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The maximum of the local time of a diffusion process in a drifted Brownian potential

Alexis Devulder *

Abstract

We consider a one-dimensional diffusion process in a drifted Brownian potential. We are interested in the maximum of its local time, and study its almost sure asymptotic behaviour, which is proved to be different from the behaviour of the maximum local time of the transient random walk in random environment.

KEY WORDS: *Random environment, diffusion in a random potential, maximum local time, Lévy class.*

AMS (2000) Classification: 60K37, 60J60, 60J55, 60F15.

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1 Introduction

1.1 Presentation of the model

Let $\kappa \in \mathbb{R}$ and

$$W_\kappa(x) := W(x) - \frac{\kappa}{2}x, \quad x \in \mathbb{R}, \quad (1.1)$$

where $(W(x), x \in \mathbb{R})$ is a standard two-sided Brownian motion. A diffusion process $(X(t), t \geq 0)$ in the random potential W_κ is formally defined by

$$X(t) = \beta(t) - \frac{1}{2} \int_0^t W'_\kappa(X(s)) ds,$$

where $(\beta(t), t \geq 0)$ is a Brownian motion independent of W_κ . The precise meaning of X is a diffusion process starting from 0, and whose conditional generator given W_κ is

$$\frac{1}{2} e^{W_\kappa(x)} \frac{d}{dx} \left(e^{-W_\kappa(x)} \frac{d}{dx} \right).$$

The law of X conditionally on the environment W_κ is denoted by P_ω , and is called the quenched law. The annealed law \mathbb{P} is defined as follows:

$$\mathbb{P}(\cdot) := \int P_\omega(\cdot) \mathbb{P}(W_\kappa \in d\omega).$$

The diffusion X is known to be a continuous time analogue of random walks in random environment (RWRE), which have been used in modeling some phenomena in physics and biology (see e.g. Le Doussal et al. [19]). See Révész [21] and Zeitouni [32] for background and general properties of RWRE. Such diffusions, introduced by Schumacher [24] and Brox [6], have been studied for example by Kawazu and Tanaka [17], Hu et al. [15], Mathieu [20], Carmona [7], Taleb [29] and Devulder [10]. For a relation between RWRE and the diffusion X , see Schumacher [24].

In this paper, we consider the transient case, that is, we assume that $\kappa \neq 0$. Notice that by symmetry we may restrict ourselves to the case $\kappa > 0$. In this case, $\lim_{t \rightarrow +\infty} X(t) = +\infty$ \mathbb{P} -almost surely.

The purpose of this paper is to study the almost sure asymptotics of the supremum L_X^* of the local time of X . See Shi [26] for some results about the upper asymptotics of L_X^* in the recurrent case $\kappa = 0$. Corresponding problems for RWRE have been studied, for example, in Révész ([21], Chapter 29), Gantert and Shi [12], Shi [26], Hu and Shi [13], Dembo et al. [9] and Andreatti [1]. In particular, the study of the maximum local time gives a better understanding of the concentration of the process on its favorite sites.

1.2 Results

We denote by $(L_X(t, x), t \geq 0, x \in \mathbb{R})$ the local time of X , which is the jointly continuous process satisfying, for any positive measurable function f ,

$$\int_0^t f(X(s))ds = \int_{-\infty}^{+\infty} f(x)L_X(t, x)dx, \quad t \geq 0. \quad (1.2)$$

We are interested in the maximum local time of X at time t , defined as

$$L_X^*(t) := \sup_{x \in \mathbb{R}} L_X(t, x), \quad t \geq 0.$$

Let

$$H(r) := \inf\{t \geq 0, \quad X(t) > r\}, \quad r \geq 0 \quad (1.3)$$

be the first hitting time of r by X . We recall that there are three different regimes for H :

Theorem 1.1 (*Kawazu and Tanaka, [17]*) *When r tends to $+\infty$,*

$$\begin{aligned} \frac{H(r)}{r^{1/\kappa}} &\xrightarrow{\mathcal{L}} c_0 S_\kappa^{ca}, & 0 < \kappa < 1, \\ \frac{H(r)}{r \log r} &\xrightarrow{P} 4, & \kappa = 1, \\ \frac{H(r)}{r} &\xrightarrow{a.s.} \frac{4}{\kappa - 1}, & \kappa > 1, \end{aligned}$$

where c_0 is a finite constant depending on κ , the symbols “ $\xrightarrow{\mathcal{L}}$ ”, “ \xrightarrow{P} ” and “ $\xrightarrow{a.s.}$ ” denote respectively convergence in law, in probability and almost sure convergence, with respect to the annealed probability \mathbb{P} . Moreover, S_κ^{ca} is a completely asymmetric stable variable of index κ , and is a positive variable for $0 < \kappa < 1$ (see (3.13) for its characteristic function).

The first set of our results gives a precise description of the almost sure asymptotics of L_X^* along the first hitting times.

Theorem 1.2 *For $\kappa > 0$,*

$$\liminf_{r \rightarrow +\infty} \frac{L_X^*(H(r))}{(r/\log \log r)^{1/\kappa}} = 4 \left(\frac{\kappa^2}{2} \right)^{1/\kappa} \quad \mathbb{P}\text{-a.s.}$$

Theorem 1.3 *Let $\kappa > 0$. For any positive nondecreasing function $a(\cdot)$, we have*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < \infty \\ = +\infty \end{cases} \iff \limsup_{r \rightarrow \infty} \frac{L_X^*(H(r))}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \quad \mathbb{P}\text{-a.s.}$$

If we consider $L_X^*(t)$ instead of $L_X^*(H(r))$, the situation is considerably more complex, and heavily depends on the value of κ . We start with the upper asymptotics of $L_X^*(t)$:

Theorem 1.4 *If $0 < \kappa < 1$, then*

$$\limsup_{t \rightarrow +\infty} \frac{L_X^*(t)}{t} = +\infty \quad \mathbb{P}\text{-a.s.}$$

Theorem 1.4 tells us that in the case $\kappa < 1$, the maximum local time of X has completely different behaviour from the maximum local time of RWRE (the latter is trivially bounded by $t/2$ for any positive integer t , for example). Such a peculiar phenomenon has already been observed by Shi [26] in the recurrent case, and is even more surprising here since X is transient.

Theorem 1.5 gives, in the case $\kappa > 1$, an integral test which completely characterizes the upper functions of $L^*(t)$, in the sense of Paul Lévy.

Theorem 1.5 *Let $a(\cdot)$ be a nondecreasing function. If $\kappa > 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \iff \limsup_{t \rightarrow \infty} \frac{L_X^*(t)}{[ta(t)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \quad \mathbb{P}\text{-a.s.}$$

This is in agreement with a result of Gantert and Shi [12] for RWRE.

We have not been able to settle the very delicate critical case $\kappa = 1$.

We now turn to the lower asymptotics of $L^*(t)$.

Theorem 1.6 *We have*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{L_X^*(t)}{t/\log \log t} &\leq \kappa^2 c_1(\kappa) \quad \mathbb{P}\text{-a.s.} \quad \text{if } 0 < \kappa < 1, \\ \liminf_{t \rightarrow \infty} \frac{L_X^*(t)}{t/[(\log t) \log \log t]} &\leq \frac{1}{2} \quad \mathbb{P}\text{-a.s.} \quad \text{if } \kappa = 1, \\ \liminf_{t \rightarrow \infty} \frac{L_X^*(t)}{(t/\log \log t)^{1/\kappa}} &= 4 \left(\frac{(\kappa - 1)\kappa^2}{8} \right)^{1/\kappa} \quad \mathbb{P}\text{-a.s.} \quad \text{if } \kappa > 1, \end{aligned}$$

where $c_1(\kappa)$ is defined in (5.12).

Theorem 1.7 *We have, for any $\varepsilon > 0$,*

$$\liminf_{t \rightarrow \infty} \frac{L_X^*(t)}{t/[(\log t)^{1/\kappa} (\log \log t)^{(2/\kappa)+\varepsilon}]} = +\infty \quad \mathbb{P}\text{-a.s.} \quad \text{if } 0 < \kappa \leq 1.$$

In the case $0 < \kappa \leq 1$, Theorems 1.6 and 1.7 give different bounds, for technical reasons.

The paper is organized as follows. We give some preliminaries on Bessel processes in Section 2. In Section 3, we present some technical lemmas which will be useful later on; the proof of one of the technical lemmas (Lemma 3.3), is postponed until Section 6. Section 4 is devoted to the study of $L_X^*(H(r))$ and the proofs of Theorems 1.2 and 1.3. In Section 5, we study $L_X^*[H(r)]/H(r)$ and prove Theorems 1.6–1.4. Finally, Section 6 is devoted to the proof of Lemma 3.3.

Throughout the paper, c_i , $0 \leq i \leq 57$, denote unimportant constants that are finite and positive.

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2 Preliminaries on Bessel processes

For any Brownian motion $(B(t), t \geq 0)$ and $r > 0$, we define the first hitting time

$$\sigma_B(r) := \inf\{t > 0, \quad B(t) = r\}.$$

Moreover, we denote by $(L_B(t, x), t \geq 0, x \in \mathbb{R})$ the local time of B , i.e., the jointly continuous process satisfying $\int_0^t f(B(s))ds = \int_{-\infty}^{+\infty} f(x)L_B(t, x)dx$ for any positive measurable function f . We define the inverse local time of B as

$$\tau_B(a) := \inf\{t \geq 0, \quad L_B(t, 0) \geq a\}, \quad a > 0.$$

Furthermore, for any $\delta \geq 0$ and $x \geq 0$, the unique strong solution of the stochastic differential equation $Z(t) = x + 2 \int_0^t \sqrt{Z(s)}d\beta(s) + \delta t$, where $(\beta(s), s \geq 0)$ is a Brownian motion, is called a δ -dimensional squared Bessel process starting from x . A δ -dimensional Bessel process starting from x is defined as the (nonnegative) square root of a δ -dimensional squared Bessel process starting from x^2 . We recall some important results.

Fact 2.1 (*first Ray–Knight theorem*) Consider $r > 0$ and a Brownian motion $(B(t), t \geq 0)$. The process $(L_B(\sigma_B(r), r - x), x \geq 0)$ is a continuous inhomogeneous Markov process, starting from 0. It is a 2-dimensional squared Bessel process for $x \in [0, r]$ and a 0-dimensional squared Bessel process for $x \geq r$.

Fact 2.2 (*second Ray–Knight theorem*) Let $r > 0$ and $(B(t), t \geq 0)$ be a Brownian motion. Then $(L_B(\tau_B(r), x), x \geq 0)$ is a 0-dimensional squared Bessel process starting from r .

Fact 2.3 (*Lamperti representation theorem [18]*) Consider $W_\kappa(x) = W(x) - \kappa x/2$ as in (1.1), where $(W(x), x \geq 0)$ is a Brownian motion. There exists a $(2 - 2\kappa)$ -dimensional Bessel process $(\rho(t), t \geq 0)$, starting from $\rho(0) = 2$, such that $\exp[W_\kappa(t)/2] = \rho(A(t))/2$ for all $t \geq 0$, where $A(t) = \int_0^t e^{W_\kappa(s)} ds$.

See e.g. Revuz and Yor ([22], chap. XI) for more details about Ray–Knight theorems and Bessel processes. We also recall the following extension to Bessel processes of Williams’ time reversal theorem (see Yor [31], p. 80).

Fact 2.4 *One has, for $\delta < 2$,*

$$(R_\delta(T_0 - s), s \leq T_0) \stackrel{\mathcal{L}}{=} (R_{4-\delta}(s), s \leq \gamma_a),$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law, $(R_\delta(s), s \geq 0)$ denotes a δ -dimensional Bessel process starting from $a > 0$, $T_0 := \inf\{s \geq 0, R_\delta(s) = 0\}$, $(R_{4-\delta}(s), s \geq 0)$ is a $(4 - \delta)$ -dimensional Bessel process starting from 0, and $\gamma_a := \sup\{s \geq 0, R_{4-\delta}(s) = a\}$.

3 Technical estimates

We first introduce

$$A(x) := \int_0^x e^{W_\kappa(y)} dy, \quad x \in \mathbb{R}.$$

This is a scaling function of X . Notice that, since $\kappa > 0$, $A(x) \rightarrow A_\infty < \infty$ when $x \rightarrow +\infty$.

For technical reasons, we introduce the random function F , defined as follows. Notice that the function $x \mapsto A_\infty - A(x)$ is almost surely continuous and (strictly) decreasing. Hence for every $r > 0$, there exists a unique $F(r) \in \mathbb{R}$, depending only on the process W_κ , such that

$$A_\infty - A(F(r)) = \exp(-\kappa r/2) = \delta(r). \quad (3.1)$$

The following lemma is proved in Devulder [10]. It describes how close $F(r)$ is to r for large r .

Lemma 3.1 *Let $\kappa > 0$, $0 < \delta_0 < 1/2$ and*

$$E_1(r) := \left\{ \left(1 - \frac{5}{\kappa} r^{-\delta_0}\right) r \leq F(r) \leq \left(1 + \frac{5}{\kappa} r^{-\delta_0}\right) r \right\}. \quad (3.2)$$

Then for all large r ,

$$\mathbb{P}(E_1(r)^c) \leq \exp(-r^{1-2\delta_0}/4). \quad (3.3)$$

Hence, for any $\varepsilon > 0$, we have, almost surely for all large r ,

$$(1 - \varepsilon)r \leq F(r) \leq (1 + \varepsilon)r. \quad (3.4)$$

Let $r \geq 0$. With an abuse of notation, we denote by $X \circ \Theta_{H(r)}$ the process $(X(H(r) + t) - r, t \geq 0)$. Conditionally on W_κ , this is a diffusion in the potential $(W_\kappa(x+r) - W_\kappa(r), x \in \mathbb{R})$, starting from 0. Moreover, we introduce $H_{X \circ \Theta_{H(r)}}(s) := H(r + s) - H(r)$. Similarly, we denote respectively by $F_{X \circ \Theta_{H(r)}}$, $L_{X \circ \Theta_{H(r)}}^*$, $(H \circ F)_{X \circ \Theta_{H(r)}}$ the processes F , L^* and $H \circ F$ for the diffusion $X \circ \Theta_{H(r)}$, with $(L^*)_X := L_X^*$. The following lemma is a modification of the Borel–Cantelli lemma.

Lemma 3.2 *Let $\kappa > 0$. Let $f : (0, +\infty)^2 \rightarrow \mathbb{R}$ be a continuous function, and $(\Delta_n)_{n \geq 1}$ be a sequence of open sets in \mathbb{R} . Let $\alpha > 0$, $r_n := \exp(n^\alpha)$ and $R_n := \sum_{k=1}^n r_k$. If*

$$\sum_{n \geq 1} \mathbb{P} \{f[(H \circ F)(r_{2n}), (L_X^* \circ H \circ F)(r_{2n})] \in \Delta_n\} = +\infty, \quad (3.5)$$

then for any $\varepsilon > 0$, there exist almost surely infinitely many n such that for some $t_n \in [(1 - \varepsilon)r_{2n}, (1 + \varepsilon)r_{2n}]$,

$$f[H_{X \circ \Theta_H(R_{2n-1})}(t_n), (L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(t_n)] \in \Delta_n.$$

In the rest of the paper, we define, for $\delta_1 > 0$,

$$\lambda := 4(1 + \kappa), \quad c_2 := 2(\lambda/\kappa)^{\delta_1}, \quad \psi_{\pm}(r) := 1 \pm \frac{c_2}{r^{\delta_1}}, \quad t_{\pm}(r) := \frac{\kappa \psi_{\pm}(r)r}{\lambda}. \quad (3.6)$$

Moreover, if $(\beta(s), s \geq 0)$ is a Brownian motion and $v > 0$, we define the Brownian motion $(\beta_v(s), s \geq 0)$ by $\beta_v(s) := (1/v)\beta(v^2s)$. We also introduce

$$K_{\beta}(\kappa) := \int_0^{+\infty} x^{1/\kappa-2} L_{\beta}(\tau_{\beta}(\lambda), x) dx, \quad 0 < \kappa < 1, \quad (3.7)$$

$$C_{\beta} := \int_0^1 \frac{L_{\beta}(\tau_{\beta}(8), x) - 8}{x} dx + \int_1^{+\infty} \frac{L_{\beta}(\tau_{\beta}(8), x)}{x} dx. \quad (3.8)$$

We prove in Section 6 the following approximation.

Lemma 3.3 *Let $\varepsilon \in (0, 1)$. For $\delta_1 > 0$ small enough, there exist a Brownian motion $(\beta(t), t \geq 0)$ and a constant $c_3 > 0$ such that the following holds:*

(i) *For $\kappa > 0$, some $\alpha > 0$ and all large r , we have $\mathbb{P}\{E_2(r)\} \geq 1 - r^{-\alpha}$, where*

$$E_2(r) := \left\{ (1 - \varepsilon)\widehat{L}_-(r) \leq L_X^*[H(F(r))] \leq (1 + \varepsilon)\widehat{L}_+(r) \right\}, \quad (3.9)$$

$$\widehat{L}_{\pm}(r) := 4[\kappa t_{\pm}(r)]^{1/\kappa} \sup_{0 \leq u \leq \tau_{\beta_{t_{\pm}(r)}}(\lambda)} [\beta_{t_{\pm}(r)}(u)]^{1/\kappa} = 4 \sup_{0 \leq u \leq \tau_{\beta}(\lambda t_{\pm}(r))} [\kappa \beta(u)]^{1/\kappa}. \quad (3.10)$$

(ii) *For $0 < \kappa \leq 1$, some $\alpha > 0$ and all large r , we have $\mathbb{P}\{E_3(r)\} \geq 1 - r^{-\alpha}$, where*

$$E_3(r) := \left\{ (1 - \varepsilon)\widehat{I}_-(r) \leq H(F(r)) \leq (1 + \varepsilon)\widehat{I}_+(r) \right\}, \quad (3.11)$$

$$\widehat{I}_{\pm}(r) := \begin{cases} 4\kappa^{1/\kappa-2} t_{\pm}(r)^{1/\kappa} \{K_{\beta_{t_{\pm}(r)}}(\kappa) \pm c_3 t_{\pm}(r)^{1-1/\kappa}\}, & 0 < \kappa < 1, \\ 4t_{\pm}(r) \{C_{\beta_{t_{\pm}(r)}} + 8 \log t_{\pm}(r)\}, & \kappa = 1. \end{cases} \quad (3.12)$$

In the proof of Theorems 1.6, 1.7 and 1.4, we will frequently need to study the almost sure asymptotics of the first hitting times $H(\cdot)$. The following results are proved in Devulder [10].

Theorem 3.4 *Let $a(\cdot)$ be a positive nondecreasing function. If $0 < \kappa < 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{na(n)} \begin{cases} < +\infty \\ = +\infty \end{cases} \iff \limsup_{r \rightarrow \infty} \frac{H(r)}{[ra(r)]^{1/\kappa}} = \begin{cases} 0 \\ +\infty \end{cases} \quad \mathbb{P}\text{-a.s.}$$

If $\kappa = 1$, the statement holds under the additional assumption that $\limsup_{r \rightarrow +\infty} \frac{\log r}{a(r)} < \infty$.

Theorem 3.5 *If $0 < \kappa < 1$, then*

$$\liminf_{r \rightarrow +\infty} \frac{H(r)}{r^{1/\kappa} / (\log \log r)^{(1/\kappa)-1}} = c_4(\kappa) \quad \mathbb{P}\text{-a.s.},$$

where the value of $c_4(\kappa)$ is given in Devulder [10].

If $\kappa = 1$, then

$$\liminf_{r \rightarrow +\infty} \frac{H(r)}{r \log r} = 4 \quad \mathbb{P}\text{-a.s.}$$

We also recall the following formulas. For $0 \leq \kappa < 1$, we denote by S_κ^{ca} a (positive) completely asymmetric stable variable of index κ . Moreover, let C_8^{ca} be a completely asymmetric Cauchy variable of parameter 8. Their characteristic functions are given by:

$$\begin{aligned} \mathbb{E} \exp(itS_\kappa^{ca}) &= \exp \left[-|t|^\kappa \left(1 - i \operatorname{sgn}(t) \tan\left(\frac{\pi\kappa}{2}\right) \right) \right], \\ \mathbb{E} \exp(itC_8^{ca}) &= \exp \left[-8 \left(|t| + it \frac{2}{\pi} \log |t| \right) \right]. \end{aligned} \quad (3.13)$$

The rest of this section is devoted to the proof of Lemma 3.2. The proof of Lemma 3.3 is postponed to Section 6.

Proof of Lemma 3.2. We follow the proof of Lemma 4 in Devulder [10]. We divide \mathbb{R}_+ into some regions in which the diffusion X will behave ‘‘independently’’, in order to apply the Borel–Cantelli lemma.

Let $n \geq 1$ and let

$$E_4(n) := \left\{ \inf_{t: H(R_{2n-1}) \leq t \leq H(R_{2n} + r_{2n+1}/2)} X(t) > R_{2n-2} + \frac{1}{2}r_{2n-1} \right\}.$$

It is proved in [10] that

$$\sum_{n=1}^{+\infty} \mathbb{P}(E_4(n)^c) < \infty. \quad (3.14)$$

We introduce

$$\begin{aligned} \mathcal{D}_n &:= \left\{ \exists t_n \in [(1 - \varepsilon)r_{2n}, (1 + \varepsilon)r_{2n}], \right. \\ &\quad \left. f[H_{X \circ \Theta_{H(R_{2n-1})}}(t_n), (L^* \circ H)_{X \circ \Theta_{H(R_{2n-1})}}(t_n)] \in \Delta_n \right\}, \\ \mathcal{E}_n &:= \left\{ \left(1 - \frac{5}{\kappa} r_{2n}^{-\delta_0} \right) r_{2n} \leq F_{X \circ \Theta_{H(R_{2n-1})}}(r_{2n}) \leq \left(1 + \frac{5}{\kappa} r_{2n}^{-\delta_0} \right) r_{2n} \right\}. \end{aligned}$$

Define $\tilde{t}_n := F_{X \circ \Theta_H(R_{2n-1})}(r_{2n})$, and notice that

$$\mathcal{D}_n \cap E_4(n) \supset \left\{ f[H_{X \circ \Theta_H(R_{2n-1})}(\tilde{t}_n), (L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(\tilde{t}_n)] \in \Delta_n \right\} \cap E_4(n) \cap \mathcal{E}_n. \quad (3.15)$$

By assumption, $\sum_n \mathbb{P}\{f[H_{X \circ \Theta_H(R_{2n-1})}(\tilde{t}_n), (L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(\tilde{t}_n)] \in \Delta_n\} = \infty$. Moreover, $X \circ \Theta_H(R_{2n-1})$ is a diffusion process in the potential $W_\kappa(x + R_{2n-1}) - W_\kappa(R_{2n-1})$, $x \in \mathbb{R}$, hence $\mathbb{P}(\mathcal{E}_n) = \mathbb{P}(E_1(r_{2n}))$. In view of (3.14), (3.15) and Lemma 3.1, this gives $\sum_{n \in \mathbb{N}} \mathbb{P}(\mathcal{D}_n \cap E_4(n)) = +\infty$.

Since $\mathcal{D}_n \cap E_4(n)$, $n \geq 1$, are independent events, Lemma 3.2 follows by an application of the Borel–Cantelli lemma. \square

The proof of Lemma 3.3 is postponed to Section 6.

4 Proof of Theorems 1.2 and 1.3

4.1 Proof of Theorem 1.3

Let $r_n := e^n$ and $R_n := \sum_{k=1}^n r_k$. Let $a(\cdot)$ be a positive nondecreasing function. We begin with the upper bound in Theorem 1.3.

According to Formula 4.1.2 of Borodin and Salminen [5],

$$\mathbb{P} \left(\sup_{0 \leq t \leq \tau_\beta(v)} \beta(t) < y \right) = \exp \left(-\frac{v}{2y} \right), \quad v > 0, y > 0. \quad (4.1)$$

In particular, for \widehat{L}_\pm which is defined in (3.10), and any positive y and r ,

$$\mathbb{P} \left(\widehat{L}_\pm(r) < (yr)^{1/\kappa} \right) = \mathbb{P} \left[\sup_{0 \leq u \leq \tau_{\beta_{t_\pm(r)}(\lambda)}} \beta_{t_\pm(r)}(u) < \frac{yr}{4^\kappa \kappa t_\pm(r)} \right] = \exp \left(-\frac{\kappa^2 4^\kappa \psi_\pm(r)}{2y} \right). \quad (4.2)$$

This together with Lemma 3.3 gives, for r large enough, since $1 - e^{-x} \leq x$ for all $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P} \left\{ L_X^*[H(F(r))] > (ra(e^{-2r}))^{1/\kappa} \right\} &\leq 1 - \exp \left(-\frac{(1+\varepsilon)^\kappa \kappa^2 4^\kappa \psi_+(r)}{2a(e^{-2r})} \right) + \frac{1}{(\log r)^2} \\ &\leq \frac{c_5}{a(e^{-2r})} + \frac{1}{(\log r)^2}. \end{aligned} \quad (4.3)$$

Assume $\sum_{n=1}^{+\infty} \frac{1}{na(n)} < \infty$, which is equivalent to $\sum_{n=1}^{+\infty} \frac{1}{a(r_n)} < \infty$. As a consequence, we have $\sum_{n=1}^{+\infty} \mathbb{P}\{L_X^*[H(F(r_n))] > [r_n a(r_{n-2})]^{1/\kappa}\} < \infty$. By the Borel–Cantelli lemma, almost surely for all large n , $L_X^*[H(F(r_n))] \leq [r_n a(r_{n-2})]^{1/\kappa}$. On the other hand, $r_{n-1} \leq F(r_n)$

almost surely for all large n (see (3.4)). As a consequence, almost surely for all large n , $L_X^*[H(r_{n-1})] \leq [r_n a(r_{n-2})]^{1/\kappa}$. Let $r \in [r_{n-2}, r_{n-1}]$. Then

$$L_X^*[H(r)] \leq L_X^*[H(r_{n-1})] \leq [r_n a(r_{n-2})]^{1/\kappa} \leq e^{2/\kappa} [ra(r)]^{1/\kappa}.$$

Consequently,

$$\limsup_{r \rightarrow +\infty} \frac{L_X^*[H(r)]}{[ra(r)]^{1/\kappa}} \leq e^{2/\kappa} \quad \mathbb{P}\text{-a.s.} \quad (4.4)$$

Since $\sum_{n=1}^{+\infty} \frac{1}{\varepsilon a(e^n)}$ is also finite, (4.4) holds for $a(\cdot)$ replaced by $\varepsilon a(\cdot)$. Letting $\varepsilon \rightarrow 0$ yields the “zero” part of Theorem 1.3.

Now we turn to the proof of the lower bound. Assume $\sum_{n=1}^{+\infty} \frac{1}{a(r_n)} = +\infty$. Observe that we may restrict ourselves to the case $a(x) \rightarrow +\infty$ when $x \rightarrow +\infty$.

By an argument similar to the one leading to (4.3), we have, for r large enough,

$$\mathbb{P} \left\{ L_X^*[H(F(r))] > (ra(e^2r))^{1/\kappa} \right\} \geq \frac{c_5}{2a(e^2r)} - \frac{1}{(\log r)^2},$$

which implies $\sum_{n=1}^{+\infty} \mathbb{P} \left\{ (L_X^* \circ H \circ F)(r_{2n}) > [r_{2n} a(r_{2n+2})]^{1/\kappa} \right\} = +\infty$. By Lemma 3.2, almost surely, there exist infinitely many n such that

$$\sup_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} (L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(t) > [r_{2n} a(r_{2n+2})]^{1/\kappa}.$$

For such n , we have $\sup_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} L_X^*[H(R_{2n-1} + t)] > [r_{2n} a(r_{2n+2})]^{1/\kappa}$. Consequently,

$$\sup_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{L_X^*(H(R_{2n-1} + t))}{[(R_{2n-1} + t)a(R_{2n-1} + t)]^{1/\kappa}} \geq c_6,$$

almost surely for infinitely many n , which gives

$$\limsup_{r \rightarrow +\infty} \frac{L_X^*(H(r))}{[ra(r)]^{1/\kappa}} \geq c_6 \quad \mathbb{P}\text{-a.s.}$$

Replace $a(\cdot)$ by $a(\cdot)/\varepsilon$, and let $\varepsilon \rightarrow 0$. This yields the “infinity” part of Theorem 1.3. \square

4.2 Proof of Theorem 1.2

By Lemma 3.3 and (4.2), for every positive function g and large r ,

$$\mathbb{P} \left[L_X^*[H(F(r))] < \left(\frac{r}{g(r)} \right)^{1/\kappa} \right] \leq \exp \left(-\frac{\kappa^2 4^\kappa (1-\varepsilon)^\kappa \psi_-(r) g(r)}{2} \right) + \frac{1}{(\log r)^2}. \quad (4.5)$$

We choose $g(r) := \frac{2(1+\varepsilon)}{\kappa^{24\kappa}(1-\varepsilon)^{\kappa+1}\psi_-(r)} \log \log r$. Let $s_n := \exp(n^{1-\varepsilon})$. As a consequence, $\sum_{n=1}^{\infty} \mathbb{P} \left[L_X^*[H(F(s_n))] < \left(\frac{s_n}{g(s_n)} \right)^{1/\kappa} \right] < \infty$. By the Borel–Cantelli lemma, almost surely for all large n ,

$$L_X^*[H(F(s_n))] \geq [s_n/g(s_n)]^{1/\kappa}.$$

On the other hand, by Lemma 3.1, $s_n \geq F(s_{n-1})$ almost surely for all large n , which implies that, for $r \in [s_n, s_{n+1}]$,

$$L_X^*[H(r)] \geq L_X^*[H(F(s_{n-1}))] \geq [s_{n-1}/g(s_{n-1})]^{1/\kappa} \geq (1-\varepsilon)[r/g(r)]^{1/\kappa},$$

since $s_{n-1}/s_{n+1} \rightarrow 1$ as $n \rightarrow +\infty$. Consequently,

$$\liminf_{r \rightarrow \infty} \frac{L_X^*(H(r))}{(r/\log \log r)^{1/\kappa}} \geq 4 \left(\frac{\kappa^2}{2} \right)^{1/\kappa} \quad \mathbb{P}\text{-a.s.}$$

Now we prove the inequality “ \leq ”. Let $r_n := \exp(n^{1+\varepsilon})$, $R_n := \sum_{k=1}^n r_k$ and $\tilde{g}(r) := \frac{2(1-\varepsilon)}{\kappa^{24\kappa}(1+\varepsilon)^{\kappa+1}\psi_+(r)} \log \log r$. By Lemma 3.3 and (4.2), for all large r ,

$$\mathbb{P} \left[L_X^*[H(F(r))] < \left(\frac{r}{\tilde{g}(r)} \right)^{1/\kappa} \right] \geq \exp \left(-\frac{\kappa^{24\kappa}(1+\varepsilon)^\kappa \psi_+(r) \tilde{g}(r)}{2} \right) - \frac{1}{(\log r)^2}.$$

Therefore, $\sum_{n \geq 1} \mathbb{P} \left[L_X^*[H(F(r_{2n}))] < \left(\frac{r_{2n}}{\tilde{g}(r_{2n})} \right)^{1/\kappa} \right] = +\infty$. It follows from Lemma 3.2 that, almost surely, there are infinitely many n such that

$$\inf_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} (L^* \circ H)_{X \circ \Theta_{H(R_{2n-1})}}(t) < \left(\frac{r_{2n}}{\tilde{g}(r_{2n})} \right)^{1/\kappa}.$$

On the other hand, an application of Theorem 1.3 gives that almost surely for large n , $L_X^*(H(R_{2n-1})) \leq [R_{2n-1} \log^2 R_{2n-1}]^{1/\kappa} \leq \varepsilon \left(\frac{r_{2n}}{\tilde{g}(r_{2n})} \right)^{1/\kappa}$, since $R_p \leq pr_p \leq p \exp(-p^\varepsilon) r_{p+1}$ for p large enough. Therefore, $\inf_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} L_X^*[H(R_{2n-1} + t)] \leq (1+\varepsilon) \left(\frac{r_{2n}}{\tilde{g}(r_{2n})} \right)^{1/\kappa}$ almost surely, for infinitely many n . Hence, for such n ,

$$\inf_{t \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{L_X^*[H(R_{2n-1} + t)]}{[(R_{2n-1} + t)/\log \log (R_{2n-1} + t)]^{1/\kappa}} \leq (1 + c_7\varepsilon) \left(\frac{\kappa^{24\kappa}\psi_+(r_{2n})}{2} \right)^{1/\kappa}.$$

This yields

$$\liminf_{r \rightarrow +\infty} \frac{L_X^*(H(r))}{(r/\log \log r)^{1/\kappa}} \leq 4 \left(\frac{\kappa^2}{2} \right)^{1/\kappa} \quad \mathbb{P}\text{-a.s.},$$

proving Theorem 1.2. □

5 Proof of Theorems 1.6–1.4

Proof of Theorem 1.6: case $\kappa > 1$. Follows from Theorems 1.2 and 1.1. \square

Proof of Theorem 1.5. Follows from Theorems 1.3 and 1.1. \square

We now assume $0 < \kappa \leq 1$, and prove Theorems 1.6, 1.7 and 1.4. Unfortunately, there is no almost sure convergence result for $H(r)$ in this case due to strong fluctuations; hence a joint study of $L_X^*(H(r))$ and $H(r)$ is useful. Section 5.1 is devoted to the proof of Theorems 1.6, 1.7 and 1.4 in the case $0 < \kappa < 1$, whereas Section 5.2 to the proof of Theorems 1.6 and 1.7 in the case $\kappa = 1$.

We first prove a lemma which will be needed later on. Let $\delta_1 > 0$ and recall $\widehat{L}_\pm(r)$ from (3.10).

Lemma 5.1 *Let $E_5(r) := \{\widehat{L}_-(r) = \widehat{L}_+(r)\}$. For all $\delta_2 \in (0, \delta_1)$ and all large r , we have*

$$\mathbb{P}[E_5(r)^c] \leq r^{-\delta_2}.$$

Proof. Observe that

$$1 \leq \left(\frac{\widehat{L}_+(r)}{\widehat{L}_-(r)} \right)^\kappa \leq \max \left(1, \frac{\sup_{0 \leq u \leq \tau_{\widetilde{\beta}}\{[\psi_+(r) - \psi_-(r)]\kappa r\}} \widetilde{\beta}(u)}{\sup_{0 \leq u \leq \tau_\beta(\psi_-(r)\kappa r)} \beta(u)} \right), \quad (5.1)$$

where $\widetilde{\beta}(u) := \beta(u + \tau_\beta[\psi_-(r)\kappa r])$, $u \geq 0$, is a Brownian motion independent of the random variable $\sup_{0 \leq u \leq \tau_\beta(\psi_-(r)\kappa r)} \beta(u)$. By (4.1) and the trivial inequality $1 - e^{-x} \leq x$ (for $x \geq 0$),

$$\begin{aligned} \mathbb{P} \left(\sup_{0 \leq u \leq \tau_{\widetilde{\beta}}\{[\psi_+(r) - \psi_-(r)]\kappa r\}} \widetilde{\beta}(u) > [\psi_+(r) - \psi_-(r)]\kappa r^{1+\delta_2} \right) &\leq \frac{1}{2r^{\delta_2}}, \\ \mathbb{P} \left(\sup_{0 \leq u \leq \tau_\beta(\psi_-(r)\kappa r)} \beta(u) < \frac{\psi_-(r)\kappa r}{4\delta_2 \log r} \right) &= e^{-2\delta_2 \log r} = \frac{1}{r^{2\delta_2}}. \end{aligned}$$

By definition, $\psi_\pm(r) = 1 \pm c_2 r^{-\delta_1}$ (see (3.6)). Therefore, we have, for large r , with probability greater than $1 - r^{-\delta_2}$,

$$\frac{\sup_{0 \leq u \leq \tau_{\widetilde{\beta}}\{[\psi_+(r) - \psi_-(r)]\kappa r\}} \widetilde{\beta}(u)}{\sup_{0 \leq u \leq \tau_\beta(\psi_-(r)\kappa r)} \beta(u)} \leq \frac{[\psi_+(r) - \psi_-(r)]\kappa r^{1+\delta_2}}{\psi_-(r)\kappa r / (4\delta_2 \log r)} = \frac{8c_2\delta_2 r^{-(\delta_1 - \delta_2)} \log r}{1 - c_2 r^{-\delta_1}} < 1,$$

which, combined with (5.1), yields the lemma. \square

5.1 Case $0 < \kappa < 1$

This section is devoted to the proof of Theorems 1.6, 1.7 and 1.4 in the case $0 < \kappa < 1$.

For any Brownian motion β , let

$$N_\beta := \frac{\int_0^{+\infty} x^{1/\kappa-2} L_\beta(\tau_\beta(\lambda), x) dx}{\{\sup_{0 \leq u \leq \tau_\beta(\lambda)} \beta(u)\}^{1/\kappa}},$$

so that in the notation of (3.7), (3.6) and (3.10), $N_{\beta_{t_\pm(r)}} = 4[\kappa t_\pm(r)]^{1/\kappa} \frac{K_{\beta_{t_\pm(r)}(\kappa)}}{\widehat{L}_\pm(r)}$.

On $E_2(r) \cap E_3(r) \cap E_5(r)$ (the events $E_2(r)$ and $E_3(r)$ are defined in Lemma 3.3, whereas $E_5(r)$ in Lemma 5.1), we have, for some constant c_8 and all large r ,

$$\begin{aligned} \frac{H(F(r))}{L_X^*[H(F(r))]} &\geq \frac{4\kappa^{1/\kappa-2} t_-(r)^{1/\kappa} (1-\varepsilon) \{K_{\beta_{t_-(r)}}(\kappa) - c_3 t_-(r)^{1-1/\kappa}\}}{(1+\varepsilon) \widehat{L}_-(r)} \\ &\geq (1-3\varepsilon) \frac{N_{\beta_{t_-(r)}}}{\kappa^2} - c_8 \frac{t_-(r)}{\widehat{L}_-(r)}. \end{aligned} \quad (5.2)$$

Similarly, on $E_2(r) \cap E_3(r) \cap E_5(r)$, for some constant c_9 and all large r ,

$$\frac{H(F(r))}{L_X^*[H(F(r))]} \leq (1+3\varepsilon) \frac{N_{\beta_{t_+(r)}}}{\kappa^2} + c_9 \frac{t_+(r)}{\widehat{L}_+(r)}. \quad (5.3)$$

Define $E_6(r) := \{\frac{c_8 t_-(r)}{\widehat{L}_-(r)} \leq \varepsilon, \frac{c_9 t_+(r)}{\widehat{L}_+(r)} \leq \varepsilon\}$. By (4.2), $\mathbb{P}(E_6(r)^c) \leq 1/r^2$ for large r . Thus $\mathbb{P}\{E_2(r) \cap E_3(r) \cap E_5(r) \cap E_6(r)\} \geq 1 - r^{-\alpha_1}$ for some $\alpha_1 > 0$ and all large r . In view of (5.2) and (5.3), we have, for some $\alpha_1 > 0$ and all large r ,

$$\mathbb{P} \left((1-3\varepsilon) \frac{N_{\beta_{t_-(r)}}}{\kappa^2} - \varepsilon \leq \frac{H(F(r))}{L_X^*[H(F(r))]} \leq (1+3\varepsilon) \frac{N_{\beta_{t_+(r)}}}{\kappa^2} + \varepsilon \right) \geq 1 - \frac{1}{r^{\alpha_1}}. \quad (5.4)$$

We now proceed to the study of the law of N_β . By the second Ray–Knight theorem (Fact 2.2), there exists a 0-dimensional Bessel process U , starting from $\sqrt{\lambda}$, such that

$$(L_\beta(\tau_\beta(\lambda), x), x \geq 0) = (U^2(x), x \geq 0), \quad (5.5)$$

$$\sup_{0 \leq u \leq \tau_\beta(\lambda)} \beta(u) = \inf\{x \geq 0, U(x) = 0\} =: \zeta_U, \quad (5.6)$$

$$N_\beta = \zeta_U^{-1/\kappa} \int_0^{\zeta_U} x^{1/\kappa-2} U^2(x) dx. \quad (5.7)$$

By Williams' time reversal theorem (Fact 2.4), there exists a 4-dimensional Bessel process R , starting from 0, such that

$$(U(\zeta_U - t), t \leq \zeta_U) \stackrel{\mathcal{L}}{=} (R(t), t \leq \gamma_a), \quad a := \sqrt{\lambda}, \quad \gamma_a := \sup\{t \geq 0, R(t) = \sqrt{\lambda}\}. \quad (5.8)$$

Therefore,

$$N_\beta \stackrel{\mathcal{L}}{=} \gamma_a^{-1/\kappa} \int_0^{\gamma_a} x^{1/\kappa-2} R^2(\gamma_a - x) dx = \int_0^1 (1-v)^{1/\kappa-2} \left(\frac{R(\gamma_a v)}{\sqrt{\gamma_a}} \right)^2 dv.$$

Recall (Yor [31], p. 52) that for any bounded measurable functional G ,

$$\mathbb{E} \left[G \left(\frac{R(\gamma_a u)}{\sqrt{\gamma_a}}, u \leq 1 \right) \right] = \mathbb{E} \left(\frac{2}{R^2(1)} G(R(u), u \leq 1) \right). \quad (5.9)$$

In particular, for $x > 0$,

$$\mathbb{P}(N_\beta > x) = \mathbb{E} \left(\frac{2}{R^2(1)} \mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2} R^2(v) dv > x\}} \right). \quad (5.10)$$

5.1.1 Proof of Theorem 1.7 (case $0 < \kappa < 1$)

Fix $y > 0$. By (5.10),

$$\begin{aligned} \mathbb{P}(N_\beta > y \log \log r) &\leq \mathbb{E} \left(\frac{2}{R^2(1)} \mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2} R^2(v) dv > y \log \log r, R^2(1) \leq 1\}} \right) \\ &\quad + 2\mathbb{P} \left(\int_0^1 (1-v)^{1/\kappa-2} R^2(v) dv > y \log \log r \right) \\ &:= \Pi_1(r) + \Pi_2(r) \end{aligned} \quad (5.11)$$

with obvious notation.

We first consider $\Pi_2(r)$. Let \mathcal{C}_0 denote the set of continuous functions $\phi : [0, 1] \rightarrow \mathbb{R}$, such that $\phi(0) = 0$. By Schilder's theorem (see Dembo and Zeitouni [8], p. 185),

$$\begin{aligned} &\lim_{r \rightarrow +\infty} \frac{1}{y \log \log r} \log \mathbb{P} \left(\int_0^1 (1-v)^{1/\kappa-2} R^2(v) dv > y \log \log r \right) \\ &= -\inf \left\{ \frac{1}{2} \int_0^1 \phi'(v)^2 dv : \phi \in \mathcal{C}_0, \int_0^1 (1-v)^{1/\kappa-2} \phi^2(v) dv \geq 1 \right\} := -c_1(\kappa). \end{aligned} \quad (5.12)$$

For $\phi \in \mathcal{C}_0$, $\phi^2(v) = (\int_0^v \phi'(u) du)^2 \leq v \int_0^1 \phi'(u)^2 du$; thus $\int_0^1 (1-v)^{1/\kappa-2} \phi^2(v) dv \leq [\int_0^1 (1-v)^{1/\kappa-2} v dv] \int_0^1 \phi'(v)^2 dv$, which implies $c_1(\kappa) > 0$.

By (5.12), for large r ,

$$\Pi_2(r) \leq \frac{1}{(\log r)^{(1-\varepsilon)yc_1(\kappa)}}. \quad (5.13)$$

Now, we consider $\Pi_1(r)$. As R is the Euclidean norm of a 4-dimensional Brownian motion $(\gamma(t), t \geq 0)$, we have

$$\Pi_1(r) = \mathbb{E} \left(\frac{2}{\|\gamma(1)\|^2} \mathbf{1}_{\{\|\gamma(1)\| \leq 1\}} \mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2} \|\gamma(v)\|^2 dv > y \log \log r\}} \right),$$

where $\|\cdot\|$ denotes the Euclidean norm. By the triangular inequality, for any positive measure μ on $[0, 1]$, $\sqrt{\int_0^1 \|\gamma(v)\|^2 d\mu(v)} \leq \sqrt{\int_0^1 \|\gamma(v) - v\gamma(1)\|^2 d\mu(v)} + \sqrt{\int_0^1 v^2 d\mu(v)} \|\gamma(1)\|$. Therefore,

$$\Pi_1(r) \leq \mathbb{E} \left(\frac{2}{\|\gamma(1)\|^2} \mathbf{1}_{\{\int_0^1 (1-v)^{1/\kappa-2} \|\gamma(v) - v\gamma(1)\|^2 dv > (\sqrt{y \log \log r} - c_{10})^2\}} \right) := \mathbb{E} \left(\frac{2}{\|\gamma(1)\|^2} \mathbf{1}_E \right).$$

By the independence of $\gamma(1)$ and $(\gamma(v) - v\gamma(1), v \in [0, 1])$, the expectation on the right hand side is $= \mathbb{E}(\frac{2}{\|\gamma(1)\|^2}) \mathbb{P}(E) = \mathbb{P}(E)$ (the last identity being a consequence of (5.9) by taking $G := 1$ there). Therefore, $\Pi_1(r) \leq \mathbb{P}(E)$.

Again, by the independence of $\gamma(1)$ and $(\gamma(v) - v\gamma(1), v \in [0, 1])$, we see that, by writing $c_{11} := 1/\mathbb{P}(\|\gamma(1)\| \leq 1)$, $\Pi_1(r) \leq c_{11} \mathbb{P}(E, \|\gamma(1)\| \leq 1)$. By another application of the triangular inequality, this leads to:

$$\Pi_1(r) \leq c_{11} \mathbb{P} \left(\int_0^1 (1-v)^{1/\kappa-2} \|\gamma(v)\|^2 dv > (\sqrt{y \log \log r} - 2c_{10})^2 \right).$$

In view of (5.12), we have, for all large r , $\Pi_1(r) \leq (\log r)^{-(1-\varepsilon)yc_1(\kappa)}$. Plugging this into (5.11) and (5.13) yields that, for any $y > 0$, $\varepsilon > 0$ and all large r ,

$$\mathbb{P}(N_\beta > y \log \log r) \leq \frac{2}{(\log r)^{(1-\varepsilon)yc_1(\kappa)}}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. In view of (5.4), we have proved that $\sum_{n=1}^{+\infty} \mathbb{P}\left\{\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} > \frac{1+4\varepsilon}{(1-\varepsilon)^2 \kappa^2 c_1(\kappa)} \log \log s_n\right\} < \infty$. By the Borel–Cantelli lemma, almost surely, for all large n ,

$$\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} \leq \frac{1+4\varepsilon}{(1-\varepsilon)^2 \kappa^2 c_1(\kappa)} \log \log s_n. \quad (5.14)$$

We now bound $\frac{H(F(s_{n+1}))}{H(F(s_n))}$. Observe that for large n , $s_{n+1} - s_n \leq n^{-\varepsilon} s_n$. By Lemma 3.1, almost surely for all large n ,

$$\begin{aligned} H(F(s_{n+1})) - H(F(s_n)) &\leq H \left[\left(1 + \frac{5}{\kappa} s_{n+1}^{-\delta_0}\right) s_{n+1} \right] - H \left[\left(1 - \frac{5}{\kappa} s_n^{-\delta_0}\right) s_n \right] \\ &\leq H \left[\left(1 - \frac{5}{\kappa} s_n^{-\delta_0}\right) s_n + (2 - \varepsilon) n^{-\varepsilon} s_n \right] - H \left[\left(1 - \frac{5}{\kappa} s_n^{-\delta_0}\right) s_n \right] \\ &= \inf \left\{ u \geq 0 : \widehat{X}_n(u) > (2 - \varepsilon) n^{-\varepsilon} s_n \right\}, \end{aligned} \quad (5.15)$$

where $(\widehat{X}_n(u), u \geq 0)$ is a diffusion process in the random potential $\widehat{W}_\kappa(x) := W_\kappa(x + (1 - \frac{5}{\kappa} s_n^{-\delta_0}) s_n) - W_\kappa((1 - \frac{5}{\kappa} s_n^{-\delta_0}) s_n)$, $x \in \mathbb{R}$. It is natural to write the above identity as

$$\inf \left\{ u \geq 0 : \widehat{X}_n(u) > (2 - \varepsilon) n^{-\varepsilon} s_n \right\} = \widehat{H}_n[(2 - \varepsilon) n^{-\varepsilon} s_n]. \quad (5.16)$$

Note that for any $r > 0$, under \mathbb{P} , $\widehat{H}_n(r)$ is distributed as $H(r)$. Recall from Devulder ([10] equation (3.5)) that

$$\mathbb{P}[H(F(r)) > (a(e^{-2r})t_+(r))^{1/\kappa}] \leq c_{12}/a(e^{-2r}) + (\log r)^{-2}.$$

Therefore, applying this and Lemma 3.1 to $r = 2n^{-\varepsilon}s_n$ yields that, for any $0 < \delta_0 < \frac{1}{2}$,

$$\sum_n \mathbb{P} \left[\widehat{H}_n \left(\left[1 - \frac{5}{\kappa} (2n^{-\varepsilon}s_n)^{-\delta_0} \right] 2n^{-\varepsilon}s_n \right) > [n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)]^{1/\kappa} \right] < \infty.$$

Since $[1 - \frac{5}{\kappa}(2n^{-\varepsilon}s_n)^{-\delta_0}]2n^{-\varepsilon}s_n \geq (2-\varepsilon)n^{-\varepsilon}s_n$ (for large n), it follows from the Borel–Cantelli lemma that, almost surely, for all large n , $\widehat{H}_n((2-\varepsilon)n^{-\varepsilon}s_n) \leq [n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)]^{1/\kappa}$. This, together with (5.15) and (5.16), yields that, almost surely, for all large n ,

$$H(F(s_{n+1})) - H(F(s_n)) \leq [n(\log n)^{1+\varepsilon}t_+(2n^{-\varepsilon}s_n)]^{1/\kappa} \leq c_{13}[n^{1-\varepsilon}(\log n)^{1+\varepsilon}s_n]^{1/\kappa}.$$

Recall from Lemma 3.1 and Theorem 3.5 that, almost surely, for all large n , $H[F(s_n)] \geq H[(1-\varepsilon)s_n] \geq \frac{c_{14}s_n^{1/\kappa}}{(\log \log s_n)^{1/\kappa-1}}$, which yields

$$\frac{H(F(s_{n+1}))}{H(F(s_n))} \leq 1 + \frac{c_{13}[n^{1-\varepsilon}(\log n)^{1+\varepsilon}s_n]^{1/\kappa}}{c_{14}s_n^{1/\kappa}/(\log \log s_n)^{1/\kappa-1}} \leq c_{15}(\log s_n)^{1/\kappa}(\log \log s_n)^{(2+\varepsilon)/\kappa-1}.$$

In view of (5.14), this yields that, almost surely, for large n and $t \in [H(F(s_n)), H(F(s_{n+1}))]$,

$$\frac{t}{L_X^*(t)} \leq \frac{H[F(s_n)]}{L_X^*[H(F(s_n))]} \frac{H(F(s_{n+1}))}{H(F(s_n))} < c_{16}(\log s_n)^{1/\kappa}(\log \log s_n)^{(2+\varepsilon)/\kappa}.$$

Since, almost surely for all large n , $\log H(F(s_n)) \geq \log H((1-\varepsilon)s_n) \geq \frac{1-\varepsilon}{\kappa} \log s_n$ (this is seen first by Lemma 3.1, and then by Theorem 3.5), we have proved that

$$\liminf_{t \rightarrow +\infty} \frac{L_X^*(t)}{t(\log t)^{-1/\kappa}(\log \log t)^{-(2+\varepsilon)/\kappa}} \geq c_{17} \quad \mathbb{P}\text{-a.s.}$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, this proves Theorem 1.7 in the case $0 < \kappa < 1$. □

5.1.2 Proof of Theorem 1.6 (case $0 < \kappa < 1$)

By (5.10), for any $s > 0$ and $u > 0$,

$$\begin{aligned} \mathbb{P}(N_\beta > s) &\geq \frac{2}{u} \mathbb{P} \left(\int_0^1 (1-v)^{1/\kappa-2} R^2(v) dv > s, R^2(1) \leq u \right) \\ &\geq \frac{2}{u} \mathbb{P} \left(\int_0^1 (1-v)^{1/\kappa-2} R^2(v) dv > s \right) - \frac{2}{u} \mathbb{P} (R^2(1) > u). \end{aligned}$$

The first probability term on the right hand side is taken care of by (5.12), whereas for the second, we have $\frac{1}{u} \log \mathbb{P}(R^2(1) > u) \rightarrow -\frac{1}{2}$, for $u \rightarrow \infty$. Taking $u := \exp(\sqrt{\log \log r})$ leads to: for any $y > 0$,

$$\liminf_{r \rightarrow \infty} \frac{\log \mathbb{P}(N_\beta > y \log \log r)}{\log \log r} \geq -yc_1(\kappa).$$

Plugging this into (5.4) yields that, for $r_n := \exp(n^{1+\varepsilon})$,

$$\sum_{n \geq 1} \mathbb{P} \left(\frac{(H \circ F)(r_{2n})}{(L_X^* \circ H \circ F)(r_{2n})} > \frac{(1 - 3\varepsilon) \log \log r_{2n}}{\kappa^2 c_1(\kappa)(1 + \varepsilon)^3} - \varepsilon \right) = +\infty.$$

Let $R_n := \sum_{k=1}^n r_k$. By Lemma 3.2 (in its notation), almost surely, for infinitely many n ,

$$\sup_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{H_{X \circ \Theta_H(R_{2n-1})}(u)}{(L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u)} > \frac{(1 - 8\varepsilon) \log \log r_{2n}}{\kappa^2 c_1(\kappa)}. \quad (5.17)$$

Observe that

$$(L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u) = \sup_{x \in \mathbb{R}} L_{\tilde{X}_n}(\tilde{H}_n(u), x) =: L_{\tilde{X}_n}^*(\tilde{H}_n(u)) \quad (5.18)$$

where $(\tilde{X}_n(v), v \geq 0)$ is a diffusion process in the random potential $W_\kappa(x + R_{2n-1}) - W_\kappa(R_{2n-1})$, $x \in \mathbb{R}$, $(L_{\tilde{X}_n}(t, x), t \geq 0, x \in \mathbb{R})$ is its local time and $\tilde{H}_n(r) := \inf\{t > 0, \tilde{X}_n(t) > r\}$, $r > 0$. Hence, for any $u > 0$, under \mathbb{P} , the left hand side of (5.18) is distributed as $L_X^*(H(u))$. Applying (4.5) and Lemma 3.1 to $\tilde{r}_{2n} := (1 - \varepsilon)^2 r_{2n}$, there exists $c_{18} > 0$ such that

$$\sum_n \mathbb{P} \left[L_{\tilde{X}_n}^* \left(\tilde{H}_n \left[\left(1 + \frac{5}{\kappa} (\tilde{r}_{2n})^{-\delta_0}\right) \tilde{r}_{2n} \right] \right) < c_{18} [r_{2n} / \log \log r_{2n}]^{1/\kappa} \right] < \infty.$$

Since $(1 + \frac{5}{\kappa} (\tilde{r}_{2n})^{-\delta_0}) \tilde{r}_{2n} \leq (1 - \varepsilon) r_{2n}$ for large n , the Borel–Cantelli lemma gives that, almost surely, for all large n ,

$$c_{18} [r_{2n} / \log \log r_{2n}]^{1/\kappa} \leq L_{\tilde{X}_n}^* \left(\tilde{H}_n[(1 - \varepsilon)r_{2n}] \right) \leq L_{\tilde{X}_n}^* \left(\tilde{H}_n(u) \right) \quad (5.19)$$

for any $u \in [(1 - \varepsilon)r_{2n}, (1 + \varepsilon)r_{2n}]$. Applying Theorem 1.3, we have almost surely for large n ,

$$L_X^*[H(R_{2n-1})] \leq [R_{2n-1} \log^2 R_{2n-1}]^{1/\kappa} \leq \varepsilon [r_{2n} / \log \log r_{2n}]^{1/\kappa} \leq (\varepsilon / c_{18}) L_{\tilde{X}_n}^* \left(\tilde{H}_n(u) \right)$$

for $u \in [(1 - \varepsilon)r_{2n}, (1 + \varepsilon)r_{2n}]$, since $R_k \leq k \exp(-k^\varepsilon) r_{k+1}$ for large k . Hence,

$$L_X^*[H(R_{2n-1} + u)] \leq (1 + \varepsilon / c_{18}) L_{\tilde{X}_n}^* \left(\tilde{H}_n(u) \right). \quad (5.20)$$

On the other hand, by Theorem 3.4 and Lemma 3.1, we have, almost surely, for all large n ,

$$\log \log r_{2n} \geq (1 - \varepsilon) \log \log H(R_{2n-1} + u), \quad u \in [(1 - \varepsilon)r_{2n}, (1 + \varepsilon)r_{2n}].$$

Consequently, almost surely for infinitely many n , by (5.20) and (5.17),

$$\begin{aligned} & \inf_{v \in [R_{2n-1} + (1-\varepsilon)r_{2n}, R_{2n-1} + (1+\varepsilon)r_{2n}]} \frac{L_X^*[H(v)]}{H(v)/\log \log H(v)} \\ & \leq (1 + c_{19}\varepsilon) \inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{(L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u)}{H_{X \circ \Theta_H(R_{2n-1})}(u)/\log \log r_{2n}} \leq (1 + c_{20}\varepsilon)\kappa^2 c_1(\kappa), \end{aligned}$$

proving Theorem 1.6 in the case $0 < \kappa < 1$. \square

5.1.3 Proof of Theorem 1.4

Assume $0 < \kappa < 1$. Fix $x > 0$, and let $r_n := \exp(n^{1+\varepsilon})$. Since $\mathbb{P}(N_\beta < x) > 0$, (5.4) implies $\sum_{n \in \mathbb{N}} \mathbb{P}\left(\frac{(H \circ F)(r_{2n})}{(L_X^* \circ H \circ F)(r_{2n})} < \frac{(1+3\varepsilon)x}{\kappa^2} + \varepsilon\right) = +\infty$. By Lemma 3.2, almost surely, for infinitely many n ,

$$\inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{H_{X \circ \Theta_H(R_{2n-1})}(u)}{(L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u)} < \frac{(1+3\varepsilon)x}{\kappa^2} + \varepsilon. \quad (5.21)$$

With the same notation as in (5.18), $H_{X \circ \Theta_H(R_{2n-1})}(u) = H(R_{2n-1} + u) - H(R_{2n-1})$ is the hitting time $\tilde{H}_n(u)$ of u by the diffusion \tilde{X}_n . For any u , under \mathbb{P} , it has the same distribution as $H(u)$. Recall from Devulder ([10] equation (3.10)) that

$$\mathbb{P}[H(F(r)) < t_-(r)^{1/\kappa} f(r)] \leq \exp \left[- (c_{21} - \varepsilon) \left(\frac{(1-\varepsilon)c_{22}}{f(r) + c_{23}r^{1-1/\kappa}} \right)^{\kappa/(1-\kappa)} \right] + \log^{-2} r.$$

Hence, applying this and Lemma 3.1 to $\tilde{r}_{2n} = (1-\varepsilon)^2 r_{2n}$ leads to

$$\sum_n \mathbb{P} \left[\tilde{H}_n \left(\left(1 + \frac{5}{\kappa} (\tilde{r}_{2n})^{-\delta_0}\right) \tilde{r}_{2n} \right) < r_{2n}^{1/\kappa} / \log r_{2n} \right] < \infty$$

(for $0 < \delta_0 < 1/2$). Since $(1 + \frac{5}{\kappa} (\tilde{r}_{2n})^{-\delta_0}) \tilde{r}_{2n} < (1-\varepsilon)r_{2n}$ for large n , it follows from the Borel–Cantelli lemma that, almost surely, for all large n ,

$$\frac{r_{2n}^{1/\kappa}}{\log r_{2n}} \leq \tilde{H}_n[(1-\varepsilon)r_{2n}] \leq \inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} H_{X \circ \Theta_H(R_{2n-1})}(u).$$

On the other hand, by Theorem 3.4, $H(R_{2n-1}) \leq [R_{2n-1} \log^2 R_{2n-1}]^{1/\kappa} \leq \varepsilon \frac{r_{2n}^{1/\kappa}}{\log r_{2n}}$ almost surely for all large n . Hence, for $u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]$, $H(R_{2n-1} + u) \leq (1+\varepsilon)H_{X \circ \Theta_H(R_{2n-1})}(u)$. Plugging this into (5.21) yields that, almost surely, for infinitely many n ,

$$\inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{H(R_{2n-1} + u)}{L_X^*(H(R_{2n-1} + u))} < \frac{(1+\varepsilon)(1+3\varepsilon)x}{\kappa^2} + \varepsilon(1+\varepsilon).$$

Hence $\limsup_{t \rightarrow +\infty} \frac{L_X^*(t)}{t} \geq \frac{\kappa^2}{x}$, a.s. Sending $x \rightarrow 0$ completes the proof of Theorem 1.4. \square

5.2 Case $\kappa = 1$

In this section we prove Theorems 1.6 and 1.7 in the case $\kappa = 1$ (thus $\lambda = 8$).

Let

$$N_\beta(t) := \frac{1}{\sup_{0 \leq u \leq \tau_\beta(8)} \beta(u)} \left[\int_0^1 \frac{L_\beta(\tau_\beta(8), x) - 8}{x} dx + \int_1^{+\infty} \frac{L_\beta(\tau_\beta(8), x)}{x} dx + 8 \log t \right].$$

Exactly as in (5.4), we have, for some $\alpha_1 > 0$, any $\varepsilon \in (0, \frac{1}{3})$, and all large r ,

$$\mathbb{P} \left((1 - 3\varepsilon) N_{\beta_{t_-(r)}}[t_-(r)] \leq \frac{H(F(r))}{L_X^*[H(F(r))]} \leq (1 + 3\varepsilon) N_{\beta_{t_+(r)}}[t_+(r)] \right) \geq 1 - \frac{1}{r^{\alpha_1}}, \quad (5.22)$$

where $t_\pm(\cdot)$ are defined in (3.6). (Compared to (5.4), we no longer have the extra “ $\pm\varepsilon$ ” terms, since they are already taken care of by the presence of $8 \log t$ in the definition of $N_\beta(t)$).

With the same notation as in (5.5) and (5.6), the second Ray–Knight theorem (Fact 2.2) gives

$$\begin{aligned} N_\beta(t) &= \frac{1}{\zeta_U} \left[\int_0^1 \frac{U^2(x) - 8}{x} dx + \int_1^{+\infty} \frac{U^2(x)}{x} dx + 8 \log t \right] \\ &= \frac{1}{\zeta_U} \left[\int_0^{\zeta_U} \frac{U^2(x) - 8}{x} dx + 8 \log \zeta_U + 8 \log t \right]. \end{aligned}$$

Proof of Theorem 1.7 (case $\kappa = 1$)

We have $\lambda = 8$ in the case $\kappa = 1$. Since $\sup_{x>0} \frac{\log x}{x} < \infty$, we have

$$N_\beta(t) \leq c_{24} + \frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x) - 8|}{x} dx + \frac{8 \log t}{\zeta_U}.$$

We claim that for some constant $c_{25} > 0$,

$$\limsup_{y \rightarrow +\infty} \frac{1}{y} \log \mathbb{P} \left(\frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x) - 8|}{x} dx > y \right) \leq -c_{25}. \quad (5.23)$$

Indeed, $\zeta_U = \sup_{0 \leq u \leq \tau_\beta(8)} \beta(u)$ by definition (see (5.6)), which, in view of (4.1), implies that $\mathbb{P}(\zeta_U > z) = 1 - e^{-4/z} \leq \frac{4}{z}$ for $z > 0$. Therefore, if we write $p(y)$ for the probability expression at (5.23), we have, for any $z > 0$,

$$p(y) \leq \frac{4}{z} + \mathbb{P} \left(\frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x) - 8|}{x} dx > y, \zeta_U \leq z \right).$$

In the notation of (5.8)–(5.9), this yields

$$\begin{aligned}
p(y) &\leq \frac{4}{z} + \mathbb{P} \left(\frac{1}{\gamma_a} \int_0^1 \frac{|R^2(\gamma_a v) - 8|}{1-v} dv > y, \gamma_a \leq z \right) \\
&= \frac{4}{z} + \mathbb{E} \left(\frac{2}{R^2(1)} \mathbf{1}_{\{\int_0^1 \frac{|R^2(v) - R^2(1)|}{1-v} dv > y, R^2(1) \geq 8/z\}} \right) \\
&\leq \frac{4}{z} + \frac{z}{4} \mathbb{P} \left(\int_0^1 \frac{|R^2(v) - R^2(1)|}{1-v} dv > y \right). \tag{5.24}
\end{aligned}$$

In order to apply Schilder's theorem as in (5.12), let $\phi \in \mathcal{C}_0$. By the Cauchy–Schwarz inequality, $|\phi(t)| \leq \sqrt{t} [\int_0^1 \phi'(u)^2 du]^{1/2}$, and $|\phi(1-t) - \phi(1)|$ satisfies a similar inequality. Hence,

$$\int_0^1 \frac{|\phi^2(u) - \phi^2(1)|}{1-u} du \leq \int_0^1 \frac{|\phi(u) - \phi(1)|}{1-u} [|\phi(u)| + |\phi(1)|] du \leq 2 \int_0^1 \frac{du}{\sqrt{1-u}} \int_0^1 \phi'(u)^2 du.$$

Consequently,

$$c_{26} := \inf \left\{ \frac{1}{2} \int_0^1 \phi'(u)^2 du : \phi \in \mathcal{C}_0, \int_0^1 \frac{|\phi^2(u) - \phi^2(1)|}{1-u} du > 1 \right\} > 0.$$

Applying Schilder's theorem gives that $\limsup_{y \rightarrow +\infty} \frac{1}{y} \log \mathbb{P} \left(\int_0^1 \frac{|R^2(v) - R^2(1)|}{1-v} dv > y \right) \leq -c_{26}$. Plugging this into (5.24), and taking $z = \exp(\frac{c_{26}}{2}y)$ there, we obtain the claimed inequality in (5.23), with $c_{25} := \frac{c_{26}}{2}$.

On the other hand, by (4.1),

$$\mathbb{P} \left(\frac{8 \log t}{\zeta_U} > 2(1 + 2\varepsilon)(\log t) \log \log t \right) = \frac{1}{(\log t)^{1+2\varepsilon}}.$$

Therefore, for all large t ,

$$\mathbb{P} \{N_\beta(t) > 2(1 + 3\varepsilon)(\log t) \log \log t\} \leq \frac{2}{(\log t)^{1+2\varepsilon}}.$$

Let $s_n := \exp(n^{1-\varepsilon})$. By (5.22), $\sum_{n=1}^{+\infty} \mathbb{P} \left(\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} > 2(1 + 3\varepsilon)^2(\log s_n) \log \log s_n \right) < \infty$, which, by means of the Borel–Cantelli lemma, implies that, almost surely, for all large n ,

$$\frac{H(F(s_n))}{L_X^*[H(F(s_n))]} \leq 2(1 + 3\varepsilon)^2(\log s_n) \log \log s_n. \tag{5.25}$$

Now we give an upper bound for $\frac{H(F(s_{n+1}))}{H(F(s_n))}$. By Lemma 3.1, almost surely for n large enough, $F(s_n) \geq (1 - \varepsilon)s_n$. An application of Theorem 3.5 yields that, almost surely, for large n ,

$$H(F(s_n)) \geq H[(1 - \varepsilon)s_n] \geq 4(1 - 2\varepsilon)s_n \log s_n. \tag{5.26}$$

With the same notation and the same arguments as in (5.15) and (5.16), almost surely for all large n , $H(F(s_{n+1})) - H(F(s_n)) \leq \widehat{H}_n[(2 - \varepsilon)n^{-\varepsilon}s_n]$. Moreover, $\widehat{H}_n(r)$ is distributed as $H(r)$ under \mathbb{P} for any $r > 0$. Recall from Devulder ([10] equation (3.8)) that

$$\mathbb{P} \left\{ H(F(r)) > 4t_+(r)(1 + \varepsilon)[8c_{27} + a(e^{-2r}) + 8 \log t_+(r)] \right\} \leq c_{28}\pi/a(e^{-2r}) + (\log r)^{-2}.$$

Hence, applying this and Lemma 3.1 to $r = \tilde{s}_n := 2n^{-\varepsilon}s_n$ and $a(e^{-2\tilde{s}_n}) = 8n(\log n)^{1+\varepsilon}$ for $0 < \delta_0 < \frac{1}{2}$, we get

$$\sum_n \mathbb{P} \left[\widehat{H}_n \left(\left(1 - \frac{5}{\kappa}(\tilde{s}_n)^{-\delta_0}\right)\tilde{s}_n \right) > 32(1 + \varepsilon)t_+(\tilde{s}_n)[c_{29} + n(\log n)^{1+\varepsilon} + \log t_+(\tilde{s}_n)] \right] < \infty.$$

Since $[1 - \frac{5}{\kappa}(\tilde{s}_n)^{-\delta_0}]\tilde{s}_n \geq (2 - \varepsilon)n^{-\varepsilon}s_n$ (for large n), the Borel–Cantelli lemma yields that, almost surely, for large n ,

$$\widehat{H}_n((2 - \varepsilon)n^{-\varepsilon}s_n) \leq 32(1 + \varepsilon)t_+(2n^{-\varepsilon}s_n)[c_{29} + n(\log n)^{1+\varepsilon} + \log t_+(2n^{-\varepsilon}s_n)].$$

Hence, $H(F(s_{n+1})) - H(F(s_n)) \leq c_{30}s_n(\log s_n)(\log n)^{1+\varepsilon}$. Consequently, almost surely for large n ,

$$\frac{H(F(s_{n+1}))}{H(F(s_n))} \leq (c_{31} + \varepsilon)(\log \log s_n)^{1+\varepsilon}.$$

Let $t \in [H(F(s_n)), H(F(s_{n+1}))]$. By (5.25),

$$\frac{t}{L_X^*(t)} \leq \frac{H[F(s_n)]}{L_X^*[H(F(s_n))]} \frac{H(F(s_{n+1}))}{H(F(s_n))} < 3c_{31}(\log s_n)(\log \log s_n)^{2+\varepsilon}.$$

Since, almost surely for large n , $\log H(F(s_n)) \geq \log H((1 - \varepsilon)s_n) \geq \log s_n$ (by Lemma 3.1 and Theorem 3.5), this yields

$$\liminf_{t \rightarrow +\infty} \frac{L_X^*(t)}{t/[(\log t)(\log \log t)^{2+\varepsilon}]} \geq \frac{1}{3c_{31}} \quad \mathbb{P}\text{-a.s.}$$

Theorem 1.7 is proved in the case $\kappa = 1$. □

Proof of Theorem 1.6 (case $\kappa = 1$)

Again, $\lambda = 8$. Recall that $N_\beta(t) = \frac{1}{\zeta_U} \left[\int_0^{\zeta_U} \frac{U^2(x)-8}{x} dx + 8 \log \zeta_U + 8 \log t \right]$. This time, we need to bound $N_\beta(t)$ from below. Since $\zeta_U = \sup_{0 \leq u \leq \tau_\beta(8)} \beta(u)$, (4.1) gives for $z > 8e$,

$$\mathbb{P} \left(8 \frac{\log \zeta_U}{\zeta_U} < -z \right) \leq \mathbb{P} \left(\zeta_U < \frac{\log(z/8)}{z/8} \right) = \exp \left(-\frac{z}{2 \log(z/8)} \right).$$

By (4.1) again,

$$\mathbb{P} \left(\frac{8 \log t}{\zeta_U} > 2(1 - \varepsilon)(\log t) \log \log t \right) = \frac{1}{(\log t)^{1-\varepsilon}}.$$

On the other hand, for all large y , $\mathbb{P}(\frac{1}{\zeta_U} \int_0^{\zeta_U} \frac{|U^2(x)-8|}{x} dx > y) \leq e^{-c_{32}y}$ (see (5.23)). Assembling these pieces yields that, for all large t ,

$$\mathbb{P}[N_\beta(t) > 2(1 - 2\varepsilon)(\log t) \log \log t] \geq \frac{1}{2(\log t)^{1-\varepsilon}}.$$

Let $r_n := \exp(n^{1+\varepsilon})$. In view of (5.22) and Lemma 3.2, we get almost surely, for infinitely many n ,

$$\sup_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{H_{X \circ \Theta_H(R_{2n-1})}(u)}{(L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u)} > 2(1 - 2\varepsilon)(1 - 3\varepsilon)(\log r_{2n}) \log \log r_{2n}. \quad (5.27)$$

The expression on the left hand side of (5.27) is “close to” $H(r_{2n})/L_X^*[H(r_{2n})]$, but we need to prove this rigorously. With the same argument as in the displays between (5.18) and (5.19), we get that, almost surely, for large n ,

$$\inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} (L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u) \geq c_{33}r_{2n}/\log \log r_{2n}.$$

Observe that $R_k \leq k \exp(-k^\varepsilon)r_{k+1}$ (for large k). Exactly as in the case $0 < \kappa < 1$, we apply Theorem 1.3, to see that, almost surely, for large n ,

$$L_X^*[H(R_{2n-1})] \leq \varepsilon r_{2n}/\log \log r_{2n} \leq (\varepsilon/c_{33}) \inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} (L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u),$$

which implies, for all $u \in [(1 - \varepsilon)r_{2n}, (1 + \varepsilon)r_{2n}]$,

$$L_X^*[H(R_{2n-1} + u)] \leq (1 + \varepsilon/c_{33})(L^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u). \quad (5.28)$$

By Theorem 3.4, almost surely for all large n , $\sup_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \log H(R_{2n-1} + u) \leq (1 + \varepsilon) \log r_{2n}$. In view of (5.27), there are almost surely infinitely many n such that

$$\begin{aligned} & \inf_{v \in [R_{2n-1} + (1-\varepsilon)r_{2n}, R_{2n-1} + (1+\varepsilon)r_{2n}]} \frac{L_X^*[H(v)]}{H(v)/[(\log H(v)) \log \log H(v)]} \\ & \leq (1 + c_{34}\varepsilon) \inf_{u \in [(1-\varepsilon)r_{2n}, (1+\varepsilon)r_{2n}]} \frac{(L_X^* \circ H)_{X \circ \Theta_H(R_{2n-1})}(u)}{H_{X \circ \Theta_H(R_{2n-1})}(u)[(\log r_{2n}) \log \log r_{2n}]^{-1}} \leq (1 + c_{35}\varepsilon)/2. \end{aligned}$$

This proves Theorem 1.6 in the case $\kappa = 1$. □

6 Proof of Lemma 3.3

The basic idea goes back to Hu et al. [15], but requires considerable refinements due to the complicated nature of the process $x \mapsto L_X(t, x)$.

Let $\kappa > 0$. Recall $A(x) = \int_0^x e^{W_\kappa(u)} du$, and $A_\infty = \lim_{x \rightarrow +\infty} A(x) < \infty$, a.s. As in Brox [6], the general diffusion theory leads to

$$X(t) = A^{-1}[B(T^{-1}(t))], \quad t \geq 0,$$

where B is a Brownian motion independent of W , $T(r) := \int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\} ds$ for $0 \leq r < \sigma_B(A_\infty)$, and A^{-1} and T^{-1} denote respectively the inverses of A and T . The local time of X can be written as

$$L_X(t, x) = e^{-W_\kappa(x)} L_B(T^{-1}(t), A(x)), \quad t \geq 0, \quad x \in \mathbb{R} \quad (6.1)$$

(see e.g. Shi [26]). Let $H(\cdot)$ be the first hitting time of X as in (1.3). Then

$$H(r) = T[\sigma_B(A(r))] = \int_{-\infty}^{A(r)} e^{-2W_\kappa[A^{-1}(x)]} L_B(\sigma_B[A(r)], x) dx = H_-(r) + H_+(r),$$

where

$$\begin{aligned} H_-(r) &:= \int_{-\infty}^0 e^{-2W_\kappa[A^{-1}(x)]} L_B\{\sigma_B[A(r)], x\} dx, \\ H_+(r) &:= \int_0^{A(r)} e^{-2W_\kappa[A^{-1}(x)]} L_B\{\sigma_B[A(r)], x\} dx. \end{aligned}$$

In view of the first Ray–Knight theorem (Fact 2.1), it is more convenient to study $L_X^*(H(r))$ instead of $L_X^*(t)$. We consider

$$L_X^+(H(r)) := \sup_{x \geq 0} L_X(H(r), x) = \sup_{x \geq 0} \{e^{-W_\kappa(x)} L_B[\sigma_B(A(r)), A(x)]\}.$$

Recall F from (3.1) and notice that $F(r) > 0$ on $E_1(r)$. By the first Ray–Knight theorem (Fact 2.1), there exists a squared Bessel process of dimension 2, starting from 0 and denoted by $(R_2^2(t), t \geq 0)$, independent of W_κ , such that

$$\left(\frac{L_B\{\sigma_B[A(F(r))], A(F(r)) - sA(F(r))\}}{A(F(r))}, s \in [0, 1] \right) = (R_2^2(s), s \in [0, 1]).$$

Therefore,

$$L_X^+[H(F(r))] = \sup_{x \in [0, F(r)]} \left\{ e^{-W_\kappa(x)} A(F(r)) R_2^2 \left[\frac{A(F(r)) - A(x)}{A(F(r))} \right] \right\}.$$

Moreover, by Lamperti's representation theorem (Fact 2.3), there exists a Bessel process $\rho = (\rho(t), t \geq 0)$, of dimension $(2 - 2\kappa)$, starting from $\rho(0) = 2$, such that for all $t \geq 0$, $e^{W_\kappa(t)/2} = \frac{1}{2}\rho(A(t))$. Now, let

$$\tilde{R}_{2+2\kappa}(t) := \rho(A_\infty - t), \quad 0 \leq t \leq A_\infty.$$

By Williams' time reversal theorem (Fact 2.4), $\widetilde{R}_{2+2\kappa}$ is a Bessel process of dimension $(2+2\kappa)$, starting from 0. Since $A(F(r))$ is independent of R_2 , $u \mapsto \sqrt{A(F(r))}R_2(u/A(F(r)))$ is a 2-dimensional Bessel process, starting from 0 and independent of $\widetilde{R}_{2+2\kappa}$. We still denote by R_2 this new Bessel process. We obtain

$$L_X^+[H(F(r))] = 4 \sup_{x \in [0, F(r)]} \frac{R_2^2[A(F(r)) - A(x)]}{\widetilde{R}_{2+2\kappa}^2[A_\infty - A(x)]} = 4 \sup_{v \in [0, A(F(r))]} \frac{R_2^2(v)}{\widetilde{R}_{2+2\kappa}^2[A_\infty - A(F(r)) + v]}.$$

Doing the same transformations on $H_+(r)$ and recalling that $A_\infty - A(F(r)) = \delta(r) = \exp(-\kappa r/2)$, we obtain

$$\begin{aligned} & (L_X^+[H(F(r))], H_+[F(r)]) \\ &= \left(4 \sup_{v \in [0, A(F(r))]} \frac{R_2^2(v)}{\widetilde{R}_{2+2\kappa}^2[\delta(r) + v]}, 16 \int_0^{A(F(r))} \frac{R_2^2(s)}{\widetilde{R}_{2+2\kappa}^4[\delta(r) + s]} ds \right) \\ &= \left(4 \sup_{v \in [0, \delta(r)^{-1}A(F(r))]} \frac{R_2^2[\delta(r)v]}{\widetilde{R}_{2+2\kappa}^2[\delta(r)(1+v)]}, 16 \int_0^{\delta(r)^{-1}A(F(r))} \frac{R_2^2[\delta(r)u]\delta(r)du}{\widetilde{R}_{2+2\kappa}^4[\delta(r)(1+u)]} \right). \end{aligned}$$

We still denote by R_2 the 2-dimensional Bessel process $u \mapsto \frac{1}{\sqrt{\delta(r)}}R_2(\delta(r)u)$. We define

$$\widehat{R}_{2+2\kappa}(u) = \frac{1}{\sqrt{\delta(r)}}\widetilde{R}_{2+2\kappa}[\delta(r)(1+u)], \quad u \geq 0.$$

Notice that $(\widehat{R}_{2+2\kappa}(u), u \geq 0)$ is a $(2+2\kappa)$ -dimensional Bessel process, starting from $\widetilde{R}_{2+2\kappa}(\delta(r))/\sqrt{\delta(r)}$ and independent of R_2 .

Recall (Karlin and Taylor [16] p. 335) that a Jacobi process of dimension (d_1, d_2) is a solution of the stochastic differential equation

$$dY(t) = 2\sqrt{Y(t)(1-Y(t))}d\widehat{\beta}(t) + [d_1 - (d_1 + d_2)Y(t)]dt, \quad (6.2)$$

where $\widehat{\beta}$ is a standard Brownian motion.

According to Warren and Yor [30], there exists a Jacobi process $(Y(t), t \geq 0)$ of dimension $(2, 2+2\kappa)$, starting from 0, independent of $(R_2^2(t) + \widehat{R}_{2+2\kappa}^2(t), t \geq 0)$, such that

$$\forall u \geq 0, \quad \frac{R_2^2(u)}{R_2^2(u) + \widehat{R}_{2+2\kappa}^2(u)} = Y \circ \Lambda_Y(u), \quad \Lambda_Y(u) := \int_0^u \frac{ds}{R_2^2(s) + \widehat{R}_{2+2\kappa}^2(s)}. \quad (6.3)$$

In particular, $(\Lambda_Y(t), t \geq 0)$ is independent of Y . As a consequence, for all $r \geq 0$,

$$\begin{aligned} & (L_X^+[H(F(r))], H_+[F(r)]) \\ &= \left(4 \sup_{u \in [0, \delta(r)^{-1}A(F(r))]} \frac{Y \circ \Lambda_Y(u)}{1 - Y \circ \Lambda_Y(u)}, 16 \int_0^{\delta(r)^{-1}A(F(r))} \frac{[Y \circ \Lambda_Y(u)]\Lambda_Y'(u)du}{[1 - Y \circ \Lambda_Y(u)]^2} \right) \\ &= \left(4 \sup_{u \in [0, \gamma(r)]} \frac{Y(u)}{1 - Y(u)}, 16 \int_0^{\gamma(r)} \frac{Y(u)}{(1 - Y(u))^2} du \right), \end{aligned}$$

where

$$\gamma(r) := \Lambda_Y[\delta(r)^{-1}A(F(r))]. \quad (6.4)$$

Let $\alpha_\kappa := 1/(4 + 2\kappa)$ and let $T_Y(\alpha_\kappa) := \inf\{t > 0, Y(t) = \alpha_\kappa\}$ be the hitting time of α_κ by Y . Define

$$\begin{aligned} \bar{L}(r) &:= 4 \sup_{u \in [0, T_Y(\alpha_\kappa)]} \frac{Y(u)}{1 - Y(u)}, & \bar{H}(r) &:= 16 \int_0^{T_Y(\alpha_\kappa)} \frac{Y(u)}{(1 - Y(u))^2} du, \\ L_0(r) &:= 4 \sup_{u \in [T_Y(\alpha_\kappa), \gamma(r)]} \frac{Y(u)}{1 - Y(u)}, & H_0(r) &:= 16 \int_{T_Y(\alpha_\kappa)}^{\gamma(r)} \frac{Y(u)}{(1 - Y(u))^2} du. \end{aligned} \quad (6.5)$$

We have

$$(L_X^+[H(F(r))], H_+(F(r))) = (\max\{\bar{L}(r), L_0(r)\}, \bar{H}(r) + H_0(r)). \quad (6.6)$$

Moreover, on the event $\{T_Y(\alpha_\kappa) \leq 64 \log r\}$, we have

$$\bar{L}(r) \leq \frac{4\alpha_\kappa}{1 - \alpha_\kappa} \quad \text{and} \quad \bar{H}(r) \leq \frac{2^{10}\alpha_\kappa}{(1 - \alpha_\kappa)^2} \log r. \quad (6.7)$$

Observe that $S(y) := \int_{\alpha_\kappa}^y \frac{dx}{x(1-x)^{1+\kappa}}$ is a scale function of Y . Hence, there exists a Brownian motion $(\beta(t), t \geq 0)$ such that for all $t \geq 0$,

$$Y[t + T_Y(\alpha_\kappa)] = S^{-1}\{\beta[U(t)]\}, \quad U(t) := 4 \int_0^t \frac{ds}{Y[s + T_Y(\alpha_\kappa)]\{1 - Y[s + T_Y(\alpha_\kappa)]\}^{1+2\kappa}}.$$

We now introduce some more technical estimates, stated as Lemmas 6.1–6.4 below. We admit these estimates for the moment, and then complete the proof of Lemma 3.3.

The proof of Lemmas 6.1–6.4 are given in Devulder [10]. Notice that (6.9) is proved during the proof of Lemma 8 in Devulder [10].

Lemma 6.1 *Let $(R(t), t \geq 0)$ be a Bessel process of dimension $d > 4$, starting from $R_0 \stackrel{\mathcal{L}}{=} \tilde{R}_{d-2}(1)$, where $(\tilde{R}_{d-2}(t), t \in [0, 1])$ is $(d-2)$ -dimensional Bessel process. For any $\delta_3 \in (0, \frac{1}{2})$ and all large t ,*

$$\mathbb{P} \left\{ \left| \frac{1}{\log t} \int_0^t \frac{ds}{R^2(s)} - \frac{1}{d-2} \right| > \frac{1}{(\log t)^{(1/2)-\delta_3}} \right\} \leq \exp(-c_{36} (\log t)^{2\delta_3}).$$

Lemma 6.2 *Let $\delta_1 > 0$ and define*

$$E_7 := \left\{ \tau_\beta[(1 - v^{-\delta_1})\lambda v] \leq U(v) \leq \tau_\beta[(1 + v^{-\delta_1})\lambda v] \right\}. \quad (6.8)$$

If δ_1 is small enough, then for all large v , $\mathbb{P}(E_7^c) \leq v^{-1/4+5\delta_1}$.

Lemma 6.3 *Let $\kappa > 0$ and define*

$$L_X^{*-}(+\infty) := \sup_{r \geq 0} \sup_{x < 0} L_X(H(r), x) = \sup_{t \geq 0} \sup_{x < 0} L_X(t, x), \quad H_-(+\infty) := \lim_{r \rightarrow +\infty} H_-(r).$$

There exist $c_{37} > 0$ and $c_{38} > 0$ such that for all large z ,

$$\mathbb{P}(L_X^{*-}(+\infty) > z) \leq \frac{c_{37}}{z^{\kappa/(\kappa+2)}}, \quad (6.9)$$

$$\mathbb{P}(H_-(+\infty) > z) \leq \frac{c_{38}}{(z/\log z)^{\kappa/(\kappa+2)}}. \quad (6.10)$$

Lemma 6.4 *Let $(\beta(t), t \geq 0)$ be a Brownian motion, and let $\lambda = 4(1 + \kappa)$. We define*

$$J_\beta(\kappa, t, \lambda) := \int_0^1 y(1-y)^{\kappa-2} L_\beta(\tau_\beta(\lambda), \frac{S(y)}{t}) dy, \quad 0 < \kappa \leq 1, t \geq 0. \quad (6.11)$$

Let $0 < d < 1$ and let $0 < \varepsilon < 1$.

(i) *Case $0 < \kappa < 1$: recall $K_\beta(\kappa)$ from (3.7). There exist $c_{39} > 0$ and $c_{40} > 0$ such that for all large t , on an event E_8 of probability greater than $1 - c_{39}/t^d$, we have*

$$(1 - \varepsilon)K_\beta(\kappa) - c_{40}t^{1-1/\kappa} \leq \kappa^{2-1/\kappa}t^{1-1/\kappa} J_\beta(\kappa, t, \lambda) \leq (1 + \varepsilon)K_\beta(\kappa) + c_{40}t^{1-1/\kappa}. \quad (6.12)$$

(ii) *Case $\kappa = 1$: recall C_β from (3.8). There exists $c_{39} > 0$ such that for t large enough, on an event E_8 of probability greater than $1 - c_{39}/t^d$,*

$$(1 - \varepsilon)[C_\beta + 8 \log t] \leq J_\beta(1, t, 8) \leq (1 + \varepsilon)[C_\beta + 8 \log t]. \quad (6.13)$$

By admitting Lemmas 6.1–6.4, we can now complete the proof of Lemma 3.3.

Proof of Lemma 3.3: part (i). First, observe that

$$S(y) \underset{y \rightarrow 1}{\sim} \int_{\alpha_\kappa}^y \frac{ds}{(1-s)^{1+\kappa}} \underset{y \rightarrow 1}{\sim} \frac{1}{\kappa} \frac{1}{(1-y)^\kappa}, \quad \frac{y}{1-y} \underset{y \rightarrow 1}{\sim} [\kappa S(y)]^{1/\kappa}. \quad (6.14)$$

Let $\tilde{L}_0(r) := 4 \sup_{u \in [T_Y(\alpha_\kappa), \gamma(r)]} [\kappa S(Y(u))]^{1/\kappa}$. We have,

$$\tilde{L}_0(r) = 4 \sup_{u \in [0, \gamma(r) - T_Y(\alpha_\kappa)]} [\kappa \beta(U(u))]^{1/\kappa} = 4 \sup_{t \in [0, U(\gamma(r) - T_Y(\alpha_\kappa))]} [\kappa \beta(t)]^{1/\kappa}. \quad (6.15)$$

Let $\varepsilon > 0$ and recall L_0 from (6.5). By (6.14), there exists a constant $c_{41} > 0$ depending on ε such that

$$\left\{ \tilde{L}_0(r) > c_{41} \right\} \subset \left\{ (1 - \varepsilon)\tilde{L}_0(r) \leq L_0(r) \leq (1 + \varepsilon)\tilde{L}_0(r) \right\}. \quad (6.16)$$

We look for an estimate of $U[\gamma(r) - T_Y(\alpha_\kappa)]$ appearing in the expression of $\tilde{L}_0(r)$ in (6.15). Recall (Dufresnes [11]) that $A_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter κ , i.e., γ_κ has density $\frac{1}{\Gamma(\kappa)}e^{-x}x^{\kappa-1}$ for positive x .

Notice that $A(F(r)) \leq A_\infty$. Hence,

$$\mathbb{P}[A(F(r)) > r^{2/\kappa}] \leq \mathbb{P}[\gamma_\kappa < 2r^{-2/\kappa}] \leq \frac{2^\kappa r^{-2}}{\kappa \Gamma(\kappa)}.$$

On the other hand, by definition, $A(F(r)) = A_\infty - \delta(r) = A_\infty - e^{-\kappa r/2}$ (see (3.1)). Hence,

$$\mathbb{P}\left[A(F(r)) < \frac{1}{2 \log r}\right] \leq \mathbb{P}\left[\frac{2}{\gamma_\kappa} < \frac{1}{2 \log r} + \delta(r)\right] \leq \frac{1}{\Gamma(\kappa)r^2}.$$

Consequently, for large r ,

$$\mathbb{P}\left\{\frac{\kappa}{2}r - 2 \log \log r \leq \log[\delta(r)^{-1}A(F(r))] \leq \frac{\kappa}{2}r + \frac{2}{\kappa} \log r\right\} \geq 1 - \frac{c_{42}}{r^2}.$$

Recall that $\gamma(r) = \Lambda_Y[\delta(r)^{-1}A(F(r))]$, see (6.4). Thus, for large r ,

$$\mathbb{P}\left\{\Lambda_Y[\exp(\frac{\kappa}{2}r - 2 \log \log r)] \leq \gamma(r) \leq \Lambda_Y[\exp(\frac{\kappa}{2}r + \frac{2}{\kappa} \log r)]\right\} \geq 1 - \frac{c_{42}}{r^2}.$$

By definition, $\Lambda_Y(u) = \int_0^u \frac{ds}{R_2^2(s) + \hat{R}_{2+2\kappa}^2(s)}$. Notice that $(R_2^2(t) + \hat{R}_{2+2\kappa}^2(t), t \geq 0)$ is a $(4 + 2\kappa)$ -dimensional squared Bessel process starting from $\hat{R}_{2+2\kappa}^2(\delta(r))/\delta(r)$. Hence, by Lemma 6.1, there exist constants $\delta_3 \in (0, \frac{1}{2})$, $c_{43} > 0$ and $c_{44} > 0$, such that for all large r ,

$$\mathbb{P}\left\{\frac{\kappa}{\lambda}r - c_{43}r^{1/2+\delta_3} \leq \gamma(r) \leq \frac{\kappa}{\lambda}r + c_{43}r^{1/2+\delta_3}\right\} \geq 1 - \frac{c_{44}}{r^2}, \quad (6.17)$$

where $\lambda = 4(1 + \kappa)$, as before.

To study the behaviour of $T_Y(\alpha_\kappa)$, we go back to the stochastic differential equation in (6.2) satisfied by the Jacobi process $Y(\cdot)$, with $d_1 = 2$ and $d_2 = 2 + 2\kappa$. By the Dubins–Schwarz theorem, there exists a Brownian motion $(\hat{B}(t), t \geq 0)$ such that

$$Y(t) = \hat{B}\left(4 \int_0^t Y(s)(1 - Y(s))ds\right) + \int_0^t [2 - (4 + 2\kappa)Y(s)]ds, \quad t \geq 0.$$

Recall that $\alpha_\kappa = 1/(4 + 2\kappa)$ and consider $t \geq 2\alpha_\kappa$. Since $Y(s) \in (0, 1)$ for any $s \geq 0$, we have, on the event $\{T_Y(\alpha_\kappa) \geq t\}$,

$$\inf_{0 \leq s \leq 4t} \hat{B}(s) \leq \hat{B}\left(4 \int_0^t Y(s)(1 - Y(s))ds\right) \leq \alpha_\kappa - t \leq -\frac{t}{2}.$$

Consequently, for $t \geq 2\alpha_\kappa$,

$$\mathbb{P}(T_Y(\alpha_\kappa) > t) \leq \mathbb{P}\left(\inf_{0 \leq s \leq 4t} \widehat{B}(s) \leq -\frac{t}{2}\right) \leq 2 \exp\left(-\frac{t}{32}\right). \quad (6.18)$$

In particular, this yields $\mathbb{P}[T_Y(\alpha_\kappa) > 64 \log r] \leq \frac{2}{r^2}$ for large r . Plug this into (6.17) and define $\underline{\gamma} = \underline{\gamma}(r) := \frac{\kappa}{\lambda}r - c_{45}r^{1/2+\delta_3}$ and $\overline{\gamma} = \overline{\gamma}(r) := \frac{\kappa}{\lambda}r + c_{43}r^{1/2+\delta_3}$. For $c_{45} > c_{43}$, this gives

$$\mathbb{P}\{U(\underline{\gamma}) \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq U(\overline{\gamma})\} \geq 1 - \frac{c_{46}}{r^2}$$

for large r . By Lemma 6.2, we obtain for small $\delta_1 > 0$ and large r ,

$$\mathbb{P}\left\{\tau_\beta \left[\left(1 - \frac{1}{\underline{\gamma}^{\delta_1}}\right) \lambda \underline{\gamma} \right] \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta \left[\left(1 + \frac{1}{\overline{\gamma}^{\delta_1}}\right) \lambda \overline{\gamma} \right] \right\} \geq 1 - \frac{1}{r^{c_{47}}}.$$

We choose δ_1 such that $\delta_1 < 1/2 - \delta_3$. Hence, for large r , we have $(1 - \frac{1}{\underline{\gamma}^{\delta_1}}) \lambda \underline{\gamma} \geq [1 - 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}] \kappa r = \lambda t_-(r)$ and $(1 + \frac{1}{\overline{\gamma}^{\delta_1}}) \lambda \overline{\gamma} \leq [1 + 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}] \kappa r = \lambda t_+(r)$. (see (3.6)). Thus,

$$\mathbb{P}\{\tau_\beta[\lambda t_-(r)] \leq U[\gamma(r) - T_Y(\alpha_\kappa)] \leq \tau_\beta[\lambda t_+(r)]\} \geq 1 - \frac{1}{r^{c_{47}}}. \quad (6.19)$$

Since $\widehat{L}_\pm(r) = 4 \sup_{t \in [0, \tau_\beta(\lambda t_\pm(r))]} [\kappa \beta(t)]^{1/\kappa}$ (see (3.10)), (6.19) and (6.15) give, for large r ,

$$\mathbb{P}\left\{\widehat{L}_-(r) \leq \widetilde{L}_0(r) \leq \widehat{L}_+(r)\right\} \geq 1 - \frac{1}{r^{c_{47}}}.$$

By (4.2), $\mathbb{P}\{\widehat{L}_-(r) > r^{(1-\delta_1)/\kappa}\} \geq 1 - \frac{1}{r}$, for all large r . Applying (6.16), we obtain for large r ,

$$\mathbb{P}\left\{(1 - \varepsilon)r^{(1-\delta_1)/\kappa} < (1 - \varepsilon)\widehat{L}_-(r) \leq L_0(r) \leq (1 + \varepsilon)\widehat{L}_+(r)\right\} \geq 1 - \frac{1}{r^{c_{48}}}.$$

Recall that $\mathbb{P}[T_Y(\alpha_\kappa) > 64 \log r] \leq \frac{2}{r^2}$ for large r . In view of (6.6) and (6.7), for large r ,

$$\mathbb{P}\left\{(1 - \varepsilon)r^{(1-\delta_1)/\kappa} < (1 - \varepsilon)\widehat{L}_-(r) \leq L_X^+[H(F(r))] \leq (1 + \varepsilon)\widehat{L}_+(r)\right\} \geq 1 - \frac{1}{r^{c_{49}}}.$$

On the other hand, applying Lemma 6.3 to $z = r^{(1-2\delta_1)/\kappa}$ gives $\mathbb{P}(\sup_{x < 0} L_X(H(F(r)), x) > r^{(1-2\delta_1)/\kappa}) \leq \mathbb{P}(L_X^{*-}(+\infty) > r^{(1-2\delta_1)/\kappa}) \leq c_{37}/r^{(1-2\delta_1)/(\kappa+2)}$ for large r , which implies

$$\mathbb{P}\left\{(1 - \varepsilon)\widehat{L}_-(r) \leq L_X^*[H(F(r))] \leq (1 + 2\varepsilon)\widehat{L}_+(r)\right\} \geq 1 - \frac{1}{r^{c_{49}}} - \frac{c_{37}}{r^{(1-2\delta_1)/(\kappa+2)}},$$

proving the first part of Lemma 3.3. \square

Proof of Lemma 3.3: part (ii). In this part, we assume $0 < \kappa \leq 1$.

Recall that $H_0(r) = 16 \int_0^{\gamma(r)-T_Y(\alpha_\kappa)} \frac{Y(u+T_Y(\alpha_\kappa))}{[1-Y(u+T_Y(\alpha_\kappa))]^2} du$, see (6.5). As in Hu et al. ([15], p. 3923), this leads to:

$$H_0(r) = 4 \int_0^1 x(1-x)^{\kappa-2} L_\beta[U(\gamma(r) - T_Y(\alpha_\kappa)), S(x)] dx.$$

By (6.19), we have, for large r ,

$$\mathbb{P}[I'_-(r) \leq H_0(r) \leq I'_+(r)] \geq 1 - r^{-c_{47}}, \quad (6.20)$$

where

$$\begin{aligned} I'_\pm(r) &:= 4 \int_0^1 x(1-x)^{\kappa-2} L_\beta\{\tau_\beta[\lambda t_\pm(r)], S(x)\} dx \\ &= 4t_\pm(r) \int_0^1 x(1-x)^{\kappa-2} L_{\beta_{t_\pm(r)}}\{\tau_{\beta_{t_\pm(r)}}(\lambda), S(x)/t_\pm(r)\} dx, \end{aligned}$$

and as before, $t_\pm(r) = [1 \pm 2(\frac{\lambda}{\kappa})^{\delta_1} r^{-\delta_1}] \frac{\kappa}{\lambda} r$ and $\beta_v(s) = \beta(v^2 s)/v$. Let J_β be as in (6.11). Then

$$I'_\pm(r) = 4t_\pm(r) J_{\beta_{t_\pm(r)}}[\kappa, t_\pm(r), \lambda].$$

Now, apply Lemma 6.4 to $d = 1/2$. We obtain for large r ,

$$\mathbb{P}\left\{(1-\varepsilon)\widehat{I}_-(r) \leq H_0(r) \leq (1+\varepsilon)\widehat{I}_+(r)\right\} \geq 1 - \frac{1}{r^{c_{50}}}, \quad (6.21)$$

where $\widehat{I}_\pm(r)$ is defined in (3.12).

In the case $0 < \kappa < 1$, we use once again the inequality $\mathbb{P}[T_Y(\alpha_\kappa) \leq 64 \log r] \geq 1 - 2r^{-2}$ (for large r) and the estimate (6.7), to see that $\mathbb{P}[\overline{H}(r) \leq c_{51} \log r] \geq 1 - 2r^{-2}$, for some c_{51} and all large r . On the other hand, by Lemma 6.3, $\mathbb{P}[H_-(F(r)) \leq \varepsilon r] \geq \mathbb{P}[H_-(+\infty) \leq \varepsilon r] \geq 1 - \frac{c_{52}}{r^{(1-\delta_1)\kappa/(\kappa+2)}}$, for all large r . Consequently, by (6.21) and (6.6), for large r ,

$$\mathbb{P}\left[(1-\varepsilon)\widehat{I}_-(r) \leq H(F(r)) \leq (1+\varepsilon)\widehat{I}_+(r) + \frac{4\varepsilon\lambda}{\kappa} t_+(r)\right] \geq 1 - \frac{1}{r^{c_{53}}}.$$

Changing the value of c_3 , this proves Lemma 3.3 (ii) in the case $0 < \kappa < 1$.

Now we consider the case $\kappa = 1$. As before, $\mathbb{P}[H_-(F(r)) + \overline{H}(r) \leq 2\varepsilon r] \geq 1 - \frac{1}{r^{c_{54}}}$ (for large r). By Biane and Yor [4], there exists $c_{55} > 0$ such that $C_{\beta_{t_\pm(r)}} \stackrel{\mathcal{L}}{=} \frac{\pi}{2} C_8^{ca} + c_{55}$. Recall from Devulder ([10], equation (3.12)) that $\mathbb{P}[C_8^{ca} \leq -x \log r] \leq r^{-x}$ for $x > 0$. Hence, $\mathbb{P}[C_{\beta_{t_\pm(r)}} > -\pi \log r] \geq 1 - r^{-2}$. Therefore, (3.12) gives $\mathbb{P}\left\{\widehat{I}_+(r) \geq 16t_+(r) \log r\right\} \geq 1 - r^{-2}$. Consequently, for large r ,

$$\mathbb{P}[0 \leq H_-(F(r)) + \overline{H}(r) \leq \varepsilon \widehat{I}_+(r)] \geq 1 - \frac{1}{r^{c_{56}}},$$

which, in view of (6.21), yields that, for large r ,

$$\mathbb{P}[(1-\varepsilon)\widehat{I}_-(r) \leq H(F(r)) \leq (1+2\varepsilon)\widehat{I}_+(r)] \geq 1 - \frac{1}{r^{c_{57}}}.$$

This proves Lemma 3.3 (ii) in the case $\kappa = 1$. □

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