

# The minimality of the map $\frac{x}{\|x\|}$ for weighted energy.

Jean-Christophe Bourgain

LABORATOIRE DE MATHÉMATIQUES ET PHYSIQUES THÉORIQUE, UNIVERSITÉ DE TOURS, PARC GRANDMONT 37200 TOURS

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## abstract

In this paper, we investigate the minimality of the map  $\frac{x}{\|x\|}$  from the euclidean unit ball  $\mathbf{B}^n$  to its boundary  $\mathbb{S}^{n-1}$  for weighted energy functionals of the type  $E_{p,f} = \int_{\mathbf{B}^n} f(r) \|\nabla u\|^p dx$ , where  $f$  is a non-negative function. We prove that in each of the two following cases :

i)  $p = 1$  and  $f$  is non-decreasing,

ii)  $p$  is an integer,  $p \leq n - 1$  and  $f = r^\alpha$  with  $\alpha \geq 0$ ,

the map  $\frac{x}{\|x\|}$  minimizes  $E_{p,f}$  among the maps in  $W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$  which coincide with  $\frac{x}{\|x\|}$  on  $\partial\mathbf{B}^n$ . We also study the case where  $f(r) = r^\alpha$  with  $-n+2 < \alpha < 0$  and prove that  $\frac{x}{\|x\|}$  does not minimize  $E_{p,f}$  for  $\alpha$  close to  $-n + 2$  and when  $n \geq 6$ , for  $\alpha$  close to  $4 - n$ .

**Keys Words** : minimizing map,  $p$ -harmonic map,  $p$ -energy, weighted energy.

## 0.1 Introduction and statement of results

For  $n \geq 3$ , the map  $u_0(x) = \frac{x}{\|x\|} : \mathbf{B}^n \longrightarrow \mathbb{S}^{n-1}$  from the unit ball  $\mathbf{B}^n$  of  $\mathbb{R}^n$  to its boundary  $\mathbb{S}^{n-1}$  plays a crucial role in the study of certain natural energy functionals. In particular, since the works of Hildebrandt, Kaul and Widman ([13]), this map is considered as a natural candidate to realize, for each real number  $p \in [1, n)$  the minimum of the  $p$ -energy functional,

$$E_p(u) = \int_{\mathbf{B}^n} \|\nabla u\|^p dx$$

among the maps  $u \in W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1}) = \{u \in W^{1,p}(\mathbf{B}^n, \mathbb{R}^n; \|u\| = 1 \text{ a.e.}\}$  satisfying  $u(x) = x$  on  $\mathbb{S}^{n-1}$ .

This question was first treated in the case  $p = 2$ . Indeed, the minimality of  $u_0$  for  $E_2$  was established by Jäger and Kaul ([16]) in dimension  $n \geq 7$  and by Brezis, Coron and Lieb in dimension 3 ([2]). In [5], Coron and Gulliver proved the minimality of  $u_0$  for  $E_p$  for any integer  $p \in \{1, \dots, n-1\}$  and any dimension  $n \geq 3$ .

Lin ([17]) has introduced the use of the elegant null Lagrangian method (or calibration method) in this topic. Avellaneda and Lin showed the efficiency of this method in [1] where they give a simpler alternative proof to the Coron-Gulliver result. Note that several results concerning the minimizing properties of  $p$ -harmonic diffeomorphisms were also obtained in this way in particular by Coron, Helein and El Soufi, Sandier ([4], [12], [7] and [6]).

The case of non-integer  $p$  seemed to be rather difficult. It is only ten years after the Coron-Gulliver article [5], that Hardt, Lin and Wang ([10]) succeeded to prove that, for all  $n \geq 3$ , the map  $u_0$  minimizes  $E_p$  for  $p \in [n-1, n)$ . Their proof is based on a deep studies of singularities of harmonic and minimizing maps made in the last two decades. In dimension  $n \geq 7$ , Wang ([20]) and Hong ([14]) have independently proved the minimality of  $u_0$  for any  $p \geq 2$  satisfying  $p + 2\sqrt{p} \leq n - 2$ .

In [15], Hong remarked that the minimality of the  $p$ -energy  $E_p$ ,  $p \in (2, n-1]$ , is related to the minimization of the following weighted 2-energy :

$$\tilde{E}_p(u) = \int_{\mathbf{B}^n} r^{2-p} \|\nabla u\|^2 dx$$

where  $r = \|x\|$ . Indeed, using Hölder inequality, it is easy to see that if the map  $u_0$  minimizes  $\tilde{E}_p$ , then it also minimizes  $E_p$  (see [15], p.465). Unfortunately, as we will see in Corollary 1.1 below, for many values of  $p \in (2, n)$ , the map  $u_0$  is not a minimizer of  $\tilde{E}_p$ . Therefore, Theorem 6 of ([15]), asserting that  $u_0$  minimizes  $\tilde{E}_p$  seems to be not correct and the question of whether  $u_0$  is a minimizing map of the  $p$ -energy  $E_p$  for non-integer  $p \in (2, n-1)$  is still open <sup>1</sup>

The aim of this paper is to study the minimizing properties of the map  $u_0$  in regard to some weighted energy functionals of the form :

$$E_{p,f}(u) = \int_{\mathbf{B}^n} f(r) \|\nabla u\|^p dx,$$

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<sup>1</sup>We suspect a problem in Theorem 6 p.464 of [15]. Indeed the author claims that the quantity  $G_{\varphi_1^0, \dots, \varphi_{n-1}^0}(v, p)$ , which represents a weighted energy of the map  $v$  on the 3-dimensional cone  $\mathcal{C}_0$  in  $\mathbf{B}^n$ , is uniformly proportional to the weighted energy on the euclidian ball  $\mathbf{B}^3$ . There is no reason for this fact to be true, the orthogonal projection of  $\mathcal{C}_0$  on to  $\mathbf{B}^n$  being not homothetic.

where  $p \in \{1, \dots, n-1\}$  and  $f: [0, 1] \rightarrow \mathbb{R}$  is a non-negative non-decreasing continuous function. For  $p = 1$ , the map  $u_0$  minimizes  $E_{1,f}$  for a large class of weights. Indeed, we have the following

**Theorem 0.1** *Suppose that  $f$  is a non-negative differentiable non-decreasing function. Then the map  $u_0 = \frac{x}{\|x\|}$  is a minimizer of the energy  $E_{1,f}$ , that is, for any  $u$  in  $W^{1,1}(\mathbf{B}^n, \mathbb{S}^{n-1})$  with  $u(x) = x$  on  $\mathbb{S}^{n-1}$ , we have*

$$\int_{\mathbf{B}^n} f(r) \|\nabla u_0\| dx \leq \int_{\mathbf{B}^n} f(r) \|\nabla u\| dx,$$

*Moreover, if  $f$  has no critical points in  $(0, 1)$ , then the map  $u_0 = \frac{x}{\|x\|}$  is the unique minimizer of the energy  $E_{1,f}$ , that is, the equality in the last inequality holds if and only if  $u = u_0$ .*

For  $p \geq 2$ , we restrict ourselves to power functions  $f(r) = r^\alpha$ ,

**Theorem 0.2** *For any  $\alpha \geq 0$  and any integer  $p \in \{1, \dots, n-1\}$ , the map  $u_0 = \frac{x}{\|x\|}$  is a minimizer of the energy  $E_{p,r^\alpha}$  that is, for any  $u$  in  $W^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$  with  $u(x) = x$  on  $\mathbb{S}^{n-1}$ , we have,*

$$\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^p dx \leq \int_{\mathbf{B}^n} r^\alpha \|\nabla u\|^p dx.$$

*Moreover, if  $\alpha > 0$ , then the map  $u_0 = \frac{x}{\|x\|}$  is the unique minimizer of the energy  $E_{p,r^\alpha}$ , that is the equality in the last inequality holds if and only if  $u = u_0$ .*

The proof of these two theorems is given in section 2. It is based on a construction of an adapted null-Lagrangian. The case of  $p = 1$  can be obtained passing through more direct ways and will be treated independently.

The case of weights of the form  $f(r) = r^\alpha$ , with  $\alpha < 0$ , is treated in section 3. The weighted energy  $\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx$  of  $u_0 = \frac{x}{\|x\|}$  is finite for  $\alpha > -n + 2$ . Hence we consider the family of maps,

$$u_a(x) = a + \lambda_a(x)(x - a), \quad a \in \mathbf{B}^n,$$

where  $\lambda_a(x) \in \mathbb{R}$  is chosen such that  $u_a(x) \in \mathbb{S}^{n-1}$  (that is  $u_a(x)$  is the intersection point of  $\mathbb{S}^{n-1}$  with the half-line of origin  $a$  passing by  $x$ ).

We study the energy  $E_{2,r^\alpha}(u_a)$  of these maps and deduce the following theorem.

**Theorem 0.3** Suppose that  $n \geq 3$ .

(i) For any  $a \in \mathbf{B}^n, a \neq 0$ , there exists a negative real number  $\alpha_0 \in (-n+2, 0)$ , such that, for any  $\alpha \in (-n+2, \alpha_0]$  we have

$$\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx \quad .$$

(ii) For any integer  $n \geq 6$ , there exists  $\alpha_0 \in (4-n, 5-n)$  such that, for any  $\alpha \in (4-n, \alpha_0)$ , there exists  $a \in \mathbf{B}^n$  such that,

$$\int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx \quad .$$

Replacing in Theorem 0.3  $\alpha$  by  $2-p$ ,  $p \in (2, n)$ , we obtain the following corollary :

**Corollary 0.1** For any  $n \geq 6$ , there exists  $p_0 \in (n-3, n-2)$  such that, for any  $p \in (p_0, n-2)$  the map  $u_0 = \frac{x}{\|x\|}$  does not minimize the functional  $\int_{\mathbf{B}^n} r^{2-p} \|\nabla u\|^2 dx$  among the maps  $u \in W^{1,2}(\mathbf{B}^n, \mathbb{S}^{n-1})$  satisfying  $u(x) = x$  on  $\mathbb{S}^{n-1}$ .

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## 0.2 Proof of theorems 0.1 and 0.2

Consider an integer  $p \in \{1, \dots, n-1\}$  and  $f$  a differentiable, non-negative, increasing, and non-identically zero map. We can suppose without loss of generality, that  $f(1) = 1$ .

For any subset  $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n-1\}$  with  $i_1 < i_2 \dots < i_p$  and for any map,

$$u = (u_1, \dots, u_n) : \mathbf{B}^n \longrightarrow \mathbb{S}^{n-1} \quad \text{in } \mathcal{C}^\infty(\mathbf{B}^n, \mathbb{S}^{n-1}) \quad \text{with } u(x) = x \text{ on } \mathbb{S}^{n-1},$$

we consider the n-form :

$$\omega_I(u) = dx_1 \wedge \dots \wedge d(f(r)u_{i_1}) \wedge \dots \wedge d(f(r)u_{i_k}) \wedge \dots \wedge dx_n$$

**Lemma 0.1** We have the identity :

$$\int_{\mathbf{B}^n} \omega_I(u) = \int_{\mathbf{B}^n} \omega_I(Id) \quad \forall x \in \mathbf{B}^n \quad \text{where } Id(x) = x.$$

**Proof** By Stokes theorem, we have :

$$\begin{aligned}
\int_{\mathbf{B}^n} \omega_I(u) &= \int_{\mathbf{B}^n} dx_1 \wedge \cdots \wedge d(f(r)u_{i_1}) \wedge \cdots \wedge d(f(r)u_{i_p}) \wedge \cdots \wedge dx_n \\
&= \int_{\mathbf{B}^n} (-1)^{i_1-1} d\left(f(r)u_{i_1} dx_1 \wedge \cdots \wedge d(\widehat{f(r)u_{i_1}}) \wedge \right. \\
&\quad \left. \cdots \wedge d(f(r)u_{i_p}) \wedge \cdots \wedge dx_n\right) \\
&= \int_{\mathbb{S}^{n-1}} (-1)^{i_1-1} x_{i_1} dx_1 \wedge \cdots \wedge d(\widehat{f(r)u_{i_1}}) \wedge \\
&\quad \cdots \wedge d(f(r)u_{i_p}) \wedge \cdots \wedge dx_n.
\end{aligned}$$

Indeed, on  $\mathbb{S}^{n-1}$ , we have  $f(r)u_{i_1} = x_{i_1}$  ( $r = 1, f(1) = 1$  and  $u(x) = x$ ). Iterating, we get the designed identities. Consider the n-form :

$$S(u) = \sum_{|I|=p} w_I(u)$$

By Lemma 0.1, we have :

$$\int_{\mathbf{B}^n} S(u) = \sum_{|I|=p} \int_{\mathbf{B}^n} w_I(u) = \sum_{|I|=p} \int_{\mathbf{B}^n} dx = C_n^p \frac{|\mathbb{S}^{n-1}|}{n},$$

where  $|\mathbb{S}^{n-1}|$  is the Lebesgue measure of the sphere.

**Lemma 0.2** *The n-form  $S(u)$  is  $O(n)$ -equivariant, that is, for any rotation  $R$  in  $O(n)$ , we have :*

$$S({}^tRuR)({}^tRx) = S(u)(x) \quad \forall x \in \mathbf{B}^n.$$

**Proof** Consider  $S(u)(x)(e_1, \dots, e_n)$  where  $(e_1, \dots, e_n)$  is the standard basis of  $\mathbb{R}^n$  and notice that it is equal to  $(-1)^n$  times the  $(p+1)^{th}$  coefficient of the polynomial  $P(\lambda) = \det(\text{Jac}(fu)(x) - \lambda Id)$  which does not change when we replace  $fu$  by  ${}^tRfuR$ .

For any  $x \in \mathbf{B}^n$ , let  $R \in O(n)$  be such that  ${}^tRu(x) = e_n = (0, \dots, 0, 1)$ . Consider  $y = {}^tRx$ ,  $v = {}^tRuR$ , so that :

$$v(y) = e_n, \quad d({}^tRuR)(y)(\mathbb{R}^n) \subset e_n^\perp \quad \text{that is} \quad \frac{\partial v_n}{\partial x_j}(y) = 0 \quad \forall j \in \{1, \dots, n\}.$$

**Lemma 0.3** *Let  $a_1, \dots, a_n$  be  $n$  non-negative numbers, and  $p \in \{1, \dots, n-1\}$ . Then :*

$$\sum_{i_1 < \dots < i_p} a_{i_1} \cdots a_{i_p} \leq \frac{1}{(n-1)^p} C_{n-1}^p \left( \sum_{j=1}^{n-1} a_j \right)^p.$$

**Proof** See for instance Hardy coll.[4], theorem 52.

Let  $I = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ . We have :  
if  $i_p \neq n$ ,

$$\begin{aligned} \omega_I(v)(y) &= (dx_1 \wedge \cdots \wedge d(f(r)v_{i_1}) \wedge \cdots \wedge d(f(r)v_{i_k}) \wedge \cdots \wedge dx_n)(y) \\ &= |f(r)|^p (dx_1 \wedge \cdots \wedge dv_{i_1} \wedge \cdots \wedge dv_{i_k} \wedge \cdots \wedge dx_n)(y). \end{aligned}$$

Indeed,  $\forall j \leq n-1$ ,  $d(f(r)v_j(y)) = d(f(r))v_j(y) + f(r)dv_j(y) = f(r)dv_j(y)$   
since  $v(y) = e_n$ .

If  $i_p = n$ ,

$$\omega_I(v)(y) = |f(r)|^{p-1} (dx_1 \wedge \cdots \wedge dv_{i_1} \wedge \cdots \wedge df)(y).$$

Indeed,  $d(f(r)v_n)(y) = df(y)v_n(y) + f(r)dv_n(y) = df(y)$  (as  $dv(y) \subset e_n^\perp$ ). The Hadamard inequality gives :

$$\begin{aligned} |S(v)(y)| &= \left| \sum_{|I|=p} \omega_I(v)(y) \right| \leq |f(r)|^p \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n-1} \|dx_1\| \cdots \|dv_{i_1}\| \\ &\quad \cdots \|dv_{i_p}\| \cdots \|dx_n\|(y) \\ &+ |f(r)|^{p-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{p-1} \leq n-1} \|dx_1\| \cdots \|dv_{i_1}\| \\ &\quad \cdots \|dv_{i_p}\| \cdots \|df\|(y) \\ &\leq |f(r)|^p \left( \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n-1} \|dx_1\|^2 \cdots \|dv_{i_1}\|^2 \cdots \right. \\ &\quad \left. \cdots \|dv_{i_p}\|^2 \cdots \|dx_n\|^2(y) \right)^{\frac{1}{2}} (C_n^p)^{\frac{1}{2}} \\ &+ |f(r)|^{p-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{p-1} \leq n-1} \|dx_1\| \cdots \|dv_{i_1}\| \\ &\quad \cdots \|dv_{i_p}\|(y). \end{aligned}$$

The Hardy inequality gives, after integration and using the fact that  $\|\nabla u\| = \|\nabla v\|$ ,

$$\begin{aligned} \frac{C_n^p}{n} |\mathbb{S}^{n-1}| &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} f^p(r) \|\nabla u\|^p dx \\ &\quad + \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \int_{\mathbf{B}^n} f'(r) f^{p-1}(r) \|\nabla u\|^{p-1} dx. \quad (1) \end{aligned}$$

**Remark :** If  $f'$  is positive and if equality holds in (1), then,  $\forall i \leq n-1$ ,  $y_i = 0$  and  $y_n = \pm \frac{x}{\|x\|}$ , which implies that  $u(x) = \pm \frac{x}{\|x\|}$ .

**Proof of the Theorem 1.1** Inequality (1) give

$$|\mathbb{S}^{n-1}| \leq \sqrt{n-1} \int_{\mathbf{B}^n} f(r) \|\nabla u\| dx + \int_{\mathbf{B}^n} f'(r) dx.$$

Hence :

$$\begin{aligned} \int_{\mathbf{B}^n} f \|\nabla u\| dx &\geq \frac{|\mathbb{S}^{n-1}|}{\sqrt{n-1}} \left(1 - \int_0^1 f'(r) r^{n-1} dr\right) \\ \int_{\mathbf{B}^n} f \|\nabla u\| dx &\geq \sqrt{n-1} |\mathbb{S}^{n-1}| \int_0^1 f(r) r^{n-2} dr = \int_{\mathbf{B}^n} f(r) \|\nabla u_0\| dx. \end{aligned}$$

To see the uniqueness it suffices to refer to the remark above. It gives that for any  $x \in \mathbf{B}^n$ ,  $u(x) = \frac{x}{\|x\|}$  or  $u(x) = -\frac{x}{\|x\|}$ . As  $u(x) = x$  on the unit sphere, we have, for any  $x \in \mathbf{B}^n \setminus \{0\}$ ,  $u(x) = \frac{x}{\|x\|}$ .  $\blacksquare$

**Proof of the Theorem 1.2.** Let  $\alpha$  be a positive real number. From inequality (1) we have :

$$\frac{C_n^p}{n} |\mathbb{S}^{n-1}| \leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \int_{\mathbf{B}^n} r^{\alpha p-1} \|\nabla u\|^{p-1} dx.$$

By Hölder inequality, we have, setting  $q = \frac{p}{p-1}$  :

$$\begin{aligned} \frac{C_n^p}{n} |\mathbb{S}^{n-1}| &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \\ &\quad + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \left( \int_{\mathbf{B}^n} r^{p(\alpha-1)} dx \right)^{1/p} \left( \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \right)^{1/q} \\ &\leq \frac{C_{n-1}^p}{(n-1)^{p/2}} \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \\ &\quad + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \frac{|\mathbb{S}^{n-1}|^{1/p}}{(n+p(\alpha-1))^{1/p}} \left( \int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p dx \right)^{1/q} \end{aligned}$$

Consider the polynomial function :

$$P(t) = \frac{C_{n-1}^p}{(n-1)^{p/2}} t^q + \alpha \frac{C_{n-1}^{p-1}}{(n-1)^{\frac{p-1}{2}}} \frac{|\mathbb{S}^{n-1}|^{1/p}}{(n+p(\alpha-1))^{1/p}} t - \frac{C_n^p}{n} |\mathbb{S}^{n-1}|.$$

Setting  $A = \left(\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u\|^p\right)^{1/q}$  and  $B = \left(\int_{\mathbf{B}^n} r^{\alpha p} \|\nabla u_0\|^p\right)^{1/q}$ , we get  $P(A) \geq 0$  while

$$\begin{aligned} P(B) &= \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| + \alpha \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| - \frac{C_n^p}{n} |\mathbb{S}^{n-1}| \\ &= \frac{C_{n-1}^{p-1}}{n+p(\alpha-1)} |\mathbb{S}^{n-1}| \left( \frac{n-p}{n} + \alpha - \frac{C_n^p}{nC_{n-1}^{p-1}} (n+p(\alpha-1)) \right) \\ &= 0. \end{aligned}$$

On the other hand,  $\forall t \geq 0$ ,  $P'(t) > 0$ . Hence,  $P$  is increasing in  $[0, +\infty)$  and is equal to zero only for  $B$ . Necessarily, we have  $A \geq B$ .

Moreover, if  $\alpha > 0$ ,  $A = B$  implies that equality in the inequality (1) holds. Referring to the remark above, and as  $u_0(x) = x$  on the sphere, we have  $u = u_0 = \frac{x}{\|x\|}$ . Replacing  $\alpha$  by  $\alpha/p$  we finish the prove of the theorem.  $\blacksquare$

### 0.3 The energy of a natural family of maps.

Let  $a = (\theta, \dots, 0)$  be a point of  $\mathbf{B}^n$  with  $0 < \theta < 1$  and consider the map,

$$u_a(x) = a + \lambda_a(x)(x - a),$$

where  $\lambda_a(x) > 0$  is chosen so that  $u_a(x) \in \mathbb{S}^{n-1}$  for any  $x \in \mathbf{B}^n \setminus \{0\}$ ,

$$\lambda_a(x) = \frac{\sqrt{\Delta_a(x)} - (a|x - a)}{\|x - a\|^2}$$

and

$$\Delta_a(x) = (1 - \|a\|^2)\|x - a\|^2 + (a|x - a)^2.$$



Notice that  $u_a(x) = x$  as soon as  $x$  is on the sphere. If we denote by  $\{e_i\}_{i \in \{1, \dots, n\}}$  the standard basis of  $\mathbb{R}^n$ , then,  $\forall i \leq n$ , we have,

$$\begin{aligned}
\|du_a(x).e_i\|^2 &= \left( \frac{\sqrt{\Delta_a} - (a|x-a)}{\|x-a\|^2} \right)^2 \\
&+ \left[ -2 \frac{(x-a|e_i)}{\|x-a\|^4} \left( \sqrt{\Delta_a} - (a|x-a) \right) \right. \\
&+ \frac{(1-\|a\|^2)(x-a|e_i) + (x-a|a)(a|e_i)}{\sqrt{\Delta_a}\|x-a\|^2} \\
&\left. - \frac{(a|e_i)}{\|x-a\|^2} \right]^2 \|x-a\|^2 \\
&+ 2 \left( \frac{\sqrt{\Delta_a} - (a|x-a)}{\|x-a\|^2} \right) \left( -2 \frac{(x-a|e_i)}{\|x-a\|^4} \left( \sqrt{\Delta_a} - (a|x-a) \right) \right. \\
&+ \frac{(1-\|a\|^2)(x-a|e_i) + (x-a|a)(a|e_i)}{\sqrt{\Delta_a}\|x-a\|^2} \\
&\left. - \frac{(a|e_i)}{\|x-a\|^2} \right) (x-a|e_i).
\end{aligned}$$

Let us prove that, for each  $\alpha \in (-n, 0)$ ,  $\int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\| dx$  is finite. Consider the map :

$$\begin{aligned}
F : \mathbb{R}^+ \times \mathbb{S}^{n-1} &\longrightarrow \mathbb{R}^n \\
(r, s) &\longmapsto a + rs = x.
\end{aligned}$$

Then, we have,

$$F^*(\|\nabla u_a\|^2 dx) = \frac{1}{r^2} \sum_{i=1}^n H_{i,a}(s) r^{n-1} dr \wedge ds,$$

where  $H_{i,a}(s)$  is given on the sphere by,

$$\begin{aligned}
H_{i,a}(s) &= \left( (1 - \|a\|^2 + (a|s)^2)^{1/2} - (a|s) \right)^2 \\
&+ \left[ -2(s|e_i) \left( (1 - \|a\|^2 + (a|s)^2)^{1/2} - (s|a) \right) \right. \\
&\quad \left. + \frac{(1 - \|a\|^2)(s|e_i) + (a|e_i)(s|a)}{(1 - \|a\|^2 + (a|s)^2)^{1/2}} - (a|e_i) \right]^2 \\
&+ 2 \left( (1 - \|a\|^2 + (a|s)^2)^{1/2} - (a|s) \right) \\
&\quad \left( -2(s|e_i) \left( (1 - \|a\|^2 + (a|s)^2)^{1/2} - (s|a) \right) \right. \\
&\quad \left. + \frac{(1 - \|a\|^2)(s|e_i) + (a|e_i)(s|a)}{(1 - \|a\|^2 + (a|s)^2)^{1/2}} - (a|e_i) \right) (s|e_i).
\end{aligned}$$

It is clear that  $H_{i,a}(s)$  is continuous on  $\mathbb{S}^{n-1}$ . Therefore, near the point  $a$ , as  $n \geq 3$ , the map  $\|x\|^\alpha \|\nabla u_a\|$  is integrable. Furthermore, near the point  $0$ , as  $\alpha > -n$ , this map is also integrable. In conclusion, for any  $\alpha \in (-n, 0)$ , the energy  $E_{r^\alpha, 2}(u_a)$  is finite.

**Proof of Theorem 1.3(i).** Since we have

$$E_{2,r^\alpha}(u_0) = \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_0\|^2 dx = \frac{|\mathbb{S}^{n-1}|(n-1)}{n+\alpha-2},$$

the energy  $E_{2,r^\alpha}(u_0)$  goes to infinity as  $\alpha \rightarrow -n+2$ . On the other hand, as the energy  $E_{2,r^\alpha}(u_a)$  is continuous in  $\alpha$ , there exists a real number  $\alpha_0 \in (-n+2, 0)$  such that,  $\forall \alpha, 2-n < \alpha \leq \alpha_0$ ,

$$\int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_0\|^2 dx > \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_a\|^2 dx.$$

**Proof of Theorem 1.3(ii).** Since  $a = (\theta, 0, \dots, 0)$ , we will study the function,

$$G(\theta) = E_{2,r^\alpha}(u_a) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx.$$

Precisely, we will show that for any  $\alpha \in (5-n, 4-n)$ ,  $G$  is two times differentiable at  $\theta = 0$  with  $\frac{dG}{d\theta}(0) = 0$  and, when  $\alpha$  is sufficiently close to  $4-n$ ,  $\frac{d^2G}{d\theta^2}(0) < 0$ . Assertion (ii) of Theorem 1.3 then follows immediately.

We have,

$$\begin{aligned}
H_{i,a}(s) = H_{i,\theta}(s) &= \left( \sqrt{1-\theta^2+\theta^2s_1^2} - \theta s_1 \right)^2 \\
&+ \left( -2s_i(\sqrt{1-\theta^2+\theta^2s_1^2} - \theta s_1) + \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2s_1}{\sqrt{1-\theta^2+\theta^2s_1^2}} - \delta_{i1}\theta \right)^2 \\
&+ 2 \left( \sqrt{1-\theta^2+\theta^2s_1^2} - \theta s_1 \right) \left( -2s_i(\sqrt{1-\theta^2+\theta^2s_1^2} - \theta s_1) \right. \\
&\quad \left. + \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2s_1}{\sqrt{1-\theta^2+\theta^2s_1^2}} - \delta_{i1}\theta \right) s_i,
\end{aligned}$$

where  $\delta_{ij} = 0$  if  $i \neq j$  and 1 else.

We notice that  $H_{i,\theta}(s)$  is bounded on  $[0, 1] \times \mathbb{S}^{n-1}$ . Indeed, for all  $x, y, z \in [0, 1]$ , excepting  $(x, y) = (0, 1)$ , we have,

$$\left| \frac{x}{\sqrt{1-y^2+y^2x^2}} \right| \leq 1 \quad \text{and} \quad \left| \frac{(1-y^2)z}{\sqrt{1-y^2+y^2x^2}} \right| \leq 1.$$

Then, for almost all  $(s, \theta) \in \mathbb{S}^{n-1} \times [0, 1]$ , we have,

$$\left| \frac{(1-\theta^2)s_i + \delta_{i1}\theta^2s_1}{\sqrt{1-\theta^2+\theta^2s_1^2}} \right| \leq 1,$$

and the others terms are continuous in  $[0, 1] \times \mathbb{S}^{n-1}$ .

We have,

$$\begin{aligned}
E_{2,r^\alpha}(u_a) &= \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u_a\|^2 dx = \int_{\mathbf{B}^n} \|a + rs\|^\alpha r^{n-3} H(\theta, s) dr ds \\
&= \int_{\mathbb{S}^{n-1}} H(\theta, s) \left( \int_0^{\gamma_\theta(s)} ((r + \theta s_1)^2 + \theta^2(1-s_1^2))^{\alpha/2} r^{n-3} dr \right) ds,
\end{aligned}$$

where  $\gamma_\theta(s) = \sqrt{1-\theta^2+\theta^2s_1^2} - \theta s_1$  and  $H(\theta, s) = \sum_{i=1}^n H_{i,\theta}(s)$ . We notice that  $H(\theta, s)$  is indefinitely differentiable in  $(-1/2, 1/2) \times \mathbb{S}^{n-1}$ . Let  $C_n$  be a positive real number so that,  $\forall(\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1}$

$$|H(\theta, s)| \leq C_n, \quad \left| \frac{\partial H(\theta, s)}{\partial \theta} \right| \leq C_n, \quad \left| \frac{\partial^2 H(\theta, s)}{\partial \theta^2} \right| \leq C_n.$$

Furthermore, we have,

$$H(\theta, s) = (n-1) - 2(n-1)s_1\theta + ((2n-3)s_1^2 - n + 2)\theta^2 + o(\theta^2). \quad (A)$$

Let us set  $\rho = r + \theta s_1$ ,  $\beta(\theta, s) = \sqrt{1 - \theta^2 + \theta^2 s_1^2}$  and

$$F(\theta, s) = \int_{\theta s_1}^{\beta(\theta, s)} (\rho - \theta s_1)^{n-3} (\rho^2 + \theta^2 (1 - s_1^2))^{\alpha/2} d\rho.$$

Notice that  $\rho \in [-1, 3]$ . Then,  $G(\theta) = \int_{\mathbb{S}^{n-1}} H(\theta, s) F(\theta, s) ds$ . Let us set  $g(\rho, \theta, s) = (\rho - \theta s_1)^{n-3} (\rho^2 + \theta^2 (1 - s_1^2))^{\alpha/2}$ .

**Lemma 0.4** *The map  $\theta \mapsto G(\theta)$  is continuous on  $(-1/2, 1/2)$  and continuously differentiable on  $(-1/2, 1/2) \setminus \{0\}$  for any  $\alpha > 3 - n$ .*

**Proof** We have,  $\forall s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$\frac{(\rho - \theta s_1)^2}{(\rho^2 + \theta^2 (1 - s_1^2))} \leq \frac{2}{1 - s_1^2} \quad (1.1)$$

Indeed,  $(1 - s_1^2)(\rho - \theta s_1)^2 \leq 2(1 - s_1^2)(\rho^2 + \theta^2) \leq 2(\rho^2 + \theta^2(1 - s_1^2))$ . And then,

$$g(\rho, \theta, s) \leq \frac{2^{\frac{n-3}{2}}}{(1 - s_1^2)^{\frac{n-3}{2}}} (\rho^2 + \theta^2 (1 - s_1^2))^{\frac{\alpha+n-3}{2}}. \quad (1.2)$$

Since  $\alpha > 3 - n$  we deduce that the map  $(\rho, \theta) \rightarrow g(\rho, \theta, s)$  is continuous on  $(-1/2, 1/2) \times [-1, 3]$ . Hence, the map  $z \mapsto \int_0^z g(\rho, \theta, s) d\rho$  is differentiable on  $[-1, 3]$  and,

$$\frac{\partial}{\partial z} \int_0^z g(\rho, \theta, s) d\rho = g(z, \theta, s).$$

Furthermore, for any  $\rho \in [-1, 3]$ , the map  $\theta \mapsto g(\rho, \theta, s)$  is differentiable and

$$\begin{aligned} \frac{\partial g}{\partial \theta}(\rho, \theta, s) &= -(n-3)s_1(\rho - \theta s_1)^{n-4} (\rho^2 + \theta^2 (1 - s_1^2))^{\frac{\alpha}{2}} \\ &+ \frac{\alpha}{2} (\rho - \theta s_1)^{n-3} 2\theta (1 - s_1^2) (\rho^2 + \theta^2 (1 - s_1^2))^{\frac{\alpha}{2}-1}. \end{aligned}$$

Let  $a, b$  be two real in  $(0, 1/2)$  with  $a < b$ . We have for any  $|\theta| \in (a, b)$ , for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$\begin{aligned} \left| \frac{\partial g}{\partial \theta}(\rho, \theta, s) \right| &\leq (n-3)4^{n-4} (a^2(1 - s_1^2))^{\frac{\alpha}{2}} \\ &+ |\alpha|4^{n-3} (1 - s_1^2) (a^2(1 - s_1^2))^{\frac{\alpha}{2}-1}. \quad (1.3) \end{aligned}$$

This shows that  $\theta \mapsto \int_0^z g(\rho, \theta, s) d\rho$  is differentiable on  $(-1/2, 1/2) \setminus \{0\}$  and

$$\frac{\partial}{\partial \theta} \int_0^z g(\rho, \theta, s) d\rho = \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho.$$

Moreover the map  $(z, \theta) \mapsto \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho$  is continuous in  $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$ . Indeed,  $\theta \mapsto \frac{\partial g}{\partial \theta}(\rho, \theta, s)$  is clearly continuous on  $(-1/2, 1/2) \setminus \{0\}$  and from (1.3) and by Lebesgue Theorem,  $\theta \mapsto \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho$  is continuous on  $(-1/2, 1/2) \setminus \{0\}$ . Then, for any  $\epsilon > 0$ , we will have for any sufficiently small  $h, k$ ,

$$\begin{aligned} \left| \int_0^{z+h} \frac{\partial g}{\partial \theta}(\rho, \theta + k, s) d\rho - \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho \right| &\leq \left| \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta + k, s) d\rho \right. \\ &\quad \left. - \int_0^z \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho \right| \\ &\quad + \left| \int_z^{z+h} \frac{\partial g}{\partial \theta}(\rho, \theta + k, s) d\rho \right| \\ &\leq \epsilon. \end{aligned}$$

The map  $(z, \theta) \mapsto \int_0^z g(\rho, \theta, s) d\rho$  is differentiable on  $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$  and the map  $\theta \mapsto F(\theta, s)$  is differentiable in  $(-1/2, 1/2) \setminus \{0\}$  and for any  $\theta \in (-1/2, 1/2) \setminus \{0\}$ ,

$$\begin{aligned} \frac{\partial F}{\partial \theta}(\theta, s) &= \frac{\partial \beta}{\partial \theta}(\theta, s) g(\beta(\theta, s), \theta, s) - s_1 g(\theta s_1, \theta, s) + \int_{\theta s_1}^{\beta(\theta, s)} \frac{\partial g}{\partial \theta}(\rho, \theta, s) d\rho \\ &= \frac{\theta(s_1^2 - 1)}{(1 - \theta^2 + \theta^2 s_1^2)^{1/2}} ((1 - \theta^2 + \theta^2 s_1^2)^{1/2} - \theta s_1)^{n-3} \\ &\quad + \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho + \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho, \end{aligned}$$

where,

$$g_1(\rho, \theta, s) = -(n-3)s_1(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha}{2}}$$

and

$$g_2(\rho, \theta, s) = \frac{\alpha}{2}(\rho - \theta s_1)^{n-3} 2\theta(1 - s_1^2)(\rho^2 + \theta^2(1 - s_1^2))^{\frac{\alpha}{2}-1}.$$

Now, the map  $\theta \mapsto F(\theta, s)$  is continuous on  $(-1/2, 1/2)$ . Indeed, since the map  $\theta \mapsto g(\rho, \theta, s) d\rho$  is continuous on  $(-1/2, 1/2)$  and from (1.2)  $\theta \mapsto \int_0^z g(\rho, \theta, s) d\rho$  is continuous on  $(-1/2, 1/2)$ . Then, for any  $\epsilon > 0$ , we have  $\forall h, k$  sufficiently small,

$$\begin{aligned} \left| \int_0^{z+h} g(\rho, \theta + k, s) d\rho - \int_0^z g(\rho, \theta, s) d\rho \right| &\leq \left| \int_0^z g(\rho, \theta + k, s) d\rho \right. \\ &\quad \left. - \int_0^z g(\rho, \theta, s) d\rho \right| \\ &\quad + \left| \int_z^{z+h} g(\rho, \theta + k, s) d\rho \right| \\ &\leq \epsilon. \end{aligned}$$

Then, the map  $(z, \theta) \mapsto \int_0^z g(\rho, \theta, s) d\rho$  is continuous on  $[-1, 3] \times (-1/2, 1/2)$  and consequently  $\theta \mapsto F(\theta, s)$  is continuous on  $(-1/2, 1/2)$ .

Now, we know that  $\theta \mapsto H(\theta, s)F(\theta, s)$  is continuous on  $(-1/2, 1/2)$  and differentiable on  $(-1/2, 1/2) \setminus \{0\}$ . Furthermore from (1.2), we have, for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$|H(\theta, s)F(\theta, s)| \leq 3.2^{\frac{n-3}{2}} 10^{\frac{\alpha+n-3}{2}} C_n \cdot \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}. \quad (1.4)$$

$$\left| \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) \right| \leq 3.2^{\frac{n-3}{2}} 10^{\frac{\alpha+n-3}{2}} C_n \cdot \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}. \quad (1.5)$$

Consider the map  $\eta : (\theta, s) \mapsto \eta(\theta, s) = \frac{\theta(s_1^2-1)}{\sqrt{1-\theta^2+\theta^2 s_1^2}} ((1-\theta^2+\theta^2 s_1^2)^{1/2} - \theta s_1)^{n-3}$ . This map is indefinitely differentiable on  $(-1/2, 1/2) \times \mathbb{S}^{n-1}$ . Let  $B_n$  be a positive real number so that,  $\forall (\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1}$ ,

$$|\eta(\theta, s)| \leq B_n \quad \left| \frac{\partial \eta}{\partial \theta}(\theta, s) \right| \leq B_n.$$

Considering  $a, b \in (0, 1/2)$  with  $a < b$  we have, for any  $\theta \in (a, b)$ , for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$\begin{aligned} \left| H(\theta, s) \frac{\partial F}{\partial \theta}(\theta, s) \right| &\leq \left( B_n + 3(n-3) \cdot 4^{n-4} \cdot a^\alpha (1-s_1^2)^{\frac{\alpha}{2}} \right. \\ &\quad \left. + |3\alpha| \cdot 4^{n-3} a^{\alpha-1} (1-s_1^2)^{\frac{\alpha}{2}} \right) C_n. \end{aligned} \quad (1.6)$$

Since the maps  $s \mapsto \frac{1}{(1-s_1^2)^{\frac{n-3}{2}}}$  and  $s \mapsto (1-s_1^2)^{\frac{\alpha}{2}}$  are integrable on  $\mathbb{S}^{n-1}$ , we deduce that  $\theta \mapsto G(\theta)$  is continuous on  $(-1/2, 1/2)$  and continuously differentiable on  $(-1/2, 1/2) \setminus \{0\}$ .

**Lemma 0.5** *The map  $\theta \mapsto G(\theta)$  is differentiable at 0 and  $\frac{dG}{d\theta}(0) = 0$ .*

**Proof** Since for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,  $\theta \mapsto F(\theta, s)$  is continuous on  $(-1/2, 1/2)$  from (A) we have,

$$\frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) \xrightarrow{\theta \rightarrow 0} \frac{\partial H}{\partial \theta}(0, s)F(0, s) = -2(n-1)s_1 \int_0^1 \rho^{n-3+\alpha} d\rho = \frac{-2(n-1)s_1}{n-2+\alpha}.$$

From (1.5) and Lebesgue Theorem we have,

$$\int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s) ds \xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} \frac{-2(n-1)s_1}{n-2+\alpha} ds = 0.$$

Moreover, it is clear that,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s)\eta(\theta, s)ds \xrightarrow{\theta \rightarrow 0} 0.$$

Let  $J(m, n)$  be the integral,

$$J(m, n) = \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (\sqrt{1-s_1^2}t - s_1)^m (t^2 + 1)^n dt.$$

Notice that  $J(m, n)$  converges as  $\theta$  goes to 0 if and only if  $m + 2n < -1$ . Consider the change of variables  $\rho = t\theta\sqrt{1-s_1^2}$  if  $\theta > 0$ . If  $\theta < 0$ , then we set  $\rho = -t\theta\sqrt{1-s_1^2}$  and conclusion will be the same. Hence, we assume that  $\theta > 0$ . Then,

$$\int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho = -(n-3)s_1(1-s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-3+\alpha} J(n-4, \frac{\alpha}{2}).$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho = \alpha \theta^{n-3+\alpha} (1-s_1^2)^{\frac{1+\alpha}{2}} J(n-3, \frac{\alpha}{2} - 1).$$

**First case :**  $\alpha \geq 4 - n$ .

$J(n-4, \frac{\alpha}{2})$  and  $J(n-3, \frac{\alpha}{2} - 1)$  go to  $+\infty$  as  $\theta \rightarrow 0$ . Furthermore, we have,

$$J(n-4, \frac{\alpha}{2}) \underset{0}{\sim} (1-s_1^2)^{\frac{n-4}{2}} \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} t^{n-4+\alpha} dt$$

$$J(n-4, \frac{\alpha}{2}) \underset{0}{\sim} \frac{1}{n-3+\alpha} \frac{1}{\theta^{n-3+\alpha}} (1-s_1^2)^{\frac{-1-\alpha}{2}}.$$

Since  $t^{n+\alpha-5}$  may be equal to zero at zero, we write,

$$J(n-3, \frac{\alpha}{2} - 1) = \int_{\frac{s_1}{\sqrt{1-s_1^2}}}^1 (\sqrt{1-s_1^2}t - s_1)^{n-3} (t^2 + 1)^{\frac{\alpha}{2}-1} dt$$

$$+ \int_1^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (\sqrt{1-s_1^2}t - s_1)^{n-3} (t^2 + 1)^{\frac{\alpha}{2}-1} dt.$$

We have,

$$J(n-3, \frac{\alpha}{2} - 1) \underset{0}{\sim} (1-s_1^2)^{\frac{n-3}{2}} \int_1^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} t^{n-5+\alpha} dt.$$

Then, if  $\alpha \neq 4 - n$ ,

$$J(n-3, \frac{\alpha}{2} - 1) \underset{0}{\sim} \frac{1}{n-4+\alpha} \frac{1}{\theta^{n-4+\alpha}} (1-s_1^2)^{\frac{1-\alpha}{2}},$$

and note that if  $\alpha = 4 - n$ ,  $J(n-3, \frac{\alpha}{2} - 1) \sim_0 -(1-s_1^2)^{\frac{n-3}{2}} \ln(\theta^2(1-s_1^2))$ . Hence, by (A) we have,

$$\begin{aligned} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho &= -H(\theta, s)(n-3)s_1(1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} I_1 \\ &\xrightarrow{\theta \rightarrow 0} -\frac{(n-3)(n-1)}{n-3+\alpha} s_1, \end{aligned}$$

and

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho = H(\theta, s)\alpha(1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} I_2 \xrightarrow{\theta \rightarrow 0} 0.$$

Observe that  $\frac{|s_1|}{\sqrt{1-s_1^2}} \leq \sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}$ . Indeed,  $s_1^2 \theta^2 \leq 1 - \theta^2 + \theta^2 s_1^2$ . It follows from (1.1) that

$$(\rho - \theta s_1)^{n-4} \leq \frac{2^{\frac{n-4}{2}}}{(1-s_1^2)^{\frac{n-4}{2}}} (\rho^2 + \theta^2(1-s_1^2))^{\frac{n-4}{2}}.$$

Recall that  $\rho = t\theta\sqrt{1-s_1^2}$ . Since  $\alpha \geq 4 - n$ , we have, for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$\begin{aligned} \left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho \right| &\leq 2C_n(n-3)2^{\frac{n-4}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1}} (t^2 + 1)^{\frac{n-4+\alpha}{2}} dt \\ &\leq C_n(n-3)2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \sqrt{\frac{1}{\theta^2(1-s_1^2)} - 1} \left( \frac{1}{\theta^2(1-s_1^2)} \right)^{\frac{n-4+\alpha}{2}} \\ &\leq C_n(n-3)2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{-n+4}{2}} \sqrt{1 - \theta^2(1-s_1^2)} \\ &\leq C_n(n-3)2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{-n+4}{2}}. \end{aligned}$$

Since  $s \mapsto (1-s_1^2)^{\frac{-n+4}{2}}$  is integrable on  $\mathbb{S}^{n-1}$ , by Lebesgue Theorem we have,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} - \int_{\mathbb{S}^{n-1}} \frac{(n-3)(n-1)}{n-3+\alpha} s_1 ds = 0.$$



Moreover, we have, for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ , since  $\alpha + n - 5 \geq 0$ ,

$$\begin{aligned}
\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho \right| &\leq 2C_n |\alpha| 2^{\frac{n-3}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (t^2+1)^{\frac{n-5+\alpha}{2}} dt \\
&\leq C_n |\alpha| 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} \frac{1}{(t^2+1)} dt \left( \frac{1}{\theta^2(1-s_1^2)} \right)^{\frac{n-3+\alpha}{2}} \\
&\leq C_n |\alpha| 2^{\frac{n-2}{2}} \frac{\pi}{2} (1-s_1^2)^{\frac{-n+4}{2}}.
\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} 0.$$

**Second case :**  $3 - n < \alpha < 4 - n$ .

For the same reasons that when  $\alpha \geq 4 - n$ , we have,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} -\frac{(n-3)(n-1)}{n-3+\alpha} s_1.$$

Furthermore, as  $4 - n > \alpha > 3 - n$ ,  $\forall s \in \mathbb{S}^{n-1} \setminus \{(-1, 0, \dots, 0), (1, 0, \dots, 0)\}$ ,

$$\begin{aligned}
\left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho \right| &\leq 2C_n (n-3) 2^{\frac{n-4}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (t^2+1)^{\frac{n-4+\alpha}{2}} dt \\
&\leq C_n (n-3) 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\
&\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (t^2)^{\frac{n-4+\alpha}{2}} dt \\
&\leq \frac{C_n (n-3) 2^{\frac{n-2}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha}}{n-3+\alpha} \left( \frac{1}{\theta^2(1-s_1^2)} - 1 \right)^{\frac{n-3+\alpha}{2}}. \\
&\leq \frac{C_n (n-3) 2^{\frac{2n-7+\alpha}{2}} (1-s_1^2)^{\frac{4-n}{2}}}{n-3+\alpha}.
\end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} -\int_{\mathbb{S}^{n-1}} \frac{(n-3)(n-1)}{n-3+\alpha} s_1 ds = 0.$$

Moreover,  $J(n-3, \frac{\alpha}{2}-1)$  is finite when  $\theta \rightarrow 0$  then, as  $\alpha > 3-n$ , Furthermore,

$$H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} 0.$$

$$\begin{aligned} \left| H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho \right| &\leq 2C_n |\alpha| 2^{\frac{n-3}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} (t^2+1)^{\frac{n+\alpha-5}{2}} dt \\ &\leq C_n |\alpha| 2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{\alpha+1}{2}} \theta^{n-3+\alpha} \\ &\quad \times \int_0^{\sqrt{\frac{1}{\theta^2(1-s_1^2)}-1}} \frac{1}{(t^2+1)} dt \left( \frac{1}{\theta^2(1-s_1^2)} \right)^{\frac{n-3+\alpha}{2}} \\ &\leq C_n |\alpha| 2^{\frac{n-1}{2}} (1-s_1^2)^{\frac{-n+4}{2}} \int_0^{+\infty} \frac{1}{(t^2+1)} dt. \end{aligned}$$

Then, by Lebesgue Theorem,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho ds \xrightarrow{\theta \rightarrow 0} 0.$$

Finally, we have

$$\frac{dG}{d\theta}(\theta) \xrightarrow{\theta \rightarrow 0} 0.$$

By Lemma 1.4 we deduce that  $G$  is differentiable at 0 and  $\frac{dG}{d\theta}(0) = 0$ .

**Lemma 0.6** *The map  $\theta \rightarrow G(\theta)$  is two times differentiable on  $(-1/2, 1/2) \setminus \{0\}$ .*

**Proof** We know that the map  $\theta \rightarrow \frac{\partial H}{\partial \theta}(\theta, s)F(\theta, s)$  is differentiable on  $(-1/2, 1/2) \setminus \{0\}$ . The maps  $\theta \rightarrow \eta(\theta, s)$ ,  $\theta \rightarrow g_1(\rho, \theta, s)$ ,  $\theta \rightarrow g_2(\rho, \theta, s)$  are differentiable on  $(-1/2, 1/2) \setminus \{0\}$ . We have,

$$\begin{aligned} \frac{\partial \eta}{\partial \theta}(\theta, s) &= \frac{(s_1^2-1)\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta(s_1^2-1)\frac{\theta(s_1^2-1)}{\sqrt{1-\theta^2+\theta^2 s_1^2}}}{1-\theta^2+\theta^2 s_1^2} \\ &\quad \times (\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1)^{n-3} \\ &\quad + \frac{(n-3)\theta(s_1^2-1)}{\sqrt{1-\theta^2+\theta^2 s_1^2}} \left( \frac{\theta(s_1^2-1)}{\sqrt{1-\theta^2+\theta^2 s_1^2}} - s_1 \right) \\ &\quad \times (\sqrt{1-\theta^2+\theta^2 s_1^2} - \theta s_1)^{n-4}. \end{aligned}$$

$$\begin{aligned}\frac{\partial g_1}{\partial \theta}(\rho, \theta, s) &= (n-3)(n-4)s_1^2(\rho - \theta s_1)^{n-5}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}} \\ &\quad - \alpha(n-3)s_1(1-s_1^2)\theta(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}.\end{aligned}$$

$$\begin{aligned}\frac{\partial g_2}{\partial \theta}(\rho, \theta, s) &= -\alpha(n-3)s_1(1-s_1^2)\theta(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1} \\ &\quad + \alpha(\alpha-2)(1-s_1^2)^2\theta^2(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-2} \\ &\quad + \alpha(1-s_1^2)(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}.\end{aligned}$$

We set,

$$\begin{aligned}g_{11}(\rho, \theta, s) &= (n-3)(n-4)s_1^2(\rho - \theta s_1)^{n-5}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}}, \\ g_{12}(\rho, \theta, s) &= -2\alpha(n-3)s_1(1-s_1^2)\theta(\rho - \theta s_1)^{n-4}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}, \\ g_{21}(\rho, \theta, s) &= \alpha(\alpha-2)(1-s_1^2)^2\theta^2(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-2}, \\ g_{22}(\rho, \theta, s) &= \alpha(1-s_1^2)(\rho - \theta s_1)^{n-3}(\rho^2 + \theta^2(1-s_1^2))^{\frac{\alpha}{2}-1}.\end{aligned}$$

Let  $a, b \in (0, 1/2)$  with  $a < b$ . We have,  $\forall s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$\left| \frac{\partial g_1}{\partial \theta}(\rho, \theta, s) \right| \leq (n-3)(n-4)4^{n-5}a^\alpha(1-s_1^2)^{\frac{\alpha}{2}} + |\alpha|(n-3)4^{n-4}a^{\alpha-1}(1-s_1^2)^{\frac{\alpha}{2}}. \quad (1.7)$$

$$\begin{aligned}\left| \frac{\partial g_2}{\partial \theta}(\rho, \theta, s) \right| &\leq |\alpha(\alpha-2)|4^{n-3}a^{\alpha-2}(1-s_1^2)^{\frac{\alpha}{2}} \\ &\quad + |\alpha|4^{n-3}a^{\alpha-1}(1-s_1^2)^{\frac{\alpha}{2}} \\ &\quad + |\alpha|(n-3)4^{n-4}a^{\alpha-1}(1-s_1^2)^{\frac{\alpha}{2}}. \quad (1.8)\end{aligned}$$

Then, for any  $i \in \{1, 2\}$ , the maps  $\theta \mapsto \int_0^z g_i(\rho, \theta, s)d\rho$  is differentiable on  $(0, 1/2)$ , and

$$\frac{\partial}{\partial \theta} \int_0^z g_i(\rho, \theta, s)d\rho = \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho.$$

Furthermore, for any  $i \in \{1, 2\}$ ,  $\theta \mapsto \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)$  is continuous on  $(-1/2, 1/2) \setminus \{0\}$ , then,  $\theta \mapsto \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho$ , is continuous on  $(-1/2, 1/2) \setminus \{0\}$ . Hence, for any  $i \in \{1, 2\}$  and for any  $\epsilon > 0$ , we have  $\forall h, k$  two sufficiently small,

$$\begin{aligned}\left| \int_0^{z+h} \frac{\partial g_i}{\partial \theta}(\rho, \theta+k, s)d\rho - \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho \right| &\leq \left| \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta+k, s)d\rho \right. \\ &\quad \left. - \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s)d\rho \right| \\ &\quad + \left| \int_z^{z+h} \frac{\partial g_i}{\partial \theta}(\rho, \theta+k, s)d\rho \right| \\ &\leq \epsilon.\end{aligned}$$

This proves that for any  $i \in \{1, 2\}$ ,  $(z, \theta) \mapsto \int_0^z \frac{\partial g_i}{\partial \theta}(\rho, \theta, s) d\rho$  is continuous on  $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$ . Moreover, for any  $i \in \{1, 2\}$  the map  $\rho \mapsto g_i(\rho, \theta, s)$  is continuous on  $[-1, 3]$  for any  $\theta \in (-1/2, 1/2) \setminus \{0\}$ . Then,  $z \mapsto \int_0^z g_i(\rho, \theta, s) d\rho$  is differentiable on  $[-1, 3]$  for any  $\theta \in (-1/2, 1/2) \setminus \{0\}$  and  $\frac{\partial}{\partial z} \int_0^z g_i(\rho, \theta, s) d\rho = g_i(z, \theta, s)$ .

Since  $(z, \theta) \mapsto g_i(z, \theta)$  is continuous on  $[-1, 3] \times (-1/2, 1/2) \setminus \{0\}$  we finally deduce that for any  $i \in \{1, 2\}$ ,  $\theta \mapsto \int_{\theta s_1}^{\beta(\theta, s)} g_i(\rho, \theta, s) d\rho$  is differentiable on  $(-1/2, 1/2) \setminus \{0\}$  and,

$$\begin{aligned} \sum_{i=1}^2 \frac{\partial}{\partial \theta} \int_{\theta s_1}^{\beta(\theta, s)} g_i(\rho, \theta, s) d\rho &= \frac{\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}} \\ &\quad \times \left( -(n-3)s_1(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-4} \right. \\ &\quad \left. + \alpha\theta(1 - s_1^2)(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-3} \right) \\ &\quad + \sum_{i=1}^2 \int_{\theta s_1}^{\beta(\theta, s)} \frac{\partial^2 g_i}{\partial^2 \theta}(\rho, \theta, s) d\rho. \end{aligned}$$

We deduce that  $\theta \mapsto \frac{\partial F}{\partial \theta}$  is differentiable in  $(-1/2, 1/2) \setminus \{0\}$ . Moreover, we see that the map,

$$\begin{aligned} \theta \mapsto \lambda(\theta, s) &= \frac{\theta(s_1^2 - 1)}{\sqrt{1 - \theta^2 + \theta^2 s_1^2}} \left( -(n-3)s_1(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-4} \right. \\ &\quad \left. + \alpha\theta(1 - s_1^2)(\sqrt{1 - \theta^2 + \theta^2 s_1^2} - \theta s_1)^{n-3} \right) \end{aligned}$$

is indefinitely differentiable on  $(-1/2, 1/2) \times \mathbb{S}^{n-1}$ . Then, by (1.1), (1.2), (1.8), (1.7), (1.3) and (A), for any  $a, b \in (0, 1/2)$ ,  $a < b$  there exists constants  $K_{1,n,ab,\alpha}$ ,  $K_{2,n,ab,\alpha}$ ,  $K_{3,n,ab,\alpha}$  so that, for any  $|\theta| \in (a, b)$ , for any  $s \in \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$\left| \frac{\partial^2 HF}{\partial \theta^2}(\theta, s) \right| \leq K_{1,n,ab,\alpha}(1 - s_1^2)^{\frac{\alpha}{2}} + K_{2,n,ab,\alpha}(1 - s_1^2)^{\frac{3-n}{2}} + K_{3,n,ab,\alpha}.$$

We deduce by Lebesgue Theorem that the map  $\theta \mapsto E(\theta)$  is two times differentiable on  $(-1/2, 1/2) \setminus \{0\}$  and,

$$\frac{d^2 G}{d\theta^2}(\theta) = \int_{\mathbb{S}^{n-1}} \frac{\partial^2 HF}{\partial \theta^2}(\theta, s) ds.$$

**Lemma 0.7** *If  $5 - n > \alpha > 4 - n$ , the map  $\theta \mapsto G(\theta)$  is two times differentiable at 0.*

**Proof** Suppose that  $\alpha \in (4 - n, 5 - n)$ . As in Lemma 1.5, we can see that,

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \frac{\partial^2 H}{\partial \theta^2}(\theta, s) F(\theta, s) ds &\xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} \frac{1}{2} \frac{(2n-3)s_1^2 - (n-2)}{n-2+\alpha} ds = \frac{-n^2 + 4n - 3}{2n(n-2+\alpha)} |\mathbb{S}^{n-1}|, \\ \int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \eta(\theta, s) ds &\xrightarrow{\theta \rightarrow 0} 0, \\ \int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_1(\rho, \theta, s) d\rho &\xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} \frac{2(n-3)(n-1)}{n-3+\alpha} s_1^2 = \frac{2(n-3)(n-1)}{n(n-3+\alpha)} |\mathbb{S}^{n-1}|, \\ \int_{\mathbb{S}^{n-1}} \frac{\partial H}{\partial \theta}(\theta, s) \int_{\theta s_1}^{\beta(\theta, s)} g_2(\rho, \theta, s) d\rho &\xrightarrow{\theta \rightarrow 0} 0, \\ \int_{\mathbb{S}^{n-1}} H(\theta, s) \frac{\partial \eta}{\partial \theta}(\theta, s) ds &\xrightarrow{\theta \rightarrow 0} \int_{\mathbb{S}^{n-1}} (n-1)(s_1^2 - 1) ds = \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}|, \end{aligned}$$

and

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \lambda(\theta, s) ds \xrightarrow{\theta \rightarrow 0} 0.$$

As in Lemma 1.5, we set  $\rho = \sqrt{1 - s_1^2} \theta t$  if  $\theta > 0$ . Hence,

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{11}(\rho, \theta, s) d\rho = (n-3)(n-4) s_1^2 (1 - s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\left(n-5, \frac{\alpha}{2}\right).$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{12}(\rho, \theta, s) d\rho = -2\alpha(n-3) s_1 (1 - s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\left(n-4, \frac{\alpha}{2} - 1\right)$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{21}(\rho, \theta, s) d\rho = \alpha(\alpha-2) (1 - s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\left(n-3, \frac{\alpha}{2} - 2\right)$$

$$\int_{\theta s_1}^{\beta(\theta, s)} g_{22}(\rho, \theta, s) d\rho = \alpha(1 - s_1^2)^{\frac{1+\alpha}{2}} \theta^{n-4+\alpha} J\left(n-3, \frac{\alpha}{2} - 1\right)$$

Since  $\alpha \in (4 - n, 5 - n)$ , the integrals  $J\left(n-5, \frac{\alpha}{2}\right)$  and  $J\left(n-3, \frac{\alpha}{2} - 1\right)$  are infinite and we have,

$$J\left(n-5, \frac{\alpha}{2}\right) \underset{0}{\sim} \frac{(1 - s_1^2)^{-\frac{1-\alpha}{2}} \theta^{-n-\alpha+4}}{n-4+\alpha}, \quad J\left(n-3, \frac{\alpha}{2} - 1\right) \underset{0}{\sim} \frac{(1 - s_1^2)^{\frac{1-\alpha}{2}} \theta^{-n-\alpha+4}}{n-4+\alpha}$$

And the integrals

$J(n-4, \frac{\alpha}{2}-1)$  and  $J(n-3, \frac{\alpha}{2}-2)$  are finite. Then,

$$\int_{\theta_{s_1}}^{\beta(\theta,s)} g_{11}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} \frac{(n-3)(n-4)s_1^2}{n-4+\alpha}, \quad \int_{\theta_{s_1}}^{\beta(\theta,s)} g_{22}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} \frac{\alpha(1-s_1^2)}{n-4+\alpha}.$$

$$\int_{\theta_{s_1}}^{\beta(\theta,s)} g_{12}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} 0, \quad \int_{\theta_{s_1}}^{\beta(\theta,s)} g_{21}(\rho, \theta, s) d\rho \xrightarrow{\theta \rightarrow 0} 0$$

Moreover, we can see that, for any  $i, j \in \{1, 2\}$ , for any  $(\theta, s) \in (-1/2, 1/2) \times \mathbb{S}^{n-1} \setminus \{(\pm 1, 0, \dots, 0)\}$ ,

$$H(\theta, s) \int_{\theta_{s_1}}^{\beta(\theta,s)} g_{ij}(\rho, \theta, s) d\rho \leq C_{n,\alpha}(1-s_1^2)^{\frac{5-n}{2}} + D_{n,\alpha}(1-s_1^2)^{\frac{\alpha+1}{2}}.$$

where  $C_{n,\alpha}$  and  $D_{n,\alpha}$  are two constants independent of  $\theta$ . By Lebesgue Theorem we deduce that,

$$\int_{\mathbb{S}^{n-1}} H(\theta, s) \frac{\partial^2 F}{\partial \theta^2}(\theta, s) ds \xrightarrow{\theta \rightarrow 0} \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}|$$

$$+ (n-1) \frac{(n-3)(n-4) + \alpha(n-1)}{n(n-4+\alpha)} |\mathbb{S}^{n-1}|.$$

By Lemmas 1.1, 1.2, 1.3,  $\theta \mapsto G(\theta) \in \mathcal{C}^1((-1/2, 1/2), \mathbb{R})$  and is two times differentiable on  $(-1/2, 1/2) \setminus \{0\}$ . Furthermore, when  $\alpha \in (4-n, 5-n)$ , as the limit of  $\frac{d^2 G}{d\theta^2}(\theta)$  exists as  $\theta \rightarrow 0$ , we have  $\theta \mapsto G(\theta)$  is two times differentiable on  $(-1/2, 1/2)$ .

**Proof of ii).** Assume that  $\alpha \in (4-n, 5-n)$ , by Lemma 1.1, 1.2, 1.3, 1.4, we have,

$$G(\theta) = G(0) + \frac{1}{2} \frac{d^2 G}{d\theta^2}(0) + o(\theta^2).$$

Furthermore we have,

$$\frac{d^2 G}{d\theta^2}(0) = \frac{-n^2 + 4n - 3}{2n(n-2+\alpha)} |\mathbb{S}^{n-1}| + \frac{2(n-3)(n-1)}{n(n-3+\alpha)} |\mathbb{S}^{n-1}|$$

$$+ \frac{-(n-1)^2}{n} |\mathbb{S}^{n-1}| + (n-1) \frac{(n-3)(n-4) + \alpha(n-1)}{n(n-4+\alpha)} |\mathbb{S}^{n-1}|.$$

We have, for any  $n \geq 6$ .

$$(n-3)(n-4) + \alpha(n-1) \xrightarrow{\alpha \rightarrow 4-n} -2(n-4) < 0.$$

Then,

$$\frac{(n-3)(n-4) + \alpha(n-1)}{n(n-4+\alpha)} \xrightarrow{\alpha \rightarrow > 4-n} -\infty, \quad \text{and} \quad \frac{d^2 G}{d^2 \theta}(0) \xrightarrow{\alpha \rightarrow > 4-n} -\infty.$$

Hence, there is  $\alpha_0$  such that, for any  $\alpha \in (4 - n, \alpha_0)$ ,  $G(\theta) < G(0)$  for  $\theta$  sufficiently small, that is,

$$G(\theta) = E_{2,r^\alpha}(u_a) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_a\|^2 dx < G(0) = \int_{\mathbf{B}^n} r^\alpha \|\nabla u_0\|^2 dx. \quad \blacksquare$$

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