)32t_-(r_{2n})[\log t_-(r_{2n})+c_3]\big]+2r_{2n}^{-\alpha_1}\\ \le\ &\mathbb{P}\big[C_8^{ca}<-\varepsilon(16/\pi)\log r_{2n}\big]+2r_{2n}^{-\alpha_1}\le2r_{2n}^{-16\varepsilon/\pi}.\end{aligned}$$

Hence, thanks to the Borel Cantelli lemma, almost surely for large n,

$$\begin{aligned} H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n) &\geq H_{X\circ\Theta_{H(Z_{2n-1})}}((1-\varepsilon)r_{2n}) \\ &\geq (1-10\varepsilon)32t_-(r_{2n})[\log t_-(r_{2n})+c_3] \geq (1-11\varepsilon)4r_{2n}\log r_{2n}. \end{aligned}$$

This together with (5.36) and (5.37) gives $\mathbb{P}$ almost surely for infinitely many n,

$$\begin{aligned} \frac{L_X^*[H(Z_{2n-1}+t_n)]}{H(Z_{2n-1}+t_n)} &\geq \frac{(L^*\circ H)_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)}{H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)}\frac{H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)}{H(Z_{2n-1})+H_{X\circ\Theta_{H(Z_{2n-1})}}(t_n)} \\ &\geq [(1+3\varepsilon)y]^{-1}(1+\varepsilon)^{-1} \geq (1-10\varepsilon)8/[c_{17}\pi], \end{aligned}$$

for small ε. As before, let $t\to+\infty$, and then $\varepsilon\to 0$. This proves (5.35) since $c_{17}=\frac{16}{\pi}$ as before (4.7).

6. Proof of Lemma 2.7

This section is devoted to the proof of Lemma 2.7. The basic idea goes back to Hu et al. [33], but requires considerable refinements due to the complicated nature of the process $x\mapsto L_X(t,x)$ and to the fact that we are interested in the joint law of $\big(L_X^*[H(.)],H(.)\big)$. Throughout the proof we consider the annealed probability $\mathbb{P}$.

Let $\kappa>0$ and $\varepsilon\in(0,1)$. We fix $r>1$. Recall that $A(x)=\int_0^x e^{W_\kappa(u)}\mathrm{d}u$, and $A_\infty=\lim_{x\to+\infty}A(x)<\infty$, a.s. As in Brox ([11], eq. (1.1)), the general diffusion theory leads to

$$X(t)=A^{-1}[B(T^{-1}(t))], \qquad t\geq 0, \tag{6.1}$$

where $(B(s),\ s\geq 0)$ is a Brownian motion independent of W, and for $0\leq u<\sigma_B(A_\infty)$, $T(u):=\int_0^u\exp\{-2W_\kappa[A^{-1}(B(s))]\}\mathrm{d}s$ (A^{-1} and T^{-1} denote respectively the inverses of A and T). The local time of X can be written as (see Shi [44], eq. (2.5))

$$L_X(t,x)=e^{-W_\kappa(x)}L_B\big(T^{-1}(t),A(x)\big), \qquad t\geq 0,\ x\in\mathbb{R}. \tag{6.2}$$

As in (1.5), $H(\cdot)$ denotes the first hitting time of X. Then as in Shi ([45], eq. (4.3) to (4.6)),

$$H(u)=T[\sigma_B(A(u))]=\int_{-\infty}^0+\int_0^{A(u)}e^{-2W_\kappa[A^{-1}(x)]}L_B(\sigma_B[A(u)],x)\mathrm{d}x=:H_-(u)+H_+(u) \tag{6.3}$$

for $u\geq 0$. Recall F from (2.7) and notice that $F(r)>0$ on $E_1(r)$ if r is large enough. By scaling since W_κ and then $A(F(r))$ are independent of B, and then by the first Ray–Knight theorem (Fact 2.1), there exists a squared Bessel process of dimension 2, starting from 0 and denoted by $\big(R_2^2(s),\ s\geq 0\big)$, independent of W_κ, such that

$$\left(\frac{L_B\{\sigma_B[A(F(r))],A(F(r))-sA(F(r))\}}{A(F(r))},\ s\in[0,1]\right)=\big(R_2^2(s),\ s\in[0,1]\big)\,.$$

Hence, it is more convenient to study $L_X^*[H(.)]$ instead of $L_X^*(t)$. We consider

$$L_X^+[H(u)]:=\sup_{x\geq 0}L_X(H(u),x)=\sup_{0\leq x\leq u}\big\{e^{-W_\kappa(x)}L_B[\sigma_B(A(u)),A(x)]\big\}, \qquad u\geq 0.$$

In particular,

$$L_X^+[H(F(r))]=\sup_{x\in[0,F(r)]}\left\{e^{-W_\kappa(x)}A(F(r))R_2^2\left[\frac{A(F(r))-A(x)}{A(F(r))}\right]\right\}.$$

Moreover, by Lamperti's representation theorem (Fact 2.3), there exists a Bessel process $\rho=(\rho(t),\ t\geq 0)$, of dimension $(2-2\kappa)$, starting from $\rho(0)=2$, such that for all $t\geq 0$, $e^{W_\kappa(t)/2}=\frac{1}{2}\rho(A(t))$. Now, let

$$\widetilde{R}_{2+2\kappa}(t):=\rho(A_\infty-t), \qquad 0\leq t\leq A_\infty.$$

By Williams' time reversal theorem (Fact 2.4), $\widetilde{R}_{2+2\kappa}$ is a Bessel process of dimension $(2+2\kappa)$, starting from 0. Since W_κ and $A(F(r))$ are independent of R_2, $u \mapsto \sqrt{A(F(r))}R_2(u/A(F(r)))$ is a 2–dimensional Bessel process, starting from 0 and independent of $\widetilde{R}_{2+2\kappa}$. We still denote by R_2 this new Bessel process. We obtain

$$L_X^+[H(F(r))] = 4 \sup_{x\in[0,F(r)]} \frac{R_2^2[A(F(r))-A(x)]}{\widetilde{R}_{2+2\kappa}^2[A_\infty - A(x)]} = 4 \sup_{v\in[0,A(F(r))]} \frac{R_2^2(v)}{\widetilde{R}_{2+2\kappa}^2[A_\infty - A(F(r))+v]}.$$

Doing the same transformations on $H_+(F(r))$ (see (6.3)) and recalling that $A_\infty - A(F(r)) = \delta(r) = \exp(-\kappa r/2)$ and so is deterministic thanks to the random function F, we obtain

$$\begin{aligned}
&\big(L_X^+[H(F(r))], H_+[F(r)]\big)\\
&= \left(4 \sup_{v\in[0,A(F(r))]} \frac{R_2^2(v)}{\widetilde{R}_{2+2\kappa}^2[\delta(r)+v]}, 16\int_0^{A(F(r))} \frac{R_2^2(s)}{\widetilde{R}_{2+2\kappa}^4[\delta(r)+s]}\mathrm{d}s\right)\\
&= \left(4 \sup_{u\in[0,\delta(r)^{-1}A(F(r))]} \frac{R_2^2[\delta(r)u]}{\widetilde{R}_{2+2\kappa}^2[\delta(r)(1+u)]}, 16\int_0^{\delta(r)^{-1}A(F(r))} \frac{R_2^2[\delta(r)u]\delta(r)\mathrm{d}u}{\widetilde{R}_{2+2\kappa}^4[\delta(r)(1+u)]}\right).
\end{aligned}$$

We still denote by R_2 the 2-dimensional Bessel process $u \mapsto \frac{1}{\sqrt{\delta(r)}}R_2(\delta(r)u)$. We define

$$\widehat{R}_{2+2\kappa}(u) = \frac{1}{\sqrt{\delta(r)}}\widetilde{R}_{2+2\kappa}[\delta(r)(1+u)], \qquad u \geq 0. \tag{6.4}$$

Notice that $\widehat{R}_{2+2\kappa}(u)$ is a $(2+2\kappa)$–dimensional Bessel process, starting from $\widetilde{R}_{2+2\kappa}(\delta(r))/\sqrt{\delta(r)}$ and independent of R_2.

Recall (see e.g. Karlin and Taylor [34] p. 335) that a Jacobi process $(Y(t),\ t\geq 0)$ of dimensions (d_1, d_2) is a solution of the stochastic differential equation

$$\mathrm{d}Y(t) = 2\sqrt{Y(t)(1-Y(t))}\,\mathrm{d}\widehat{\beta}(t) + \big[d_1 - (d_1+d_2)Y(t)\big]\mathrm{d}t, \tag{6.5}$$

where $\big(\widehat{\beta}(t),\ t\geq 0\big)$ is a standard Brownian motion.

According to Warren and Yor ([53] p. 337), there exists a Jacobi process $(Y(t),\ t\geq 0)$ of dimensions $(2, 2+2\kappa)$, starting from 0, independent of $\big(R_2^2(t)+\widehat{R}_{2+2\kappa}^2(t),\ t\geq 0\big)$, such that

$$\forall u\geq 0, \qquad \frac{R_2^2(u)}{R_2^2(u)+\widehat{R}_{2+2\kappa}^2(u)} = Y\circ\Lambda_Y(u), \qquad \Lambda_Y(u) := \int_0^u \frac{\mathrm{d}s}{R_2^2(s)+\widehat{R}_{2+2\kappa}^2(s)}. \tag{6.6}$$

In particular, $(\Lambda_Y(t),\ t\geq 0)$ is independent of Y. As a consequence, for all $r\geq 0$,

$$\begin{aligned}
&(L_X^+[H(F(r))], H_+[F(r)])\\
&= \left(4 \sup_{u\in[0,\delta(r)^{-1}A(F(r))]} \frac{Y\circ\Lambda_Y(u)}{1-Y\circ\Lambda_Y(u)}, 16\int_0^{\delta(r)^{-1}A(F(r))} \frac{[Y\circ\Lambda_Y(u)]\Lambda_Y'(u)\mathrm{d}u}{[1-Y\circ\Lambda_Y(u)]^2}\right)\\
&= \left(4 \sup_{u\in[0,\varphi(r)]} \frac{Y(u)}{1-Y(u)}, 16\int_0^{\varphi(r)} \frac{Y(u)}{(1-Y(u))^2}\mathrm{d}u\right),
\end{aligned}$$

where

$$\varphi(r) := \Lambda_Y\big[\delta(r)^{-1}A(F(r))\big]. \tag{6.7}$$

Define $\alpha_\kappa := 1/(4+2\kappa)$ and let $T_Y(\alpha_\kappa) := \inf\{t>0, Y(t)=\alpha_\kappa\}$ be the hitting time of α_κ by Y. We introduce

$$\begin{aligned} \overline{L}(r) &:= 4 \sup_{u\in[0,T_Y(\alpha_\kappa)]} \frac{Y(u)}{1-Y(u)}, & \overline{H}(r) &:= 16 \int_0^{T_Y(\alpha_\kappa)} \frac{Y(u)}{(1-Y(u))^2}\mathrm{d}u, \\ L_0(r) &:= 4 \sup_{u\in[T_Y(\alpha_\kappa),\varphi(r)]} \frac{Y(u)}{1-Y(u)}, & H_0(r) &:= 16 \int_{T_Y(\alpha_\kappa)}^{\varphi(r)} \frac{Y(u)}{(1-Y(u))^2}\mathrm{d}u. \end{aligned} \tag{6.8}$$

We have on $E_9 := \{T_Y(\alpha_\kappa) \le 64\log r \le \kappa r/(2\lambda) \le \varphi(r)\}$,

$$\left(L_X^+[H(F(r))],\ H_+(F(r))\right) = \left(\max\{\overline{L}(r), L_0(r)\},\ \overline{H}(r)+H_0(r)\right), \tag{6.9}$$

$$\overline{L}(r) \le \frac{4\alpha_\kappa}{1-\alpha_\kappa} \qquad \text{and} \qquad \overline{H}(r) \le \frac{16\alpha_\kappa}{(1-\alpha_\kappa)^2} T_Y(\alpha_\kappa) \le \frac{2^{10}\alpha_\kappa}{(1-\alpha_\kappa)^2}\log r. \tag{6.10}$$

Observe that $S(y) := \int_{\alpha_\kappa}^y \frac{\mathrm{d}x}{x(1-x)^{1+\kappa}}$, $y\in(0,1)$ is a scale function of Y, as in Hu et al. ([33], eq. (2.27)). Hence $t \mapsto S[Y(t+T_Y(\alpha_\kappa))]$ is a continuous local martingale, so by Dubins-Schwarz theorem, there exists a Brownian motion $(\beta(t),\ t\ge 0)$ such that for all $t\ge 0$,

$$Y[t+T_Y(\alpha_\kappa)] = S^{-1}\{\beta[U_Y(t)]\}, \qquad U_Y(t) := 4\int_0^t \frac{\mathrm{d}s}{Y[s+T_Y(\alpha_\kappa)]\{1-Y[s+T_Y(\alpha_\kappa)]\}^{1+2\kappa}}. \tag{6.11}$$

The rest of the proof of Lemma 2.7 requires some more estimates, stated as Lemmas 6.1–6.4 below. Lemmas 6.1–6.3 deal only with Bessel processes, Jacobi processes and Brownian motion, and may be of independent interest, whereas Lemma 6.4 gives an upper bound for the total time spent by X on $(-\infty,0)$, and for the maximum local time of X in $(-\infty,0)$. We defer the proofs of Lemmas 6.1–6.3 to Section 7, and we complete the proof of Lemma 2.7.

Lemma 6.1. *Let $(R(t),\ t\ge 0)$ be a Bessel process of dimension $d>4$, starting from $R_0 \overset{\mathcal{L}}{=} \widetilde{R}_{d-2}(1)$, where $\left(\widetilde{R}_{d-2}(t),\ t\in[0,1]\right)$ is $(d-2)$–dimensional Bessel process. For any $\delta_3\in(0,\frac{1}{2})$ and all large t,*

$$\mathbb{P}\left\{\left|\frac{1}{\log t}\int_0^t \frac{\mathrm{d}s}{R^2(s)} - \frac{1}{d-2}\right| > \frac{1}{(\log t)^{(1/2)-\delta_3}}\right\} \le \exp\left(-c_{46}\,(\log t)^{2\delta_3}\right).$$

Lemma 6.2. *Let $\delta_1>0$ and define*

$$E_{10} := \left\{\tau_\beta\left[\left(1-v^{-\delta_1}\right)\lambda v\right] \le U_Y(v) \le \tau_\beta\left[\left(1+v^{-\delta_1}\right)\lambda v\right]\right\}. \tag{6.12}$$

If δ_1 is small enough, then for all large v, $\mathbb{P}(E_{10}^c) \le \frac{1}{v^{1/4-5\delta_1}}$.

In the two previous lemmas, taking respectively $\frac{1}{(\log t)^{(1/2)-\delta_3}}$ and $v^{-\delta_1}$ instead of simply some fixed $\tilde\varepsilon>0$ is necessary to obtain Lemma 2.7 with $\psi_\pm(r)$ instead of simply $1\pm\tilde\varepsilon$ in the definition of $\widehat{L}_\pm(r)$ and $\widehat{I}_\pm(r)$, which itself is necessary for example to prove Lemma 5.1.

Lemma 6.3. *Let $(\beta(s),\ s\ge 0)$ be a Brownian motion, and $\lambda = 4(1+\kappa)$ as before. We define*

$$J_\beta(\kappa,t) := \int_0^1 y(1-y)^{\kappa-2} L_\beta\left[\tau_\beta(\lambda), S(y)/t\right] dy, \qquad 0<\kappa\le 1,\ t\ge 0. \tag{6.13}$$

Let $0<d<1$ and recall that $0<\varepsilon<1$.

(i) *Case $0<\kappa<1$: recall $K_\beta(\kappa)$ from (2.2). There exist $c_{47}>0$ and $c_{48}>0$ such that for all large t, on an event E_{11} of probability greater than $1-c_{47}/t^d$, we have*

$$(1-\varepsilon)K_\beta(\kappa)-c_{48}t^{1-1/\kappa}\le \kappa^{2-1/\kappa}t^{1-1/\kappa}J_\beta(\kappa,t)\le(1+\varepsilon)K_\beta(\kappa)+c_{48}t^{1-1/\kappa}. \tag{6.14}$$

(ii) *Case $\kappa=1$: recall C_β from (2.3). There exists $c_{47}>0$ such that for t large enough, on an event E_{11} of probability greater than $1-c_{47}/t^d$,*

$$(1-\varepsilon)[C_\beta+8\log t]\le J_\beta(1,t)\le(1+\varepsilon)[C_\beta+8\log t]. \tag{6.15}$$

Lemma 6.4. *Let $\kappa>0$ and define*

$$L_X^{*-}(+\infty):=\sup_{r\ge0}\sup_{x<0}L_X(H(r),x)=\sup_{t\ge0}\sup_{x<0}L_X(t,x),\qquad H_-(+\infty):=\lim_{r\to+\infty}H_-(r).$$

There exist $c_{49}>0$ and $c_{50}>0$ such that for all large z,

$$\mathbb{P}\big[L_X^{*-}(+\infty)>z\big]\ \le\ c_{49}z^{-\kappa/(\kappa+2)}, \tag{6.16}$$

$$\mathbb{P}\big[H_-(+\infty)>z\big]\ \le\ c_{50}[(\log z)/z]^{\kappa/(\kappa+2)}. \tag{6.17}$$

Proof of Lemma 6.4: This lemma is proved in Andreoletti et al. ([3], Lemma 3.5, which is true for every $\kappa>0$). More precisely, (6.17) is proved in ([3], eq. (3.29)), whereas (6.16) is proved in ([3], eq. (3.31)). □

By admitting Lemmas 6.1–6.3, we can now complete the proof of Lemma 2.7.

Proof of Lemma 2.7: part (i). Notice that

$$S(y)\underset{y\to1}{\sim}\int_{\alpha_\kappa}^{y}\frac{\mathrm{d}s}{(1-s)^{1+\kappa}}\underset{y\to1}{\sim}\frac{1}{\kappa}\frac{1}{(1-y)^\kappa}.\qquad \frac{y}{1-y}\underset{y\to1}{\sim}[\kappa S(y)]^{1/\kappa}. \tag{6.18}$$

Define $\widetilde{L}_0(r):=4\big[\sup_{u\in[T_Y(\alpha_\kappa),\varphi(r)]}\kappa S(Y(u))\big]^{1/\kappa}$. We have,

$$\widetilde{L}_0(r)=4\Big[\sup_{u\in[0,\varphi(r)-T_Y(\alpha_\kappa)]}\kappa\beta\big(U_Y(u)\big)\Big]^{1/\kappa}=4\Big[\sup_{s\in[0,U_Y(\varphi(r)-T_Y(\alpha_\kappa))]}\kappa\beta(s)\Big]^{1/\kappa}. \tag{6.19}$$

Recall L_0 from (6.8). By (6.18), there exists a constant $c_{51}>0$ depending on ε such that

$$\Big\{\widetilde{L}_0(r)>c_{51}\Big\}\subset\Big\{(1-\varepsilon)\widetilde{L}_0(r)\le L_0(r)\le(1+\varepsilon)\widetilde{L}_0(r)\Big\}. \tag{6.20}$$

We look for an estimate of $U_Y[\varphi(r)-T_Y(\alpha_\kappa)]$ appearing in the expression of $\widetilde{L}_0(r)$ in the right hand side of (6.19). Recall (see Dufresne [24], or Borodin et al. [10] IV.48 p. 78) that $A_\infty\overset{\mathcal{L}}{=}2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter $(\kappa,1)$, with density $e^{-x}x^{\kappa-1}\mathbf{1}_{(0,\infty)}(x)/\Gamma(\kappa)$. Since $A(F(r))\le A_\infty$, we have

$$\mathbb{P}\big[A(F(r))>r^{2/\kappa}\big]\le\mathbb{P}\big[\gamma_\kappa<2r^{-2/\kappa}\big]\le 2^\kappa r^{-2}/[\kappa\Gamma(\kappa)].$$

On the other hand, by definition, $A(F(r))=A_\infty-\delta(r)=A_\infty-e^{-\kappa r/2}$ (see 2.7), which implies

$$\mathbb{P}\left[A(F(r))<\frac{1}{2\log r}\right]=\mathbb{P}\left[\frac{2}{\gamma_\kappa}<\frac{1}{2\log r}+\delta(r)\right]\le\frac{1}{r^2}$$

for large r. Consequently,

$$\mathbb{P}\left\{\kappa r/2-2\log\log r\le\log\left[\delta(r)^{-1}A(F(r))\right]\le\kappa r/2+(2/\kappa)\log r\right\}\ge1-c_{52}r^{-2}.$$

Recall that $\varphi(r)=\Lambda_Y[\delta(r)^{-1}A(F(r))]$ by (6.7). Thus, for large r,

$$\mathbb{P}\left\{\Lambda_Y\big[\exp(\kappa r/2-2\log\log r)\big]\le\varphi(r)\le\Lambda_Y\big[\exp(\kappa r/2+(2/\kappa)\log r)\big]\right\}\ge1-c_{52}r^{-2}.$$

By definition, $\Lambda_Y(u) = \int_0^u \frac{\mathrm{d}s}{R_2^2(s)+\widehat{R}_{2+2\kappa}^2(s)}$. Notice that $\big(R_2^2(t) + \widehat{R}_{2+2\kappa}^2(t),\ t \geq 0\big)$ is a $(4+2\kappa)$–dimensional squared Bessel process starting from $\widetilde{R}_{2+2\kappa}^2[\delta(r)]/\delta(r)$ by the additivity property of squared Bessel processes (see e.g. Revuz et al. [41], XI th. 1.2). So, it follows from Lemma 6.1 applied with $d = 4+2\kappa$ and $\delta_3 = 1/4$, that there exist constants $c_{53} > 0$ and $c_{54} > 0$, such that

$$\mathbb{P}\big\{\kappa r/\lambda - c_{53} r^{1/2+\delta_3} \leq \varphi(r) \leq \kappa r/\lambda + c_{53} r^{1/2+\delta_3}\big\} \geq 1 - c_{54} r^{-2}, \tag{6.21}$$

for large r, where $\lambda = 4(1+\kappa)$, as before.

In order to study $T_Y(\alpha_\kappa)$, we go back to the stochastic differential equation in (6.5) satisfied by the Jacobi process $Y(\cdot)$, with $d_1 = 2$ and $d_2 = 2+2\kappa$. Note that $Y(s) \in (0,1)$ for any $s > 0$. By the Dubins–Schwarz theorem, there exists a Brownian motion $\big(\widehat{B}(s),\ s \geq 0\big)$ such that

$$Y(t) = \widehat{B}\left(4\int_0^t Y(s)(1-Y(s))\mathrm{d}s\right) + \int_0^t [2-(4+2\kappa)Y(s)]\mathrm{d}s, \qquad t \geq 0.$$

Recall that $\alpha_\kappa = 1/(4+2\kappa)$, and let $t \geq 2\alpha_\kappa$. We have, on the event $\{T_Y(\alpha_\kappa) \geq t\}$,

$$\inf_{0\leq s\leq 4t} \widehat{B}(s) \leq \widehat{B}\left(4\int_0^t Y(s)(1-Y(s))\mathrm{d}s\right) \leq \alpha_\kappa - t \leq -\frac{t}{2},$$

since $Y(s) \leq \alpha_\kappa \leq 1$ if $0 \leq s \leq t \leq T_Y(\alpha_\kappa)$. As a consequence, for $t \geq 2\alpha_\kappa$,

$$\mathbb{P}[T_Y(\alpha_\kappa) > t] \leq \mathbb{P}\left(\inf_{0\leq s\leq 4t} \widehat{B}(s) \leq -\frac{t}{2}\right) = \mathbb{P}\left(\big|\widehat{B}(4t)\big| \geq \frac{t}{2}\right) \leq 2\exp\left(-\frac{t}{32}\right). \tag{6.22}$$

In particular, $\mathbb{P}[T_Y(\alpha_\kappa) > 64\log r] \leq 2r^{-2}$ for large r. Plug this into (6.21), let $c_{55} > c_{53}$ and define $\underline{\varphi} = \underline{\varphi}(r) := \kappa r/\lambda - c_{55} r^{1/2+\delta_3}$ and $\overline{\varphi} = \overline{\varphi}(r) := \kappa r/\lambda + c_{53} r^{1/2+\delta_3}$. This gives,

$$\mathbb{P}\left\{U_Y(\underline{\varphi}) \leq U_Y[\varphi(r) - T_Y(\alpha_\kappa)] \leq U_Y(\overline{\varphi})\right\} \geq 1 - c_{56} r^{-2}$$

for large r. By Lemma 6.2, for small $\delta_1 > 0$ and all large r,

$$\mathbb{P}\left\{\tau_\beta\left[\big(1-(\underline{\varphi})^{-\delta_1}\big)\lambda\underline{\varphi}\right] \leq U_Y[\varphi(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta\left[\big(1+(\overline{\varphi})^{-\delta_1}\big)\lambda\overline{\varphi}\right]\right\} \geq 1 - r^{-c_{57}}.$$

We choose δ_1 such that $\delta_1 < 1/2 - \delta_3$. Then for large r, we have $\big(1-(\underline{\varphi})^{-\delta_1}\big)\lambda\underline{\varphi} \geq \big[1 - 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}\big]\kappa r = \lambda t_-(r)$, and $\big(1+(\overline{\varphi})^{-\delta_1}\big)\lambda\overline{\varphi} \leq \big[1 + 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}\big]\kappa r = \lambda t_+(r)$ (see (2.12)). Thus,

$$\mathbb{P}\left\{\tau_\beta[\lambda t_-(r)] \leq U_Y[\varphi(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta[\lambda t_+(r)]\right\} \geq 1 - r^{-c_{57}}. \tag{6.23}$$

With $\widehat{L}_\pm(r) = 4\big[\sup_{s\in[0,\tau_\beta(\lambda t_\pm(r))]} \kappa\beta(s)\big]^{1/\kappa}$ (see (2.14)), (6.23) and (6.19) give for large r,

$$\mathbb{P}\left\{\widehat{L}_-(r) \leq \widetilde{L}_0(r) \leq \widehat{L}_+(r)\right\} \geq 1 - r^{-c_{57}}. \tag{6.24}$$

By (3.1), $\mathbb{P}\big\{\widehat{L}_-(r) > r^{(1-\delta_1)/\kappa}\big\} \geq 1 - r^{-1}$ for large r. Applying (6.20) and (6.24), this yields

$$\mathbb{P}\left\{(1-\varepsilon)r^{(1-\delta_1)/\kappa} < (1-\varepsilon)\widehat{L}_-(r) \leq L_0(r) \leq (1+\varepsilon)\widehat{L}_+(r)\right\} \geq 1 - r^{-c_{58}}.$$

Recall that $\mathbb{P}[T_Y(\alpha_\kappa) > 64\log r] \leq 2r^{-2}$ for large r, which together with (6.21) gives $\mathbb{P}(E_9^c) \leq (c_{54}+2)r^{-2}$. In view of (6.9) and (6.10), for large r,

$$\mathbb{P}\left\{(1-\varepsilon)r^{(1-\delta_1)/\kappa} < (1-\varepsilon)\widehat{L}_-(r) \leq L_X^+[H(F(r))] \leq (1+\varepsilon)\widehat{L}_+(r)\right\} \geq 1 - r^{-c_{59}}. \tag{6.25}$$

On the other hand, applying Lemma 6.4 to $z = r^{(1-2\delta_1)/\kappa}$ gives $\mathbb{P}[\sup_{x<0} L_X[H(F(r)), x] > r^{(1-2\delta_1)/\kappa}] \leq \mathbb{P}[L_X^{*-}(+\infty) > r^{(1-2\delta_1)/\kappa}] \leq c_{49}/r^{(1-2\delta_1)/(\kappa+2)}$ for large r. This implies

$$\mathbb{P}\left\{(1-\varepsilon)\widehat{L}_-(r) \leq L_X^*[H(F(r))] \leq (1+\varepsilon)\widehat{L}_+(r)\right\} \geq 1 - \frac{1}{r^{c_{59}}} - \frac{c_{49}}{r^{(1-2\delta_1)/(\kappa+2)}},$$

proving the first part of Lemma 2.7. □

Proof of Lemma 2.7: part (ii). In this part, we assume $0 < \kappa \le 1$.

Recall that $H_0(r) = 16\int_0^{\varphi(r)-T_Y(\alpha_\kappa)} \frac{Y[u+T_Y(\alpha_\kappa)]}{(1-Y[u+T_Y(\alpha_\kappa)])^2}\mathrm{d}u$ and $Y[t+T_Y(\alpha_\kappa)] = S^{-1}\{\beta[U_Y(t)]\}$, see (6.8) and (6.11). As in Hu et al. ([33] p. 3923, calculation of Υ_μ), this and again (6.11) lead to:

$$\begin{aligned} H_0(r) &= 4\int_0^{\varphi(r)-T_Y(\alpha_\kappa)} (Y[u+T_Y(\alpha_\kappa)])^2(1-Y[u+T_Y(\alpha_\kappa)])^{2\kappa-1}\mathrm{d}U_Y(u) \\ &= 4\int_0^1 x(1-x)^{\kappa-2}L_\beta[U_Y(\varphi(r)-T_Y(\alpha_\kappa)), S(x)]\mathrm{d}x. \end{aligned} \tag{6.26}$$

Recall that $t_\pm(r) = \left[1 \pm 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}\right]\frac{\kappa}{\lambda}r$, $\beta_v(s) = \beta(v^2 s)/v$ and let J_β be as in (6.13). We have,

$$\begin{aligned} \int_0^1 x(1-x)^{\kappa-2}L_\beta\{\tau_\beta[\lambda t_\pm(r)], S(x)\}\mathrm{d}x &= t_\pm(r)\int_0^1 x(1-x)^{\kappa-2}L_{\beta_{t_\pm(r)}}\{\tau_{\beta_{t_\pm(r)}}(\lambda), \frac{S(x)}{t_\pm(r)}\}\mathrm{d}x \\ &= t_\pm(r)J_{\beta_{t_\pm(r)}}[\kappa, t_\pm(r)]. \end{aligned}$$

By (6.23) and (6.26), we have for large r,

$$\mathbb{P}\left[4t_-(r)J_{\beta_{t_-(r)}}[\kappa, t_-(r)] \le H_0(r) \le 4t_+(r)J_{\beta_{t_+(r)}}[\kappa, t_+(r)]\right] \ge 1 - r^{-c_{57}}.$$

Now, apply Lemma 6.3 to $d = 1/2$. So there exist $c_6 > 0$ and $c_{60} > 0$ such that for large r,

$$\mathbb{P}\left\{(1-\varepsilon)\widehat{I}_-(r) \le H_0(r) \le (1+\varepsilon)\widehat{I}_+(r)\right\} \ge 1 - r^{-c_{60}}, \tag{6.27}$$

where $\widehat{I}_\pm(r)$ is defined in (2.16).

In the case $0 < \kappa < 1$, we know that $\mathbb{P}(E_9^c) \le (c_{54}+2)r^{-2}$ for large r as proved before (6.25), so by (6.10), $\mathbb{P}\big[\overline{H}(r) \le c_{61}\log r\big] \ge \mathbb{P}(E_9) \ge 1 - (c_{54}+2)r^{-2}$ for some c_{61} and all large r. On the other hand, by Lemma 6.4, $\mathbb{P}[H_-(F(r)) \le \varepsilon r] \ge \mathbb{P}[H_-(+\infty) \le \varepsilon r] \ge 1 - c_{62}r^{-(1-\delta_1)\kappa/(\kappa+2)}$, for all large r. Consequently, by (6.27) and (6.9), for large r,

$$\mathbb{P}\big\{(1-\varepsilon)\widehat{I}_-(r) \le H(F(r)) \le (1+\varepsilon)\widehat{I}_+(r) + (4\varepsilon\lambda/\kappa)t_+(r)\big\} \ge 1 - r^{-c_{63}}.$$

Changing the value of c_6, this proves Lemma 2.7 (ii) in the case $0 < \kappa < 1$.

Now we consider the case $\kappa = 1$. As before, $\mathbb{P}[H_-(F(r)) + \overline{H}(r) \le 2\varepsilon r] \ge 1 - r^{-c_{64}}$ (for large r). Moreover, $\mathbb{P}\big[C_{\beta_{t_\pm(r)}} > -\pi\log r\big] \ge 1 - r^{-2}$ by Fact 2.5 and (4.11). Therefore, (2.16) gives $\mathbb{P}\big\{\widehat{I}_+(r) \ge 16t_+(r)\log r\big\} \ge 1 - r^{-2}$. Consequently, for large r,

$$\mathbb{P}\big[0 \le H_-(F(r)) + \overline{H}(r) \le \varepsilon\widehat{I}_+(r)\big] \ge 1 - r^{-c_{65}},$$

which, in view of (6.27), yields that, for large r,

$$\mathbb{P}\big[(1-\varepsilon)\widehat{I}_-(r) \le H(F(r)) \le (1+2\varepsilon)\widehat{I}_+(r)\big] \ge 1 - r^{-c_{66}}.$$

This proves Lemma 2.7 (ii) in the case $\kappa = 1$. □

7. Proof of lemmas 6.1–6.3

This section is devoted to the proof of Lemmas 6.1–6.3. For the sake of clarity, the proofs of these lemmas are presented in separated subsections.

7.1. **Proof of Lemma 6.1.** First, notice that we can not apply Talet ([51], Lem. 3.2 eq. (3.4)) since her constant c_3 depends on her (fixed) δ, whereas we would like to take her $\delta = (\log t)^{\delta_3 - 1/2} \to_{t\to+\infty} 0$, which is necessary for example for our Lemma 5.1. A similar remark applies for Talet ([51], Prop. 5.1) and our Lemma 6.2. So we need different estimates than in her paper.

Let $d > 4$ and $R_0 \stackrel{\mathcal{L}}{=} \widetilde{R}_{d-2}(1)$, where $\widetilde{R}_{d-2}$ is a $(d-2)$–dimensional Bessel process. We consider a d–dimensional Bessel process R, starting from R_0. We introduce $\theta(t) := \int_0^t R^{-2}(s)\mathrm{d}s$. Itô's formula gives $\log R(t) = \log R_0 + M(t) + \frac{d-2}{2}\theta(t)$, where $M(t) := \int_0^t R(s)^{-1}\mathrm{d}\widehat{\beta}(s)$ and $\big(\widehat{\beta}(t),\ t \geq 0\big)$ is a Brownian motion. By the Dubins–Schwarz theorem, there exists a Brownian motion $\big(\widetilde{\beta}(t),\ t \geq 0\big)$ such that $M(t) = \widetilde{\beta}(\theta(t))$ for all $t \geq 0$. Accordingly,

$$(d-2)\theta(t)/2 = \log R(t) - \log R_0 - \widetilde{\beta}(\theta(t)), \qquad t \geq 0. \tag{7.1}$$

Let $\delta_3 \in (0, \frac{1}{2})$, $0 < \varepsilon < 1$, and $x = x(t) := \frac{d-2}{6}\frac{1}{(\log t)^{(1/2)-\delta_3}}$. We have (see e.g. Göing-Jaeschke et al. [28], eq. (50)), $\mathbb{P}\big(R_0^2 \in \mathrm{d}u\big) = u^{d/2-2}e^{-u/2}\mathbf{1}_{(0,\infty)}(u)/\big[\Gamma(d/2-1)2^{d/2-1}\big]$. So for large t,

$$\mathbb{P}\left(\left|\frac{\log R_0}{\log t}\right| > x\right) = \mathbb{P}\left(\frac{\log R_0}{\log t} > x\right) + \mathbb{P}\left(\frac{\log R_0}{\log t} < -x\right) \leq \exp\left(-(1-\varepsilon)\frac{t^{2x}}{2}\right) + \frac{c_{67}}{t^{x(d/2-1)}}. \tag{7.2}$$

Denote by $n := \lceil d \rceil$ the smallest integer such that $n \geq d$. Since an n-dimensional Bessel process can be realized as the Euclidean modulus of an $\mathbb{R}^n$-valued Brownian motion, it follows from the triangular inequality that $R(t) \leq_{\mathcal{L}} R_0 + \widehat{R}_n(t)$, where $(\widehat{R}_n(t),\ t \geq 0)$ is an n-dimensional Bessel process starting from 0. Consequently, for large t, $\mathbb{P}\big(R(t) > t^{(1/2)+x}\big) \leq \mathbb{P}\big(\widehat{R}_n(t) > t^{(1/2)+x}/2\big) + \mathbb{P}\big(R_0 > t^x\big) \leq 2\exp(-(1-\varepsilon)t^{2x}/8)$, and $\mathbb{P}\big(R(t) < t^{(1/2)-x}\big) \leq 2t^{-x}$, e.g. since $\big|\widetilde{\beta}(t)\big| \leq_{\mathcal{L}} R(t)$. Therefore, for large t,

$$\mathbb{P}\left(\left|\frac{\log R(t)}{\log t} - \frac{1}{2}\right| > x\right) \leq 2\exp\left(-(1-\varepsilon)\frac{t^{2x}}{8}\right) + 2t^{-x}. \tag{7.3}$$

Define $E_{12} := \Big\{\Big|\frac{\log R(t)}{\log t} - \frac{1}{2}\Big| \leq x\Big\} \cap \Big\{\Big|\frac{\log R_0}{\log t}\Big| \leq x\Big\}$ and

$$E_{13} := \left\{\frac{d-2}{2}\theta(t) < \log t\right\}, \qquad E_{14} := \left\{\sup_{0\leq s\leq 2(\log t)/(d-2)} \big|\widetilde{\beta}(s)\big| \leq x\log t\right\}.$$

By (7.2) and (7.3), we have for large t,

$$\mathbb{P}(E_{12}^c) \leq 3\exp\big[-(1-\varepsilon)t^{2x}/8\big] + 3t^{-x}. \tag{7.4}$$

We now estimate $\mathbb{P}(E_{12} \cap E_{13}^c)$. We first observe that on E_{12}, we have, by (7.1),

$$\big|\widetilde{\beta}(\theta(t)) + (d-2)\theta(t)/2 - (\log t)/2\big| \leq 2x\log t.$$

We claim that $E_{12} \cap E_{13}^c \subset \big\{\big|\widetilde{\beta}(\theta(t))\big| > \frac{d-2}{6}\theta(t)\big\}$ for large t. Indeed, on the event $E_{12} \cap E_{13}^c \cap \big\{\big|\widetilde{\beta}(\theta(t))\big| \leq \frac{d-2}{6}\theta(t)\big\}$,

$$(d-2)\theta(t)/2 \leq (2x+1/2)\log t - \widetilde{\beta}(\theta(t)) \leq (2x+1/2)\log t + (d-2)\theta(t)/6,$$

which implies $\frac{d-2}{2}\theta(t) \le (\frac{3}{4}+3x)\log t$. This, for large t, contradicts $\frac{d-2}{2}\theta(t) \ge \log t$ on E_{13}^c. Therefore, $E_{12}\cap E_{13}^c \subset \big\{\big|\widetilde{\beta}(\theta(t))\big| > \frac{d-2}{6}\theta(t)\big\}$ holds for all large t, from which it follows that

$$\begin{aligned}\mathbb{P}\big(E_{12}\cap E_{13}^c\big) &\le \mathbb{P}\bigg(\sup_{s\ge 2(\log t)/(d-2)} \frac{\big|\widetilde{\beta}(s)\big|}{s} > \frac{d-2}{6}\bigg) = \mathbb{P}\bigg(\sup_{u\ge 1}\frac{\big|\widetilde{\beta}(u)\big|}{u} > \sqrt{\frac{(d-2)\log t}{18}}\bigg)\\ &= \mathbb{P}\bigg(\sup_{0\le v\le 1}\big|\widetilde{\beta}(v)\big| > \sqrt{(d-2)(\log t)/18}\bigg) \le 4\exp\big[-(d-2)(\log t)/36\big],\end{aligned}$$

because $u\mapsto u\widetilde{\beta}(1/u)$ is a Brownian motion and $\sup_{0\le v\le 1}\widetilde{\beta}(v) \overset{\mathcal{L}}{=} \big|\widetilde{\beta}(1)\big|$. Since $\mathbb{P}\big(E_{14}^c\big) \le 4\exp\big[-\frac{d-2}{4}x^2\log t\big]$ for large t, this and (7.4) give for large t,

$$\mathbb{P}\big(E_{12}^c\cup E_{13}^c\cup E_{14}^c\big) \le \mathbb{P}\big(E_{12}^c\big)+\mathbb{P}\big(E_{12}\cap E_{13}^c\big)+P\big(E_{12}\cap E_{13}\cap E_{14}^c\big) \le \exp\big(-c_{68}\,x^2\log t\big).$$

Since $E_{12}\cap E_{13}\cap E_{14} \subset \big\{\big|\frac{\theta(t)}{\log t}-\frac{1}{d-2}\big| \le \frac{6x}{d-2}\big\}$ by (7.1), this completes the proof of Lemma 6.1.□

7.2. **Proof of Lemma 6.2.** Let $v>0$. Recall that for every $x\ge 0$, $\beta_v(x)=(1/v)\beta(v^2x)$, and notice that $v^2\tau_{\beta_v}(x)=\tau_\beta(xv)$ almost surely. Then,

$$E_{10} = \big\{\tau_{\beta_v}\big[\big(1-v^{-\delta_1}\big)\lambda\big] \le U_Y(v)/v^2 \le \tau_{\beta_v}\big[\big(1+v^{-\delta_1}\big)\lambda\big]\big\}. \tag{7.5}$$

For $\delta_1>0$, define $E_{15} := \big\{\sup_{0\le s\le\tau_{\beta_v}(2\lambda)}|\varepsilon_1(v,s)| < v^{-\delta_1}\big\}$, where

$$\varepsilon_1=\varepsilon_1(v,s) := \frac{1}{4}\int_0^1 (1-x)^\kappa\bigg[L_{\beta_v}\bigg(s,\frac{S(x)}{v}\bigg)-L_{\beta_v}(s,0)\bigg]\,\mathrm{d}x, \qquad s\ge 0.$$

By Hu et al. ([33] eq. (2.34) p. 3924), $E_{15}\subset E_{10}$. Thus it remains to prove that for δ_1 small enough, $\mathbb{P}(E_{15}^c)\le 1/v^{1/4-5\delta_1}$ for large v. Notice that for $s\ge 0$,

$$\begin{aligned}|\varepsilon_1| &\le \bigg(\int_{\{S(x)>\sqrt{v}\}}+\int_{\{S(x)<-\sqrt{v}\}}+\int_{\{|S(x)|\le\sqrt{v}\}}\bigg)\frac{(1-x)^\kappa}{4}\bigg|L_{\beta_v}\bigg(s,\frac{S(x)}{v}\bigg)-L_{\beta_v}(s,0)\bigg|\,\mathrm{d}x\\ &=: \varepsilon_2(v,s)+\varepsilon_3(v,s)+\varepsilon_4(v,s).\end{aligned} \tag{7.6}$$

Since $S(y)=\int_{\alpha_\kappa}^y \frac{\mathrm{d}x}{x(1-x)^{1+\kappa}}$, we have $1-S^{-1}(u) \underset{u\to+\infty}{\sim} (\kappa u)^{-1/\kappa}$. So, we have

$$\begin{aligned}\sup_{0\le s\le\tau_{\beta_v}(2\lambda)}\varepsilon_2(v,s) &\le \frac{1}{4}\int_{1-\left(\frac{2}{\kappa\sqrt{v}}\right)^{1/\kappa}}^1 (1-x)^\kappa \sup_{0\le s\le\tau_{\beta_v}(2\lambda)}\sup_{u\ge 0}\big[L_{\beta_v}(s,u)+L_{\beta_v}(s,0)\big]\,\mathrm{d}x\\ &\le \big[2/(\kappa\sqrt{v})\big]^{\frac{1}{\kappa}+1}\sup_{u\ge 0}\big[L_{\beta_v}(\tau_{\beta_v}(2\lambda),u)+2\lambda\big],\end{aligned}$$

for all large v. By the second Ray–Knight theorem (Fact 2.2), $Q:=(L_{\beta_v}(\tau_{\beta_v}(2\lambda),u),\ u\ge 0)$ is a 0–dimensional squared Bessel process starting from 2λ. Moreover, $x\mapsto x$ is a scale function of Q (see e.g. Revuz et al. [41] p. 442). Hence, for large v,

$$\mathbb{P}\bigg(\sup_{0\le s\le\tau_{\beta_v}(2\lambda)}\varepsilon_2(v,s)\ge\big[2/(\kappa\sqrt{v})\big]^{1/\kappa+1}\sqrt{v}\bigg) \le \mathbb{P}\bigg(\sup_{u\ge 0}Q(u)\ge\frac{\sqrt{v}}{2}\bigg)=\frac{4\lambda}{\sqrt{v}}. \tag{7.7}$$

Similarly (this time, using $S(x)\sim\log x$, $x\to 0$), we have, for large v,

$$\mathbb{P}\Big[\sup_{0\le s\le\tau_{\beta_v}(2\lambda)}\varepsilon_3(v,s)\ge\exp(-\sqrt{v}/2)\sqrt{v}\Big]\le 4\lambda/\sqrt{v}. \tag{7.8}$$

To estimate $\varepsilon_4(v,s)$, we note that

$$\varepsilon_4(v,s)\le\sup_{|u|\le 1/\sqrt{v}}\big|L_{\beta_v}(s,u)-L_{\beta_v}(s,0)\big|. \tag{7.9}$$

Let $\varepsilon\in(0,1/2)$, $t_v>0$, $\gamma\ge 1$ and define $(M)_t^*:=\sup_{0\le s\le t}|M(s)|$ for $t>0$ and any Brownian motion $(M(s),\ s\ge 0)$. Applying Barlow and Yor ([6], (ii) p. 199) to the continuous martingale

$\beta_v(.\wedge t_v)$ and its jointly continuous local time $(L_{\beta_v}(u\wedge t_v,a),\ u\geq 0,\ a\in\mathbb{R})$, we see that for some constant $C_{\gamma,\varepsilon}>0$,

$$\left\|\sup_{0\leq s\leq t_v, a\neq b}\frac{\left|L_{\beta_v}(s,b)-L_{\beta_v}(s,a)\right|}{|b-a|^{1/2-\varepsilon}}\right\|_\gamma \leq C_{\gamma,\varepsilon}\left\|[(\beta_v)^*_{t_v}]^{1/2+\varepsilon}\right\|_\gamma = C_{\gamma,\varepsilon}\left\|[(\beta)^*_{t_v}]^{1/2+\varepsilon}\right\|_\gamma,$$

where $\|.\|_\gamma=\mathbb{E}(|.|^\gamma)^{1/\gamma}$. Then, by Chebyshev's inequality and a change of scale, for $\alpha>0$,

$$\mathbb{P}\left(\sup_{0\leq s\leq t_v,\, a\neq b}\frac{\left|L_{\beta_v}(s,b)-L_{\beta_v}(s,a)\right|}{|b-a|^{1/2-\varepsilon}}\geq\alpha\right)\leq\frac{(\sqrt{t_v})^{(1/2+\varepsilon)\gamma}}{\alpha^\gamma}\left[C_{\gamma,\varepsilon}\left\|[(\beta)^*_1]^{1/2+\varepsilon}\right\|_\gamma\right]^\gamma. \tag{7.10}$$

On $E_{16}:=\left\{\sup_{0\leq s\leq\tau_{\beta_v}(2\lambda),a\neq b}\frac{\left|L_{\beta_v}(s,b)-L_{\beta_v}(s,a)\right|}{|b-a|^{1/2-\varepsilon}}\leq v^{\frac12(\frac12-2\varepsilon)}\right\}$, we have by (7.9),

$$\sup_{0\leq s\leq\tau_{\beta_v}(2\lambda)}\varepsilon_4(v,s)\ \leq\ v^{-\frac12(\frac12-\varepsilon)}v^{\frac12(\frac12-2\varepsilon)}=v^{-\varepsilon/2}. \tag{7.11}$$

We now choose $\gamma:=2$ and $t_v:=v^{\frac{1/4-\varepsilon}{1/2+\varepsilon}}$. Since $\mathbb{P}[\tau_{\beta_v}(2\lambda)>t_v]=\mathbb{P}[L_{\beta_v}(t_v,0)<2\lambda]=\mathbb{P}\big[\sup_{0\leq s\leq t_v}\beta(s)<2\lambda\big]=\mathbb{P}[|\beta(t_v)|<2\lambda]\leq 4\lambda/\sqrt{t_v}$ by Lévy's theorem (see e.g. Revuz et al. [41] VI th. 2.3), we get for all large v (if ε is small enough).

$$\begin{aligned}\mathbb{P}\big[E_{16}(v)^c\big]\ &\leq\ \mathbb{P}\big[\tau_{\beta_v}(2\lambda)>t_v\big]+\mathbb{P}\left(\sup_{0\leq s\leq t_v,a\neq b}\frac{|L_{\beta_v}(s,b)-L_{\beta_v}(s,a)|}{|b-a|^{1/2-\varepsilon}}\geq v^{\frac12(\frac12-2\varepsilon)}\right)\\ &\leq\ 4\lambda\, v^{\frac{\varepsilon-1/4}{1+2\varepsilon}}+c_{69}v^{-1/4+\varepsilon}\leq v^{-1/4+2\varepsilon}/2.\end{aligned}$$

Combining this with (7.6), (7.7), (7.8) and (7.11), we obtain that, for $\varepsilon>0$ small enough,

$$\mathbb{P}\left(\sup_{0\leq s\leq\tau_{\beta_v}(2\lambda)}|\varepsilon_1(v,s)|\geq 2v^{-\varepsilon/2}\right)\leq v^{-1/4+2\varepsilon}.$$

This gives, with the choice of $\delta_1:=2\varepsilon/5$, $\mathbb{P}(E_{10}^c)\leq\mathbb{P}(E_{15}^c)\leq v^{-1/4+5\delta_1}$ (for large v). □

7.3. **Proof of Lemma 6.3.** Assume $0<\kappa\leq 1$. Consider $0<d<1$, $\varepsilon\in(0,1/2)$ such that $d(1/2+\varepsilon)+(\varepsilon-1)(1/2-\varepsilon)<0$, $M_\varepsilon>0$, and a Brownian motion $(\beta(t),\ t\geq 0)$. We can write for $t>0$,

$$\begin{aligned}J_\beta(\kappa,t)\ &=\ \left(\int_0^{S^{-1}(-t^\varepsilon)}+\int_{S^{-1}(-t^\varepsilon)}^{\alpha_\kappa}+\int_{\alpha_\kappa}^{S^{-1}(M_\varepsilon)}+\int_{S^{-1}(M_\varepsilon)}^1\right)y(1-y)^{\kappa-2}L_\beta\left(\tau_\beta(\lambda),\frac{S(y)}{t}\right)\mathrm{d}y\\ &:=\ J_1+J_2+J_3+J_4.\end{aligned}$$

We begin by estimating J_1. Since $S(x)\sim_{x\to 0}\log x$, we have $J_1\leq\exp(-t^\varepsilon/2)\sup_{s\geq 0}\tilde{Q}(s)$ for large t, where $\tilde{Q}$ is a 0–dimensional squared Bessel process starting from λ (by the second Ray–Knight theorem stated in Fact 2.2, applied to $-\beta$). Hence, we get $\mathbb{P}\left[J_1\geq\exp(-t^\varepsilon/2)t^d\right]\leq\lambda/t^d$. Fix a large constant $\gamma>0$ such that $d(1/2+\varepsilon+1/\gamma)+(\varepsilon-1)(1/2-\varepsilon)<0$, and define

$$E_{17}:=\left\{\tau_\beta(\lambda)\leq t^{2d}\right\},\qquad E_{18}:=\left\{\sup_{0\leq s\leq t^{2d},\,a\neq b}\frac{|L_\beta(s,b)-L_\beta(s,a)|}{|b-a|^{1/2-\varepsilon}}\leq t^{d(1/2+\varepsilon+1/\gamma)}\right\}.$$

Recall that $S(\alpha_\kappa)=0$. To estimate J_2, we note that, on $E_{17}\cap E_{18}$, uniformly for all large t,

$$J_2\leq\left[\int_0^{\alpha_k}\frac{y\mathrm{d}y}{(1-y)^{2-\kappa}}\right]\sup_{-t^{\varepsilon-1}\leq b\leq 0}L_\beta(\tau_\beta(\lambda),b)\leq\frac{\alpha_\kappa\left[\lambda+t^{d(1/2+\varepsilon+1/\gamma)}(t^{\varepsilon-1})^{\frac12-\varepsilon}\right]}{(1-\alpha_k)^{2-\kappa}}\leq\frac{2\alpha_\kappa\lambda}{(1-\alpha_k)^{2-\kappa}}.$$

Notice that $\mathbb{P}(E_{17}^c) \le 2\lambda t^{-d}$ as proved after (7.11), and that $\mathbb{P}(E_{18}^c) \le c_{70} t^{-d}$ (by (7.10) with t^{2d} instead of t_v). Therefore, there exists $c_{71} > 0$ such that for large t,

$$\mathbb{P}\left(J_2 \le c_{71}\right) \ge \mathbb{P}(E_{17} \cap E_{18}) \ge 1 - c_{72} t^{-d}. \tag{7.12}$$

We now turn to J_3. As already noticed after (7.6), we have $1 - S^{-1}(u) \underset{u\to+\infty}{\sim} (\kappa u)^{-1/\kappa}$. Therefore, we can choose $M_\varepsilon > 0$ such that

$$\forall u \ge M_\varepsilon, \qquad \frac{[1 - S^{-1}(u)]^{2\kappa-1}}{(\kappa u)^{1/\kappa-2}} \in (1-\varepsilon, 1+\varepsilon) \quad \text{and} \quad S^{-1}(u) \ge 1 - \varepsilon. \tag{7.13}$$

On the event $E_{17} \cap E_{18}$, uniformly for all large t,

$$\begin{aligned} J_3 &\le \sup_{0\le x\le M_\varepsilon/t} L_\beta(\tau_\beta(\lambda), x) \int_{\alpha_\kappa}^{S^{-1}(M_\varepsilon)} y(1-y)^{\kappa-2} \mathrm{d}y \\ &\le c_{73}\left[\lambda + t^{d(1/2+\varepsilon+1/\gamma)} (M_\varepsilon/t)^{\frac{1}{2}-\varepsilon}\right] \le 2\lambda c_{73}. \end{aligned}$$

Consequently, $\mathbb{P}[J_3 \le 2\lambda c_{73}] \ge \mathbb{P}(E_{17} \cap E_{18}) \ge 1 - c_{72} t^{-d}$ for large t.

Now we write

$$J_4 = \kappa^{1/\kappa-2}\, t^{1/\kappa-1} \int_{M_\varepsilon/t}^{+\infty} \left[S^{-1}(tx)\right]^2 \frac{\left[1 - S^{-1}(tx)\right]^{2\kappa-1}}{(\kappa t)^{1/\kappa-2}} L_\beta(\tau_\beta(\lambda), x) \mathrm{d}x.$$

Therefore, (7.13) leads to

$$(1-\varepsilon)^3 \int_{M_\varepsilon/t}^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x) \mathrm{d}x \le \kappa^{2-1/\kappa} t^{1-1/\kappa} J_4 \le (1+\varepsilon) \int_{M_\varepsilon/t}^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x) \mathrm{d}x. \tag{7.14}$$

Proof of Lemma 6.3: part (i). We first assume $0 < \kappa < 1$.

On $E_{17} \cap E_{18}$, for large t, we have $\int_0^{M_\varepsilon/t} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x) \mathrm{d}x \le c_{74} t^{1-1/\kappa}$. Recall K_β from (2.2). It follows from (7.14) and (7.12) that, for large t,

$$\mathbb{P}\left[(1-\varepsilon)^3 K_\beta(\kappa) - (1-\varepsilon)^3 c_{74} t^{1-1/\kappa} \le \kappa^{2-1/\kappa} t^{1-1/\kappa} J_4 \le (1+\varepsilon) K_\beta(\kappa)\right] \ge 1 - c_{72} t^{-d}.$$

Since $J_\beta(\kappa, t) = J_1 + J_2 + J_3 + J_4$, we get for large t,

$$\mathbb{P}\left\{(1-\varepsilon)^3 K_\beta(\kappa) - c_{48} t^{1-1/\kappa} \le \kappa^{2-1/\kappa} t^{1-1/\kappa} J_\beta(\kappa, t) \le (1+\varepsilon) K_\beta(\kappa) + c_{48} t^{1-1/\kappa}\right\} \ge 1 - c_{75} t^{-d},$$

for some $c_{48} > 0$, proving the lemma in the case $0 < \kappa < 1$. □

Proof of Lemma 6.3: part (ii). We assume $\kappa = 1$, thus $\lambda = 8$.

By the definition of C_β (see (2.3)), we have

$$\int_{M_\varepsilon/t}^{\infty} \frac{L_\beta(\tau_\beta(8), x)}{x} \mathrm{d}x = C_\beta - \int_0^{M_\varepsilon/t} \frac{L_\beta(\tau_\beta(8), x) - 8}{x} \mathrm{d}x + 8\log t - 8\log M_\varepsilon.$$

On $E_{17} \cap E_{18}$, for large t,

$$\int_0^{M_e/t} \frac{|L_\beta(\tau_\beta(8), x) - 8|}{x} \mathrm{d}x \le \int_0^{M_e/t} \frac{t^{d(1/2+\varepsilon+1/\gamma)} x^{1/2-\varepsilon}}{x} \mathrm{d}x \le \varepsilon.$$

As in (4.11), $\mathbb{P}(C_\beta + 8\log t < \log t) \le t^{-7}$. Therefore, by (7.14) and (7.12), we have for large t,

$$\mathbb{P}\left\{(1-\varepsilon)^4 [C_\beta + 8\log t] \le J_4 \le (1+\varepsilon)^2 [C_\beta + 8\log t]\right\} \ge 1 - c_{76} t^{-d}.$$

Since $J_\beta(1,t) = J_1 + J_2 + J_3 + J_4$, we get for large t,

$$\mathbb{P}\left\{(1-\varepsilon)^4[C_\beta + 8\log t] \le J_\beta(1,t) \le (1+\varepsilon)^3[C_\beta + 8\log t]\right\} \ge 1 - c_{77}t^{-d}.$$

This proves the lemma in the case $\kappa = 1$. □

Acknowledgements

I would like to thank Zhan Shi for many helpful discussions. I am also grateful to an anonymous referee for a very careful reading of the first version of the paper and for valuable comments which helped improve the presentation of the paper, including a recommendation to merge the first version of this paper with [19].

References

[1] Andreoletti, P.: Almost sure estimates for the concentration neighborhood of Sinai's walk. *Stoch. Proc. Appl.* **117** (2007), 1473–1490.

[2] Andreoletti, P.: On the concentration of Sinai's walk. *Stoch. Proc. Appl.* **116** (2007), 1377–1408.

[3] Andreoletti, P. and Devulder A.: Localization and number of visited valleys for a transient diffusion in random environment. *Electron. J. Probab.* **20** (2015), 1–58.

[4] Andreoletti, P., Devulder A. and Vechambre G.: Renewal structure and local time for diffusions in random environment. Preprint, arXiv:1506.02895 (2015).

[5] Andreoletti, P. and Diel R.: Limit law of the local time for Brox's diffusion. *J. Theoretical. Probab.* **24** (2011), 634–656.

[6] Barlow, M. T. and Yor, M.: Semimartingale inequalities via the Garsia-Rodemich-Rumsey lemma, and applications to local times. *J. Funct. Anal.* **49** (1982), 198–229.

[7] Bertoin, J.: *Lévy Processes.* Cambridge University Press, Cambridge, 1996.

[8] Biane, Ph. and Yor, M.: Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math.* **111** (1987), 23–101.

[9] Bingham, N. H. and Teugels, J. L.: Duality for regularly varying functions. *Quart. J. Math. Oxford Ser. (2)* **26** (1975), 333–353.

[10] Borodin, A. N. and Salminen, P.: *Handbook of Brownian Motion—Facts and Formulae.* Second edition, Birkhäuser, Boston, 2002.

[11] Brox, Th.: A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.* **14** (1986), 1206–1218.

[12] Carmona, P.: The mean velocity of a Brownian motion in a random Lévy potential. *Ann. Probab.* **25** (1997), 1774–1788.

[13] Cheliotis, D.: One-dimensional diffusion in an asymmetric random environment. *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2006), 715–726.

[14] Cheliotis, D.: Localization of favorite points for diffusion in a random environment. *Stoch. Proc. Appl.* **118** (2008), 1159–1189.

[15] Comets, F., Falconnet, M., Loukianov, O., Loukianova, D. and Matias, C.: Maximum likelihood estimator consistency for a ballistic random walk in a parametric random environment. *Stoch. Proc. Appl.* **124** (2014), 268–288.

[16] Deheuvels, P. and Révész, P.: Simple random walk on the line in random environment. *Probab. Theory Related Fields* **72** (1986), 215–230.

[17] Dembo, A., Gantert, N., Peres, Y. and Shi, Z.: Valleys and the maximum local time for random walk in random environment. *Probab. Theory Related Fields* **137** (2007), 443–473.

[18] Dembo, A. and Zeitouni, O.: *Large Deviations Techniques and Applications.* Second edition, Springer-Verlag, New York, 1998.

[19] Devulder, A.: Almost sure asymptotics for a diffusion process in a drifted Brownian potential. Preprint arXiv:math/0511053 (2005).

[20] Devulder, A.: Some properties of the rate function of quenched large deviations for random walk in random environment. *Markov Processes. Related Fields* **12** (2006), 27–42.

[21] Devulder, A.: Persistence of some additive functionals of Sinai's walk. Preprint arXiv:1402.2267 (2014), to appear in *Ann. Inst. H. Poincaré Probab. Stat.*

[22] Diel, R. Almost sure asymptotics for the local time of a diffusion in Brownian environment. *Stoch. Proc. Appl.* **121** (2011), 2303–2330.

[23] Diel, R, Voisin, G. Local time of a diffusion in a stable Lévy environment. *Stochastics* **83** (2011), 127–152.

[24] Dufresne, D.: Laguerre series for Asian and other options. *Math. Finance* **10** (2000), 407–428.

[25] Faraud, G.: Estimates on the speedup and slowdown for a diffusion in a drifted brownian potential. *J. Theoret. Probab.* **24** (2009), 194–239.

[26] Gantert, N., Peres, Y. and Shi, Z.: The infinite valley for a recurrent random walk in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010), 525–536.

[27] Gantert, N. and Shi, Z.: Many visits to a single site by a transient random walk in random environment. *Stoch. Proc. Appl.* **99** (2002), 159–176.

[28] Göing-Jaeschke, A. and Yor, M.: A survey and some generalizations of Bessel processes. *Bernoulli* **9** (2003), 313–349.

[29] Hu, Y. and Shi, Z.: The local time of simple random walk in random environment. *J. Theoretical. Probab.* **11** (1998), 765–793.

[30] Hu, Y. and Shi, Z.: The limits of Sinai's simple random walk in random environment. *Ann. Probab.* **26** (1998), 1477–1521.

[31] Hu, Y. and Shi, Z.: The problem of the most visited site in random environment. *Probab. Theory Related Fields* **116** (2000), 273–302.

[32] Hu, Y. and Shi, Z.: Moderate deviations for diffusions with Brownian potential. *Ann. Probab.* **32** (2004), 3191–3220.

[33] Hu, Y., Shi, Z. and Yor, M.: Rates of convergence of diffusions with drifted Brownian potentials. *Trans. Amer. Math. Soc.* **351** (1999), 3915–3934.

[34] Karlin, S. and Taylor, H. M.: *A Second Course in Stochastic Processes.* Academic Press, New York, 1981.

[35] Kawazu, K. and Tanaka, H.: On the maximum of a diffusion process in a drifted Brownian environment. *Séminaire de Probabilités XXVII.* Lecture Notes in Mathematics **1557**, 78–85, Springer, Berlin, 1993.

[36] Kawazu, K. and Tanaka, H.: A diffusion process in a Brownian environment with drift. *J. Math. Soc. Japan* **49** (1997), 189–211.

[37] Lamperti, J.: Semi-stable Markov processes, I. *Z. Wahrsch. Verw. Gebiete* **22** (1972), 205–225.

[38] Le Doussal, P., Monthus, C. and Fisher, D.: Random walkers in one-dimensional random environments: exact renormalization group analysis. *Phys. Rev. E 3*, **59** (1999), 4795–4840.

[39] Mathieu, P.: On random perturbations of dynamical systems and diffusions with a Brownian potential in dimension one. *Stoch. Proc. Appl.* **77** (1998), 53–67.

[40] Révész, P.: *Random Walk in Random and Non-Random Environments.* World Scientific, Singapore, 1990.

[41] Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion.* Third edition, Springer, Berlin, 1999.

[42] Samorodnitsky, G. and Taqqu, M. S.: *Stable Non-Gaussian Random Processes.* Chapman & Hall, New York, 1994.

[43] Schumacher, S.: Diffusions with random coefficients. *Contemp. Math.* **41** (1985), 351–356.

[44] Shi, Z.: A local time curiosity in random environment. *Stoch. Proc. Appl.* **76** (1998), 231–250.

[45] Shi, Z.: Sinai's walk via stochastic calculus. *Panoramas et Synthèses* (Eds: F.Comets & E. Pardoux) **12** (2001), 53–74, Société Mathématique de France.

[46] Sinai, Ya. G.: The limiting behavior of a one-dimensional random walk in a random medium, (English translation). *Theory Probab. Appl.* **27** (1982), 256–268.

[47] Singh, A.: Limiting behavior of a diffusion in an asymptotically stable environment. *Ann. Inst. H. Poincaré Probab. Statist.* **43** (2007), 101–138.

[48] Singh, A.: Rates of convergence of a transient diffusion in a spectrally negative lévy potential. *Ann. Probab.* **36** (2008), 279–318.

[49] Solomon, F.: Random walks in a random environment. *Ann. Probab.* **3** (1975), 1–31.

[50] Taleb, M.: Large deviations for a Brownian motion in a drifted Brownian potential. *Ann. Probab.* **29** (2001), 1173–1204.

[51] Talet, M.: Annealed tail estimates for a Brownian motion in a drifted Brownian potential. *Ann. Probab.* **35** (2007), 32–67.

[52] Tanaka, H.: Limit distribution for 1-dimensional diffusion in a reflected Brownian medium. *Séminaire de Probabilités XXI.* Lecture Notes in Mathematics **1247**, 246–261, Springer 1987.

[53] Warren, J. and Yor M.: The Brownian burglar: conditioning Brownian motion by its local time process. *Séminaire de Probabilités XXXII.* Lecture Notes in Mathematics **1686**, 328–342, Springer, Berlin, 1998.

[54] Yor M.: Sur certaines fonctionnelles exponentielles du mouvement brownien réel. *J. Appl. Probab.* **29** (1992), 202–208.

[55] Yor M.: *Local Times and Excursions for Brownian Motion: A Concise Introduction.* Lecciones en Mathemàticas, Universidad Central de Venuezuela, **1**, 1995.

[56] Zeitouni, O.: Lecture notes on random walks in random environment. *École d'été de probabilités de Saint-Flour 2001.* Lecture Notes in Math. **1837**, 189–312. Springer, Berlin, 2004.

[57] Zindy, O.: Upper limits of Sinai's walk in random scenery. *Stoch. Proc. Appl.* **118** (2008), 981–1003.

Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 78035 Versailles, France

E-mail address: `devulder@math.uvsq.fr`