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# Phase transitions for the multifractal analysis of self-similar measures

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**Abstract.** We are interested in the multifractal analysis of a class of self-similar measures with overlaps. This class, for which we obtain explicit formulae for the  $L^q$ -spectrum  $\tau(q)$  as well as the singularity spectrum  $f(\alpha)$ , is sufficiently large to point out new phenomena in the multifractal structure of self-similar measures. We show that, unlike the classical quasi-Bernoulli case, the  $L^q$ -spectrum  $\tau(q)$  of the measures studied can have an arbitrarily large number of non-differentiability points (phase transitions). These singularities occur only for the negative values of  $q$  and yield to measures that do not satisfy the usual multifractal formalism. The weak quasi-Bernoulli property is the key point of most of the arguments.

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## 1. Introduction

Let us begin with some notation. For an integer  $\ell \geq 2$ , we denote by  $\mathcal{F} = \cup_n \mathcal{F}_n$  where  $\mathcal{F}_n$  is the set of the  $\ell$ -adic intervals of the  $n$ th generation included in the interval  $[0, 1)$ . In other terms,  $\mathcal{F}_n = \{I = [k/\ell^n, (k+1)/\ell^n), 0 \leq k < \ell^n\}$ . For every  $x \in [0, 1)$ ,  $I_n(x)$  stands for the unique interval among  $\mathcal{F}_n$  containing  $x$ .

Let  $m$  be a probability measure on the interval  $[0, 1)$ . For  $x \in [0, 1)$ , we define the *local dimension* (also called Hölder exponent) of  $m$  at  $x$  by

$$\alpha(x) = \lim_{n \rightarrow +\infty} -\frac{\log m(I_n(x))}{n \log \ell},$$

provided this limit exists. The aim of multifractal analysis is to find the Hausdorff dimension,  $\dim(E_\alpha)$ , of the level set  $E_\alpha = \{x : \alpha(x) = \alpha\}$  for  $\alpha > 0$ . The function  $f(\alpha) = \dim(E_\alpha)$  is called the *singularity spectrum* (or multifractal spectrum) of  $m$  and we say that  $m$  is a *multifractal measure* when  $f(\alpha) > 0$  for several  $\alpha$ 's.

The concepts underlying the multifractal decomposition of a measure go back to an early paper of Mandelbrot [24]. In the 80's multifractal measures were used by physicists

to study various models arising from natural phenomena. In fully developed turbulence they were used by Frisch and Parisi [14] to investigate the intermittent behaviour in the regions of high vorticity. In dynamical system theory they were used by Benzi et al. [3] to measure how often a given region of the attractor is visited. In diffusion-limited aggregation (DLA) they were used by Meakin et al. [25] to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate.

In order to determine the function  $f(\alpha)$ , Hentschel and Procaccia [18] used ideas based on Renyi entropies [34] to introduce the generalized dimensions  $D_q$  defined by

$$D_q = \lim_{n \rightarrow +\infty} \frac{1}{q-1} \frac{\log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right)}{n \log \ell},$$

(see also [15, 16]). From a physical and heuristical point of view, Halsey et al. [17] showed that the singularity spectrum  $f(\alpha)$  and the generalized dimensions  $D_q$  can be derived from each other. The Legendre transform turned out to be a useful tool linking  $f(\alpha)$  and  $D_q$ . More precisely, it was suggested that

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q), \quad q \in \mathbb{R}), \quad (1.1)$$

where

$$\tau(q) = \limsup_{n \rightarrow +\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n \log \ell} \log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right).$$

(The sum runs over the  $\ell$ -adic intervals  $I$  such that  $m(I) \neq 0$ .) The function  $\tau(q)$  is called the  $L^q$ -spectrum of  $m$  and if the limit exists  $\tau(q) = (q-1)D_q$ . Note that there may be problems of stability or invariance in the definition of  $\tau(q)$  for negative  $q$  and Riedi [35] propose an improvement of this definition. In what follows, these difficulties will be avoided by restricting the sums over convenient  $\ell$ -adic intervals defining the measure. Of course, this way is not an option in many applications where the structure of the measure is not known in advance. For more information on the  $L^q$ -spectrum and the singularity spectrum we refer the reader to [1, 2, 5, 10, 11, 19, 28, 31, 32, 37, 38, 39, 41].

Relation (1.1) is called the multifractal formalism and in many aspects it is analogous to the well-known thermodynamic formalism developed by Bowen [4] and Ruelle [36].

For number of measures, relation (1.1) can be verified rigorously. In particular, under some separation conditions, self-similar measures satisfy the multifractal formalism (e.g. [6, 7, 9, 23, 30]). Despite all the investigations mentioned, the exact range of the validity of the multifractal formalism is still not known. Furthermore, it is easy to construct measures that do not satisfy (1.1) (e.g [35]). It is thus interesting to find conditions ensuring the validity of (1.1). The main difficulty is often to get a lower bound of  $\dim(E_\alpha)$ . Usually, such a minoration relies on the existence of an auxiliary measure  $m_q$ , the so-called Gibbs measure, supported on the level set  $E_\alpha$ . Recall that  $m_q$  is a *Gibbs measure* at state  $q$  for the measure  $m$  if

$$\forall n, \forall I \in \mathcal{F}_n, \quad \frac{1}{C} m(I)^q \ell^{-n\tau(q)} \leq m_q(I) \leq C m(I)^q \ell^{-n\tau(q)},$$

where the constant  $C > 0$  is independent of  $n$  and  $I$ . If  $\tau$  is differentiable at  $q$ , the measure  $m_q$ —if it exists—will be supported by  $E_{-\tau'(q)}$ . In this case, Brown, Michon and Peyrière established [5, 32] that

$$\dim(E_{-\tau'(q)}) = \tau^*(-\tau'(q)) = -q\tau'(q) + \tau(q).$$

In general, to prove the existence of Gibbs measures we need some homogeneity hypotheses on the measure. This is, for instance, the case of quasi-Bernoulli measures : a probability measure  $m$  is said to be *quasi-Bernoulli* if there exists a constant  $C > 0$  such that

$$\forall (n, p) \in \mathbb{N}^2, \forall I \in \mathcal{F}_n, \forall J \in \mathcal{F}_p, \quad \frac{1}{C}m(I)m(J) \leq m(IJ) \leq Cm(I)m(J), \quad (1.2)$$

where  $IJ = I \cap \sigma^{-n}(J)$  and  $\sigma(x) = \ell x \pmod{1}$  is the shift map on the interval  $[0, 1)$ . In this situation, Brown, Michon and Peyrière [5, 27, 32] proved the existence of a Gibbs measure at every state  $q$ . A few years later, Heurteaux [19] showed that  $\tau$  is differentiable on  $\mathbb{R}$ . Therefore, for quasi-Bernoulli measures, we have

$$\forall \alpha \in (-\tau'(+\infty), -\tau'(-\infty)), \quad \dim(E_\alpha) = \tau^*(\alpha).$$

Recently, in [37, 39] we introduced a more general condition that we call the weak quasi-Bernoulli property. More precisely, we say that a measure  $m$  satisfies the *weak quasi-Bernoulli* property if there exists a constant  $C > 0$  and some integers  $r_1, r_2, p_1, p_2, s_1, s_2$  such that

$$\left\{ \begin{array}{l} \exists C > 0, \forall n, \forall p, \forall I \in \mathcal{F}_n, \forall J \in \mathcal{F}_p, \\ C^{-1}m(I) \sum_{k=r_1}^{r_2} m(\sigma^{-k}(J)) \leq \sum_{k=p_1}^{p_2} m(I \cap \sigma^{-(n+k)}(J)) \leq Cm(I) \sum_{k=s_1}^{s_2} m(\sigma^{-k}(J)). \end{array} \right. \quad (1.3)$$

At first sight, this new condition may seem artificial but is in fact natural. Indeed, in [37, 39] we showed that many self-similar measures with overlaps are not quasi-Bernoulli but are weak quasi-Bernoulli and may be used to estimate the dimension of self-affine graphs.

Furthermore, under this condition, we proved in [39] the existence of Gibbs measures at every positive state  $q$  and the differentiability of  $\tau$  on  $\mathbb{R}^+$ . For weak quasi-Bernoulli measures, we deduced that

$$\forall \alpha \in (-\tau'(+\infty), -\tau'(0)), \quad \dim(E_\alpha) = \tau^*(\alpha).$$

Now, it is natural to ask whether or not these results still hold for negative  $q$  when the measure only satisfies the weak quasi-Bernoulli property. In particular, in this setting, we would like to know if

- (i) the  $L^q$ -spectrum  $\tau(q)$  is differentiable on  $(-\infty, 0)$ ,
- (ii) there exists Gibbs measures for negative  $q$ ,
- (iii) we have  $\dim(E_\alpha) = \tau^*(\alpha)$  for  $\alpha > -\tau'(0)$ .

Note that the tools used in this context for  $q \geq 0$  cannot be applied for  $q < 0$ . In particular, to prove the existence of Gibbs measures for  $q \geq 0$  we use some multiplicative

properties of the sequence  $\ell^{n\tau_n(q)}$  which are no longer verified for  $q < 0$ . In what follows, we show that the multifractal formalism may break down for weak quasi-Bernoulli measures. Therefore, for these measures, the answer to the above questions could be no.

Let us precise these examples. For an integer  $\ell \geq 2$ , we consider the  $2\ell$  similitudes  $S_i : [0, 1] \mapsto [0, 1]$  defined by

$$\forall 0 \leq i \leq \ell - 1, \quad S_i(x) = \frac{1}{\ell}x + \frac{i}{\ell} \quad \text{and} \quad S_{i+\ell}(x) = -\frac{1}{\ell}x + \frac{i+1}{\ell}.$$

For a given probability weight  $\{p_i\}_{i=0}^{2\ell-1}$ , it is well known (e.g [8, 21]) that there exists a unique probability measure  $\mu$  on  $[0, 1]$  verifying

$$\mu = \sum_{i=0}^{2\ell-1} p_i \mu \circ S_i^{-1}. \quad (1.4)$$

This measure is often called the self-similar measure generated by  $\{S_i\}_{i=0}^{2\ell-1}$ . In this paper we establish that  $\mu$  satisfies the weak quasi-Bernoulli property. Moreover, we show that there exists a *Frostman measure*  $\mu_q$  at every negative state  $q$ , i.e. a measure  $\mu_q$  such that

$$\forall n, \forall I \in \mathcal{F}_n, \quad \mu_q(I) \leq C\mu(I)^q \ell^{-n\tau(q)}, \quad (1.5)$$

where the constant  $C > 0$  is independent of  $n$  and  $I$ . Thus, for  $\alpha = -\tau'(q)$ , we have

$$\forall x \in E_\alpha, \quad \mu_q(I_n(x)) \leq (\ell^{-n})^{\tau^*(\alpha)},$$

if  $n$  is large enough. The mass distribution principle or Frostman Lemma (e.g. [8]) implies that  $\dim(E_\alpha) \geq \tau^*(\alpha)$ . Thus, the values of  $\alpha$  for which the multifractal formalism may fail lie in intervals  $(-\tau'_+(q), -\tau'_-(q))$  where  $q$  is a point of non-differentiability of  $\tau$  ( $\tau'_-(q)$  and  $\tau'_+(q)$  stand for the left and the right derivative respectively). Such a point  $q$  will be called a *phase transition*.

We assume that the weights  $p_i$  associated to the measure  $\mu$  verifying (1.4) are positive for every  $0 \leq i \leq \ell - 1$ . We set  $B = \{0 \leq i \leq \ell - 1, p_{i+\ell} = 0\}$  and  $\tilde{\tau}(q) = \log_\ell(\sum_{i \in B} p_i^q)$ . In this case, the  $L^q$ -spectrum  $\tau_\mu(q)$  of  $\mu$  is given by  $\tau_\mu(q) = \max(\tau_\nu(q), \tilde{\tau}(q))$  where  $\nu = (\mu + \mu \circ T)/2$  and  $T(x) = 1 - x$ . In order to get phase transitions for the function  $\tau_\mu$ , it is thus enough to find conditions on the  $p_i$ 's ensuring that the equation  $\tau_\nu(q) = \tilde{\tau}(q)$  has isolated solutions.

Let  $K$  be the compact set defined by  $K = \cup_{i \in B} S_i(K)$ . The attractor  $K$  plays an important role to determine the local dimensions of  $\mu$ . Indeed, we can link the level sets of  $\mu$  and  $\nu$  in the following way :  $E_\alpha(\mu) = (E_\alpha(\nu) \cap ([0, 1] \setminus K)) \cup (E_\alpha(\mu) \cap K)$ . If  $\nu$  satisfies the quasi-Bernoulli property, we get  $\dim(E_\alpha(\mu)) = \max(\tau_\nu^*(\alpha), \tilde{\tau}^*(\alpha))$ . Using the expression of  $\tau_\mu$ , we deduce that each phase transition corresponds to an interval in which the multifractal formalism does not hold. More precisely, we have the following.

- (i) If  $\tau'_\mu(q)$  exists and if  $\alpha = -\tau'_\mu(q)$ , then  $\dim(E_\alpha(\mu)) = \tau_\mu^*(\alpha)$ .
- (ii) If  $\tau'_\mu(q)$  does not exist and if  $-(\tau_\mu)'_+(q) < \alpha < -(\tau_\mu)'_-(q)$ , then  $\dim(E_\alpha(\mu)) < \tau_\mu^*(\alpha)$ .

The class of measures studied may appear rather restrictive but is in fact sufficiently large to point out new and interesting phenomena. In particular, we can observe the following facts.

- The existence of an isolated point in the set of the local dimensions  $D_\mu$  defined by  $D_\mu = \{\alpha, E_\alpha(\mu) \neq \emptyset\}$ . This situation has already been obtained for the Erdős measure and for the 3-time convolution of the Cantor measure (e.g [13, 20]).
- The existence of non-concave multifractal spectra eventually supported by a union of mutually disjoint intervals. To the best of our knowledge, it is the first time that such multifractal structures are obtained for self-similar measures.
- The existence of an arbitrarily large number of phase transitions for the  $L^q$ -spectrum  $\tau(q)$ .

The paper is organized as follows. In section 2 we prove that the measure  $\mu$ , given by (1.4), satisfies the weak quasi-Bernoulli property. In section 3 we establish the existence of Gibbs measures at every negative state  $q$  for the measure  $\mu$ . In section 4 we determine the  $L^q$ -spectrum  $\tau_\mu(q)$ . In section 5 we are interested in the singularity spectrum of  $\mu$ . The paper ends with a range of examples.

## 2. The weak quasi-Bernoulli property

Let us introduce some notation. In what follows, except contrary mention,  $\ell \geq 2$  is an integer,  $I$  a  $\ell$ -adic interval of  $n$ th generation and  $J$  a  $\ell$ -adic interval. For every  $(\epsilon_1, \dots, \epsilon_n) \in \{0, \dots, \ell - 1\}^n$ ,  $I_{\epsilon_1 \dots \epsilon_n}$  stands for the element of  $\mathcal{F}_n$  defined by

$$I_{\epsilon_1 \dots \epsilon_n} = \left[ \sum_{i=1}^n \frac{\epsilon_i}{\ell^i}, \sum_{i=1}^n \frac{\epsilon_i}{\ell^i} + \frac{1}{\ell^n} \right).$$

If  $I = I_{\epsilon_1 \dots \epsilon_n}$  and  $\epsilon \in \{0, \dots, \ell - 1\}$ , we shall write  $\epsilon I$  instead of  $I_{\epsilon \epsilon_1 \dots \epsilon_n}$ . If  $f$  and  $g$  are positive functions of the same parameter,  $f \approx g$  means there exists a constant  $C > 0$  such that  $C^{-1}g \leq f \leq Cg$ . Moreover, for any matrices  $M$  and  $N$ , we shall write  $M > 0$  (and we shall say that  $M$  is positive) if all the digits of  $M$  are positive and  $M > N$  if  $M - N > 0$ . The matrix relations  $<$ ,  $\geq$  and  $\leq$  are similarly defined. Finally, for a  $2 \times 2$  nonnegative matrix  $M$ , we define  $\|M\|_1$  and  $\|M\|$  by

$$\|M\|_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} M \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \|M\| = \frac{1}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let  $\mu$  be the measure verifying (1.4). For convenience, we suppose that  $\mu$  is supported on the interval  $[0, 1]$ . That is equivalent to the condition :  $p_i + p_{i+\ell} > 0$ , for every  $i \in \{0, \dots, \ell - 1\}$ . The relation (1.4) implies that

$$\forall \epsilon \in \{0, \dots, \ell - 1\}, \quad \mu(\epsilon I) = p_\epsilon \mu(I) + p_{\epsilon+\ell} \mu(I^*),$$

where  $I^* = T(I)$  and  $T(x) = 1 - x$ . Since  $(I^*)^* = I$ , we have

$$\begin{pmatrix} \mu(\epsilon I) \\ \mu \circ T(\epsilon I) \end{pmatrix} = M_\epsilon \begin{pmatrix} \mu(I) \\ \mu \circ T(I) \end{pmatrix} \quad \text{where} \quad M_\epsilon = \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_{2\ell-1-\epsilon} & p_{\ell-1-\epsilon} \end{pmatrix}. \quad (2.1)$$

By iterating this relation, we get

$$\forall I = I_{\epsilon_1 \dots \epsilon_n} \in \mathcal{F}_n, \quad \begin{pmatrix} \mu(I) \\ \mu \circ T(I) \end{pmatrix} = M_{\epsilon_1} \cdots M_{\epsilon_n} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and we deduce that

$$\mu(I) = \|M_I\|_1 \quad \text{and} \quad \nu(I) = \|M_I\|, \quad \text{where} \quad M_I = M_{\epsilon_1} \cdots M_{\epsilon_n}. \quad (2.2)$$

These relations will be used to prove that  $\mu$  satisfies the weak quasi-Bernoulli property (1.3). More precisely, we have the following.

**Proposition 2.1.** *Let  $\mu$  be the measure verifying (1.4). Then,*

$$\begin{cases} \exists C > 0, \forall n, \forall p, \forall I \in \mathcal{F}_n, \forall J \in \mathcal{F}_p, \\ C^{-1} \mu(I) \mu(J) \leq \mu(I \cap \sigma^{-(n+1)}(J)) \leq C \mu(I) \mu(\sigma^{-2}(J)), \end{cases}$$

where  $\sigma : [0, 1] \mapsto [0, 1]$  is the shift map on the  $\ell$ -adic basis given by  $\sigma(x) = \ell x \pmod{1}$ .

**Remarks.** 1. In general, the measure  $\mu$  does not satisfy the quasi-Bernoulli property (1.2). To see that, take for example  $\ell = 2$  and suppose that  $p_0 > p_1$ ,  $p_0 p_1 p_2 > 0$  and  $p_3 = 0$ . Using (1.4), we get for  $J = I_{1\dots 1} \in \mathcal{F}_n$ ,

$$\mu(0J) = \mu(I_{01\dots 1}) = p_0 \mu(J) + p_2 \mu(J^*) = p_0 \mu(I_{1\dots 1}) + p_2 \mu(I_{0\dots 0}).$$

From (2.1) and (2.2),  $\mu(J) = \|M_1^n\|_1 = p_1^n$  and  $\mu(J^*) = \|M_0^n\|_1 \approx p_0^n$ . Therefore, if  $I = I_0$ , we have  $\mu(IJ) \approx p_0^n$  and  $\mu(I)\mu(J) \approx p_1^n$ , which proves that  $\mu$  is not quasi-Bernoulli.

2. If for every  $\epsilon \in \{0, \dots, \ell-1\}$ ,  $p_\epsilon p_{\epsilon+\ell} = 0$ , the Open Set Condition of Hutchinson [21] is verified. In this case,  $\mu$  is quasi-Bernoulli and proposition 2.1 easily follows.

**Proof of proposition 2.1.** According to the above remark, we can suppose that there exists  $\tilde{\epsilon} \in \{0, \dots, \ell-1\}$  such that  $p_{\tilde{\epsilon}} p_{\tilde{\epsilon}+\ell} > 0$ . By (2.1),  $M_{\tilde{\epsilon}} + M_{\ell-1-\tilde{\epsilon}} > 0$ . Thus, we can find a constant  $C > 0$  such that

$$\frac{1}{C} E \leq \sum_{\epsilon=0}^{\ell-1} M_\epsilon \quad \text{and} \quad I_2 \leq C E \sum_{\epsilon=0}^{\ell-1} M_\epsilon,$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from (2.2) that

$$\mu(I \cap \sigma^{-(n+1)}(J)) = \left\| M_I \left( \sum_{\epsilon=0}^{\ell-1} M_\epsilon \right) M_J \right\|_1 \geq \frac{1}{C} \|M_I E M_J\|_1 = \frac{1}{C} \mu(I) \mu(J).$$

On the other hand, we obtain in a similar way that

$$\mu(I \cap \sigma^{-n}(J)) = \|M_I M_J\|_1 \leq C \left\| M_I E \left( \sum_{\epsilon=0}^{\ell-1} M_\epsilon \right) M_J \right\|_1 = C \mu(I) \mu(\sigma^{-1}(J)).$$

A monotone class argument implies that this relation still holds if we replace  $J$  by any Borel set  $B$ . Thus, by taking  $B = \sigma^{-1}(J)$ , we have

$$\mu(I \cap \sigma^{-(n+1)}(J)) \leq C\mu(I)\mu(\sigma^{-2}(J)),$$

which completes the proof of proposition 2.1.  $\square$

### 3. Frostman measures

In this section we establish the existence of Frostman measures (1.5) at every negative state  $q$  for the measure  $\mu$  defined by (1.4). We begin with a preliminary result that gives conditions ensuring the existence of Frostman measures.

For a probability measure  $m$  on the interval  $[0, 1]$ , let us define the series  $Z(s)$  by

$$\forall s \in \mathbb{R}, \quad Z(s) = \sum_{n \geq 1} u_n \ell^{-ns}, \quad \text{where } u_n = \ell^{n\tau_n(q)}.$$

If  $m(I) > 0$ ,  $Z_I(s)$  denotes the series associated to the measure  $m_I$  verifying  $m_I(J) = m(IJ)/m(I)$ .

**Proposition 3.1.** *Let  $m$  be a probability measure on the interval  $[0, 1]$  and  $q \in \mathbb{R}$ . With the above notation, suppose that there exists a constant  $C > 0$  such that*

- (i)  $\forall n, \forall p, \quad u_{n+p} \leq C u_n u_p,$
- (ii)  $\forall I \in \mathcal{F}, \forall s \in \mathbb{R}, \quad Z_I(s) \leq CZ(s)$

*Then, there exists a Frostman measure at state  $q$  for the measure  $m$ .*

**Proof.** We adapt to our situation the arguments used by Michon and Peyrière [27, 32] in another context. The submultiplicativity property of the sequence  $v_n = C\ell^{n\tau_n(q)}$  implies that the sequence  $v_n^{1/n}$  tends to its lower bound. As a consequence,  $\tau_n(q)$  converges and if we call  $\tau(q)$  its limit, we have

$$\forall n \in \mathbb{N}, \quad C\ell^{n\tau_n(q)} \geq \ell^{n\tau(q)}.$$

Therefore, the series  $Z(s)$  converges for  $s > \tau(q)$  and diverges for  $s = \tau(q)$ . Let us consider, for  $s > \tau(q)$ , the function  $\phi_s$  defined by

$$\phi_s(x) = \sum_{n \geq 1} m(I_n(x))^q (\ell^{-n})^{-1+s}.$$

Since  $\int_0^1 \phi_s(x) dx = Z(s)$ , we can define a probability measure  $\nu_s$  on the interval  $[0, 1]$  by

$$\forall I \in \mathcal{F}, \quad \nu_s(I) = \frac{\int_I \phi_s(x) dx}{Z(s)}.$$

For every  $I \in \mathcal{F}_n$ , we find that

$$Z(s)\nu_s(I) = \int_I \phi_s(x) dx = \ell^{-n} \sum_{1 \leq k \leq n} m(I_k)^q (\ell^{-k})^{-1+s} + m(I)^q \ell^{-ns} Z_I(s),$$

where  $I_k$  denotes the element of  $\mathcal{F}_k$  containing  $I$ .



Let  $m_q$  be a weak\*-limit of  $\nu_s$  as  $s$  goes to  $\tau(q)$ . The divergence of the series  $Z(s)$  for  $s = \tau(q)$  and the inequality  $Z_I(s) \leq CZ(s)$  imply that  $m_q(I) \leq Cm(I)^q \ell^{-n\tau(q)}$ , which completes the proof of proposition 3.1.  $\square$

We easily deduce the following result.

**Corollary 3.2.** *Let  $m$  be a probability measure on the interval  $[0, 1]$  and  $q \in \mathbb{R}$ . Suppose that there exists a constant  $C > 0$  such that*

$$\forall I, \forall J, \quad m(IJ)^q \leq Cm(I)^q m(J)^q. \quad (3.1)$$

*Then, there exists a Frostman measure at state  $q$  for the measure  $m$ .*

*In particular, the condition (3.1) is satisfied if  $m(IJ) \leq Cm(I)m(J)$  and  $q > 0$  or if  $m(IJ) \geq Cm(I)m(J)$  and  $q < 0$ .*

We will use the following lemma to prove that the measure  $\mu$  satisfies the hypotheses of proposition 3.1.

**Lemma 3.3.** *Let  $\mu$  be the measure defined by (1.4). For every  $I \in \mathcal{F}$ , one of the following is satisfied.*

$$(i) \quad \forall J \in \mathcal{F}, \quad \mu(I)\mu(J) \leq 2\mu(IJ),$$

$$(ii) \quad \forall J \in \mathcal{F}, \quad \mu(I)\mu \circ T(J) \leq 2\mu(IJ), \text{ where } T(x) = 1 - x.$$

**Proof.** For  $I \in \mathcal{F}$  and  $\epsilon \in \{0, \dots, \ell - 1\}$ , we have  $S_i^{-1}(\epsilon I) = I$  or  $S_i^{-1}(\epsilon I) = T(I)$  or  $S_i^{-1}(\epsilon I) = \emptyset$ . Thus, by iterating (1.4), we can find two non-negative real numbers  $A(I)$  and  $B(I)$  (depending only on  $I$ ) such that

$$\forall J \in \mathcal{F}, \quad \mu(IJ) = A(I)\mu(J) + B(I)\mu \circ T(J). \quad (3.2)$$

We then obtain that either

$$\forall J \in \mathcal{F}, \quad \mu(IJ) \geq C(I)\mu(J),$$

or

$$\forall J \in \mathcal{F}, \quad \mu(IJ) \geq C(I)\mu \circ T(J),$$

where  $C(I) = \max(A(I), B(I)) > 0$ .

By taking  $J = [0, 1]$  in (3.2), we get  $\mu(I) \leq 2C(I)$ . Hence, either

$$\forall J \in \mathcal{F}, \quad 2\mu(IJ) \geq \mu(I)\mu(J),$$

or

$$\forall J \in \mathcal{F}, \quad 2\mu(IJ) \geq \mu(I)\mu \circ T(J),$$

which completes the proof of lemma 3.3.  $\square$

**Theorem 3.4.** *There exists a Frostman measure at every state  $q < 0$  for the measure  $\mu$  verifying (1.4).*

**Proof.** Let  $I \in \mathcal{F}$ . If for every  $J \in \mathcal{F}$ ,  $\mu(I)\mu(J) \leq 2\mu(IJ)$  (respectively,  $\mu(I)\mu \circ T(J) \leq 2\mu(IJ)$ ), we set  $\tilde{\mu}_I = \mu$  (respectively,  $\tilde{\mu}_I = \mu \circ T$ ). By lemma 3.3, we have

$$\forall J \in \mathcal{F}, \quad \mu(I)\tilde{\mu}_I(J) \leq 2\mu(IJ).$$

Since  $q$  is negative, we obtain that

$$u_{n+p} = \sum_{I \in \mathcal{F}_n} \sum_{J \in \mathcal{F}_p} \mu(IJ)^q \leq \left(\frac{1}{2}\right)^q \sum_{I \in \mathcal{F}_n} \sum_{J \in \mathcal{F}_p} \mu(I)^q \tilde{\mu}_I(J)^q = \left(\frac{1}{2}\right)^q u_n u_p$$

and

$$Z_I(s) = \sum_{n \geq 1} \sum_{J \in \mathcal{F}_n} \left(\frac{\mu(IJ)}{\mu(I)}\right)^q \ell^{-ns} \leq \left(\frac{1}{2}\right)^q \sum_{n \geq 1} \sum_{J \in \mathcal{F}_n} \tilde{\mu}_I(J)^q \ell^{-ns} = \left(\frac{1}{2}\right)^q Z(s).$$

Thus,  $\mu$  satisfies the hypotheses of proposition 3.1 and theorem 3.4 follows.  $\square$

#### 4. The function $\tau_\mu$

In this section we determine the  $L^q$ -spectrum  $\tau_\mu(q)$  of the measure  $\mu$  verifying (1.4). Let us start with the following easy lemma.

**Lemma 4.1.** *Let  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  be two sequences of real positive numbers such that*

$$\lim_{n \rightarrow \infty} u_n^{1/n} = u \quad \text{et} \quad \lim_{n \rightarrow \infty} v_n^{1/n} = v.$$

Let

$$w_n = \sum_{k=0}^n u_k v_{n-k}.$$

Then, the sequence  $(w_n^{1/n})_{n \in \mathbb{N}}$  converges to  $w = \max\{u, v\}$ .

The proof is elementary and therefore omitted.

**Theorem 4.2.** *Let  $\mu$  be the probability measure verifying (1.4). Suppose that for every  $0 \leq i \leq \ell - 1$ ,  $p_i > 0$ , and set  $B = \{0 \leq i \leq \ell - 1, p_{i+\ell} = 0\}$ . Then, by denoting  $\tilde{\tau}(q) = \log_\ell \left(\sum_{i \in B} p_i^q\right)$ , we have*

$$\forall q \in \mathbb{R}, \quad \tau_\mu(q) = \max(\tau_\nu(q), \tilde{\tau}(q)), \quad (4.1)$$

where  $\nu = (\mu + \mu \circ T)/2$  and  $T(x) = 1 - x$ . By convention,  $\tilde{\tau}(q) = -\infty$  if  $B = \emptyset$ .

- Remarks.**
1. We obtain a similar result replacing the hypothesis “ $p_i > 0$ , for every  $0 \leq i \leq \ell - 1$ ” by “ $p_i + p_{i+\ell} > 0$ , for every  $0 \leq i \leq \ell - 1$ ”.
  2. For positive  $q$ , the two  $L^q$ -spectra  $\tau_\mu(q)$  and  $\tau_\nu(q)$  are obviously the same. If  $q < 0$ , the situation is inverted : “small become big”. In fact, sets with negligible mass determine the function  $\tau_\mu$  ; therefore it suffices to consider the sum over indices  $i \in B$  in the expression of  $\tilde{\tau}$ .
  3. If  $B = \{0, \dots, \ell - 1\}$ ,  $\mu$  is a multinomial measure (also called Bernoulli product). The calculation of  $\tau_\mu$  is then straightforward and we have  $\tau_\mu = \tilde{\tau}$  (e.g [9]).
  4. To obtain phase transitions for the function  $\tau_\mu$ , it is thus enough to find conditions on the  $p_i$ 's ensuring that the equation  $\tau_\nu(q) = \tilde{\tau}(q)$  has many isolated solutions.

**Proof of theorem 4.2.** We fix  $q \in \mathbb{R}$ ,  $n \in \mathbb{N}^*$  and we write  $C$  for each constant which depends on  $q$  but not on  $n$ . Using (1.4), we get

$$\begin{aligned} w_n &= \sum_{I \in \mathcal{F}_n} \mu(I)^q = \sum_{i=0}^{\ell-1} \sum_{I \in \mathcal{F}_{n-1}} \mu(iI)^q \\ &= \sum_{i \notin B} \sum_{I \in \mathcal{F}_{n-1}} (p_i \mu(I) + p_{i+\ell} \mu(I^*))^q + \sum_{i \in B} \sum_{I \in \mathcal{F}_{n-1}} p_i^q \mu(I)^q \\ &\leq C \sum_{I \in \mathcal{F}_{n-1}} \nu(I)^q + \sum_{i \in B} p_i^q w_{n-1} := C v_{n-1} + \sum_{i \in B} p_i^q w_{n-1}. \end{aligned}$$

By induction, we obtain

$$w_n \leq C \sum_{k=0}^{n-1} \left( \sum_{i \in B} p_i^q \right)^k v_{n-(k+1)} := C \sum_{k=0}^{n-1} u_k v_{n-(k+1)}.$$

We can find a minoration of the same type in a similar way.

Furthermore, the definition of  $\|\cdot\|$  in (2.2) implies that  $\nu(IJ) \leq 2\nu(I)\nu(J)$ . Thus, the sequence  $v_n^{1/n}$  converges and theorem 4.2 easily follows from lemma 4.1.  $\square$

## 5. The level sets $E_\alpha$

In this section we link the level sets  $E_\alpha(\mu)$  and  $E_\alpha(\nu)$  associated to the measures  $\mu$  and  $\nu$ .

**Proposition 5.1.** *The hypotheses are the same as in theorem 4.2. Let  $K$  be the compact set defined by  $K = \bigcup_{i \in B} S_i(K)$  with the convention that  $K = \emptyset$  if  $B = \emptyset$ . Then,*

$$E_\alpha(\mu) = (E_\alpha(\nu) \cap ([0, 1] \setminus K)) \cup (E_\alpha(\mu) \cap K).$$

**Remarks.** (i) If  $B = \{0, \dots, \ell - 1\}$ ,  $K = [0, 1]$  and proposition 5.1 is immediate.

(ii) If  $B = \emptyset$ ,  $K = \emptyset$  and by proposition 5.1,  $E_\alpha(\mu) = E_\alpha(\nu)$ . In fact, in this case, it is easy to prove that the measures  $\mu$  and  $\nu$  are strongly equivalent.

(iii) If  $B$  is reduced to a single element,  $K$  is a singleton.

(iv) In all other cases,  $K$  is a Cantor set.

**Proof of proposition 5.1.** According to the above remark, we can suppose that  $B \neq \{0, \dots, \ell - 1\}$  and  $B \neq \emptyset$ . Fix  $x \notin K$  and  $\alpha > 0$ . To prove our claim, it is sufficient to show that  $x \in E_\alpha(\mu)$  if and only if  $x \in E_\alpha(\nu)$ . Since  $x \notin K$ , there exists  $n(x) \in \mathbb{N}$  and  $\epsilon \notin B$  such that

$$\forall n \geq n(x), \quad I_n(x) = I_{\epsilon_1 \dots \epsilon_{n(x)} \epsilon \dots \epsilon_n}$$

where  $(\epsilon_1, \dots, \epsilon_{n(x)}) \in B^{n(x)}$ . With obvious notation, it results from (2.1) and (2.2) that

$$\frac{\mu(I_n(x))}{\nu(I_n(x))} = \frac{\|M_{I_n(x)} M_\epsilon M_{x,n}\|_1}{\|M_{I_n(x)} M_\epsilon M_{x,n}\|} = \frac{\left\| \begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_{2\ell-1-\epsilon} & p_{\ell-1-\epsilon} \end{pmatrix} \begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix} \right\|_1}{\left\| \begin{pmatrix} a(x) & 0 \\ b(x) & c(x) \end{pmatrix} \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_{2\ell-1-\epsilon} & p_{\ell-1-\epsilon} \end{pmatrix} \begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix} \right\|_1}$$

$$\begin{aligned}
&\geq C(x) \frac{\left\| \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_\epsilon + p_{2\ell-1-\epsilon} & p_{\epsilon+\ell} + p_{\ell-1-\epsilon} \end{pmatrix} \begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix} \right\|_1}{\left\| \begin{pmatrix} p_\epsilon & p_{\epsilon+\ell} \\ p_\epsilon + p_{2\ell-1-\epsilon} & p_{\epsilon+\ell} + p_{\ell-1-\epsilon} \end{pmatrix} \begin{pmatrix} \tilde{a}_{x,n} & \tilde{b}_{x,n} \\ \tilde{c}_{x,n} & \tilde{d}_{x,n} \end{pmatrix} \right\|} \\
&= C(x) \frac{p_\epsilon(\tilde{a}_{x,n} + \tilde{b}_{x,n}) + p_{\epsilon+\ell}(\tilde{c}_{x,n} + \tilde{d}_{x,n})}{(2p_\epsilon + p_{2\ell-1-\epsilon})(\tilde{a}_{x,n} + \tilde{b}_{x,n}) + (2p_{\epsilon+\ell} + p_{\ell-1-\epsilon})(\tilde{c}_{x,n} + \tilde{d}_{x,n})}.
\end{aligned}$$

if  $n$  is large enough. Using  $p_\epsilon p_{\epsilon+\ell} > 0$ , we can find another constant  $C'(x) > 0$  which does not depend on  $n$  such that  $\mu(I_n(x)) \geq C'(x)\nu(I_n(x))$ . Since  $\mu \leq 2\nu$ , we then deduce that

$$(E_\alpha(\mu) \cap ([0, 1] \setminus K)) = (E_\alpha(\nu) \cap ([0, 1] \setminus K))$$

and proposition 5.1 follows easily.  $\square$

**Theorem 5.2.** *The notation are the same as in theorem 4.2. Suppose that  $B \cap B^* = \emptyset$  where  $B^* = \{\ell - 1 - \epsilon, \epsilon \in B\}$ . Then,*

$$(i) \ D_\mu = D_\nu \cup \left[ -\log_\ell \left( \max_{i \in B} p_i \right), -\log_\ell \left( \min_{i \in B} p_i \right) \right] \quad \text{where } D_\mu = \{\alpha, E_\alpha(\mu) \neq \emptyset\}.$$

(By convention,  $[-\log_\ell(\max_{i \in B} p_i), -\log_\ell(\min_{i \in B} p_i)] = \emptyset$  if  $B = \emptyset$ .)

$$(ii) \ \forall \alpha \in D_\mu, \quad \dim(E_\alpha(\mu)) = \max(\dim(E_\alpha(\nu)), \tilde{\tau}^*(\alpha)).$$

**Proof.** If  $B = \emptyset$ , the measures  $\mu$  and  $\nu$  are strongly equivalent and theorem 5.2 is easy. Now assume that  $B \neq \emptyset$ . Let  $\pi$  be the self-similar measure supported on  $K$  verifying

$$\pi = \frac{1}{\sum_{i \in B} p_i} \sum_{i \in B} p_i \pi \circ S_i^{-1}. \quad (5.1)$$

The family  $(S_i)_{i \in B}$  satisfying the Open Set Condition [21], the calculation of the  $L^q$ -spectrum  $\tau_\pi(q)$  is then straightforward :

$$\forall q \in \mathbb{R}, \quad \tau_\pi(q) = \log_\ell \left( \sum_{i \in B} \left( \frac{p_i}{\sum_{i \in B} p_i} \right)^q \right), \quad (5.2)$$

(e.g. [6, 30]). Moreover, we have

$$\begin{cases} \dim(E_\alpha(\pi)) = \tau_\pi^*(\alpha) & \text{if } \alpha \in \left[ -\log_\ell \left( \frac{\max_{i \in B} p_i}{\sum_{i \in B} p_i} \right), -\log_\ell \left( \frac{\min_{i \in B} p_i}{\sum_{i \in B} p_i} \right) \right], \\ E_\alpha(\pi) = \emptyset & \text{otherwise.} \end{cases} \quad (5.3)$$

Furthermore, for every  $(\epsilon_1 \cdots \epsilon_n) \in B^n$  and  $I = I_{\epsilon_1 \cdots \epsilon_n}$ , we find that

$$\pi(I) = \left( \sum_{i \in B} p_i \right)^{\log_\ell(|I|)} \mu(I).$$

We thus deduce that

$$\forall \alpha, \quad E_\alpha(\pi) = E_{\alpha - \log_\ell(\sum_{i \in B} p_i)}(\mu) \cap K,$$

or equivalently

$$\forall \alpha, \quad E_{\alpha + \log_\ell(\sum_{i \in B} p_i)}(\pi) = E_\alpha(\mu) \cap K.$$

It follows from (5.3) that

$$E_\alpha(\mu) \cap K \neq \emptyset \Leftrightarrow \alpha \in \left[ -\log_\ell \left( \max_{i \in B} p_i \right), -\log_\ell \left( \min_{i \in B} p_i \right) \right],$$

and, if  $-\log_\ell(\max_{i \in B} p_i) \leq \alpha \leq -\log_\ell(\min_{i \in B} p_i)$ , then

$$\dim(E_\alpha(\mu) \cap K) = \tau_\pi^* \left( \alpha + \log_\ell \left( \sum_{i \in B} p_i \right) \right) = \tilde{\tau}^*(\alpha). \quad (5.4)$$

Proposition 5.1 also leads to estimate  $\dim(E_\alpha(\nu) \cap ([0, 1] \setminus K))$ . The hypothesis  $B \cap B^* = \emptyset$  implies that  $T(K) \subset [0, 1] \setminus K$ . Therefore,

$$\begin{aligned} \dim(E_\alpha(\nu) \cap K) &= \dim(T(E_\alpha(\nu) \cap K)) \\ &= \dim(E_\alpha(\nu) \cap T(K)) \leq \dim(E_\alpha(\nu) \cap ([0, 1] \setminus K)), \end{aligned}$$

and we conclude that

$$\dim(E_\alpha(\nu)) = \dim(E_\alpha(\nu) \cap ([0, 1] \setminus K)).$$

Theorem 5.2 then follows from proposition 5.1 and (5.4).  $\square$

**Remarks.** 1. Using the same ideas, we can also obtain that

$$\forall \alpha, \quad \dim(V_\alpha(\mu)) = \max(\dim(V_\alpha(\nu)), \tilde{\tau}^*(\alpha)),$$

where  $V_\alpha$  is defined as  $E_\alpha$  replacing  $\lim$  by  $\liminf$ . In other terms,

$$V_\alpha(m) = \left\{ x \in [0, 1], \liminf_{n \rightarrow +\infty} -\frac{\log m(I_n(x))}{n \log \ell} = \alpha \right\}.$$

2. Similar results can also be established replacing the Hausdorff dimension  $\dim(E_\alpha)$  by the Packing dimension  $\text{Dim}(E_\alpha)$ .

We deduce the following.

**Corollary 5.3.** *Suppose that  $B \cap B^* = \emptyset$  and that  $\nu$  satisfies the quasi-Bernoulli property. Then,*

$$\forall \alpha, \quad \dim(E_\alpha(\mu)) = \text{Dim}(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \text{Dim}(V_\alpha(\mu)) = \max(\tau_\nu^*(\alpha), \tilde{\tau}^*(\alpha)).$$

According to theorem 4.2, the function  $\tau_\mu^*$  is the Legendre transform of the maximum of  $\tau_\nu$  and  $\tilde{\tau}$ . On the other hand, by corollary 5.3, the dimension of the level sets  $E_\alpha(\mu)$  is given by the maximum of the Legendre transform of  $\tau_\nu$  and the Legendre transform of  $\tilde{\tau}_\nu$ . Since we cannot invert Legendre transform and maximum, we have the following.

**Theorem 5.4.** *Suppose that  $B \cap B^* = \emptyset$  and that  $\nu$  satisfies the quasi-Bernoulli property. Then, we have the following.*

(i) *If  $\tau_\mu'(q)$  exists and if  $\alpha = -\tau_\mu'(q)$ , then*

$$\dim(E_\alpha(\mu)) = \text{Dim}(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \text{Dim}(V_\alpha(\mu)) = \tau_\mu^*(\alpha).$$

(ii) If  $\tau_\mu'(q)$  does not exist and if  $-(\tau_\mu)'_+(q) < \alpha < -(\tau_\mu)'_-(q)$ , then

$$\dim(E_\alpha(\mu)) = \text{Dim}(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \text{Dim}(V_\alpha(\mu)) < \tau_\mu^*(\alpha).$$

Hence, each phase transition  $q$  gives rise to an interval  $(-(\tau_\mu)'_+(q), -(\tau_\mu)'_-(q))$  in which the multifractal formalism breaks down.

**Remark.** It is possible to prove that  $\nu$  satisfies the quasi-Bernoulli property if and only if

$$\text{either } \forall i \in B, p_i < p_{\ell-1-i}, \quad \text{or } \forall i \in B, p_i > p_{\ell-1-i}. \quad (5.5)$$

The multifractal formalism fails for the measure  $\mu$  only in the first case. Indeed, if for every  $i \in B$ ,  $p_i > p_{\ell-1-i}$ , it is easy to check that the measures  $\mu$  and  $\nu$  are strongly equivalent.

## 6. Examples

In this section we construct measures with non-differentiable  $L^q$ -spectra  $\tau(q)$  for which previous results apply. Furthermore, based on these examples, we point out new phenomena in the multifractal structure of self-similar measures.

### 6.1. An isolated point in the set of local dimensions

Let us take  $\ell = 2$  and consider the probability measure  $\mu$  verifying

$$\mu = p_0\mu \circ S_0^{-1} + p_1\mu \circ S_1^{-1} + p_2\mu \circ S_2^{-1}, \quad (6.1)$$

where  $S_0(x) = x/2$ ,  $S_1(x) = x/2 + 1/2$  and  $S_2(x) = -x/2 + 1/2$ . We assume that  $p_0p_1p_2 > 0$  and  $p_1 < p_0$ . With the notation previously introduced, we have  $B = \{1\}$  and  $K = \{1\}$ . Moreover, by theorem 4.2,

$$\forall q \in \mathbb{R}, \quad \tau_\mu(q) = \max(\tau_\nu(q), q \log_2(p_1)).$$

Thus, in order to get a phase transition for the function  $\tau_\mu$ , we have to compare  $\tau_\nu'(-\infty)$  and  $\log_2(p_1)$ . For every  $I \in \mathcal{F}_n$ , by iterating (6.1), we get  $\nu(I) \geq (p_-)^n$  where  $p_- = \min(p_0, p_1 + p_2)$ . We easily deduce that  $-\tau_\nu'(-\infty) \leq -\log_2(p_-) < -\log_2(p_1)$ . Since  $\tau_\nu(q) \geq q \log_2(p_1)$  for  $q = 0$ , we conclude that there exists  $q_0 < 0$  such that

$$\tau_\mu(q) = \begin{cases} q \log_2(p_1) & \text{if } q \leq q_0, \\ \tau_\nu(q) & \text{if } q \geq q_0, \end{cases} \quad (6.2)$$

and the  $L^q$ -spectrum  $\tau_\mu(q)$  is not differentiable at  $q = q_0$  (see figure 1(a)). Furthermore, by (5.5),  $\nu$  satisfies the quasi-Bernoulli property. Using theorem 5.2 and corollary 5.3 we deduce the following .

**Theorem 6.1.** *Let  $\mu$  be the measure satisfying (6.1). Then,*

$$D_\mu = D_\nu \cup \{-\log_2(p_1)\} = (-\tau_\nu'(+\infty), -\tau_\nu'(-\infty)) \cup \{-\log_2(p_1)\},$$

and

$$\dim(E_\alpha(\mu)) = \begin{cases} \tau_\nu^*(\alpha) & \text{if } \alpha \in (-\tau_\nu'(+\infty), -\tau_\nu'(-\infty)), \\ 0 & \text{if } \alpha = -\log_2(p_1). \end{cases}$$

**Remarks.** 1. Since  $-\tau_\nu'(-\infty) < -\log_2(p_1)$ ,  $D_\mu$  contains an isolated point. In this sense, the situation is close to the ones obtained for the Erdős measure and for the 3-time convolution of the Cantor measure (e.g [13, 20]). Note that in our situation, the value of  $[p_0, p_1, p_2]$  is not a matter.

2. It is easy to show that

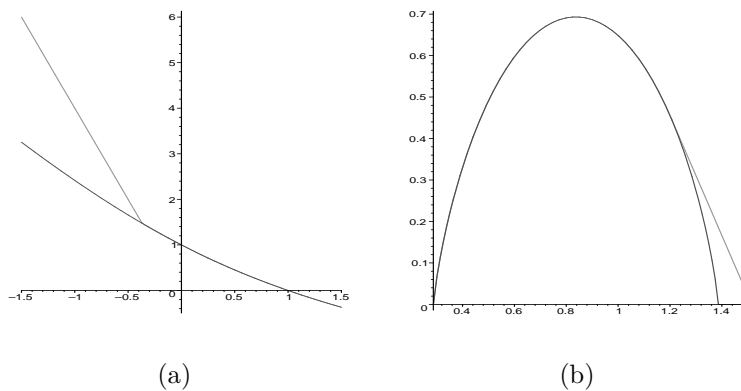
$$\tau_\mu^*(\alpha) = \begin{cases} \tau_\nu^*(\alpha) & \text{if } -\tau_\nu'(+\infty) \leq \alpha \leq -\tau_\nu'(q_0), \\ \frac{\tau_\nu^*(q_0)}{\log_2(p_1) - \tau_\nu'(q_0)} (\alpha + \log_2(p_1)) & \text{if } -\tau_\nu'(q_0) \leq \alpha \leq -\log_2(p_1). \end{cases}$$

Thus,

$$\begin{cases} \forall \alpha \in (-\tau_\nu'(+\infty), -\tau_\nu'(q_0)], & \dim(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \tau_\nu^*(\alpha) = \tau_\mu^*(\alpha), \\ \forall \alpha \in (-\tau_\nu'(q_0), -\tau_\nu'(-\infty)], & \dim(E_\alpha(\mu)) = \dim(V_\alpha(\mu)) = \tau_\nu^*(\alpha) < \tau_\mu^*(\alpha). \end{cases}$$

Contrary to the usual situation, the singularity spectrum of  $\mu$  is not given by the Legendre transform of  $\tau_\mu$  but instead by the Legendre transform of an auxiliary function. Figure 1(b) illustrates this phenomenon.

3. The measure  $\mu$  may be used to estimate the Hausdorff dimension of self-affine graphs studied by McMullen [26], Prsytycki and Urbański [33, 40]. More details can be found in [37, 39].



**Figure 1.** (a)  $\tau_\mu$  is not differentiable. (b) The singularity spectrum of  $\mu$ , given by  $\tau_\nu^*$ , differs from  $\tau_\mu^*$ .

## 6.2. Non-concave spectra

Subsection 3.1 and several papers deal with measures for which the  $L^q$ -spectrum  $\tau(q)$  is not differentiable at a single point  $q = q_0$  and is linear for  $q \leq q_0$  (e.g. [12, 13, 20, 22, 29]). In this part we construct measures with non-differentiable and strictly concave  $L^q$ -spectra. That leads to new situations for the multifractal analysis of self-similar measures.

Let us take  $\ell = 4$  and consider the probability measure  $\mu$  satisfying

$$\mu = \sum_{i=0}^5 p_i \mu \circ S_i^{-1}, \quad (6.3)$$

where

$$\begin{aligned} S_0(x) &= \frac{x}{4}, & S_1(x) &= \frac{x}{4} + \frac{1}{4}, & S_2(x) &= \frac{x}{4} + \frac{1}{2}, \\ S_3(x) &= \frac{x}{4} + \frac{3}{4}, & S_4(x) &= -\frac{x}{4} + \frac{1}{4} & \text{and} & S_5(x) = -\frac{x}{4} + \frac{1}{2}. \end{aligned}$$

In this case,  $B = \{2, 3\}$  and  $K$  is the Cantor set whose points only contains digits 2 and 3 in their 4-adic expression, i.e.  $K = \{x = \sum \epsilon_i/4^i, \epsilon_i = 2 \text{ or } 3, \forall i \in \mathbb{N}^*\}$ .

By theorem 4.2,  $\tau_\mu(q) = \max(\tau_\nu(q), \log_4(p_2^q + p_3^q))$ . In order to compute  $\tau_\nu$ , we assume that the  $p_i$ 's verify  $p_0 = p_3 + p_4$  and  $p_1 = p_2 + p_5$ . In this situation, it is easy to show that  $\nu$  is a multinomial measure (see [37]). The calculation of  $\tau_\nu$  is then straightforward :  $\tau_\nu(q) = 1/2 + \log_4(p_0^q + p_1^q)$ . Therefore, there exists  $q_0 < 0$  such that

$$\tau_\mu(q) = \begin{cases} \log_4(p_2^q + p_3^q) & \text{if } q \leq q_0, \\ \frac{1}{2} + \log_4(p_0^q + p_1^q) & \text{if } q \geq q_0, \end{cases} \quad (6.4)$$

and  $\tau_\mu(q)$  is not differentiable at  $q = q_0$  (see figure 2(a)).

Moreover, if we denote  $p_0 \vee p_1$  ( $p_0 \wedge p_1$ ) the maximum (minimum) of  $p_0$  and  $p_1$ , we get  $D_\nu = [-\log_4(p_0 \vee p_1), -\log_4(p_0 \wedge p_1)]$  and  $\dim(E_\alpha(\nu)) = \tau_\nu^*(\alpha) \geq 1/2$ , for all  $\alpha \in D_\nu$ . It follows from theorem 5.2 that

$$D_\mu = [-\log_4(p_0 \vee p_1), -\log_4(p_0 \wedge p_1)] \cup [-\log_4(p_2 \vee p_3), -\log_4(p_2 \wedge p_3)],$$

and

$$\dim(E_\alpha(\mu)) = \begin{cases} \tau_\nu^*(\alpha) & \text{if } -\log_4(p_0) \leq \alpha \leq -\log_4(p_1), \\ \tilde{\tau}^*(\alpha) & \text{if } -\log_4(p_2 \vee p_3) \leq \alpha \leq -\log_4(p_2 \wedge p_3). \end{cases}$$

Thus, if  $p_3 < p_1 \leq p_0$ , the singularity spectrum of  $\mu$  is supported by a union of mutually disjoint intervals and differs from  $\tau_\mu^*(\alpha)$  for  $-(\tau_\mu)'_+(q_0) < \alpha < -(\tau_\mu)'_-(q_0)$  (see figure 2(b)). To the best of our knowledge, self-similar measures with such multifractal structures have not previously appeared in the litterature.

### 6.3. Two phase transitions

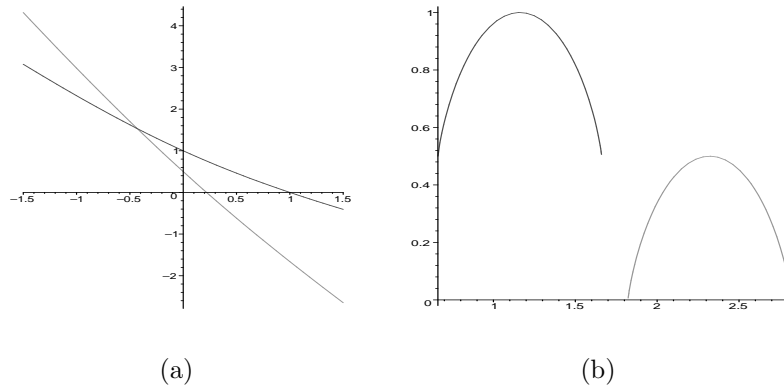
Until now we have studied measures for which the  $L^q$ -spectrum  $\tau(q)$  is not differentiable at one single point  $q_0 < 0$ . In this part we propose examples with two phase transitions. Let us take  $\ell = 5$  and consider the probability measure  $\mu$  satisfying

$$\mu = \sum_{i=0}^7 p_i \mu \circ S_i^{-1}, \quad (6.5)$$

where

$$S_0(x) = \frac{x}{5}, \quad S_1(x) = \frac{x}{5} + \frac{1}{5}, \quad S_2(x) = \frac{x}{5} + \frac{2}{5}, \quad S_3(x) = \frac{x}{5} + \frac{3}{5}$$





**Figure 2.** (a)  $\tau_\mu$  is not differentiable. (b) The singularity spectrum of  $\mu$  is supported by a union of two disjoint intervals.

$$S_4(x) = \frac{x}{5} + \frac{4}{5}, \quad S_5(x) = -\frac{x}{5} + \frac{1}{5}, \quad S_6(x) = -\frac{x}{5} + \frac{2}{5} \quad \text{and} \quad S_7(x) = -\frac{x}{5} + \frac{3}{5}.$$

In this case,  $B = \{3, 4\}$  and  $K = \{x = \sum \epsilon_i/5^i, \epsilon_i = 3 \text{ or } 4, \forall i \in \mathbb{N}^*\}$ . We suppose that the coefficients  $p_i$ 's verify  $p_0 = p_4 + p_5$ ,  $p_1 = p_3 + p_6$  and  $p_2 = p_7$ . As in section 6.2, we get  $\tau_\mu(q) = \max(\log_5(2p_0^q + 2p_1^q + (2p_2)^q), \log_5(p_3^q + p_4^q))$ , and

$$\forall \alpha \in D_\nu = [-\log_5(p_0 \vee p_1 \vee 2p_2), -\log_5(p_0 \wedge p_1 \wedge 2p_2)], \quad \dim(E_\alpha(\nu)) = \tau_\nu^*(\alpha).$$

In order to have  $\tau_\mu(q) = \tau_\nu(q)$  for large negative  $q$ , we choose  $p_2$  sufficiently small. For example, if we take  $p_0 = 0.35$ ,  $p_1 = 0.14$ ,  $p_2 = 0.01$ ,  $p_3 = 0.03$  and  $p_4 = 0.025$ , the equation  $\tau_\nu(q) = \tilde{\tau}(q)$  has two solutions  $q_0$  and  $q_1$  corresponding to the points of non-differentiability of  $\tau_\mu(q)$ . By theorem 5.2,  $D_\mu = [-\log_5(p_0), -\log_5(2p_2)]$  and

$$\dim(E_\alpha(\mu)) = \begin{cases} \tau_\nu^*(\alpha) & \text{if } -\log_5(p_0) \leq \alpha \leq \alpha_0, \\ \tilde{\tau}^*(\alpha) & \text{if } \alpha_0 \leq \alpha \leq \alpha_1, \\ \tau_\nu^*(\alpha) & \text{if } \alpha_1 \leq \alpha \leq -\log_5(2p_2), \end{cases}$$

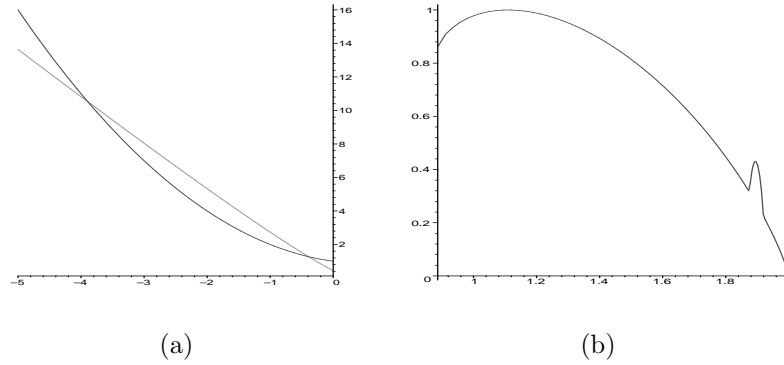
where  $\alpha_0$  and  $\alpha_1$  denote the solutions of the equation  $\tau_\nu^*(\alpha) = \tilde{\tau}^*(\alpha)$ . From the expression of the Legendre transform of  $\tau_\mu = \max(\tau_\nu, \tilde{\tau})$ , it follows that

$$\forall \alpha \in (-(\tau_\mu)'_+(q_0), -(\tau_\mu)'_-(q_0)) \cup (-(\tau_\mu)'_+(q_1), -(\tau_\mu)'_-(q_1)), \quad \dim(E_\alpha(\mu)) < \tau_\mu^*(\alpha).$$

#### 6.4. More phase transitions

In this part we describe a way to construct measures with an arbitrarily large number  $N$  of phase transitions. Theorem 4.2 leads us to find conditions on the  $p_i$ 's such that the equation  $\tau_\nu(q) = \tilde{\tau}(q)$  has  $N$  solutions. Since  $\tau_\nu(0) \geq \tilde{\tau}(0)$ , we have to distinguish the case where  $N$  is odd from the case where  $N$  is even.

First, assume that  $N$  is odd. Let us take  $\ell = 2N$ ,  $B = \{\ell/2, \dots, \ell-1\}$  and suppose that  $p_i = p_{i+\ell} + p_{\ell-1-i}$ , for all  $0 \leq i \leq \ell/2 - 1$ . In this case, the arguments developed in



**Figure 3.**  $\tau_\mu = \max(\tau_\nu, \tilde{\tau})$  is not differentiable at two points. The singularity spectrum of  $\mu$  is not concave and differs from  $\tau_\mu^*$ .

section 6.2 imply that

$$\tau_\mu(q) = \max(\tau_\nu(q), \tilde{\tau}(q)) = \max \left( \log_{2N} \left( \sum_{i=0}^{N-1} 2p_i^q \right), \log_{2N} \left( \sum_{i=N}^{2N-1} p_i^q \right) \right).$$

Moreover, since  $\ell = 2N$ , we can choose the  $p_i$ 's such that the equation

$$\sum_{i=0}^{N-1} 2p_i^q = \sum_{i=N}^{2N-1} p_i^q$$

has  $N$  solutions. These solutions correspond to the phase transitions for the  $L^q$ -spectrum  $\tau_\mu(q)$ .

Assume now that  $N$  is even. To ensure that  $\tau_\mu(q) = \tau_\nu(q)$  for large negative  $q$ , tools used in section 6.3 suggest to take  $\ell$  odd. Let  $\ell = 2N + 1$  and  $B = \{N + 1, \dots, 2N\}$ . Under the conditions, for all  $0 \leq i \leq N - 1$ ,  $p_i = p_{i+\ell} + p_{\ell-1-i}$  and  $p_N = p_{N+\ell}$ , we get

$$\tau_\mu(q) = \max \left( \log_{2N+1} \left( \sum_{i=0}^{N-1} 2p_i^q + (2p_N)^q \right), \log_{2N+1} \left( \sum_{i=N+1}^{2N} p_i^q \right) \right).$$

Thus, in order to have  $\tau_\mu(q) = \tau_\nu(q)$  for large negative  $q$ , we also suppose that  $2p_N < \min(p_i, N + 1 \leq i \leq 2N)$ . Once again, we can choose the  $p_i$ 's such that the equation

$$\sum_{i=0}^{N-1} 2p_i^q + (2p_N)^q = \sum_{i=N+1}^{2N} p_i^q$$

has  $N$  solutions. They correspond to the phase transitions for the function  $\tau_\mu$ .

More details about these examples can be found in [37].

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