phase transitions for the multifractal analysis of self-similar measures

Benoit Testud

▶ **To cite this version:**

Benoit Testud. phase transitions for the multifractal analysis of self-similar measures. 2006. hal-00022128

HAL Id: hal-00022128
https://hal.science/hal-00022128

Preprint submitted on 3 Apr 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ccsd-00022128, version 1 - 3 Apr 2006

Phase transitions for the multifractal analysis of self-similar measures

B Testud

MAPMO UMR 6628, Fédération Denis Poisson, Université d'Orléans, BP 6759, 45067 Orléans cedex 2, France

E-mail: benoit.testud@univ-orleans.fr

Abstract. We are interested in the multifractal analysis of a class of self-similar measures with overlaps. This class, for which we obtain explicit formulae for the L^q-spectrum $\tau(q)$ as well as the singularity spectrum $f(\alpha)$, is sufficiently large to point out new phenomena in the multifractal structure of self-similar measures. We show that, unlike the classical quasi-Bernoulli case, the L^q-spectrum $\tau(q)$ of the measures studied can have an arbitrarily large number of non-differentiability points (phase transitions). These singularities occur only for the negative values of q and yield to measures that do not satisfy the usual multifractal formalism. The weak quasi-Bernoulli property is the key point of most of the arguments.

Submitted to: *Nonlinearity*

AMS classification scheme numbers: 28A80, 28A78

1. Introduction

Let us begin with some notation. For an integer $\ell \geq 2$, we denote by $\mathcal{F} = \cup_n \mathcal{F}_n$ where $\mathcal{F}_n$ is the set of the ℓ-adic intervals of the nth generation included in the interval $[0,1)$. In other terms, $\mathcal{F}_n = \{I = [k/\ell^n, (k+1)/\ell^n)\,,\ 0 \leq k < \ell^n\}$. For every $x \in [0,1)$, $I_n(x)$ stands for the unique interval among $\mathcal{F}_n$ containing x.

Let m be a probability measure on the interval $[0,1)$. For $x \in [0,1)$, we define the *local dimension* (also called Hölder exponent) of m at x by

$$\alpha(x) = \lim_{n \to +\infty} -\frac{\log m(I_n(x))}{n \log \ell},$$

provided this limit exists. The aim of multifractal analysis is to find the Hausdorff dimension, $\dim(E_\alpha)$, of the level set $E_\alpha = \{x : \alpha(x) = \alpha\}$ for $\alpha > 0$. The function $f(\alpha) = \dim(E_\alpha)$ is called the *singularity spectrum* (or multifractal spectrum) of m and we say that m is a *multifractal measure* when $f(\alpha) > 0$ for several $\alpha's$.

The concepts underlying the multifractal decomposition of a measure go back to an early paper of Mandelbrot [24]. In the 80's multifractal measures were used by physicists

to study various models arising from natural phenomena. In fully developped turbulence they were used by Frisch and Parisi [14] to investigate the intermittent behaviour in the regions of high vorticity. In dynamical system theory they were used by Benzi et al. [3] to measure how often a given region of the attractor is visited. In diffusion-limited aggregation (DLA) they were used by Meakin et al. [25] to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate.

In order to determine the function $f(\alpha)$, Hentschel and Procaccia [18] used ideas based on Renyi entropies [34] to introduce the generalized dimensions D_q defined by

$$D_q = \lim_{n\to+\infty} \frac{1}{q-1} \frac{\log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right)}{n\log\ell},$$

(see also [15, 16]). From a physical and heuristical point of view, Halsey et al. [17] showed that the singularity spectrum $f(\alpha)$ and the generalized dimensions D_q can be derived from each other. The Legendre transform turned out to be a useful tool linking $f(\alpha)$ and D_q. More precisely, it was suggested that

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q),\ q\in\mathbb{R}), \tag{1.1}$$

where

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n\log\ell}\log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right).$$

(The sum runs over the ℓ-adic intervals I such that $m(I)\neq 0$.) The function $\tau(q)$ is called the L^q-spectrum of m and if the limit exists $\tau(q) = (q-1)D_q$. Note that there may be problems of stability or invariance in the definition of $\tau(q)$ for negative q and Riedi [35] propose an improvement of this definition. In what follows, these difficulties will be avoided by restricting the sums over convenient ℓ-adic intervals defining the measure. Of course, this way is not an option in many applications where the structure of the measure is not known in advance. For more information on the L^q-spectrum and the singularity spectrum we refer the reader to [1, 2, 5, 10, 11, 19, 28, 31, 32, 37, 38, 39, 41].

Relation (1.1) is called the multifractal formalism and in many aspects it is analogous to the well-known thermodynamic formalism developed by Bowen [4] and Ruelle [36].

For number of measures, relation (1.1) can be verified rigorously. In particular, under some separation conditions, self-similar measures satisfy the multifractal formalism (e.g. [6, 7, 9, 23, 30]). Despite all the investigations mentioned, the exact range of the validity of the multifractal formalism is still not known. Furthermore, it is easy to construct measures that do not satisfy (1.1) (e.g [35]). It is thus interesting to find conditions ensuring the validity of (1.1). The main difficulty is often to get a lower bound of $\dim(E_\alpha)$. Usually, such a minoration relies on the existence of an auxiliary measure m_q, the so-called Gibbs measure, supported on the level set E_α. Recall that m_q is a *Gibbs measure* at state q for the measure m if

$$\forall n,\ \forall I\in\mathcal{F}_n,\quad \frac{1}{C} m(I)^q \ell^{-n\tau(q)} \le m_q(I) \le C m(I)^q \ell^{-n\tau(q)},$$

where the constant $C > 0$ is independent of n and I. If τ is differentiable at q, the measure m_q –if it exists– will be supported by $E_{-\tau'(q)}$. In this case, Brown, Michon and Peyrière established [5, 32] that

$$\dim(E_{-\tau'(q)}) = \tau^*(-\tau'(q)) = -q\tau'(q) + \tau(q).$$

In general, to prove the existence of Gibbs measures we need some homogeneity hypotheses on the measure. This is, for instance, the case of quasi-Bernoulli measures : a probability measure m is said to be *quasi-Bernoulli* if there exists a constant $C > 0$ such that

$$\forall (n,p) \in \mathbb{N}^2,\ \forall I \in \mathcal{F}_n,\ \forall J \in \mathcal{F}_p, \quad \frac{1}{C} m(I)m(J) \leq m(IJ) \leq Cm(I)m(J), \tag{1.2}$$

where $IJ = I \cap \sigma^{-n}(J)$ and $\sigma(x) = \ell x \,(\mathrm{mod}\, 1)$ is the shift map on the interval $[0,1)$. In this situation, Brown, Michon and Peyrière [5, 27, 32] proved the existence of a Gibbs measure at every state q. A few years later, Heurteaux [19] showed that τ is differentiable on $\mathbb{R}$. Therefore, for quasi-Bernoulli measures, we have

$$\forall \alpha \in (-\tau'(+\infty), -\tau'(-\infty)), \quad \dim(E_\alpha) = \tau^*(\alpha).$$

Recently, in [37, 39] we introduced a more general condition that we call the weak quasi-Bernoulli property. More precisely, we say that a measure m satisfies the *weak quasi-Bernoulli* property if there exists a constant $C > 0$ and some integers $r_1, r_2, p_1, p_2, s_1, s_2$ such that

$$\begin{cases} \exists C > 0,\ \forall n, \forall p,\ \forall I \in \mathcal{F}_n, \forall J \in \mathcal{F}_p, \\ C^{-1} m(I) \displaystyle\sum_{k=r_1}^{r_2} m(\sigma^{-k}(J)) \leq \sum_{k=p_1}^{p_2} m(I \cap \sigma^{-(n+k)}(J)) \leq Cm(I) \sum_{k=s_1}^{s_2} m(\sigma^{-k}(J)). \end{cases} \tag{1.3}$$

At first sight, this new condition may seem artificial but is in fact natural. Indeed, in [37, 39] we showed that many self-similar measures with overlaps are not quasi-Bernoulli but are weak quasi-Bernoulli and may be used to estimate the dimension of self-affine graphs.

Furthermore, under this condition, we proved in [39] the existence of Gibbs measures at every positive state q and the differentiability of τ on $\mathbb{R}^+$. For weak quasi-Bernoulli measures, we deduced that

$$\forall \alpha \in (-\tau'(+\infty), -\tau'(0)), \quad \dim(E_\alpha) = \tau^*(\alpha).$$

Now, it is natural to ask whether or not these results still hold for negative q when the measure only satisfies the weak quasi-Bernoulli property. In particular, in this setting, we would like to know if

(i) the L^q-spectrum $\tau(q)$ is differentiable on $(-\infty, 0)$,

(ii) there exists Gibbs measures for negative q,

(iii) we have $\dim(E_\alpha) = \tau^*(\alpha)$ for $\alpha > -\tau'(0)$.

Note that the tools used in this context for $q \geq 0$ cannot be applied for $q < 0$. In particular, to prove the existence of Gibbs measures for $q \geq 0$ we use some multiplicative

properties of the sequence $\ell^{n\tau_n(q)}$ which are no longer verified for $q<0$. In what follows, we show that the multifractal formalism may break down for weak quasi-Bernoulli measures. Therefore, for these measures, the answer to the above questions could be no.

Let us precise these examples. For an integer $\ell \geq 2$, we consider the 2ℓ similitudes $S_i : [0,1] \mapsto [0,1]$ defined by

$$\forall\, 0 \leq i \leq \ell-1, \quad S_i(x) = \frac{1}{\ell}x + \frac{i}{\ell} \quad \text{and} \quad S_{i+\ell}(x) = -\frac{1}{\ell}x + \frac{i+1}{\ell}.$$

For a given probability weight $\{p_i\}_{i=0}^{2\ell-1}$, it is well known (e.g [8, 21]) that there exists a unique probability measure μ on $[0,1]$ verifying

$$\mu = \sum_{i=0}^{2\ell-1} p_i\, \mu \circ S_i^{-1}. \tag{1.4}$$

This measure is often called the self-similar measure generated by $\{S_i\}_{i=0}^{2\ell-1}$. In this paper we establish that μ satisfies the weak quasi-Bernoulli property. Moreover, we show that there exists a *Frostman measure* μ_q at every negative state q, i.e. a measure μ_q such that

$$\forall n,\ \forall I \in \mathcal{F}_n, \quad \mu_q(I) \leq C\mu(I)^q \ell^{-n\tau(q)}, \tag{1.5}$$

where the constant $C>0$ is independent of n and I. Thus, for $\alpha = -\tau'(q)$, we have

$$\forall x \in E_\alpha, \quad \mu_q(I_n(x)) \leq (\ell^{-n})^{\tau^*(\alpha)},$$

if n is large enough. The mass distribution principle or Frostman Lemma (e.g. [8]) implies that $\dim(E_\alpha) \geq \tau^*(\alpha)$. Thus, the values of α for which the multifractal formalism may fail lie in intervals $(-\tau'_+(q), -\tau'_-(q))$ where q is a point of non-differentiability of τ ($\tau'_-(q)$ and $\tau'_+(q)$ stand for the left and the right derivative respectively). Such a point q will be called a *phase transition*.

We assume that the weights p_i associated to the measure μ verifying (1.4) are positive for every $0 \leq i \leq \ell-1$. We set $B = \{0 \leq i \leq \ell-1,\ p_{i+\ell} = 0\}$ and $\tilde{\tau}(q) = \log_\ell\left(\sum_{i\in B} {p_i}^q\right)$. In this case, the L^q-spectrum $\tau_\mu(q)$ of μ is given by $\tau_\mu(q) = \max\left(\tau_\nu(q), \tilde{\tau}(q)\right)$ where $\nu = (\mu + \mu\circ T)/2$ and $T(x) = 1-x$. In order to get phase transitions for the function τ_μ, it is thus enough to find conditions on the ${p_i}'s$ ensuring that the equation $\tau_\nu(q) = \tilde{\tau}(q)$ has isolated solutions.

Let K be the compact set defined by $K = \cup_{i\in B} S_i(K)$. The attractor K plays an important role to determine the local dimensions of μ. Indeed, we can link the level sets of μ and ν in the following way : $E_\alpha(\mu) = \left(E_\alpha(\nu) \cap ([0,1]\setminus K)\right) \cup \left(E_\alpha(\mu) \cap K\right)$. If ν satisfies the quasi-Bernoulli property, we get $\dim\left(E_\alpha(\mu)\right) = \max\left({\tau_\nu}^*(\alpha), \tilde{\tau}^*(\alpha)\right)$. Using the expression of τ_μ, we deduce that each phase transition corresponds to an interval in which the multifractal formalism does not hold. More precisely, we have the following.

(i) If ${\tau_\mu}'(q)$ exists and if $\alpha = -{\tau_\mu}'(q)$, then $\dim(E_\alpha(\mu)) = {\tau_\mu}^*(\alpha)$.

(ii) If ${\tau_\mu}'(q)$ does not exist and if $-(\tau_\mu)'_+(q) < \alpha < -(\tau_\mu)'_-(q)$, then $\dim(E_\alpha(\mu)) < {\tau_\mu}^*(\alpha)$.

The class of measures studied may appear rather restrictive but is in fact sufficiently large to point out new and interesting phenomena. In particular, we can observe the following facts.

- The existence of an isolated point in the set of the local dimensions D_μ defined by $D_\mu = \{\alpha, E_\alpha(\mu) \neq \emptyset\}$. This situation has already been obtained for the Erdös measure and for the 3-time convolution of the Cantor measure (e.g [13, 20]).
- The existence of non-concave multifractal spectra eventually supported by a union of mutually disjoint intervals. To the best of our knowledge, it is the first time that such multifractal structures are obtained for self-similar measures.
- The existence of an arbitrarily large number of phase transitions for the L^q-spectrum $\tau(q)$.

The paper is organized as follows. In section 2 we prove that the measure μ, given by (1.4), satisfies the weak quasi-Bernoulli property. In section 3 we establish the existence of Gibbs measures at every negative state q for the measure μ. In section 4 we determine the L^q-spectrum $\tau_\mu(q)$. In section 5 we are interested in the singularity spectrum of μ. The paper ends with a range of examples.

2. The weak quasi-Bernoulli property

Let us introduce some notation. In what follows, except contrary mention, $\ell \geq 2$ is an integer, I a ℓ-adic interval of nth generation and J a ℓ-adic interval. For every $(\epsilon_1, ..., \epsilon_n) \in \{0, ..., \ell-1\}^n$, $I_{\epsilon_1 \cdots \epsilon_n}$ stands for the element of $\mathcal{F}_n$ defined by

$$I_{\epsilon_1 \cdots \epsilon_n} = \left[\sum_{i=1}^{n} \frac{\epsilon_i}{\ell^i}, \sum_{i=1}^{n} \frac{\epsilon_i}{\ell^i} + \frac{1}{\ell^n} \right).$$

If $I = I_{\epsilon_1 \cdots \epsilon_n}$ and $\epsilon \in \{0, \cdots, \ell-1\}$, we shall write ϵI instead of $I_{\epsilon\epsilon_1 \cdots \epsilon_n}$. If f and g are positive functions of the same parameter, $f \approx g$ means there exists a constant $C > 0$ such that $C^{-1} g \leq f \leq C g$. Moreover, for any matrices M and N, we shall write $M > 0$ (and we shall say that M is positive) if all the digits of M are positive and $M > N$ if $M - N > 0$. The matrice relations $<$, $\geq$ and $\leq$ are similarly defined. Finally, for a 2×2 nonnegative matrix M, we define $\|M\|_1$ and $\|M\|$ by

$$\|M\|_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \|M\| = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let μ be the measure verifying (1.4). For convenience, we suppose that μ is supported on the interval $[0, 1]$. That is equivalent to the condition : $p_i + p_{i+\ell} > 0$, for every $i \in \{0, \cdots, \ell-1\}$. The relation (1.4) implies that

$$\forall \epsilon \in \{0, \cdots, \ell-1\}, \quad \mu(\epsilon I) = p_\epsilon \, \mu(I) + p_{\epsilon+\ell} \, \mu(I^*),$$

where $I^* = T(I)$ and $T(x) = 1 - x$. Since $(I^*)^* = I$, we have

$$\begin{pmatrix} \mu(\epsilon I) \\ \mu \circ T(\epsilon I) \end{pmatrix} = M_\epsilon \begin{pmatrix} \mu(I) \\ \mu \circ T(I) \end{pmatrix} \quad \text{where} \quad M_\epsilon = \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_{2\ell-1-\epsilon} & p_{\ell-1-\epsilon} \end{pmatrix}. \tag{2.1}$$

By iterating this relation, we get

$$\forall I = I_{\epsilon_1 \cdots \epsilon_n} \in \mathcal{F}_n, \quad \begin{pmatrix} \mu(I) \\ \mu \circ T(I) \end{pmatrix} = M_{\epsilon_1} \cdots M_{\epsilon_n} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and we deduce that

$$\mu(I) = \|M_I\|_1 \quad \text{and} \quad \nu(I) = \|M_I\|, \quad \text{where} \quad M_I = M_{\epsilon_1} \cdots M_{\epsilon_n}. \tag{2.2}$$

These relations will be used to prove that μ satisfies the weak quasi-Bernoulli property (1.3). More precisely, we have the following.

Proposition 2.1. *Let μ be the measure verifying (1.4). Then,*

$$\begin{cases} \exists C > 0,\, \forall n, \forall p,\, \forall I \in \mathcal{F}_n, \forall J \in \mathcal{F}_p, \\ C^{-1}\mu(I)\mu(J) \leq \mu(I \cap \sigma^{-(n+1)}(J)) \leq C\mu(I)\mu(\sigma^{-2}(J)), \end{cases}$$

where $\sigma : [0,1] \mapsto [0,1]$ is the shift map on the ℓ-adic basis given by $\sigma(x) = \ell x (mod 1)$.

Remarks. 1. In general, the measure μ does not satisfy the quasi-Bernoulli property (1.2). To see that, take for example $\ell = 2$ and suppose that $p_0 > p_1$, $p_0 p_1 p_2 > 0$ and $p_3 = 0$. Using (1.4), we get for $J = I_{1\cdots 1} \in \mathcal{F}_n$,

$$\mu(0J) = \mu(I_{01\cdots 1}) = p_0\mu(J) + p_2\mu(J^*) = p_0\mu(I_{1\cdots 1}) + p_2\mu(I_{0\cdots 0}).$$

From (2.1) and (2.2), $\mu(J) = \|{M_1}^n\|_1 = {p_1}^n$ and $\mu(J^*) = \|{M_0}^n\|_1 \approx {p_0}^n$. Therefore, if $I = I_0$, we have $\mu(IJ) \approx {p_0}^n$ and $\mu(I)\mu(J) \approx {p_1}^n$, which proves that μ is not quasi-Bernoulli.

2. If for every $\epsilon \in \{0, \cdots, \ell-1\}$, $p_\epsilon p_{\epsilon+\ell} = 0$, the Open Set Condition of Hutchinson [21] is verified. In this case, μ is quasi-Bernoulli and proposition 2.1 easily follows.

Proof of proposition 2.1. According to the above remark, we can suppose that there exists $\tilde{\epsilon} \in \{0, \cdots, \ell-1\}$ such that $p_{\tilde{\epsilon}} p_{\tilde{\epsilon}+\ell} > 0$. By (2.1), $M_{\tilde{\epsilon}} + M_{\ell-1-\tilde{\epsilon}} > 0$. Thus, we can find a constant $C > 0$ such that

$$\frac{1}{C}E \leq \sum_{\epsilon=0}^{\ell-1} M_\epsilon \quad \text{and} \quad I_2 \leq CE \sum_{\epsilon=0}^{\ell-1} M_\epsilon,$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from (2.2) that

$$\mu(I \cap \sigma^{-(n+1)}(J)) = \left\| M_I \left(\sum_{\epsilon=0}^{\ell-1} M_\epsilon \right) M_J \right\|_1 \geq \frac{1}{C}\|M_I E M_J\|_1 = \frac{1}{C}\mu(I)\mu(J).$$

On the other hand, we obtain in a similar way that

$$\mu(I \cap \sigma^{-n}(J)) = \|M_I M_J\|_1 \leq C \left\| M_I E \left(\sum_{\epsilon=0}^{\ell-1} M_\epsilon \right) M_J \right\|_1 = C\mu(I)\mu(\sigma^{-1}(J)).$$

A monotone class argument implies that this relation still holds if we replace J by any Borel set B. Thus, by taking $B = \sigma^{-1}(J)$, we have

$$\mu(I \cap \sigma^{-(n+1)}(J)) \leq C\mu(I)\mu(\sigma^{-2}(J)),$$

which completes the proof of proposition 2.1.□

3. Frostman measures

In this section we establish the existence of Frostman measures (1.5) at every negative state q for the measure μ defined by (1.4). We begin with a preliminary result that gives conditions ensuring the existence of Frostman measures.

For a probability measure m on the interval $[0,1]$, let us define the series $Z(s)$ by

$$\forall s \in \mathbb{R}, \quad Z(s) = \sum_{n\geq 1} u_n \ell^{-ns}, \quad \text{where} \quad u_n = \ell^{n\tau_n(q)}.$$

If $m(I) > 0$, $Z_I(s)$ denotes the series associated to the measure m_I verifying $m_I(J) = m(IJ)/m(I)$.

Proposition 3.1. *Let m be a probability measure on the interval $[0,1]$ and $q \in \mathbb{R}$. With the above notation, suppose that there exists a constant $C > 0$ such that*

(i) $\forall n, \forall p, \quad u_{n+p} \leq C u_n u_p,$

(ii) $\forall I \in \mathcal{F}, \forall s \in \mathbb{R}, \quad Z_I(s) \leq C Z(s)$

Then, there exists a Frostman measure at state q for the measure m.

Proof. We adapt to our situation the arguments used by Michon and Peyrière [27, 32] in another context. The submultiplicativity property of the sequence $v_n = C\ell^{n\tau_n(q)}$ implies that the sequence $v_n^{1/n}$ tends to its lower bound. As a consequence, $\tau_n(q)$ converges and if we call $\tau(q)$ its limit, we have

$$\forall n \in \mathbb{N}, \quad C\ell^{n\tau_n(q)} \geq \ell^{n\tau(q)}.$$

Therefore, the series $Z(s)$ converges for $s > \tau(q)$ and diverges for $s = \tau(q)$. Let us consider, for $s > \tau(q)$, the function ϕ_s defined by

$$\phi_s(x) = \sum_{n\geq 1} m(I_n(x))^q \left(\ell^{-n}\right)^{-1+s}.$$

Since $\int_0^1 \phi_s(x)dx = Z(s)$, we can define a probability measure ν_s on the interval $[0,1]$ by

$$\forall I \in \mathcal{F}, \quad \nu_s(I) = \frac{\int_I \phi_s(x)dx}{Z(s)}.$$

For every $I \in \mathcal{F}_n$, we find that

$$Z(s)\nu_s(I) = \int_I \phi_s(x)\,dx = \ell^{-n} \sum_{1\leq k\leq n} m(I_k)^q \left(\ell^{-k}\right)^{-1+s} + m(I)^q \ell^{-ns} Z_I(s),$$

where I_k denotes the element of $\mathcal{F}_k$ containing I.

Let m_q be a weak*-limit of ν_s as s goes to $\tau(q)$. The divergence of the series $Z(s)$ for $s = \tau(q)$ and the inequality $Z_I(s) \leq CZ(s)$ imply that $m_q(I) \leq Cm(I)^q \ell^{-n\tau(q)}$, which completes the proof of proposition 3.1.□

We easily deduce the following result.

Corollary 3.2. *Let m be a probability measure on the interval $[0,1]$ and $q \in \mathbb{R}$. Suppose that there exists a constant $C > 0$ such that*

$$\forall I, \forall J, \quad m(IJ)^q \leq Cm(I)^q m(J)^q. \tag{3.1}$$

Then, there exists a Frostman measure at state q for the measure m.

In particular, the condition (3.1) is satisfied if $m(IJ) \leq Cm(I)m(J)$ and $q > 0$ or if $m(IJ) \geq Cm(I)m(J)$ and $q < 0$.

We will use the following lemma to prove that the measure μ satisfies the hypotheses of proposition 3.1.

Lemma 3.3. *Let μ be the measure defined by (1.4). For every $I \in \mathcal{F}$, one of the following is satisfied.*

(i) $\forall J \in \mathcal{F}, \quad \mu(I)\mu(J) \leq 2\,\mu(IJ)$,

(ii) $\forall J \in \mathcal{F}, \quad \mu(I)\mu \circ T(J) \leq 2\,\mu(IJ)$, *where* $T(x) = 1 - x$.

Proof. For $I \in \mathcal{F}$ and $\epsilon \in \{0, \cdots, \ell - 1\}$, we have $S_i^{-1}(\epsilon I) = I$ or $S_i^{-1}(\epsilon I) = T(I)$ or $S_i^{-1}(\epsilon I) = \emptyset$. Thus, by iterating (1.4), we can find two non-negative real numbers $A(I)$ and $B(I)$ (depending only on I) such that

$$\forall J \in \mathcal{F}, \quad \mu(IJ) = A(I)\mu(J) + B(I)\mu \circ T(J). \tag{3.2}$$

We then obtain that either

$$\forall J \in \mathcal{F}, \quad \mu(IJ) \geq C(I)\mu(J),$$

or

$$\forall J \in \mathcal{F}, \quad \mu(IJ) \geq C(I)\mu \circ T(J),$$

where $C(I) = \max(A(I), B(I)) > 0$.

By taking $J = [0,1]$ in (3.2), we get $\mu(I) \leq 2C(I)$. Hence, either

$$\forall J \in \mathcal{F}, \quad 2\mu(IJ) \geq \mu(I)\mu(J),$$

or

$$\forall J \in \mathcal{F}, \quad 2\mu(IJ) \geq \mu(I)\mu \circ T(J),$$

which completes the proof of lemma 3.3.□

Theorem 3.4. *There exists a Frostman measure at every state $q < 0$ for the measure μ verifying (1.4).*

Proof. Let $I \in \mathcal{F}$. If for every $J \in \mathcal{F}$, $\mu(I)\mu(J) \leq 2\mu(IJ)$ (respectively, $\mu(I)\mu \circ T(J) \leq 2\mu(IJ)$), we set $\tilde{\mu_I} = \mu$ (respectively, $\tilde{\mu_I} = \mu \circ T$). By lemma 3.3, we have

$$\forall J \in \mathcal{F}, \quad \mu(I)\tilde{\mu_I}(J) \leq 2\mu(IJ).$$

Since q is negative, we obtain that

$$u_{n+p} = \sum_{I\in\mathcal{F}_n}\sum_{J\in\mathcal{F}_p}\mu(IJ)^q \le \left(\frac{1}{2}\right)^q \sum_{I\in\mathcal{F}_n}\sum_{J\in\mathcal{F}_p}\mu(I)^q\tilde{\mu}_I(J)^q = \left(\frac{1}{2}\right)^q u_n u_p$$

and

$$Z_I(s) = \sum_{n\ge 1}\sum_{J\in\mathcal{F}_n}\left(\frac{\mu(IJ)}{\mu(I)}\right)^q \ell^{-ns} \le \left(\frac{1}{2}\right)^q \sum_{n\ge 1}\sum_{J\in\mathcal{F}_n}\tilde{\mu}_I(J)^q\ell^{-ns} = \left(\frac{1}{2}\right)^q Z(s).$$

Thus, μ satisfies the hypotheses of proposition 3.1 and theorem 3.4 follows.□

4. The function τ_μ

In this section we determine the L^q-spectrum $\tau_\mu(q)$ of the measure μ verifying (1.4). Let us start with the following easy lemma.

Lemma 4.1. *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences of real positive numbers such that*

$$\lim_{n\to\infty} u_n^{1/n} = u \ \ et \ \ \lim_{n\to\infty} v_n^{1/n} = v.$$

Let

$$w_n = \sum_{k=0}^{n} u_k v_{n-k}.$$

Then, the sequence $(w_n^{1/n})_{n\in\mathbb{N}}$ converges to $w = \max\{u, v\}$.

The proof is elementary and therefore omitted.

Theorem 4.2. *Let μ be the probability measure verifying (1.4). Suppose that for every $0 \le i \le \ell-1$, $p_i > 0$, and set $B = \{0 \le i \le \ell-1, p_{i+\ell} = 0\}$. Then, by denoting $\tilde{\tau}(q) = \log_\ell\left(\sum_{i\in B} {p_i}^q\right)$, we have*

$$\forall q \in \mathbb{R}, \quad \tau_\mu(q) = \max(\tau_\nu(q), \tilde{\tau}(q)), \tag{4.1}$$

where $\nu = (\mu + \mu\circ T)/2$ and $T(x) = 1 - x$. By convention, $\tilde{\tau}(q) = -\infty$ if $B = \emptyset$.

Remarks.

1. We obtain a similar result replacing the hypothesis "$p_i > 0$, for every $0 \le i \le \ell-1$" by "$p_i + p_{i+\ell} > 0$, for every $0 \le i \le \ell-1$".
2. For positive q, the two L^q-spectra $\tau_\mu(q)$ and $\tau_\nu(q)$ are obviously the same. If $q < 0$, the situation is inverted : "small become big". In fact, sets with negligeable mass determine the function τ_μ ; therefore it suffices to consider the sum over indices $i \in B$ in the expression of $\tilde{\tau}$.
3. If $B = \{0, \cdots, \ell-1\}$, μ is a multinomial measure (also called Bernoulli product). The calculation of τ_μ is then straightforward and we have $\tau_\mu = \tilde{\tau}$ (e.g [9]).
4. To obtain phase transitions for the function τ_μ, it is thus enough to find conditions on the ${p_i}'s$ ensuring that the equation $\tau_\nu(q) = \tilde{\tau}(q)$ has many isolated solutions.

Proof of theorem 4.2. We fix $q \in \mathbb{R}$, $n \in \mathbb{N}^*$ and we write C for each constant which depends on q but not on n. Using (1.4), we get

$$\begin{aligned} w_n &= \sum_{I\in\mathcal{F}_n} \mu(I)^q = \sum_{i=0}^{\ell-1} \sum_{I\in\mathcal{F}_{n-1}} \mu(iI)^q \\ &= \sum_{i\notin B} \sum_{I\in\mathcal{F}_{n-1}} (p_i\mu(I) + p_{i+\ell}\mu(I^*))^q + \sum_{i\in B} \sum_{I\in\mathcal{F}_{n-1}} {p_i}^q \mu(I)^q \\ &\leq C \sum_{I\in\mathcal{F}_{n-1}} \nu(I)^q + \sum_{i\in B} {p_i}^q w_{n-1} := C v_{n-1} + \sum_{i\in B} {p_i}^q w_{n-1}. \end{aligned}$$

By induction, we obtain

$$w_n \leq C \sum_{k=0}^{n-1} \left(\sum_{i\in B} {p_i}^q \right)^k v_{n-(k+1)} := C \sum_{k=0}^{n-1} u_k v_{n-(k+1)}.$$

We can find a minoration of the same type in a similar way.

Furthermore, the definition of $\|.\|$ in (2.2) implies that $\nu(IJ) \leq 2\nu(I)\nu(J)$. Thus, the sequence $v_n^{1/n}$ converges and theorem 4.2 easily follows from lemma 4.1.□

5. The level sets E_α

In this section we link the level sets $E_\alpha(\mu)$ and $E_\alpha(\nu)$ associated to the measures μ and ν.

Proposition 5.1. *The hypotheses are the same as in theorem 4.2. Let K be the compact set defined by $K = \bigcup_{i\in B} S_i(K)$ with the convention that $K = \emptyset$ if $B = \emptyset$. Then,*

$$E_\alpha(\mu) = (E_\alpha(\nu) \cap ([0,1] \setminus K)) \cup (E_\alpha(\mu) \cap K).$$

Remarks. (i) If $B = \{0, \cdots, \ell-1\}$, $K = [0,1]$ and proposition 5.1 is immediate.

(ii) If $B = \emptyset$, $K = \emptyset$ and by proposition 5.1, $E_\alpha(\mu) = E_\alpha(\nu)$. In fact, in this case, it is easy to prove that the measures μ and ν are strongly equivalent.

(iii) If B is reduced to a single element, K is a singleton.

(iv) In all other cases, K is a Cantor set.

Proof of proposition 5.1. According to the above remark, we can suppose that $B \neq \{0, \cdots, \ell-1\}$ and $B \neq \emptyset$. Fix $x \notin K$ and $\alpha > 0$. To prove our claim, it is sufficient to show that $x \in E_\alpha(\mu)$ if and only if $x \in E_\alpha(\nu)$. Since $x \notin K$, there exists $n(x) \in \mathbb{N}$ and $\epsilon \notin B$ such that

$$\forall n \geq n(x), \quad I_n(x) = I_{\epsilon_1\cdots\epsilon_{n(x)}\epsilon\cdots\epsilon_n}$$

where $(\epsilon_1, \cdots, \epsilon_{n(x)}) \in B^{n(x)}$. With obvious notation, it results from (2.1) and (2.2) that

$$\frac{\mu(I_n(x))}{\nu(I_n(x))} = \frac{\|M_{I_{n(x)}} M_\epsilon M_{x,n}\|_1}{\|M_{I_{n(x)}} M_\epsilon M_{x,n}\|} = \frac{\left\| \begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_{2\ell-1-\epsilon} & p_{\ell-1-\epsilon} \end{pmatrix} \begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix} \right\|_1}{\left\| \begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_{2\ell-1-\epsilon} & p_{\ell-1-\epsilon} \end{pmatrix} \begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix} \right\|}$$

$$\geq C(x)\frac{\left\|\begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_\epsilon + p_{2\ell-1-\epsilon} & p_{\epsilon+\ell} + p_{\ell-1-\epsilon} \end{pmatrix}\begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix}\right\|_1}{\left\|\begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_\epsilon + p_{2\ell-1-\epsilon} & p_{\epsilon+\ell} + p_{\ell-1-\epsilon} \end{pmatrix}\begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix}\right\|}$$

$$= C(x)\frac{p_\epsilon(\tilde{a}_{x,n} + \tilde{b}_{x,n}) + p_{\epsilon+\ell}(\tilde{c}_{x,n} + \tilde{d}_{x,n})}{(2p_\epsilon + p_{2\ell-1-\epsilon})(\tilde{a}_{x,n} + \tilde{b}_{x,n}) + (2p_{\epsilon+\ell} + p_{\ell-1-\epsilon})(\tilde{c}_{x,n} + \tilde{d}_{x,n})}.$$

if n is large enough. Using $p_\epsilon p_{\epsilon+\ell} > 0$, we can find another constant $C'(x) > 0$ which does not depend on n such that $\mu(I_n(x)) \geq C'(x)\nu(I_n(x))$. Since $\mu \leq 2\nu$, we then deduce that

$$(E_\alpha(\mu) \cap ([0,1] \setminus K)) = (E_\alpha(\nu) \cap ([0,1] \setminus K))$$

and proposition 5.1 follows easily.□

Theorem 5.2. *The notation are the same as in theorem 4.2. Suppose that* $B \cap B^* = \emptyset$ *where* $B^* = \{\ell - 1 - \epsilon, \epsilon \in B\}$. *Then,*

(i) $D_\mu = D_\nu \cup \left[-\log_\ell\left(\max_{i\in B} p_i\right), -\log_\ell\left(\min_{i\in B} p_i\right)\right]$ *where* $D_\mu = \{\alpha,\ E_\alpha(\mu) \neq \emptyset\}$.

(By convention, $[-\log_\ell(\max_{i\in B} p_i), -\log_\ell(\min_{i\in B} p_i)] = \emptyset$ *if* $B = \emptyset$.*)*

(ii) $\forall \alpha \in D_\mu, \quad \dim(E_\alpha(\mu)) = \max\left(\dim(E_\alpha(\nu)), \tilde{\tau}^*(\alpha)\right).$

Proof. If $B = \emptyset$, the measures μ and ν are strongly equivalent and theorem 5.2 is easy. Now assume that $B \neq \emptyset$. Let π be the self-similar measure supported on K verifying

$$\pi = \frac{1}{\sum_{i\in B} p_i}\sum_{i\in B} p_i\, \pi \circ S_i^{-1}. \tag{5.1}$$

The family $(S_i)_{i\in B}$ satisfying the Open Set Condition [21], the calculation of the L^q-spectrum $\tau_\pi(q)$ is then straightforward :

$$\forall q \in \mathbb{R}, \quad \tau_\pi(q) = \log_\ell\left(\sum_{i\in B}\left(\frac{p_i}{\sum_{i\in B} p_i}\right)^q\right), \tag{5.2}$$

(e.g. [6, 30]). Moreover, we have

$$\begin{cases} \dim(E_\alpha(\pi)) = {\tau_\pi}^*(\alpha) & \text{if } \alpha \in \left[-\log_\ell\left(\frac{\max_{i\in B}(p_i)}{\sum_{i\in B} p_i}\right), -\log_\ell\left(\frac{\min_{i\in B}(p_i)}{\sum_{i\in B} p_i}\right)\right], \\ E_\alpha(\pi) = \emptyset & \text{otherwise.} \end{cases} \tag{5.3}$$

Furthermore, for every $(\epsilon_1 \cdots \epsilon_n) \in B^n$ and $I = I_{\epsilon_1\cdots\epsilon_n}$, we find that

$$\pi(I) = \left(\sum_{i\in B} p_i\right)^{\log_\ell(|I|)} \mu(I).$$

We thus deduce that

$$\forall \alpha, \quad E_\alpha(\pi) = E_{\alpha - \log_\ell\left(\sum_{i\in B} p_i\right)}(\mu) \cap K,$$

or equivalently

$$\forall \alpha, \quad E_{\alpha+\log_\ell\left(\sum_{i\in B} p_i\right)}(\pi) = E_\alpha(\mu) \cap K.$$

It follows from (5.3) that

$$E_\alpha(\mu) \cap K \neq \emptyset \Leftrightarrow \alpha \in \left[-\log_\ell \left(\max_{i\in B} p_i \right), -\log_\ell \left(\min_{i\in B} p_i \right) \right],$$

and, if $-\log_\ell \left(\max_{i\in B} p_i\right) \leq \alpha \leq -\log_\ell \left(\min_{i\in B} p_i\right)$, then

$$\dim(E_\alpha(\mu) \cap K) = {\tau_\pi}^* \left(\alpha + \log_\ell \left(\sum_{i\in B} p_i \right) \right) = \tilde{\tau}^*(\alpha). \tag{5.4}$$

Proposition 5.1 also leads to estimate $\dim(E_\alpha(\nu) \cap ([0,1] \setminus K))$. The hypothesis $B \cap B^* = \emptyset$ implies that $T(K) \subset [0,1] \setminus K$. Therefore,

$$\begin{aligned} \dim(E_\alpha(\nu) \cap K) &= \dim(T(E_\alpha(\nu) \cap K)) \\ &= \dim(E_\alpha(\nu) \cap T(K)) \leq \dim(E_\alpha(\nu) \cap ([0,1] \setminus K)), \end{aligned}$$

and we conclude that

$$\dim(E_\alpha(\nu)) = \dim(E_\alpha(\nu) \cap ([0,1] \setminus K)).$$

Theorem 5.2 then follows from proposition 5.1 and (5.4).□

Remarks. 1. Using the same ideas, we can also obtain that

$$\forall \alpha, \quad \dim(V_\alpha(\mu)) = \max\left(\dim(V_\alpha(\nu)), \tilde{\tau}^*(\alpha)\right),$$

where V_α is defined as E_α replacing lim by lim inf. In other terms,

$$V_\alpha(m) = \left\{ x \in [0,1], \liminf_{n\to+\infty} -\frac{\log m(I_n(x))}{n \log \ell} = \alpha \right\}.$$

2. Similar results can also be established replacing the Hausdorff dimension $\dim(E_\alpha)$ by the Packing dimension $\mathrm{Dim}\,(E_\alpha)$.

We deduce the following.

Corollary 5.3. *Suppose that $B \cap B^* = \emptyset$ and that ν satisfies the quasi-Bernoulli property. Then,*

$$\forall \alpha, \ \dim(E_\alpha(\mu)) = \mathrm{Dim}\,(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \mathrm{Dim}\,(V_\alpha(\mu)) = \max\left({\tau_\nu}^*(\alpha), \tilde{\tau}^*(\alpha)\right).$$

According to theorem 4.2, the function ${\tau_\mu}^*$ is the Legendre transform of the maximum of τ_ν and $\tilde{\tau}$. On the other hand, by corollary 5.3, the dimension of the level sets $E_\alpha(\mu)$ is given by the maximum of the Legendre transform of τ_ν and the Legendre transform of $\tilde{\tau}_\nu$. Since we cannot invert Legendre transform and maximum, we have the following.

Theorem 5.4. *Suppose that $B \cap B^* = \emptyset$ and that ν satisfies the quasi-Bernoulli property. Then, we have the following.*

(i) If ${\tau_\mu}'(q)$ exists and if $\alpha = -{\tau_\mu}'(q)$, then

$$\dim(E_\alpha(\mu)) = \mathrm{Dim}\,(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \mathrm{Dim}\,(V_\alpha(\mu)) = {\tau_\mu}^*(\alpha).$$

(ii) If $\tau_\mu{}'(q)$ *does not exist and if* $-(\tau_\mu)'_+(q) < \alpha < -(\tau_\mu)'_-(q)$, *then*

$$\dim(E_\alpha(\mu)) = \mathrm{Dim}\,(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \mathrm{Dim}\,(V_\alpha(\mu)) < \tau_\mu{}^*(\alpha).$$

Hence, each phase transition q *gives rise to an interval* $\big(-(\tau_\mu)'_+(q), -(\tau_\mu)'_-(q)\big)$ *in which the multifractal formalism breaks down.*

Remark. It is possible to prove that ν satisfies the quasi-Bernoulli property if and only if

$$\text{either} \quad \forall\, i \in B,\ p_i < p_{\ell-1-i}, \quad \text{or} \quad \forall i \in B,\ p_i > p_{\ell-1-i}. \tag{5.5}$$

The multifractal formalism fails for the measure μ only in the first case. Indeed, if for every $i \in B,\ p_i > p_{\ell-1-i}$, it is easy to check that the measures μ and ν are strongly equivalent.

6. Examples

In this section we construct measures with non-differentiable L^q-spectra $\tau(q)$ for which previous results apply. Furthermore, based on these examples, we point out new phenomena in the multifractal structure of self-similar measures.

6.1. An isolated point in the set of local dimensions

Let us take $\ell = 2$ and consider the probability measure μ verifying

$$\mu = p_0\mu \circ S_0^{-1} + p_1\mu \circ S_1^{-1} + p_2\mu \circ S_2^{-1}, \tag{6.1}$$

where $S_0(x) = x/2$, $S_1(x) = x/2 + 1/2$ and $S_2(x) = -x/2 + 1/2$. We assume that $p_0p_1p_2 > 0$ and $p_1 < p_0$. With the notation previously introduced, we have $B = \{1\}$ and $K = \{1\}$. Moreover, by theorem 4.2,

$$\forall q \in \mathbb{R}, \quad \tau_\mu(q) = \max(\tau_\nu(q), q\log_2(p_1)).$$

Thus, in order to get a phase transition for the function τ_μ, we have to compare $\tau_\nu{}'(-\infty)$ and $\log_2(p_1)$. For every $I \in \mathcal{F}_n$, by iterating (6.1), we get $\nu(I) \geq (p_-)^n$ where $p_- = \min(p_0, p_1 + p_2)$. We easily deduce that $-\tau_\nu{}'(-\infty) \leq -\log_2(p_-) < -\log_2(p_1)$. Since $\tau_\nu(q) \geq q\log_2(p_1)$ for $q = 0$, we conclude that there exists $q_0 < 0$ such that

$$\tau_\mu(q) = \begin{cases} q\log_2(p_1) & \text{if} \quad q \leq q_0, \\ \tau_\nu(q) & \text{if} \quad q \geq q_0, \end{cases} \tag{6.2}$$

and the L^q-spectrum $\tau_\mu(q)$ is not differentiable at $q = q_0$ (see figure 1(a)). Furthermore, by (5.5), ν satisfies the quasi-Bernoulli property. Using theorem 5.2 and corollary 5.3 we deduce the following .

Theorem 6.1. *Let* μ *be the measure satisfying (6.1). Then,*

$$D_\mu = D_\nu \cup \{-\log_2(p_1)\} = (-\tau_\nu{}'(+\infty), -\tau_\nu{}'(-\infty)) \cup \{-\log_2(p_1)\},$$

and

$$\dim(E_\alpha(\mu)) = \begin{cases} \tau_\nu{}^*(\alpha) & \text{if} \ \ \alpha \in (-\tau_\nu{}'(+\infty), -\tau_\nu{}'(-\infty))\,, \\ 0 & \text{if} \ \ \alpha = -\log_2(p_1). \end{cases}$$

Remarks. 1. Since $-{\tau_\nu}'(-\infty) < -\log_2(p_1)$, D_μ contains an isolated point. In this sense, the situation is close to the ones obtained for the Erdös measure and for the 3-time convolution of the Cantor measure (e.g [13, 20]). Note that in our situation, the value of $[p_0, p_1, p_2]$ is not a matter.

2. It is easy to show that

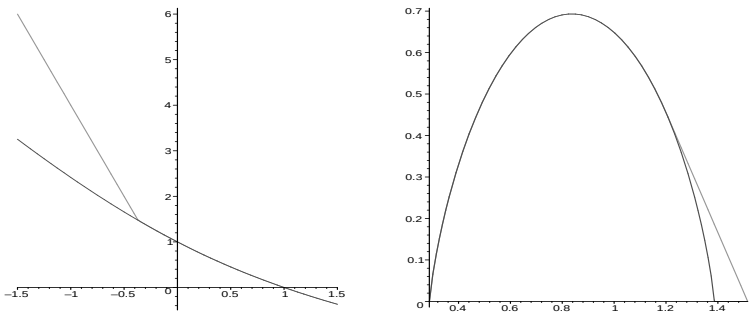
$${\tau_\mu}^*(\alpha) = \begin{cases} {\tau_\nu}^*(\alpha) & \text{if} \quad -{\tau_\nu}'(+\infty) \le \alpha \le -{\tau_\nu}'(q_0), \\ \dfrac{{\tau_\nu}^*(q_0)}{\log_2(p_1) - {\tau_\nu}'(q_0)}\,(\alpha + \log_2(p_1)) & \text{if} \quad -{\tau_\nu}'(q_0) \le \alpha \le -\log_2(p_1). \end{cases}$$

Thus,

$$\begin{cases} \forall \alpha \in (-{\tau_\nu}'(+\infty), -{\tau_\nu}'(q_0)], \quad \dim(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = {\tau_\nu}^*(\alpha) = {\tau_\mu}^*(\alpha), \\ \forall \alpha \in (-{\tau_\nu}'(q_0), -{\tau_\nu}'(-\infty)], \quad \dim(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = {\tau_\nu}^*(\alpha) < {\tau_\mu}^*(\alpha). \end{cases}$$

Contrary to the usual situation, the singularity spectrum of μ is not given by the Legendre transform of τ_μ but instead by the Legendre transform of an auxiliary function. Figure 1(b) illustrates this phenomenon.

3. The measure μ may be used to estimate the Hausdorff dimension of self-affine graphs studied by McMullen [26], Prsytycki and Urbański [33, 40]. More details can be found in [37, 39].



(a) (b)

Figure 1. (a) τ_μ is not differentiable. (b) The singularity spectrum of μ, given by ${\tau_\nu}^*$, differs from ${\tau_\mu}^*$.

6.2. Non-concave spectra

Subsection 3.1 and several papers deal with measures for which the L^q-spectrum $\tau(q)$ is not differentiable at a single point $q = q_0$ and is linear for $q \le q_0$ (e.g. [12, 13, 20, 22, 29]). In this part we construct measures with non-differentiable and strictly concave L^q-spectra. That leads to new situations for the multifractal analysis of self-similar measures.

Let us take $\ell = 4$ and consider the probability measure μ satisfying

$$\mu = \sum_{i=0}^{5} p_i \, \mu \circ S_i^{-1}, \tag{6.3}$$

where

$$S_0(x) = \frac{x}{4}, \quad S_1(x) = \frac{x}{4} + \frac{1}{4}, \quad S_2(x) = \frac{x}{4} + \frac{1}{2},$$
$$S_3(x) = \frac{x}{4} + \frac{3}{4}, \quad S_4(x) = -\frac{x}{4} + \frac{1}{4} \quad \text{and} \quad S_5(x) = -\frac{x}{4} + \frac{1}{2}.$$

In this case, $B = \{2, 3\}$ and K is the Cantor set whose points only contains digits 2 and 3 in their 4-adic expression, i.e. $K = \{x = \sum \epsilon_i/4^i,\ \epsilon_i = 2 \text{ or } 3,\ \forall i \in \mathbb{N}^*\}$.

By theorem 4.2, $\tau_\mu(q) = \max(\tau_\nu(q), \log_4({p_2}^q + {p_3}^q))$. In order to compute τ_ν, we assume that the $p_i's$ verify $p_0 = p_3 + p_4$ and $p_1 = p_2 + p_5$. In this situation, it is easy to show that ν is a multinomial measure (see [37]). The calculation of τ_ν is then straightforward : $\tau_\nu(q) = 1/2 + \log_4({p_0}^q + {p_1}^q)$. Therefore, there exists $q_0 < 0$ such that

$$\tau_\mu(q) = \begin{cases} \log_4({p_2}^q + {p_3}^q) & \text{if} \quad q \le q_0, \\ \dfrac{1}{2} + \log_4({p_0}^q + {p_1}^q) & \text{if} \quad q \ge q_0, \end{cases} \tag{6.4}$$

and $\tau_\mu(q)$ is not differentiable at $q = q_0$ (see figure 2(a)).

Moreover, if we denote $p_0 \vee p_1$ ($p_0 \wedge p_1$) the maximum (minimum) of p_0 and p_1, we get $D_\nu = [-\log_4(p_0 \vee p_1), -\log_4(p_0 \wedge p_1)]$ and $\dim(E_\alpha(\nu)) = {\tau_\nu}^*(\alpha) \ge 1/2$, for all $\alpha \in D_\nu$. It follows from theorem 5.2 that

$$D_\mu = [-\log_4(p_0 \vee p_1), -\log_4(p_0 \wedge p_1)] \cup [-\log_4(p_2 \vee p_3), -\log_4(p_2 \wedge p_3)],$$

and

$$\dim(E_\alpha(\mu)) = \begin{cases} {\tau_\nu}^*(\alpha) & \text{if} \quad -\log_4(p_0) \le \alpha \le -\log_4(p_1), \\ \tilde{\tau}^*(\alpha) & \text{if} \quad -\log_4(p_2 \vee p_3) \le \alpha \le -\log_4(p_2 \wedge p_3). \end{cases}$$

Thus, if $p_3 < p_1 \le p_0$, the singularity spectrum of μ is supported by a union of mutually disjoint intervals and differs from $\tau_\mu^*(\alpha)$ for $-(\tau_\mu)'_+(q_0) < \alpha < -(\tau_\mu)'_-(q_0)$ (see figure 2(b)). To the best of our knowledge, self-similar measures with such multifractal structures have not previously appeared in the litterature.

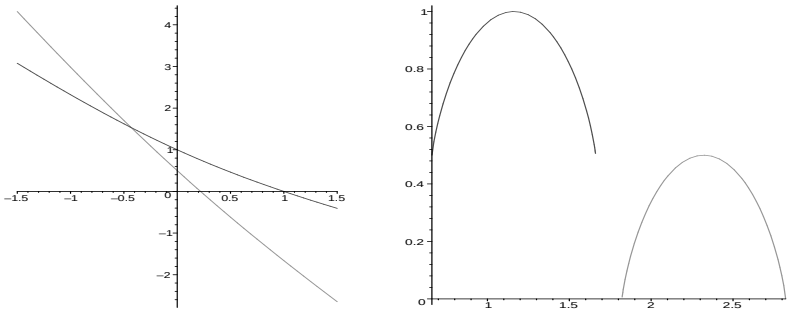
6.3. Two phase transitions

Until now we have studied measures for which the L^q-spectrum $\tau(q)$ is not differentiable at one single point $q_0 < 0$. In this part we propose examples with two phase transitions. Let us take $\ell = 5$ and consider the probability measure μ satisfying

$$\mu = \sum_{i=0}^{7} p_i \, \mu \circ S_i^{-1}, \tag{6.5}$$

where

$$S_0(x) = \frac{x}{5}, \quad S_1(x) = \frac{x}{5} + \frac{1}{5}, \quad S_2(x) = \frac{x}{5} + \frac{2}{5}, \quad S_3(x) = \frac{x}{5} + \frac{3}{5}$$



(a) (b)

Figure 2. (a) τ_μ is not differentiable. (b) The singularity spectrum of μ is supported by a union of two disjoint intervals.

$$S_4(x) = \frac{x}{5} + \frac{4}{5}, \quad S_5(x) = -\frac{x}{5} + \frac{1}{5}, \quad S_6(x) = -\frac{x}{5} + \frac{2}{5} \quad \text{and} \quad S_7(x) = -\frac{x}{5} + \frac{3}{5}.$$

In this case, $B = \{3, 4\}$ and $K = \{x = \sum \epsilon_i/5^i,\ \epsilon_i = 3 \text{ or } 4,\ \forall i \in \mathbb{N}^*\}$. We suppose that the coefficients $p_i's$ verify $p_0 = p_4 + p_5$, $p_1 = p_3 + p_6$ and $p_2 = p_7$. As in section 6.2, we get $\tau_\mu(q) = \max(\log_5(2{p_0}^q + 2{p_1}^q + (2p_2)^q), \log_5({p_3}^q + {p_4}^q))$, and

$$\forall \alpha \in D_\nu = [-\log_5(p_0 \vee p_1 \vee 2p_2), -\log_4(p_0 \wedge p_1 \wedge 2p_2)],\ \dim(E_\alpha(\nu)) = {\tau_\nu}^*(\alpha).$$

In order to have $\tau_\mu(q) = \tau_\nu(q)$ for large negative q, we choose p_2 sufficiently small. For example, if we take $p_0 = 0.35$, $p_1 = 0.14$, $p_2 = 0.01$, $p_3 = 0.03$ and $p_4 = 0.025$, the equation $\tau_\nu(q) = \tilde{\tau}(q)$ has two solutions q_0 and q_1 corresponding to the points of non-differentiability of $\tau_\mu(q)$. By theorem 5.2, $D_\mu = [-\log_5(p_0), -\log_5(2p_2)]$ and

$$\dim(E_\alpha(\mu)) = \begin{cases} {\tau_\nu}^*(\alpha) & \text{if} \quad -\log_5(p_0) \le \alpha \le \alpha_0, \\ \tilde{\tau}^*(\alpha) & \text{if} \quad \alpha_0 \le \alpha \le \alpha_1, \\ {\tau_\nu}^*(\alpha) & \text{if} \quad \alpha_1 \le \alpha \le -\log_5(p_2), \end{cases}$$

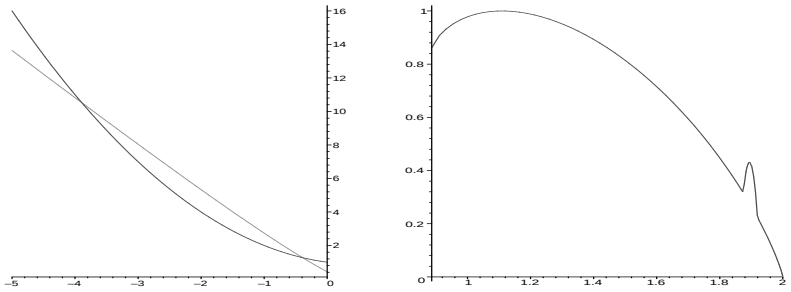
where α_0 and α_1 denote the solutions of the equation ${\tau_\nu}^*(\alpha) = \tilde{\tau}^*(\alpha)$. From the expression of the Legendre transform of $\tau_\mu = \max(\tau_\nu, \tilde{\tau})$, it follows that

$$\forall \alpha \in \left(-(\tau_\mu)'_+(q_0), -(\tau_\mu)'_-(q_0)\right) \cup \left(-(\tau_\mu)'_+(q_1), -(\tau_\mu)'_-(q_1)\right), \quad \dim(E_\alpha(\mu)) < {\tau_\mu}^*(\alpha).$$

6.4. More phase transitions

In this part we describe a way to construct measures with an arbitrarily large number N of phase transitions. Theorem 4.2 leads us to find conditions on the $p_i's$ such that the equation $\tau_\nu(q) = \tilde{\tau}(q)$ has N solutions. Since $\tau_\nu(0) \ge \tilde{\tau}(0)$, we have to distinguish the case where N is odd from the case where N is even.

First, assume that N is odd. Let us take $\ell = 2N$, $B = \{\ell/2, \cdots, \ell - 1\}$ and suppose that $p_i = p_{i+\ell} + p_{\ell-1-i}$, for all $0 \le i \le \ell/2 - 1$. In this case, the arguments developed in



(a) (b)

Figure 3. $\tau_\mu = \max(\tau_\nu, \tilde{\tau})$ is not differentiable at two points. The singularity spectrum of μ is not concave and differs from ${\tau_\mu}^*$.

section 6.2 imply that

$$\tau_\mu(q) = \max(\tau_\nu(q), \tilde{\tau}(q)) = \max\left(\log_{2N}\left(\sum_{i=0}^{N-1} 2{p_i}^q\right), \log_{2N}\left(\sum_{i=N}^{2N-1} {p_i}^q\right)\right).$$

Moreover, since $\ell = 2N$, we can choose the $p_i's$ such that the equation

$$\sum_{i=0}^{N-1} 2{p_i}^q = \sum_{i=N}^{2N-1} {p_i}^q$$

has N solutions. These solutions correspond to the phase transitions for the L^q-spectrum $\tau_\mu(q)$.

Assume now that N is even. To ensure that $\tau_\mu(q) = \tau_\nu(q)$ for large negative q, tools used in section 6.3 suggest to take ℓ odd. Let $\ell = 2N + 1$ and $B = \{N + 1, \cdots, 2N\}$. Under the conditions, for all $0 \leq i \leq N - 1$, $p_i = p_{i+\ell} + p_{\ell-1-i}$ and $p_N = p_{N+\ell}$, we get

$$\tau_\mu(q) = \max\left(\log_{2N+1}\left(\sum_{i=0}^{N-1} 2{p_i}^q + (2p_N)^q\right), \log_{2N+1}\left(\sum_{i=N+1}^{2N} {p_i}^q\right)\right).$$

Thus, in order to have $\tau_\mu(q) = \tau_\nu(q)$ for large negative q, we also suppose that $2p_N < \min(p_i,\ N + 1 \leq i \leq 2N)$. Once again, we can choose the $p_i's$ such that the equation

$$\sum_{i=0}^{N-1} 2{p_i}^q + (2p_N)^q = \sum_{i=N+1}^{2N} {p_i}^q$$

has N solutions. They correspond to the phase transitions for the function τ_μ.

More details about these examples can be found in [37].

Acknowledgments

The author had a position in Clermont-Ferrand (Université Blaise Pascal) when this work was achevied and he thanks this institution. He also would like to thank Yanick

Heurteaux and Anathassios Batakis for reading the manuscript carefully and suggesting some improvements.

References

[1] BenNasr F 1994 Analyse multifractale de mesures *C. R Acad. Sci. Paris Sér . I Math.* **319** 807–10

[2] BenNasr F, Bhouri I and Heurteaux Y 2002 The validity of the multifractal formalism : results and examples *Adv. Math.* **165** 264–84

[3] Benzi R, Paladin G, Parisi G and Vulpiani A 1984 On the multifractal nature of fully developed turbulence and chaotic system *J. Phys. A* **17** 3521–31

[4] Bowen R 1975 *Equilibrium States and the Ergodic Theory of Anasov Diffeomorphisms* (*Lecture Notes in Math* vol 470) (Springer-Verlag, New York/Berlin)

[5] Brown G, Michon G and Peyrière J 1992 On the multifractal analysis of measures *J. Stat. Phys.* **66** 775–90

[6] Cawley R and Mauldin R D 1992 Multifractal decompositions of Moran fractals *Adv. Math.* **92** 196–236

[7] Edgar G A and Mauldin R D 1992 Multifractal decompositions of digraph recursive fractals *Proc. London Math. Soc.* **65** 604–28

[8] Falconer K 1990 *Fractal Geometry, Mathematical Foundations and Applications* (New York: J. Wiley & Sons Ltd)

[9] Falconer K 1997 *Techniques in Fractal Geometry* (New York: J. Wiley & Sons Ltd)

[10] Fan A H 1994 Sur la dimension inférieure des mesures *Studia Math.* **111** 1-17

[11] Feng D J 2003 Smoothness of the L^q-spectrum of self-similar measures with overlaps *Proc. London Math. Soc.* **68** 102–18

[12] Feng D J 2005 The limit Rademacher functions and Bernoulli convolutions associated with Pisot numbers *Adv. Math.* **195** 24–101

[13] Feng D J and Olivier E 2003 Multifractal analysis of weak Gibbs measures and phase transition–application to some Bernoulli convolution *Ergod. Theory Dynam. Syst.* **23** 1751–84

[14] Frisch U and Parisi G 1985 On the singularity structure of fully developed turbulence *Appendix to Fully developed turbulence and intermittency* (*Proc. Internat. School Phys. Enrico Fermi*) ed U Frisch (North-Holland, Amsterdam) pp. 84–8

[15] Grassberger P 1983 Generalized dimension of strange attractors *Phys. Lett. A* **97** 227–30

[16] Grassberger P and Procaccia I 1983 Characterization of strange sets *Phys. Rev. Lett.* **50** 346–9

[17] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B I 1986 Fractal measures and their singularities : The characterization of strange sets *Phys. Rev. A* **33** 1141–51

[18] Hentschel H and Procaccia I 1983 The infinite number of generalized dimensions of fractals and strange attractors *Physica D* **8** 435–44

[19] Heurteaux Y 1998 Estimations de la dimension inférieure et de la dimension supérieure des mesures *Ann. Inst. H. Poincaré Probab. Statist.* **34** 309–38

[20] Hu T Y and Lau K S 2001 Multifractal structure of convolution of the Cantor measure *Adv. Appl. Math.* **27** 1–16

[21] Hutchinson J E 1981 Fractals and self similarity *Indiana Univ. Math. J.* **30** 713–47

[22] Lau K S and Ngai S M 1998 L^q-spectrum of the Bernoulli convolution associated with the golden ratio *Studia Math.* **131** 225–51

[23] Lau K S and Ngai S M 1999 Multifractal measures and a weak separation condition *Adv. Math.* **141** 45–96

[24] Mandelbrot B B 1974 Intermittent turbulence in self-similar cascades : Divergence of high moments and dimension of the carrier *J. Fluid Mech.* **62** 331–58

[25] Meakin P, Conoglio A, Stanley H and Witten T 1986 Scaling properties for the surfaces of fractal and nonfractal objects : An infinite hierarchy of critical exponents *Phys. Rev. A* **34** 3325–40

[26] McMullen C 1984 The Hausdorff dimension of general Sierpiński carpets *Nagoya Math. J.* **96** 1–9

[27] Michon G 1983 Mesures de Gibbs sur les Cantor réguliers *Ann. Inst. H. Poincaré Phys. Théor.* **58** 267–85

[28] Ngai S M 1997 A dimension result arising from the L^q-spectrum of a measure *Proc. Amer. Math.*

Soc. **125** 2943–51
[29] Olivier E, Sidorov N and Thomas A 2005 On the Gibbs properties of Bernoulli convolutions related to β-numeration in multinacci bases *Monatsh. Math.* **145** 145–74
[30] Olsen L 1995 A multifractal formalism *Adv in Math.* **116** 82-196
[31] Olsen L 2000 Dimensions inequalities of multifractal Hausdorff measures and multifractal packing measures *Math. Scand* **86** 109–129
[32] J. Peyrière 1992 Multifractal measures *Proc. of the NATO Adv. Study Inst. II Ciocco* NATO ASI Series C vol 372 (Kluwer Academic Press, Dordrecht) pp 175–86
[33] Przytycki F and Urbański M 1989 On Hausdorff dimension of some fractal sets *Studia Math.* **93** 155–86
[34] Renyi A 1970 *Probability Theory* (North-Holland, Amsterdam)
[35] Riedi R 1995 An improved multifractal formalism and self-similar measures *J. Math. Anal. Appl.* **189** 462–90
[36] Ruelle D 1978 *Thermodynamic Formalism* (Addison-Wesley, Reading, MA)
[37] Testud B 2004 Etude d'une large classe de mesures autosimilaires : calculs de dimensions et analyse multifractale *PhD Thesis* Université Blaise Pascal Clermont-Ferrand
[38] Testud B 2005 Transitions de phase dans l'analyse multifractale de mesures auto-similaires *C. R Acad. Sci. Paris Sér . I Math.* **340** 653–8
[39] Testud B 2006 Mesures quasi-Bernoulli au sens faible : résultats et exemples *Ann. Inst. H. Poincaré Probab. Statist.* **42** 1–35
[40] Urbański M 1990 The Hausdorff dimension of the graphs of continuous self-affine functions *Proc. Amer. Math. Soc.* **108** 921–30
[41] Ye Y L 2005 Multifractal of self-conformal measures *Nonlinearity* **18** 2111–33