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Shadow Fields and Local Supersymmetric Gauges

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Abstract

To control supersymmetry and gauge invariance in super-Yang–Mills theories we introduce new fields, called shadow fields, which enable us to enlarge the conventional Faddeev–Popov framework and write down a set of useful Slavnov–Taylor identities. These identities allow us to address and answer the issue of the supersymmetric Yang–Mills anomalies, and to perform the conventional renormalization programme in a fully regularization-independent way.

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1 Introduction

Renormalizing super-Yang–Mills theories is a subtle issue. One major difficulty is that of finding the right way of preserving both supersymmetry and gauge invariance, since supersymmetry involves the matrix γ_5 and no consistent regularization scheme preserving both supersymmetry and gauge invariance is available so far.

It is worth recalling that, for $\mathcal{N} = 1$ super-Yang–Mills theories, a superspace formulation is at our disposal. In this case, by means of the introduction of superfield Faddeev–Popov ghosts, the Slavnov–Taylor identity associated to super gauge transformations can be established directly in superspace [1]. This has allowed for a superspace characterization of both counterterms and gauge anomalies, by means of algebraic methods [1, 2]. However, the gauge superfield is dimensionless, and its renormalization in superspace is achieved through non-linear redefinition and involves an infinite number of parameters in the gauge sector [1, 3]. In the case of $\mathcal{N} = 2, 4$ super Yang-Mills theories, a component field framework, based on the Wess–Zumino gauge, is frequently employed. Although in this gauge the physical degrees of freedom are more transparent, gauge and supersymmetric transformations mix each other in a nonlinear way, a feature that has required a careful use of the antifield formalism to write down suitable Slavnov–Taylor identities [4, 5]. However, these identities are not suited for a check of general supersymmetry covariance: they only control the invariance of observables that are scalar under supersymmetric transformations.

In this work, these issues will be faced in a novel way. We introduce new fields in supersymmetric theories, which we call shadow fields, not to be confused with the Faddeev–Popov ghosts. This determines a system of coupled Slavnov-Taylor identities, which allow for a characterization of the possible anomalies as well as of the compensating non-invariant counterterms needed to restore both gauge invariance and supersymmetry, order by order in perturbation theory, in component formalism. Eventually, within a class of renormalizable gauges, one gets a proof that observables truly fall in supersymmetric multiplets.

The method improves the Faddeev–Popov gauge-fixing procedure. It can be applied to all models, for $\mathcal{N} = 1, 2, 4$. Despite their name, these shadow fields add light to the theory! Their introduction allows us to write down a family of local gauge-fixing terms with several gauge parameters. With some particular choice of the latter, the gauge-fixing becomes explicitly supersymmetric, while the result of the usual non-supersymmetric Faddeev–Popov procedure is recovered for another choice of the gauge parameters. Both

class of gauges determine the same set of observables, characterized by the cohomology of the BRST operator. Shadow fields are in fact of great importance, as they allow us to define the relevant Slavnov–Taylor identities that control both gauge invariance and supersymmetry, using the properties of local quantum field theory. The possible gauge and supersymmetric anomalies can then be classified. By introducing the corresponding set of shadow and ghost fields, we thus have a generalization of the case of the non-supersymmetric Yang–Mills theory, where Faddeev–Popov ghosts are sufficient.

The content of the paper is quite general.[1] The ordinary BRST symmetry s of supersymmetric theories involves the physical fields and the Faddeev–Popov ghost Ω, the antighost $\bar{\Omega}$ and the Lagrange multiplier $b = s\,\bar{\Omega}$. The BRST doublet $(\bar{\Omega}, b)$ is cohomologically trivial and is normally used to construct a BRST-invariant gauge-fixing term. Observables are defined as belonging to the cohomology sector of ghost number zero. Shadow fields are introduced in the form of two BRST doublets, $(c,\, \mu)$, and $(\bar{\mu},\, \bar{c})$. They make it possible to define a second differential operator Q that consistently anticommutes with s and carries the relevant information about supersymmetry. The introduction of the shadow doublets $(c,\, \mu)$ and $(\bar{\mu},\, \bar{c})$ is just what is needed to make the supersymmetric transformations compatible with a large enough class of local gauge functions. As we shall see, this allows us to define suitable Slavnov–Taylor identities to control both gauge invariance and supersymmetry. This is achieved by introducing classical external sources for the s, Q and $s\,Q$ transforms of the gauge and matter fields, as well as of the Faddeev–Popov and shadow fields. In fact, for Green's functions with no external legs of shadow type, internal loops of shadow fields compensate each other in a way that is specific to the class of renormalizable gauge that we choose. Shadow fields can be formally integrated out by making explicit use of their equations of motion, within the Landau gauge. However, this leads to the appearance of a highly complex and non-local supersymmetry of the ordinary Faddeev–Popov action in the Landau gauge, which cannot be used to obtain meaningful Slavnov–Taylor identities. Therefore, shadow fields have to be kept for non-ambiguously defining the quantization of supersymmetric gauge theories.

We also give an explanation of why anomalies for supersymmetry and gauge symmetry can only occur for $\mathcal{N} = 1$ models, but not for $\mathcal{N} = 2$, 4 models. The reason has to do only with the structure of the classical supersymmetric algebra, and it boils down to the existence of non-trivial supersymmetric cocycles. The classification of anomalies involves

[1] However, for pedagogical reasons and to illustrate the method, we sketch at the end the example of $\mathcal{N} = 2$ super-Yang–Mills theory.

only the supersymmetric transformations and the set of gauge-invariant local functional in the physical fields. We generalize the derivation of the Adler–Bardeen anomaly from a "Russian formula", to determine solutions of the consistency conditions including supersymmetry. This equation allows for an algebraic proof of the absence of Adler–Bardeen anomaly in extended supersymmetry. Notice that the demonstration that we display here only holds for cases where the supersymmetric algebra can be closed without the use of the equations of motion, i.e. for $\mathcal{N} = 1, 2$. In the case of $\mathcal{N} = 4$ supersymmetry, our method can be used, but it must be improved. In order to keep a linear dependence on the sources, one has to restrict to a subalgebra of the whole supersymmetric one, a question that will be detailed in a forthcoming paper [13]. Interestingly, shadow fields arise very naturally for topological quantum field theories [13].

2 Supersymmetry algebra with shadow fields

The "physical" fields of a generic supersymmetric Yang–Mills theory in four dimensions are the Yang–Mills field A and the matter fields $\varphi^{a'}$, which take values in a given representation of the gauge group. For simplicity, we adopt a notation as if the matter fields were in the adjoint representation. All our results remain true when they are in any given representation.[2] We denote by φ^a the whole set of fields $(A, \varphi^{a'})$, and by δ^{susy} the generator of the corresponding supersymmetric transformations. The way two supersymmetries commute can be generically cast in the following form:

$$(\delta^{susy})^2 \approx \delta^{\text{gauge}}(\omega(\varphi) + i_\kappa A) + \mathcal{L}_\kappa \tag{1}$$

where $\approx$ means that this equality can hold modulo equations of motion. δ^{susy} stands for a supersymmetry transformation, with a "ghostified" spinor parameter ϵ, which is commuting. The quantities $\omega(\varphi)$ and κ are bilinear functions in the parameter ϵ. For example, in the case of $\mathcal{N} = 2$ super-Yang–Mills one has (see section 8)

$$\omega(\varphi) = \big(\overline{\epsilon}[\gamma_5\phi^5 - \phi]\epsilon\big) \qquad \kappa^\mu = -i\big(\overline{\epsilon}\gamma^\mu\epsilon\big) \tag{2}$$

For $\mathcal{N} = 1, 2$, the closure relation (1) can hold off-shell with the introduction of auxiliary fields. When auxiliary fields are not used (for example in $\mathcal{N} = 4$ super-Yang–Mills) one has, for each field φ^a,

$$(\delta^{susy})^2\varphi^a + C^{ab}\frac{\delta^L S}{\delta\varphi^b} = \delta^{\text{gauge}}(\omega(\varphi) + i_\kappa A)\varphi^a + \mathcal{L}_\kappa\varphi^a \tag{3}$$

[2] In the discussion of the anomalies we will suppose, however, that the gauge group is semi-simple.

where C^{ab} are bilinear functions of the supersymmetric parameters, which do not depend on the fields.[3] From the invariance of the classical action S and from the Jacobi identity for the differential δ^{Susy}, one has :

$$\begin{aligned} C^{ab} + (-1)^{ab} C^{ba} &= 0 \\ \frac{\delta^R \, \delta^{Susy} \varphi^a}{\delta \varphi^c} C^{cb} - (-1)^{ab} \frac{\delta^R \, \delta^{Susy} \varphi^b}{\delta \varphi^c} C^{ca} &= 0 \end{aligned} \tag{4}$$

Let Ω be the Faddeev–Popov ghost for the gauge transformations, with antighost $\bar{\Omega}$ and Lagrange multiplier field b. The BRST operator s is:

$$\begin{aligned} &s\, A = -d_A \Omega \qquad && s\, \varphi^{a'} = -[\Omega, \varphi^{a'}] \\ &s\, \Omega = -\Omega^2 && \\ &s\, \bar{\Omega} = b && s\, b = 0 \end{aligned} \tag{5}$$

We introduce two s trivial doublets, $(\bar{\mu}, \bar{c})$, and (c, μ) with:

$$\begin{aligned} &s\, \bar{\mu} = \bar{c} \qquad && s\, \bar{c} = 0 \\ &s\, c = \mu \qquad && s\, \mu = 0 \end{aligned} \tag{6}$$

We call the set $(\bar{\mu}^{(-1,-1)}, \bar{c}^{(-1,0)}, c^{(0,1)}, \mu^{(1,1)})$ the shadow quartet. In analogy with the Faddeev–Popov ghost number, we define a shadow number. The upper index $\phi^{(g,s)}$ means that g and s are respectively the ghost and shadow numbers of the field ϕ. With this notation, one has for the physical fields $A^{(0,0)}$ and $\varphi^{(0,0)}$. The sum, modulo 2, of g, s, the ordinary form degree and the usual Grassmann grading associated to the spin, determines the commutation properties of any given field or operator. The transformations and the gradings of the shadow quartet and the Faddeev–Popov ghosts can be deduced from the following diagrams:

$$\begin{array}{ccccccc} & & & & \bar{\mu}^{(-1,-1)} & & \\ \Omega^{(1,0)} & & c^{(0,1)} & & \bar{c}^{(0,-1)} & & \bar{\Omega}^{(-1,0)} \\ & \mu^{(1,1)} & & & & b^{(0,0)} & \end{array} \tag{7}$$

We attribute the values $g = 0$ and $s = 1$ to each one of the supersymmetry parameters present in the operator δ^{Susy} (they are commuting spinors). Then, we define a differential graded operator Q with $g = 0$, $s = 1$, (for s, $g = 1$, $s = 0$) which represents

[3] This property holds true for all super-Yang–Mills theories

supersymmetry in a nilpotent way, as follows:

$$
\begin{aligned}
QA &= \delta^{Susy} A - d_A c & Q\varphi^{a'} &= \delta^{Susy}\varphi^{a'} - [c, \varphi^{a'}] \\
Qc &= \omega(\varphi) + i_\kappa A - c^2 & & \\
Q\Omega &= -\mu - [c, \Omega] & Q\mu &= -[\omega(\varphi), \Omega] - \mathscr{L}_\kappa \Omega - [c, \mu] \\
Q\bar{\mu} &= \bar{\Omega} & Q\bar{\Omega} &= \mathcal{L}_\kappa \bar{\mu} \\
Q\bar{c} &= -b & Qb &= -\mathcal{L}_\kappa \bar{c}
\end{aligned}
\tag{8}
$$

The operators s and Q verify

$$
\begin{gathered}
s^2 = 0 \qquad Q^2 \approx \mathcal{L}_\kappa \\
\{ s , Q \} = 0
\end{gathered}
\tag{9}
$$

These consistent closure relations hold true thanks to the introduction of the ghost and shadow fields [6]. In fact, one has the graded equation, $(d + s + Q - i_\kappa)^2 = 0$, and the s and Q transformations of the fields can be nicely condensed into a graded horizontality equation

$$
(d + s + Q - i_\kappa)\big(A + \Omega + c\big) + \big(A + \Omega + c\big)^2 = F + \delta^{Susy} A + \omega(\varphi) \tag{10}
$$

completed by $(d + s + Q - i_\kappa)\bar{\mu} = d\mu + \bar{c} + \bar{\Omega}$, and their Bianchi identities.[4] Eq. (10) is analogous to the horizontality condition that one encounters in topological quantum field theories [7]. Here, it will be used for classifying supersymmetric anomalies.

3 Slavnov–Taylor identities for s and Q symmetries

3.1 Source dependent local effective action

To extend both s and Q symmetry at the quantum level, we must introduce external sources for all s, Q and sQ transformations of the fields, which are non-linear. We shall

[4] For any matter super multiplet associated to scalar fields h^α in any given representation of the gauge group, one has also

$$
(d + s + Q - i_\kappa)h^\alpha + (A + \Omega + c)h^\alpha = d_A h^\alpha + \delta^{Susy} h^\alpha
$$

consider the following class of gauge-fixing fermion:

$$\begin{aligned}\mathbf{\Psi} &= \int \mathrm{Tr}\,\left(\bar{\Omega} d\star A - \frac{\alpha}{2}\star\bar{\Omega} b + \bar{\mu} d\star dc + (1-\zeta)\bar{\mu} d\star\big([A,c]-\delta^{susy}A\big) + \frac{\alpha}{2}\star\bar{\mu}\mathcal{L}_\kappa\bar{c}\right)\\ &= \zeta\int \mathrm{Tr}\,\left(\bar{\Omega} d\star A - \frac{\alpha}{2}\star\bar{\Omega} b + \bar{\mu} d\star dc + \frac{\alpha}{2}\star\bar{\mu}\mathcal{L}_\kappa\bar{c}\right)\\ &\qquad + (1-\zeta)\,Q\int \mathrm{Tr}\,\big(\bar{\mu} d\star A - \frac{\alpha}{2}\star\bar{\mu} b\big) \qquad (11)\end{aligned}$$

where α and ζ are gauge parameters. It leads one to the following classical gauge-fixed and BRST invariant local action Σ, which also contains all needed external sources :

$$\begin{aligned}\Sigma = S[\varphi] + {}_s\,\mathbf{\Psi} &+ \int (-1)^a\left({\varphi^{(s)}}_a\,{}_s\,\varphi^a + {\varphi^{(Q)}}_a Q\varphi^a + {\varphi^{(Q_s)}}_a\,{}_s\,Q\varphi^a\right)\\ &+ \int \mathrm{Tr}\,\left(\Omega^{(s)}\Omega^2 - \Omega^{(Q)}Q\Omega - \Omega^{(Q_s)}\,{}_s\,Q\Omega + \mu^{(Q)}Q\mu - c^{(Q)}Qc\right)\\ &+ \int \frac{1}{2}\big({\varphi^{(Q)}}_a - [{\varphi^{(Q_s)}}_a,\Omega]\big)C^{ab}\big({\varphi^{(Q)}}_b - [{\varphi^{(Q_s)}}_b,\Omega]\big)\\ &+ (1-\zeta)\int \mathrm{Tr}\,(d\bar{c} - [\Omega, d\bar{\mu}])\star C^{A\,H^I}\big(H_I^{(Q)} - [H_I^{(Q_s)},\Omega]\big) \qquad (12)\end{aligned}$$

Here, $H_I^{(Q)}$ and $H_I^{(Q_s)}$ are the sources for the Q and ${}_s\,Q$ transformations of the possible auxiliary fields H^I of the chosen supersymmetric model. The last term can be of utility only for the case $\mathcal{N}=4$.

We will shortly show that the gauge function $\mathbf{\Psi}$ is stable under renormalization, for all values of the parameters α and ζ. This determines a very interesting class of gauges, which interpolates, in particular, between the Feynman–Landau gauges, for $\zeta=1$, and a new class of explicitly supersymmetric gauges, for $\zeta=0$.

3.2 Slavnov–Taylor identity for the ${}_s$-symmetry

The ${}_s$-invariance of the action Σ implies the following Slavnov-Taylor identity, which is valid for all values of α and ζ:

$$\mathcal{S}_{(s)}(\Sigma) = 0 \qquad (13)$$

where the Slavnov–Taylor operator $\mathcal{S}_{(s)}$ is given by

$$\begin{aligned}\mathcal{S}_{(s)}(\mathscr{F}) &\equiv \int\left(\frac{\delta^R\mathscr{F}}{\delta\varphi^a}\frac{\delta^L\mathscr{F}}{\delta{\varphi^{(s)}}_a} + (-1)^a{\varphi^{(Q)}}_a\frac{\delta^L\mathscr{F}}{\delta{\varphi^{(Q_s)}}_a}\right)\\ &+ \int \mathrm{Tr}\,\left(\frac{\delta^R\mathscr{F}}{\delta\Omega}\frac{\delta^L\mathscr{F}}{\delta\Omega^{(s)}} - \Omega^{(Q)}\frac{\delta^L\mathscr{F}}{\delta\Omega^{(Q_s)}} - c^{(Q)}\frac{\delta^L\mathscr{F}}{\delta\mu^{(Q)}} + \mu\frac{\delta^L\mathscr{F}}{\delta c} + b\frac{\delta^L\mathscr{F}}{\delta\bar{\Omega}} + \bar{c}\frac{\delta^L\mathscr{F}}{\delta\bar{\mu}}\right) \qquad (14)\end{aligned}$$

Here and elsewhere, $\mathscr{F}$ is a generical functional of fields and sources. Let us introduce for further use the linearized Slavnov-Taylor operator

$$\mathcal{S}_{(s)|\mathscr{F}} \equiv \int \left(\frac{\delta^R \mathscr{F}}{\delta \varphi^a} \frac{\delta^L}{\delta \varphi^{(s)}{}_a} - \frac{\delta^R \mathscr{F}}{\delta \varphi^{(s)}{}_a} \frac{\delta^L}{\delta \varphi^a} + (-1)^a \varphi^{(Q)}{}_a \frac{\delta^L}{\delta \varphi^{(Q_s)}{}_a} \right)$$
$$+ \int \mathrm{Tr} \left(\frac{\delta^R \mathscr{F}}{\delta \Omega} \frac{\delta^L}{\delta \Omega^{(s)}} - \frac{\delta^R \mathscr{F}}{\delta \Omega^{(s)}} \frac{\delta^L}{\delta \Omega} - \Omega^{(Q)} \frac{\delta^L}{\delta \Omega^{(Q_s)}} - c^{(Q)} \frac{\delta^L}{\delta \mu^{(Q)}} + \mu \frac{\delta^L}{\delta c} + b \frac{\delta^L}{\delta \bar{\Omega}} + \bar{c} \frac{\delta^L}{\delta \bar{\mu}} \right) \quad (15)$$

In particular, the identity (13) implies that $\mathcal{S}_{(s)|\Sigma}$ is nilpotent:

$$\mathcal{S}_{(s)|\Sigma}{}^2 = 0 \quad (16)$$

The aim of perturbative gauge theory is to compute, order by order in the loop expansion, a quantum effective action Γ that satisfies $\mathcal{S}_{(s)}(\Gamma) = 0$. The property $\mathcal{S}_{(s)|\mathscr{F}} \mathcal{S}_{(s)}(\mathscr{F}) = 0$ allows us to characterize the possible gauge anomalies. However, in the present case, one also has to take into account the Q symmetry, as we will shortly see.

3.3 Gauge independence of the s-invariant observables

The spinor commuting parameters of supersymmetric transformation $\delta^{Susy} A$ enter the action Σ through a s-exact term. (Notice that these parameters have shadow number 1). As such, they can be treated as gauge parameters, precisely as the parameters α and ζ. This property, combined with the s invariance of the action, allows us to select observables that do not depend on the gauge parameters, within the class of gauges that we are considering.

In fact, the observables are defined from the cohomology of s, and they turn out to be independent from these generalized gauge parameters. To prove this property, one uses the fact that the free propagators and interaction terms of the theory depend analytically on these gauge parameters, and thus, the full generating functional also depends analytically on them. Then, the derivative of the 1PI generating functional Γ with respect to any one of the gauge parameters amounts to the insertion of a s-exact local term in Γ. It can be shown, using for instance the method [8], that these insertions yield a vanishing contribution when inserted in a Green function of s invariant observables. Moreover, the theory can be constructed, while maintaining all relevant Ward identities for the Q and s symmetries, for all possible values of the parameters α and ζ, (Section 7). In fact, the gauges defined by the gauge fermion (11) are stable under renormalization, due to some additional Ward identities stemming from field equations that are "linear" in the quantum fields. As the Green's functions of s invariant observables are independent on

all gauge parameters, their values are the same within this class of gauges. This gives a nice way of understanding the supersymmetry covariance of observables, even for a gauge-fixing that is not manifestly supersymmetric.

When the gauge parameter ζ is set to 1, the shadow fields $\bar{\mu}, \bar{c}, c$ and μ become free, since, in this case, their contribution in the action is just $\int \mathrm{Tr}\ (\bar{\mu} d \star d\mu + \bar{c} d \star dc)$. The remaining of the gauge-fixing action is the ordinary Faddeev–Popov action. In this gauge, supersymmetry is not manifest, and supersymmetry covariance of observables is very difficult to be established. On the other hand, after having introduced the shadows, one finds that the Landau–Feynman gauges are a continuous limit of more general gauges, which are parametrized by ζ and α and by the parameters of the supersymmetry transformations. These gauges can now be made explicitly supersymmetric, by choosing $\zeta = 0$, and keeping non-zero values for the supersymmetric parameters. Indeed, in this case, the gauge-fixing term is nothing but a $s\,Q$-exact term, namely

$$s\,Q \int \mathrm{Tr}\ \left(\bar{\mu} d \star A - \frac{\alpha}{2} \star \bar{\mu} b\right) . \tag{17}$$

The idea is thus to perform the renormalization programme in the gauge $\zeta = 0$, since in this case the supersymmetric Ward identities take the simplest form. One relies on the independence theorem of the observables upon changes of the gauge parameters, so that the result will be the same as in a standard Faddeev–Popov gauge-fixing, thereby ensuring the supersymmetry covariance, that is automatic in the gauges where $\zeta = 0$, provided that no anomaly occur for the Ward identities implied by the Q symmetry.

3.4 Slavnov–Taylor identity for the Q-symmetry in the $\zeta = 0$ gauge

In the gauge $\zeta = 0$, as a consequence of the Q and s invariance of the gauge-fixing term (17), the complete action Σ satisfies a second Slavnov–Taylor identity

$$\mathcal{S}_{(Q)}(\Sigma) = 0 \tag{18}$$

where $\mathcal{S}_{(Q)}$ is defined by

$$\begin{aligned}\mathcal{S}_{(Q)}(\mathscr{F}) \equiv & \int \left(\frac{\delta^R \mathscr{F}}{\delta \varphi^a} \frac{\delta^L \mathscr{F}}{\delta \varphi^{(Q)}{}_a} - (-1)^a \varphi^{(s)}{}_a \frac{\delta^L \mathscr{F}}{\delta \varphi^{(Qs)}{}_a} - (-1)^a \mathcal{L}_\kappa \varphi^{(Qs)}{}_a \frac{\delta^L \mathscr{F}}{\delta \varphi^{(s)}{}_a} + \varphi^{(Q)}{}_a \mathcal{L}_\kappa \varphi^a \right) \\ & + \int \mathrm{Tr}\ \left(\frac{\delta^R \mathscr{F}}{\delta \Omega} \frac{\delta^L \mathscr{F}}{\delta \Omega^{(Q)}} + \frac{\delta^R \mathscr{F}}{\delta c} \frac{\delta^L \mathscr{F}}{\delta c^{(Q)}} + \frac{\delta^R \mathscr{F}}{\delta \mu} \frac{\delta^L \mathscr{F}}{\delta \mu^{(Q)}} + \Omega^{(s)} \frac{\delta^L \mathscr{F}}{\delta \Omega^{(Qs)}} + \mathcal{L}_\kappa \Omega^{(Qs)} \frac{\delta^L \mathscr{F}}{\delta \Omega^{(s)}} \right. \\ & \left. + \Omega^{(Q)} \mathcal{L}_\kappa \Omega + c^{(Q)} \mathcal{L}_\kappa c + \mu^{(Q)} \mathcal{L}_\kappa \mu - b \frac{\delta^L \mathscr{F}}{\delta \bar{c}} - \mathcal{L}_\kappa \bar{c} \frac{\delta^L \mathscr{F}}{\delta b} + \bar{\Omega} \frac{\delta^L \mathscr{F}}{\delta \bar{\mu}} + \mathcal{L}_\kappa \bar{\mu} \frac{\delta^L \mathscr{F}}{\delta \bar{\Omega}} \right)\end{aligned} \tag{19}$$

Notice that the equation $\mathcal{S}_{(Q)}(\Sigma) = 0$ displays non homogeneous terms which do not depend on Σ. However, the latter are linear in the quantum fields. One defines a linearized Slavnov–Taylor operator $\mathcal{S}_{(Q)|\mathscr{F}}$:

$$\mathcal{S}_{(Q)|\mathscr{F}} \equiv \int \left(\frac{\delta^R \mathscr{F}}{\delta \varphi^a} \frac{\delta^L}{\delta \varphi^{(Q)}{}_a} - \frac{\delta^R \mathscr{F}}{\delta \varphi^{(Q)}{}_a} \frac{\delta^L}{\delta \varphi^a} - (-1)^a \varphi^{(s)}{}_a \frac{\delta^L}{\delta \varphi^{(Qs)}{}_a} - (-1)^a \mathcal{L}_\kappa \varphi^{(Qs)}{}_a \frac{\delta^L}{\delta \varphi^{(s)}{}_a} \right)$$
$$+ \int \mathrm{Tr} \left(\frac{\delta^R \mathscr{F}}{\delta \Omega} \frac{\delta^L}{\delta \Omega^{(Q)}} - \frac{\delta^R \mathscr{F}}{\delta \Omega^{(Q)}} \frac{\delta^L}{\delta \Omega} + \frac{\delta^R \mathscr{F}}{\delta c} \frac{\delta^L}{\delta c^{(Q)}} - \frac{\delta^R \mathscr{F}}{\delta c^{(Q)}} \frac{\delta^L}{\delta c} + \frac{\delta^R \mathscr{F}}{\delta \mu} \frac{\delta^L}{\delta \mu^{(Q)}} - \frac{\delta^R \mathscr{F}}{\delta \mu^{(Q)}} \frac{\delta^L}{\delta \mu} \right.$$
$$\left. + \Omega^{(s)} \frac{\delta^L}{\delta \Omega^{(Qs)}} + \mathcal{L}_\kappa \Omega^{(Qs)} \frac{\delta^L}{\delta \Omega^{(s)}} - b \frac{\delta^L}{\delta \bar{c}} - \mathcal{L}_\kappa \bar{c} \frac{\delta^L}{\delta b} + \bar{\Omega} \frac{\delta^L}{\delta \bar{\mu}} + \mathcal{L}_\kappa \bar{\mu} \frac{\delta^L}{\delta \bar{\Omega}} \right) \quad (20)$$

Provided the identities (13) and (18) hold true, the operators $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ satisfy the following anticommutation relations:

$$\mathcal{S}_{(s)|\Sigma}{}^2 = 0 \qquad \mathcal{S}_{(Q)|\Sigma}{}^2 = \mathcal{P}_\kappa$$
$$\left\{ \mathcal{S}_{(s)|\Sigma}, \mathcal{S}_{(Q)|\Sigma} \right\} = 0 \quad (21)$$

$\mathcal{P}_\kappa$ is the differential operator which acts as the Lie derivative along κ on all fields and external sources.

When $\zeta \neq 0$, the gauge-fixing term breaks the Q symmetry. However, this breaking can be kept under control by introducing an anticommuting parameter z which forms a doublet together with the parameter ζ, namely, $Qz = \zeta$, $Q\zeta = 0$, and $s\,\zeta = s\,z = 0$. As a consequence, the term

$$s\,\zeta \int \mathrm{Tr} \left(\bar{\Omega} d \star A - \frac{\alpha}{2} \star \bar{\Omega} b + \bar{\mu} d \star dc + \frac{\alpha}{2} \star \bar{\mu} \mathcal{L}_\kappa \bar{c} \right) \quad (22)$$

is changed into

$$s\,(\zeta - zQ) \int \mathrm{Tr} \left(\bar{\Omega} d \star A - \frac{\alpha}{2} \star \bar{\Omega} b + \bar{\mu} d \star dc + \frac{\alpha}{2} \star \bar{\mu} \mathcal{L}_\kappa \bar{c} \right) \quad (23)$$

Thanks to the introduction of the anticommuting parameter z, the Slavnov–Taylor identity $\mathcal{S}_{(Q)}(\Sigma) = 0$ can be maintained for $\zeta \neq 0$. Section 7 will be devoted to the study of this improved Slavnov–Taylor identity, as well as to the stability of the modified action, with its z dependence.

From now on and till section 7, we will consider that $\zeta = 0$. All results, but those involving the ghost Ward identities, can be generalized to the case $\zeta \neq 0$, by including the z and ζ dependence. Section 7 sketches the relevant modifications.

3.5 Antighost Ward identities

For all values of α, we have "linear field equations" for the fields $\bar{\mu}$, $\bar{c}$, $\bar{\Omega}$ and b. These equations imply the antighost Ward identities that can be enforced in perturbation theory. They constitute a BRST quartet:

$$\overline{\mathcal{G}}(\Sigma) = 0 \ , \quad \overline{\mathcal{G}_{(s)}}(\Sigma) = 0 \ , \quad \overline{\mathcal{G}_{(Q)}}(\Sigma) = 0 \ , \quad \overline{\mathcal{G}_{(Q_s)}}(\Sigma) = 0 \tag{24}$$

where

$$\overline{\mathcal{G}}(\mathscr{F}) \equiv \int \mathrm{Tr}\ X\left(\frac{\delta^L \mathscr{F}}{\delta b} - d \star A + \alpha \star b\right)$$

$$\overline{\mathcal{G}_{(s)}}(\mathscr{F}) \equiv \int \mathrm{Tr}\ X\left(\frac{\delta^L \mathscr{F}}{\delta \bar{\Omega}} + d \star \frac{\delta^L \mathscr{F}}{\delta A^{(s)}}\right) \qquad \overline{\mathcal{G}_{(Q)}}(\mathscr{F}) \equiv \int \mathrm{Tr}\ X\left(\frac{\delta^L \mathscr{F}}{\delta \bar{c}} - d \star \frac{\delta^L \mathscr{F}}{\delta A^{(Q)}} - \alpha \star \mathcal{L}_\kappa \bar{c}\right)$$

$$\overline{\mathcal{G}_{(Q_s)}}(\mathscr{F}) \equiv \int \mathrm{Tr}\ X\left(\frac{\delta^L \mathscr{F}}{\delta \bar{\mu}} + d \star \frac{\delta^L \mathscr{F}}{\delta A^{(Q_s)}}\right) \tag{25}$$

To obtain integrated equations, we introduced a local arbitrary function X, which takes values in the Lie algebra of the gauge group. We call the set of the four operators $\overline{\mathcal{G}_\bullet} = (\overline{\mathcal{G}},\ \overline{\mathcal{G}_{(s)}},\ \overline{\mathcal{G}_{(Q)}},\ \overline{\mathcal{G}_{(Q_s)}})$ the antighost operator quartet.

3.6 Ghost Ward identities

For the particular case of the Landau gauge, $\alpha = 0$, we have integrated Ward identities:

$$\mathcal{G}^{(Q_s)}(\Sigma) = 0 \ , \quad \mathcal{G}^{(Q)}(\Sigma) = 0 \ , \quad \mathcal{G}^{(s)}(\Sigma) = 0 \tag{26}$$

with

$$\mathcal{G}^{(Q_s)}(\mathscr{F}) = \int \mathrm{Tr}\ \left(\mathrm{x}\frac{\delta^L \mathscr{F}}{\delta \mu} - [\mathrm{x}, \bar{\mu}]\frac{\delta^L \mathscr{F}}{\delta b} + \mathrm{x}\big(-(-1)^a[\varphi^{(Q_s)}{}_a, \varphi^a] + [\Omega^{(Q_s)}, \Omega] + [\mu^{(Q)}, c]\big)\right)$$

$$\mathcal{G}^{(Q)}(\mathscr{F}) = \int \mathrm{Tr}\ \left(\mathrm{x}\frac{\delta^L \mathscr{F}}{\delta c} + [\mathrm{x}, \bar{c}]\frac{\delta^L \mathscr{F}}{\delta b} - [\mathrm{x}, \bar{\mu}]\frac{\delta^L \mathscr{F}}{\delta \bar{\Omega}} + (-1)^A[\mathrm{x}, \varphi^{(Q_s)}{}_a]\frac{\delta^L \mathscr{F}}{\delta \varphi^{(s)}{}_a} - [\mathrm{x}, \Omega^{(Q_s)}]\frac{\delta^L \mathscr{F}}{\delta \Omega^{(s)}}\right.$$
$$\left. + \mathrm{x}\big([\varphi^{(Q)}{}_a, \varphi^a] + [\Omega^{(Q)}, \Omega] + [c^{(Q)}, c] + [\mu^{(Q)}, \mu]\big)\right)$$

$$\mathcal{G}^{(s)}(\mathscr{F}) = \int \mathrm{Tr}\ \left(\mathrm{x}\frac{\delta^L \mathscr{F}}{\delta \Omega} - [\mathrm{x}, \bar{\Omega}]\frac{\delta^L \mathscr{F}}{\delta b} + [\mathrm{x}, \bar{\mu}]\frac{\delta^L \mathscr{F}}{\delta \bar{c}} - [\mathrm{x}, c]\frac{\delta^L \mathscr{F}}{\delta \mu} - (-1)^A[\mathrm{x}, \varphi^{(Q_s)}{}_a]\frac{\delta^L \mathscr{F}}{\delta \varphi^{(Q)}{}_a}\right.$$
$$\left. + [\mathrm{x}, \Omega^{(Q_s)}]\frac{\delta^L \mathscr{F}}{\delta \Omega^{(Q)}} + [\mathrm{x}, \mu^{(Q)}]\frac{\delta^L \mathscr{F}}{\delta c^{(Q)}} + \mathrm{x}[\varphi^{(s)}{}_a, \varphi^a] + \mathrm{x}[\Omega^{(s)}, \Omega]\right) \tag{27}$$

x is an arbitrary constant element of the Lie algebra. One has also the obvious property

$$\mathcal{G}(\Sigma) = 0 \tag{28}$$

where the linear operator $\mathcal{G}[\mathrm{x}]$ stands for global gauge transformations of parameter $-\mathrm{x}$, The operators $\mathcal{G}^\bullet = (\mathcal{G}^{(Q_s)},\ \mathcal{G}^{(Q)},\ \mathcal{G}^{(s)},\ \mathcal{G})$ form a quartet. We call $\mathcal{G}^\bullet$ the ghost operator quartet.

3.7 Consistency equations for Slavnov–Taylor, antighost and ghost operators

Given the operators $\overline{\mathcal{G}_\bullet}$ and $\mathcal{G}^\bullet$, we define their linearized functional operators $\overline{L\mathcal{G}_\bullet}$ and $L\mathcal{G}^\bullet$, which will complete the linearized Slavnov–Taylor operators.[5] This set of operators builds the following closed algebra:

$$\begin{gathered}
\mathcal{S}_{(s)|\mathscr{F}}\mathcal{S}_{(s)}(\mathscr{F}) = 0 \qquad \mathcal{S}_{(Q)|\mathscr{F}}\mathcal{S}_{(Q)}(\mathscr{F}) = 0 \\
\mathcal{S}_{(s)|\mathscr{F}}\mathcal{S}_{(Q)}(\mathscr{F}) + \mathcal{S}_{(Q)|\mathscr{F}}\mathcal{S}_{(s)}(\mathscr{F}) = 0 \\
\mathcal{S}_{(s)|\mathscr{F}}\overline{\mathcal{G}}(\mathscr{F}) - \overline{L\mathcal{G}}\mathcal{S}_{(s)}(\mathscr{F}) = -\overline{\mathcal{G}_{(s)}}(\mathscr{F}) \qquad \mathcal{S}_{(Q)|\mathscr{F}}\overline{\mathcal{G}}(\mathscr{F}) - \overline{L\mathcal{G}}\mathcal{S}_{(Q)}(\mathscr{F}) = \overline{\mathcal{G}_{(Q)}}(\mathscr{F}) \\
\mathcal{S}_{(s)|\mathscr{F}}\overline{\mathcal{G}_{(Q)}}(\mathscr{F}) + \overline{L\mathcal{G}_{(Q)}}\mathcal{S}_{(s)}(\mathscr{F}) = \mathcal{S}_{(Q)|\mathscr{F}}\overline{\mathcal{G}_{(s)}}(\mathscr{F}) + \overline{L\mathcal{G}_{(s)}}\mathcal{S}_{(Q)}(\mathscr{F}) = \overline{\mathcal{G}_{(Q_s)}}(\mathscr{F}) \\
\mathcal{S}_{(s)|\mathscr{F}}\mathcal{G}^{(Q_s)}(\mathscr{F}) - L\mathcal{G}^{(Q_s)}\mathcal{S}_{(s)}(\mathscr{F}) = -\mathcal{G}^{(Q)}(\mathscr{F}) \qquad \mathcal{S}_{(Q)|\mathscr{F}}\mathcal{G}^{(Q_s)}(\mathscr{F}) - L\mathcal{G}^{(Q_s)}\mathcal{S}_{(Q)}(\mathscr{F}) = \mathcal{G}^{(s)}(\mathscr{F}) \\
\mathcal{S}_{(s)|\mathscr{F}}\mathcal{G}^{(s)}(\mathscr{F}) + L\mathcal{G}^{(s)}\mathcal{S}_{(s)}(\mathscr{F}) = \mathcal{S}_{(Q)|\mathscr{F}}\mathcal{G}^{(Q)}(\mathscr{F}) + L\mathcal{G}^{(Q)}\mathcal{S}_{(Q)}(\mathscr{F}) = \mathcal{G}(\mathscr{F})
\end{gathered} \tag{29}$$

The ghost and antighost operators commute in the following way

$$\begin{aligned}
L\mathcal{G}^{(s)}[\mathrm{x}]\overline{\mathcal{G}_{(s)}}[X](\mathscr{F}) + \overline{L\mathcal{G}_{(s)}}[X]\mathcal{G}^{(s)}[\mathrm{x}](\mathscr{F}) &= -\overline{\mathcal{G}}[[\mathrm{x}, X]](\mathscr{F}) \\
L\mathcal{G}^{(Q)}[\mathrm{x}]\overline{\mathcal{G}_{(Q)}}[X](\mathscr{F}) + \overline{L\mathcal{G}_{(Q)}}[X]\mathcal{G}^{(Q)}[\mathrm{x}](\mathscr{F}) &= \overline{\mathcal{G}}[[\mathrm{x}, X]](\mathscr{F}) \\
L\mathcal{G}^{(Q_s)}[\mathrm{x}]\overline{\mathcal{G}_{(Q_s)}}[X](\mathscr{F}) - \overline{L\mathcal{G}_{(Q_s)}}[X]\mathcal{G}^{(Q_s)}[\mathrm{x}](\mathscr{F}) &= \overline{\mathcal{G}}[[\mathrm{x}, X]](\mathscr{F})
\end{aligned}$$

$$\begin{aligned}
L\mathcal{G}^{(s)}[\mathrm{x}]\overline{\mathcal{G}_{(Q_s)}}[X](\mathscr{F}) - \overline{L\mathcal{G}_{(Q_s)}}[X]\mathcal{G}^{(s)}[\mathrm{x}](\mathscr{F}) &= -\overline{\mathcal{G}_{(Q)}}[[\mathrm{x}, X]](\mathscr{F}) \\
L\mathcal{G}^{(Q)}[\mathrm{x}]\overline{\mathcal{G}_{(Q_s)}}[X](\mathscr{F}) - \overline{L\mathcal{G}_{(Q_s)}}[X]\mathcal{G}^{(Q)}[\mathrm{x}](\mathscr{F}) &= \overline{\mathcal{G}_{(s)}}[[\mathrm{x}, X]](\mathscr{F})
\end{aligned} \tag{30}$$

$$L\mathcal{G}^{(s)}[\mathrm{x}]\mathcal{G}^{(Q)}[\mathrm{y}](\mathscr{F}) + L\mathcal{G}^{(Q)}[\mathrm{y}]\mathcal{G}^{(s)}[\mathrm{x}](\mathscr{F}) = -\mathcal{G}^{(Q_s)}[[\mathrm{x}, \mathrm{y}]](\mathscr{F}) \tag{31}$$

$$\begin{aligned}
L\mathcal{G}[\mathrm{x}]\overline{\mathcal{G}_\bullet}[X](\mathscr{F}) - \overline{L\mathcal{G}_\bullet}[X]\mathcal{G}[\mathrm{x}](\mathscr{F}) &= -\overline{\mathcal{G}_\bullet}[[\mathrm{x}, X]](\mathscr{F}) \\
L\mathcal{G}[\mathrm{x}]\mathcal{G}^\bullet[\mathrm{y}](\mathscr{F}) - L\mathcal{G}^\bullet[\mathrm{y}]\mathcal{G}[\mathrm{x}](\mathscr{F}) &= -\mathcal{G}^\bullet[[\mathrm{x}, \mathrm{y}]](\mathscr{F})
\end{aligned} \tag{32}$$

[5] For example $\overline{L\mathcal{G}} = \int \mathrm{Tr}\ X \frac{\delta^L}{\delta b}$.

where we have written the dependence in the Lie algebra elements X and x. For simplicity, we shall consider cases such that

$$Q^2 = \mathcal{L}_\kappa \qquad C^{ab} = 0 \tag{33}$$

For instance, by introducing auxiliary fields, this condition holds true for $\mathcal{N} = 1, 2$ supersymmetry. We will see how the case $\mathcal{N} = 4$ can be handled in a separate publication.[13]

4 Elimination of the shadow fields and non-local supersymmetry in the Landau-gauge

In the class of gauges that we are considering, the dependence of the action Σ on the shadow fields c, $\bar{c}$ and their ghosts $\bar{\mu}$ and μ is quadratic, including the classical source dependence. Thus, they can be eliminated by gaussian integration. This defines an effective action Σ^{eff} :

$$\begin{aligned}
&\int \mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}\mu\mathcal{D}\bar{\mu}\; e^{-\Sigma - \int (Jc + \bar{J}\bar{c} + K\mu + \bar{K}\bar{\mu})} \\
&= \det{}^{-1} \begin{bmatrix} -1 & \alpha G \mathcal{L}_\kappa \\ G\big(\mathrm{ad}_{c^{(Q)}} + d^\dagger \mathrm{ad}_{d_A\Omega} G \mathrm{ad}_{\mu^{(Q)}} + \mathrm{ad}_{\mu^{(Q)}} G \mathrm{ad}^\dagger_{d_A\Omega} d\big) & 1 \end{bmatrix} e^{-\Sigma^{\text{eff}}}
\end{aligned} \tag{34}$$

The simplicity of the determinant in Eq.(34) is due to compensations between the bosonic and fermionic gaussian integrations over the shadow fields c, $\bar{c}$ and μ, $\bar{\mu}$. Here, ad_X denotes the adjoint action by a gauge Lie algebra element X, and the symbol † indicates the adjoint of the corresponding operator with respect to the metric $\big(f, g\big) \equiv \int \mathrm{Tr}\,\big(f \star g\big)$. The operator G is the inverse of the Faddeev–Popov operator, so that, one has $Gd^\dagger d_A = d^\dagger d_A G = 1$.

In the Landau gauge, $\alpha = 0$, the determinant (34) is one. Therefore, in this gauge, the functional integration over the shadow fields c, $\bar{c}$, $\bar{\mu}$ and μ reduces to the implementation of their equations of motion. Since the Q transformations depends on c and μ, the effective Q symmetry then depends on a non-local operator, namely, on the inverse of the Faddeev–Popov operator. Although perturbation theory cannot be safely controlled in presence of a symmetry implying non-local operators, it is interesting to discuss this resulting effective supersymmetry.

The classical equations of motion of the shadow fields are:

$$\begin{aligned}
d^\dagger_A d\bar{c} &\approx J + ... & d^\dagger_A d\bar{\mu} &\approx K + ... \\
c &\approx G\big(d^\dagger \delta^{Susy} A + \bar{J}\big) & \mu &\approx {}_s\; G d^\dagger \delta^{Susy} A + G\bar{K},
\end{aligned} \tag{35}$$

where the dots ... stand for terms depending on the external sources of the Q and $s\,Q$ transformations of the fields. The effective action $\Sigma^{\rm eff}$ is obtained by substituting the solution of the equations of motion for c, $\bar{c}$, $\bar{\mu}$ and μ in the action $(\Sigma + \int (Jc + \bar{J}\bar{c} + K\mu + \bar{K}\bar{\mu}))$. Upon setting the sources $\bar{J}$ and $\bar{K}$ to zero, the resulting effective action $\Sigma^{\rm eff}$ turns out to be the ordinary Faddeev–Popov action in the Landau gauge, with the addition of external sources coupled to non-local operators. In particular, this amounts to replace c by the non local expression $Gd^\dagger\delta^{susy}A$, and μ by $s\ Gd^\dagger\delta^{susy}A$. For this effective action, the external sources J, K, $c^{(Q)}$ and $\mu^{(Q)}$ have to be regarded as sources for the non local operator $Gd^\dagger\delta^{susy}A$ and its variations under s , Q and $s\,Q$, respectively.

The corresponding Slavnov–Taylor identities inherit this non-locality, and become quite complicated. However, one can compute the 1PI generating functional $\Gamma^{\rm eff}$ associated to the classical action $\Sigma^{\rm eff}$ in the framework of local quantum field theory. Let us consider at first the 1PI generating functional Γ computed from the local classical action Σ including the shadow fields. Then, we eliminate the external shadow fields by Legendre transformation, substituting to them their solutions c^*, μ^*, $\bar{c}^*$ and μ^* of the equations:[6]

$$\begin{aligned} \frac{\delta^R\Gamma}{\delta c} &= -J \qquad & \frac{\delta^R\Gamma}{\delta \mu} &= -K \\ \frac{\delta^R\Gamma}{\delta \bar{c}} &= 0 \qquad & \frac{\delta^R\Gamma}{\delta \bar{\mu}} &= 0 \end{aligned} \tag{36}$$

These solutions are non-local functionals of the sources J and K, and of other fields. The Legendre transformation,

$$\Gamma^{\rm eff}[J,K] \equiv \Gamma[c^*,\mu^*,\bar{c}^*,\bar{\mu}^*] + \int \big(Jc^* + K\mu^*\big) \tag{37}$$

defines the 1PI generating functional $\Gamma^{\rm eff}$ associated to $\Sigma^{\rm eff}$. In this way, we obtain well defined Slavnov–Taylor identities, which formally take into account the insertions of the non local operator $Gd^\dagger\delta^{susy}A$ and its transformations.

After elimination of the shadow fields, for $\alpha = 0$, one has:

$$Q\varphi^a = \delta^{susy}\varphi^a - \delta^{\rm gauge}(Gd^\dagger\delta^{susy}A)\varphi^a \tag{38}$$

This is the combination of an ordinary supersymmetry transformation and a gauge transformation with a field dependent fermionic parameter. This parameter involves the non-local inverse Faddeev–Popov operator.

[6] These equations are solvable as power series in $\hbar$ since the classical equations of motions are solvable.

This effective Q transformation provides a representation of supersymmetry on the physical fields, since one has:

$$Q^2 = \mathcal{L}_\kappa + \delta^{\rm gauge}(Gd^\dagger \mathcal{L}_\kappa A) \tag{39}$$

For the gauge field, we have:

$$QA = (1 - d_A G d^\dagger)\, \delta^{Susy} A \tag{40}$$

One notices that $1 - d_A G d^\dagger$ is a projector, since $(1 - d_A G d^\dagger)^2 = 1 - d_A G d^\dagger$. Moreover, one has:

$$d^\dagger (1 - d_A G d^\dagger) = 0 \tag{41}$$

Therefore, after the elimination of the shadow fields, Q can be regarded as a representation of supersymmetry which is compatible with the Landau gauge condition

$$d^\dagger A = 0 \qquad \Rightarrow \qquad d^\dagger (A + QA) = 0 \tag{42}$$

Furthermore, one has:

$$Q\Omega = -Gd^\dagger [\Omega, \delta^{Susy} A - d_A G d^\dagger \delta^{Susy} A] \qquad Q\bar{\Omega} = 0 \qquad Qb = 0 \tag{43}$$

so one can check that $Qd^\dagger d_A \Omega = 0$. These remarks indicate that the Faddeev–Popov determinant admits a supersymmetry in the absence of the shadow fields, that is:

$$Q\, \det\big[\, d^\dagger d_A \,\big]\ \delta\, [\, d^\dagger A \,] = 0 \tag{44}$$

However, the non-locality of this symmetry makes it of no practical use for controlling the quantum theory. In contrast, by keeping the local dependence on the shadow fields, one can safely use the conventional tools of local quantum field theory.

5 Supersymmetric anomalies

The standard method to discuss the renormalizability of the theory to all orders of perturbation theory is as follows. One assumes that the renormalized generating functional Γ_n has been computed at a given order n, in such a way that it satisfies the Slavnov–Taylor identities for the s and Q symmetry and the antighost (and possibly ghost) Ward identities. Then, one checks if one can proceed to the next order, with the same conclusion. By definition, this recursive procedure holds true if there are no obstructions,

i.e. if anomalies are absent. In the present case, since a manifest invariant regularization framework does not exist, one must check the absence of anomalies, defined as cohomological non trivial solutions of the consistency conditions of linearized operators $\mathcal{S}_{(s)|\Sigma}$, $\mathcal{S}_{(Q)|\Sigma}$, $\overline{L\mathcal{G}_\bullet}$ and $L\mathcal{G}^\bullet$, for ensuring that we can define the supersymmetric theory with all necessary Ward identities order by order in perturbation theory, through the introduction of suitable compensating local non invariant counterterms [8]. It is thus essential to classify the possible solutions of the consistency conditions of the linearized Slavnov–Taylor operators that we have constructed, within our class of renormalizable gauges.

The supersymmetric transformation δ^{Susy} is nilpotent on the set $\mathscr{O}_{\rm inv}$ of gauge-invariant functionals of physical fields φ^a and supersymmetric parameters. We can consider its restriction to this set of functionals. This permits us to introduce a differential complex, $\mathscr{O}^*_{\rm inv}(\delta^{Susy})$, which is graded by the shadow number. We shall show that the question of finding the possible anomalies for the Ward identities $\mathcal{S}_{(s)}$, $\mathcal{S}_{(Q)}$, $\overline{\mathcal{G}_\bullet}$ and $\mathcal{G}^\bullet$ is closely linked to that of finding the cohomology $\mathcal{H}^*$ of $\mathscr{O}^*_{\rm inv}(\delta^{Susy})$.

We will find that the possible anomalies are either elements with shadow number one in $\mathcal{H}^1$, or pairs built from the $(1,0)$ Adler–Bardeen anomaly and a $(0,1)$ supersymmetric counterpart. In fact, the latter possibility only occurs provided we are in a theory where the following local functional:

$$\mathcal{I} = \int \mathrm{Tr}\,\left(F_\wedge \delta^{Susy} A_\wedge \delta^{Susy} A + \omega(\varphi) F_\wedge F\right) \tag{45}$$

is such that $\mathcal{I} = \delta^{Susy}(\dots)$, that is, vanishes in $\mathcal{H}^2$.

We must proceed by steps. First, we will show that the eventual anomalies of the antighost equations can be removed without spoiling the Slavnov–Taylor identities. Then, we will be able to restrict to the search of solutions of the ordinary BRST invariance consistency conditions, the unique solution of which is the Adler–Bardeen anomaly. Afterward, we will classify the solutions of the rest of the consistency conditions associated to supersymmetry, which must be consistent with the BRST invariance. Finally, we will check that there are no solutions corresponding to the ghost operators which satisfy all the consistency conditions.

5.1 Absence of anomalies associated to the antighost operators

To demonstrate that there are no anomalies associated with the antighost operators, it is convenient to assemble the linearized operators (but the ghost ones) into the following

nilpotent differential operator δ :

$$\delta \equiv \alpha^{(s)}\mathcal{S}_{(s)|\Sigma} + \alpha^{(Q)}\mathcal{S}_{(Q)|\Sigma} + \overline{L\mathcal{G}}[X] + \overline{L\mathcal{G}_{(s)}}[X^{(s)}] + \overline{L\mathcal{G}_{(Q)}}[X^{(Q)}] + \overline{L\mathcal{G}_{(Q_s)}}[X^{(Q_s)}] + \sigma \quad (46)$$

$\alpha^{(s)}$ and $\alpha^{(Q)}$ are commuting scalars, X and $X^{(Q_s)}$ are anticommuting Lie algebra valued functions and $X^{(s)}$ and $X^{(Q)}$, commuting ones. The differential operator σ has been introduced in order that:

$$\delta^2 = (\alpha^{(Q)})^2\ \mathcal{P}_\kappa \quad (47)$$

so that δ is nilpotent, when acting on functionals. The operator σ gives zero on all quantities, but the functions $X^\bullet$, with:

$$\begin{aligned} \sigma X &= \alpha^{(Q)}\mathcal{L}_\kappa X^{(Q)} & \sigma X^{(s)} &= -\alpha^{(s)}X - \alpha^{(Q)}\mathcal{L}_\kappa X^{(Q_s)} \\ \sigma X^{(Q_s)} &= -\alpha^{(Q)}X^{(s)} - \alpha^{(s)}X^{(Q)} & \sigma X^{(Q)} &= \alpha^{(Q)}X \end{aligned} \quad (48)$$

To show that there are no anomalies for the ghost operators, one can use a standard and general method [10]. It suffices to exhibit that δ admits a trivializing homotopy k, such that the anticommutator of k and δ determines the following counting operator

$$\{\delta, k\} = e^{\int \mathrm{Tr}\ \left(-d\bar{\Omega}\star\frac{\delta^L}{\delta A^{(s)}} + d\bar{c}\star\frac{\delta^L}{\delta A^{(Q)}} + d\bar{\mu}\star\frac{\delta^L}{\delta A^{(Q_s)}}\right)} N e^{-\int \mathrm{Tr}\ \left(-d\bar{\Omega}\star\frac{\delta^L}{\delta A^{(s)}} + d\bar{c}\star\frac{\delta^L}{\delta A^{(Q)}} + d\bar{\mu}\star\frac{\delta^L}{\delta A^{(Q_s)}}\right)}$$

$$N \equiv \int \mathrm{Tr}\ \left(b\frac{\delta^L}{\delta b} + \bar{\Omega}\frac{\delta^L}{\delta\bar{\Omega}} + \bar{c}\frac{\delta^L}{\delta\bar{c}} + \bar{\mu}\frac{\delta^L}{\delta\bar{\mu}} + X\frac{\delta^L}{\delta X} + X^{(s)}\frac{\delta^L}{\delta X^{(s)}} + X^{(Q)}\frac{\delta^L}{\delta X^{(Q)}} + X^{(Q_s)}\frac{\delta^L}{\delta X^{(Q_s)}}\right) \quad (49)$$

This trivializing homotopy is simply :

$$k \equiv \int \mathrm{Tr}\ \left(b\frac{\delta^L}{\delta X} + \bar{\Omega}\frac{\delta^L}{\delta X^{(s)}} + \bar{c}\frac{\delta^L}{\delta X^{(Q)}} + \bar{\mu}\frac{\delta^L}{\delta X^{(Q_s)}}\right) \quad (50)$$

Therefore, we can always remove the anomalies of $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ in a way which preserves the antighost symmetries.

5.2 Cohomology of the BRST operator $\mathcal{S}_{(s)|\Sigma}$

We then compute the cohomology of $\mathcal{S}_{(s)|\Sigma}$ alone. This operator can be split in two parts

$$\mathcal{S}_{(s)|\Sigma} = \Sigma_{(s)} + \varsigma \quad (51)$$

where $\Sigma_{(s)}$ is the operator which acts on any functional $\mathscr{F}$ of the fields trough the following antibracket

$$\Sigma_{(s)}\mathscr{F} = \left(\Sigma, \mathscr{F}\right)_{(s)} \equiv \int\left(\frac{\delta^R\Sigma}{\delta\varphi^a}\frac{\delta^L\mathscr{F}}{\delta\varphi^{(s)}{}_a} - \frac{\delta^R\Sigma}{\delta\varphi^{(s)}{}_a}\frac{\delta^L\mathscr{F}}{\delta\varphi^a} + \mathrm{Tr}\ \frac{\delta^R\Sigma}{\delta\Omega}\frac{\delta^L\mathscr{F}}{\delta\Omega^{(s)}} - \mathrm{Tr}\ \frac{\delta^R\Sigma}{\delta\Omega^{(s)}}\frac{\delta^L\mathscr{F}}{\delta\Omega}\right) \quad (52)$$

and ς is the Σ independent part of $\mathcal{S}_{(s)|\Sigma}$. One then finds that ς admits a trivializing homotopy $k^{(s)}$ defined by

$$k^{(s)} \equiv \int (-1)^a \varphi^{(Q_s)}{}_a \frac{\delta^L}{\delta\varphi^{(Q)}} + \int \mathrm{Tr}\left(-\Omega^{(Q_s)}\frac{\delta^L}{\delta\Omega^{(Q)}} - \mu^{(Q)}\frac{\delta^L}{\delta c^{(Q)}} + c\frac{\delta^L}{\delta\mu} + \bar{\Omega}\frac{\delta^L}{\delta b} + \bar{\mu}\frac{\delta^L}{\delta\bar{c}}\right) \tag{53}$$

The anticommutator of $k^{(s)}$ with ς is the counting operator M for all fields but the physical ones, the ghost Ω and their sources for the BRST symmetry s. One can construct perturbatively a functional V, such that, the differential operator $V_{(s)} \equiv \big(V, \cdot\big)_{(s)}$ satisfies

$$e^{-V_{(s)}}\mathcal{S}_{(s)|\Sigma}e^{V_{(s)}} = \Sigma^0_{(s)} + \varsigma \tag{54}$$

Here, Σ^0 is the part of Σ which is annihilated by M and $k^{(s)}$. This uses the "Maurer–Cartan" equation :

$$\varsigma^2 = 0 \qquad \varsigma\,\Sigma + \frac{1}{2}\big(\Sigma, \Sigma\big)_{(s)} = 0 \tag{55}$$

Eq.(54) implies that the cohomology of $\mathcal{S}_{(s)|\Sigma}$ is isomorphic to that of $\Sigma^0_{(s)} + \varsigma$. The latter is itself isomorphic to the cohomology of $\Sigma^0_{(s)}$ on the set of fields made by the physical fields, the ghost Ω and the sources for their s transformations. This cohomology turns out to be the Adler–Bardeen anomaly $\mathscr{A}$ in the case of a semi-simple gauge group [9].

5.3 Supersymmetric anomalies consistent with BRST invariance

As announced, we now show that there are two possibilities for the consistent anomalies of the Ward identities. Either one has a supersymmetric term associated to the Adler-Bardeen anomaly, or one has a pure supersymmetric anomaly.

5.3.1 Possibility for the "supersymmetric" Adler–Bardeen anomaly

Consider the Adler–Bardeen anomaly $\mathscr{A}$. To make it consistent with the operator $\mathcal{S}_{(Q)|\Sigma}$, we need an integrated local functional $\mathscr{B}$, possibly depending on all fields and sources, with ghost number 0, shadow number one and canonical dimension $\frac{9}{2}$. $\mathscr{B}$ must verify the consistency conditions

$$\mathcal{S}_{(s)|\Sigma}\mathscr{B} + \mathcal{S}_{(Q)|\Sigma}\mathscr{A} = 0 \qquad \mathcal{S}_{(Q)|\Sigma}\mathscr{B} = 0 \tag{56}$$

To look for possible solutions, we use the horizontality condition Eq.(10) and its Bianchi identity, which express the s and Q transformation laws. One has :

$$\tilde{F} \equiv (d + s + Q - i_\kappa)\big(A + \Omega + c\big) + \big(A + \Omega + c\big)^2 = F + \delta^{susy}A + \omega(\varphi) \tag{57}$$

The extended curvature defined by the later equation permits one to use the formal five-dimensional Chern–Simon identity

$$\mathrm{Tr}\ \tilde{F}^3 = \tilde{d}\,\mathrm{Tr}\ \big(\tilde{A}\tilde{F}^2 - \frac{1}{2}\tilde{A}^3\tilde{F} + \frac{1}{10}\tilde{A}^5\big) \tag{58}$$

to determine both the Adler–Bardeen anomaly and its supersymmetric counter part. The term with ghost and shadow numbers $(2,0)$ shows that the Adler–Bardeen anomaly satisfies the consistency condition for s :

$$\mathscr{A} \equiv \int \mathrm{Tr}\ d\Omega\big(AdA + \frac{1}{2}A^3\big) \qquad s\,\mathscr{A} = 0 \tag{59}$$

The term with ghost and shadow numbers $(1,1)$ gives a solution for the consistency condition

$$s\,\mathscr{B}^{\mathrm{c}} + Q\mathscr{A} = 0 \tag{60}$$

It is defined by

$$\mathscr{B}^{\mathrm{c}} \equiv \mathrm{Tr}\ \Big(dc\big(AdA + \frac{1}{2}A^3\big) + \big([A,F] - \frac{1}{2}A^3\big)\delta^{Susy}A\Big) \tag{61}$$

The term with ghost and shadow numbers $(0,2)$ gives the breaking of the consistency condition $Q\mathscr{B} = 0$ for this solution. Indeed, one gets :

$$Q\mathscr{B}^{\mathrm{c}} = 3\int \mathrm{Tr}\ \big(F_\wedge\delta^{Susy}A_\wedge\delta^{Susy}A + \omega(\varphi)F_\wedge F\big) \tag{62}$$

To obtain a consistent $\mathscr{B}$, we must add a $\mathcal{S}_{(s)|\Sigma}$-closed term $\mathscr{B}^{\mathrm{inv}}$ to $\mathscr{B}^{\mathrm{c}}$, in order to have a $\mathcal{S}_{(Q)|\Sigma}$-closed functional $\mathscr{B}$, which preserves Eq. (60). Since $Q\mathscr{B}^{\mathrm{c}}$ is not $\mathcal{S}_{(s)|\Sigma}$-exact, $\mathscr{B}^{\mathrm{inv}}$ must be in the cohomology of $\mathcal{S}_{(s)|\Sigma}$. Therefore $\mathscr{B}^{\mathrm{inv}}$ is a gauge-invariant functional on the physical fields. Because the transformation $\mathcal{S}_{(Q)|\Sigma}$ acts as δ^{Susy} on gauge-invariant functionals, we obtain the following result. A functional $\mathscr{B}$, which satisfies the consistency equations (56), where $\mathscr{A}$ is the Adler–Bardeen anomaly, exists, if and only if, there is a gauge-invariant local functional $\mathscr{B}^{\mathrm{inv}}$, for which the following equation holds

$$\delta^{Susy}\mathscr{B}^{\mathrm{inv}} = -3\int \mathrm{Tr}\ \big(F_\wedge\delta^{Susy}A_\wedge\delta^{Susy}A + \omega(\varphi)F_\wedge F\big) \tag{63}$$

For the case $\mathcal{N} = 2, 4$, the right hand side in the last equation can be identified as a cohomology class of the operator δ^{Susy}. It can be associated to Donaldson–Witten invariants [13]. This gives an algebraic proof of the absence of Adler–Bardeen anomaly in extended supersymmetry. This feature is usually attributed to the fact that there are no chiral fermions in such theories.

For the case $\mathcal{N}=1$, one can construct the following antecedent:[7]

$$\delta^{\mathcal{S}usy}\frac{1}{4}\mathrm{Tr}\,\big(\overline{\epsilon}\gamma^{\mu}\lambda\big)\big(\overline{\lambda}\gamma_5\gamma_{\mu}\lambda\big)=\frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\mathrm{Tr}\;F_{\mu\nu}\big(\overline{\epsilon}\gamma_{\sigma}\lambda\big)\big(\overline{\epsilon}\gamma_{\rho}\lambda\big) \tag{66}$$

Thus, gauge anomalies can exist in this case. They can be detected as gauge anomalies and as anomalies in the supersymmetry Noether current.

5.3.2 Purely supersymmetric anomalies

Since the Adler–Bardeen is the only possible anomaly for the $\mathcal{S}_{(s)|\Sigma}$ operator, other potential anomalies for $\mathcal{S}_{(Q)|\Sigma}$ must be in the kernel of $\mathcal{S}_{(s)|\Sigma}$. In other word, we may have a term $\mathscr{C}^{(0,1)}$, with

$$\mathcal{S}_{(s)|\Sigma}\mathscr{C}=0 \qquad \mathcal{S}_{(Q)|\Sigma}\mathscr{C}=0 \tag{67}$$

As a result of [9], any element of the kernel of $\mathcal{S}_{(s)|\Sigma}$ of ghost number zero can be decomposed into a gauge-invariant functional in the physical fields and a $\mathcal{S}_{(s)|\Sigma}$-exact term. Thus, one computes the cohomology of $\mathcal{S}_{(Q)|\Sigma}$ in the image of $\mathcal{S}_{(s)|\Sigma}$. Any such element must be of the following form

$$\begin{aligned}\mathscr{C}^{\mathrm{im}}=\mathcal{S}_{(s)|\Sigma}\int\Big(&\big\langle\bar{\Omega},\,\mathscr{l}^{(0,1)}_{\frac{5}{2}}\big\rangle+\big\langle\bar{\mu},\mathscr{l}^{(0,2)}_{3}\big\rangle+\big\langle\varphi^{(s)}{}_{a},f^{(0,1)\;a}_{[a]+\frac{1}{2}}\big\rangle+\big\langle\varphi^{(Q_s)}{}_{a},\mathscr{g}^{(0,2)\;a}_{[a]+1}\big\rangle\\&+\big\langle\Omega^{(s)},f^{(1,1)}_{\frac{1}{2}}\big\rangle+\big\langle\Omega^{(Q_s)},\mathscr{g}^{(1,2)}_{1}\big\rangle+\big\langle\Omega^{(Q)},\mathscr{h}^{(0,2)}_{1}\big\rangle+\big\langle\mu^{(Q)},\mathscr{h}^{(0,3)}_{\frac{3}{2}}\big\rangle\Big)\end{aligned} \tag{68}$$

Here $\mathscr{l}^{(g,s)}_{d}$, $f^{(g,s)}_{d}$, $\mathscr{g}^{(g,s)}_{d}$ and $\mathscr{h}^{(g,s)}_{d}$ are local functions of the fields, where the sub-index d is the canonical dimension. The explicit form of $\mathscr{C}^{\mathrm{im}}$ is

$$\begin{aligned}\mathscr{C}^{\mathrm{im}}=\int\Big(&\big\langle b,\mathscr{l}^{(0,1)}_{\frac{5}{2}}\big\rangle-\big\langle\bar{\Omega},\mathcal{S}_{(s)|\Sigma}\,\mathscr{l}^{(0,1)}_{\frac{5}{2}}\big\rangle+\big\langle\bar{c},\mathscr{l}^{(0,2)}_{3}\big\rangle+\big\langle\bar{\mu},\mathcal{S}_{(s)|\Sigma}\,\mathscr{l}^{(0,2)}_{3}\big\rangle\\&+\big\langle\frac{\delta^{R}\Sigma}{\delta\varphi^{a}},f^{(0,1)\;a}_{[a]+\frac{1}{2}}\big\rangle+(-1)^{a}\big\langle\varphi^{(Q)}{}_{a},\mathscr{g}^{(0,2)\;a}_{[a]+1}\big\rangle-(-1)^{a}\big\langle\varphi^{(s)}{}_{a},\mathcal{S}_{(s)|\Sigma}f^{(0,1)\;a}_{[a]+\frac{1}{2}}\big\rangle+(-1)^{a}\big\langle\varphi^{(Q_s)}{}_{a},\mathcal{S}_{(s)|\Sigma}\mathscr{g}^{(0,2)\;a}_{[a]+1}\big\rangle\\&+\big\langle\frac{\delta^{R}\Sigma}{\delta\Omega},f^{(1,1)}_{\frac{1}{2}}\big\rangle-\big\langle\Omega^{(Q)},\mathscr{g}^{(1,2)}_{1}-\mathcal{S}_{(s)|\Sigma}\,\mathscr{h}^{(0,2)}_{1}\big\rangle+\big\langle\Omega^{(s)},\mathcal{S}_{(s)|\Sigma}f^{(1,1)}_{\frac{1}{2}}\big\rangle-\big\langle\Omega^{(Q_s)},\mathcal{S}_{(s)|\Sigma}\mathscr{g}^{(1,2)}_{1}\big\rangle\\&-\big\langle c^{(Q)},\mathscr{h}^{(0,3)}_{\frac{3}{2}}\big\rangle-\big\langle\mu^{(Q)},\mathcal{S}_{(s)|\Sigma}\,\mathscr{h}^{(0,3)}_{\frac{3}{2}}\big\rangle\Big)\end{aligned} \tag{69}$$

[7] For $\mathcal{N}=1$ in Minkowski space time

$$\delta^{\mathcal{S}usy}A_{\mu}=-i\big(\overline{\epsilon}\gamma_{\mu}\lambda\big) \qquad \delta^{\mathcal{S}usy}\lambda=\big(\not{F}+\gamma_5 H\big)\epsilon \qquad \delta^{\mathcal{S}usy}H=i\big(\overline{\epsilon}\gamma_5\not{D}\lambda\big) \tag{64}$$

with

$$(\delta^{\mathcal{S}usy})^2=i\big(\overline{\epsilon}\gamma^{\mu}\epsilon\big)\big(\partial_{\mu}+\delta^{\mathrm{gauge}}(A_{\mu})\big) \tag{65}$$

The consistency conditions imply that :

$$\overline{\mathcal{G}}_{\bullet}\mathscr{C}^{\rm im} = 0 \tag{70}$$

Thus :

$$\ell_{\frac{5}{2}}^{(0,1)} = \ell_{3}^{(0,2)} = 0 \tag{71}$$

On the other hand, the condition $\mathcal{S}_{(Q)|\Sigma}\mathscr{C}^{\rm im} = 0$ yields at zero-th order in the sources :

$$\int\Big(\big\langle\frac{\delta^L\Sigma}{\delta\varphi^a},\mathcal{S}_{(Q)|\Sigma}f_{[a]+\frac{1}{2}}^{(0,1)\ a}+g_{[a]+1}^{(0,2)\ a}\big\rangle+\big\langle\frac{\delta^L\Sigma}{\delta\Omega},\mathcal{S}_{(Q)|\Sigma}f_{\frac{1}{2}}^{(1,1)}+g_{1}^{(1,2)}-\mathcal{S}_{(s)|\Sigma}\,h_{1}^{(0,2)}\big\rangle$$
$$+\big\langle\frac{\delta^L\Sigma}{\delta c},h_{\frac{3}{2}}^{(0,3)}\big\rangle-\big\langle\frac{\delta^L\Sigma}{\delta\mu},\mathcal{S}_{(s)|\Sigma}\,h_{\frac{3}{2}}^{(0,3)}\big\rangle\Big)=\mathcal{O}(\varphi^{\bullet}) \tag{72}$$

This equation is solved, either when each term vanishes, or when the action Σ admits an accidental symmetry, with ghost and shadow numbers $(0,2)$, and canonical dimension one, which only acts on the physical fields, Ω, c and μ. It is reasonable to assume that this later possibility never arises. Therefore, one must have :

$$\begin{aligned}\mathcal{S}_{(Q)|\Sigma}f_{[a]+\frac{1}{2}}^{(0,1)\ a}+g_{[a]+1}^{(0,2)\ a}&=c_{[a]+[b]-3}^{(0,2)\ ab}\frac{\delta^R\Sigma}{\delta\varphi^b}+\mathcal{O}(\varphi^{\bullet})\\ \mathcal{S}_{(Q)|\Sigma}f_{\frac{1}{2}}^{(1,1)}+g_{1}^{(1,2)}-\mathcal{S}_{(s)|\Sigma}h_{1}^{(0,2)}&=\mathcal{O}(\varphi^{\bullet})\\ h_{\frac{3}{2}}^{(0,3)}&=\mathcal{O}(\varphi^{\bullet})\end{aligned} \tag{73}$$

This property tells us that $\mathscr{C}^{\rm im}$ is trivial at zeroth order in the sources $\varphi^{\bullet}$, that is :

$$\mathscr{C}^{\rm im}=\mathcal{S}_{(s)|\Sigma}\mathcal{S}_{(Q)|\Sigma}\int\Big(-(-1)^a\big\langle\varphi^{(Q_s)}{}_a,f_{[a]+\frac{1}{2}}^{(0,1)\ a}\big\rangle+\big\langle\Omega^{(Q_s)},f_{\frac{1}{2}}^{(1,1)}\big\rangle\Big)+\mathcal{O}(\varphi^{\bullet}) \tag{74}$$

Due to the power counting, only the functions $f_{[a]+\frac{1}{2}}^{(0,1)\ a}$ and $g_{[a]+1}^{(0,2)\ a}$ can depend on the external sources. Moreover, this dependence must be linear. One can then compute that $\mathscr{C}^{\rm im}$ must be trivial at all order in the sources, extending the equations (72,73). We thus obtain the desired result that the cohomology of $\mathcal{S}_{(Q)|\Sigma}$ in the image of $\mathcal{S}_{(s)|\Sigma}$ is empty.

We have thus shown that, the possible purely supersymmetric anomalies $\mathscr{C}$ are the elements of $\mathcal{H}^1$ with canonical dimension four. In fact, $\mathcal{H}^1$ depends on the given supersymmetric theory. As a direct consequence of results in [5], $\mathcal{H}^1=\{0\}$ in $\mathcal{N}=2$ super Yang–Mills. Preliminary computations show that there are no supersymmetric anomalies in $\mathcal{N}=4$ superYang–Mills, in agreement with [11].

To summarize, anomalies for the linearized operators $\overline{L\mathcal{G}_{\bullet}}$, $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ must be, either an element of $\mathcal{H}^1$, or the Adler–Bardeen anomaly $\mathscr{A}$, provided that a corresponding counterpart $\mathscr{B}$ exists.

5.4 Solutions of the consistency conditions associated to the ghost operators

In the Landau gauge, there are additional symmetries, expressed by the ghost operators $\mathcal{G}^\bullet$, whose homogeneous parts are the differential operators $L\mathcal{G}^\bullet$. This gauge has many advantages, but one must check that there is no anomaly for the operators $L\mathcal{G}^\bullet$. From the previous computation, one can check that the possible anomalies for $\overline{L\mathcal{G}_\bullet}$, $\mathcal{S}_{(s)|\Sigma}$ and $\mathcal{S}_{(Q)|\Sigma}$ are left invariant by the four operators $L\mathcal{G}^\bullet$'s. Therefore, the computation of possible anomalies of the ghost operators reduces to a relatively simple system of consistency equations. Moreover, because the operators $L\mathcal{G}^\bullet$ decrease the ghost and shadow numbers, their anomalies $\Delta^\bullet$ must depend on the anti-ghosts, the anti-shadows, and the sources, which have large canonical dimensions. By inspection, one finds that the consistency conditions of $\Delta^\bullet$ have no local solutions with the required canonical dimension.

Therefore, one can use the ghost Ward identities of the Landau gauge, and the anomaly problem is the same in this gauge as for all other values of α. The advantage of the Landau gauge, is that it simplifies the proof of the stability of the action. Most importantly, it is a preferred gauge for demonstrating non-renormalization theorems [8, 12].

6 General local solution of the Slavnov–Taylor identities

In this section we assume that the anomalies mentioned in the last section do not exist in the supersymmetric theory we are dealing with, or at least that these anomalies do not arise in perturbation theory. For $\mathcal{N} = 2, 4$ no anomaly can arise, but for $\mathcal{N} = 1$, the consistency conditions may have solutions, and the coupled hypermultiplet must be carefully chosen.

To prove the stability of the action, i.e. to show that the ambiguities related to the local invariant counterterms, which can be freely added at each order of perturbation theory, do not exceed the number of parameters of the classical starting action, we must compute the most general integrated local polynomial, with ghost and shadow numbers $(0, 0)$ and canonical dimension four, which satisfies all Ward identities. We will work in the Landau gauge, $\alpha = 0$.

Because of the ghost equations and power counting, the source dependence of the action must be linear, but for possible quadratic terms in the ${\varphi^{(Q)}}_a$. However, since the

Slavnov–Taylor identities mix the different sources (${\varphi^{(Q)}}_a$, ${\varphi^{(s)}}_a$ and ${\varphi^{(Qs)}}_a$), such quadratic terms must be absent. Therefore, the most general solution $\Sigma^{\mathcal{R}}$ of the Slavnov–Taylor identities must have a linear dependence on all sources. It can be written as follows :

$$\Sigma^{\mathcal{R}} = S^{\mathcal{R}}[\varphi^a, \Omega, \bar{\Omega}, c, \bar{c}, b, \bar{\mu}, \mu] + \int (-1)^a \left({\varphi^{(s)}}_a \Delta^a_{(s)} + \varphi^{(Q)} \Delta^a_{(Q)} + \varphi^{(Qs)} \Delta^a_{(Qs)} \right)$$
$$+ \int \mathrm{Tr} \left(-\Omega^{(s)} \Delta_{(s)} - \Omega^{(Q)} \Delta_{(Q)} - \Omega^{(Qs)} \Delta_{(Qs)} + \mu^{(Q)} \Lambda_{(Qs)} - c^{(Q)} \Lambda_{(Q)} \right) \quad (75)$$

Here, the $\Delta_\bullet$'s are local polynomials in the quantum fields, which can be used to define differential operators $s^{\mathcal{R}}$ and $Q^{\mathcal{R}}$.

$$\Delta^a_{(s)} = s^{\mathcal{R}} \varphi^a \qquad \Delta^a_{(Q)} = Q^{\mathcal{R}} \varphi^a$$
$$\Delta^a_{(Qs)} = s^{\mathcal{R}} Q^{\mathcal{R}} \varphi^A$$
$$\Delta_{(s)} = s^{\mathcal{R}} \Omega \qquad \Delta_{(Q)} = Q^{\mathcal{R}} \Omega$$
$$\Delta_{(Qs)} = s^{\mathcal{R}} Q^{\mathcal{R}} \Omega$$
$$\Lambda_{(Qs)} = Q^{\mathcal{R}} \mu \qquad \Lambda_{(Q)} = Q^{\mathcal{R}} c \quad (76)$$

The Slavnov–Taylor identities imply that $S^{\mathcal{R}}$ must be invariant under the action of $s^{\mathcal{R}}$ and $Q^{\mathcal{R}}$, and that these latter must verify

$$s^{\mathcal{R}2} = 0 \qquad Q^{\mathcal{R}2} = \mathcal{L}_\kappa$$
$$\{ s^{\mathcal{R}}, Q^{\mathcal{R}} \} = 0 \quad (77)$$

$s^{\mathcal{R}}$ and $Q^{\mathcal{R}}$ acts as s and Q when these ones act linearly. Modulo a rescaling of ghosts and shadows Ω, μ and c as well as of the structure constants of the gauge group, it turns out that the operators $s^{\mathcal{R}}$ and $Q^{\mathcal{R}}$ must have the same form of s and Q. Thus, the gauge symmetry can only be renormalized multiplicatively, and supersymmetric transformations must be renormalized in the most general possible way, which is compatible with power counting and their closure on a renormalized gauge transformation.

$${\delta^{Susy}}_{\mathcal{R}}{}^2 = \delta^{\mathrm{gauge}}(\omega^{\mathcal{R}}(\varphi) + i_\kappa A) + \mathcal{L}_\kappa \quad (78)$$

Then, the renormalized action decomposes into a part belonging to the cohomology of $s^{\mathcal{R}}$ and into a $s^{\mathcal{R}}$-exact part. Because $s^{\mathcal{R}}$ and s are identical modulo rescalings of ghosts and the coupling constant, the former term is a gauge invariant local functional which depends only on the fields φ^a. It must also be invariant under the renormalized supersymmetry. Since the transformations are the same on the fields of negative ghost

number, the $s^{\mathcal{R}}$-exact part must be $Q^{\mathcal{R}}$-exact in order to be $Q^{\mathcal{R}}$ invariant. The antighost equations then fix this part to be the one of the classical action

$$s^{\mathcal{R}} Q^{\mathcal{R}} \int \mathrm{Tr}\ \bar{\mu} d \star A \tag{79}$$

To go further in determining $\omega^{\mathcal{R}}(\varphi)$, one must look at the details of the particular supersymmetric theory. Therefore, the conclusion is that in a supersymmetric theory with a supersymmetric algebra which closes off-shell, and which has no anomaly, the action is stable under radiative corrections if and only if, the most general supersymmetric algebra defined on the set of fields from Eq. (78), can be related to the standard one by linear redefinitions of the fields. In fact, this is the case in all super Yang–Mills theories.

7 Interpolating gauges

In this section, we discuss the extension of the Slavnov–Taylor identity associated to supersymmetry to an arbitrary value of the gauge parameter ζ. As already mentioned, this is achieved by introducing an anticommuting parameter z which allows one to vary the gauge parameter ζ, while maintaining the desired Slavnov–Taylor identity.

The anticommuting parameter z is such that $Qz = \zeta$, $Q\zeta = 0$, and $s\,\zeta = 0$, $s\,z = 0$. For $\zeta \neq 0$, one has a non-vanishing Q-variation of the gauge-fixing term. Using the gauge-fixing term eq.(23), one has, however, an extended Slavnov–Taylor identity:

$$\mathcal{S}_{(Q)}{}'(\Sigma') \equiv \mathcal{S}_{(Q)}(\Sigma') + \zeta \frac{\partial \Sigma}{\partial z} = 0 \tag{80}$$

The antighost Ward identities can also be extended to a generic value of the gauge parameters ζ and z. In the present case, these Ward identities generalize to

$$\begin{aligned}
\overline{\mathcal{G}}(\mathscr{F}) &\equiv \int \mathrm{Tr}\ X \left(\frac{\delta^L \mathscr{F}}{\delta b} - d \star A + \alpha \star b - z d \star \frac{\delta^L \mathscr{F}}{\delta A^{(Q)}} + z d \star dc \right) \\
\overline{\mathcal{G}_{(s)}}(\mathscr{F}) &\equiv \int \mathrm{Tr}\ X \left(\frac{\delta^L \mathscr{F}}{\delta \bar{\Omega}} + d \star \frac{\delta^L \mathscr{F}}{\delta A^{(s)}} + z d \star \frac{\delta^L \mathscr{F}}{\delta A^{(Qs)}} + z d \star d\mu \right) \\
\overline{\mathcal{G}_{(Q)}}(\mathscr{F}) &\equiv \int \mathrm{Tr}\ X \left(\frac{\delta^L \mathscr{F}}{\delta \bar{c}} - (1-\zeta) d \star \frac{\delta^L \mathscr{F}}{\delta A^{(Q)}} - \zeta d \star dc - \alpha \star \mathcal{L}_\kappa \bar{c} + z d \star d \frac{\delta^L \mathscr{F}}{\delta c^{(Q)}} + z \mathcal{L}_\kappa d \star A \right) \\
\overline{\mathcal{G}_{(Qs)}}(\mathscr{F}) &\equiv \int \mathrm{Tr}\ X \left(\frac{\delta^L \mathscr{F}}{\delta \bar{\mu}} + (1-\zeta) d \star \frac{\delta^L \mathscr{F}}{\delta A^{(Qs)}} - \zeta d \star d\mu - z d \star d \frac{\delta^L \mathscr{F}}{\delta \mu^{(Q)}} - z \mathcal{L}_\kappa d \star \frac{\delta^L \mathscr{F}}{\delta A^{(s)}} \right)
\end{aligned} \tag{81}$$

The consistency conditions for the operators $\mathcal{S}_{(s)}$, $\mathcal{S}_{(Q)}$ and $\overline{\mathcal{G}_\bullet}$ remain the same. For the classification of anomalies, one can repeat the same arguments as previously, with $\zeta \neq 0$. The only change is that, the translation operator :

$$e^{\int \mathrm{Tr}\,\left(-d\bar{\Omega}\star\frac{\delta^L}{\delta A^{(s)}}+d\bar{c}\star\frac{\delta^L}{\delta A^{(Q)}}+d\bar{\mu}\star\frac{\delta^L}{\delta A^{(Q_s)}}\right)} \tag{82}$$

which appears in (49), must be modified to

$$e^{\int \mathrm{Tr}\,\left(-d\bar{\Omega}\star\frac{\delta^L}{\delta A^{(s)}}+((1-\zeta)d\bar{c}+zdb)\star\frac{\delta^L}{\delta A^{(Q)}}+((1-\zeta)d\bar{\mu}-zd\bar{\Omega})\star\frac{\delta^L}{\delta A^{(Q_s)}}-zd\star d\bar{c}\frac{\delta^L}{\delta c^{(Q)}}-zd\star d\bar{\mu}\frac{\delta^L}{\delta \mu^{(Q)}}\right)} \tag{83}$$

When checking the stability of the action, one still finds that the gauge-fixing action $_s\,\mathbf{\Psi}$, which depends on the parameters ζ and z, is stable. However, we have not been able to derive ghosts Ward identities and prove that the action depends linearly in the sources, even in the Landau gauge. As a matter of fact, more computations are needed, to obtain the stability of the whole action, including the sources.

However, ghost Ward identities are in fact not necessary to remove ambiguities. What matters is that the supersymmetric gauge, $\zeta = 0$, can be analytically linked to the other gauges $\zeta \neq 0$, and, in particular, to the ordinary Faddeev–Popov gauge-fixing. As we have just seen, one can extend all the required Ward identities necessary to define the theory in the interpolating gauge. The Green functions of quantities which are left invariant by the operators $\mathcal{S}_{(s)|\Sigma}$ are independent from the parameter ζ and z. The supersymmetric gauge $\zeta = 0,\ z = 0$ gives the simplest gauge for explicit computations and check the Slavnov–Taylor identities at each order in perturbation theory. For $\zeta \neq 0$, one must keep the $\zeta,\ z$ dependence, in order to control supersymmetric covariance.

8 An example: the case of $\mathcal{N} = 2$ super-Yang–Mills

In order to illustrate our general formalism, let us consider the example of $\mathcal{N} = 2$ super-Yang–Mills in Euclidean space. In this case the supersymmetry algebra closes off-shell with the introduction of an auxiliary field H^i in the adjoint representation of the $SU(2)$ R-symmetry group. The theory contains a gauge field A, a $SU(2)$-Majorana fermion λ (we do not write the R-symmetry index), one scalar ϕ and one pseudo-scalar ϕ^5. The classical action is

$$\int d^4x \mathrm{Tr}\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}H_iH^i+\frac{1}{2}D_\mu\phi^5\,D^\mu\phi^5-\frac{1}{2}D_\mu\phi\,D^\mu\phi-\frac{i}{2}\big(\overline{\lambda}\not{D}\lambda\big)\right.$$
$$\left.\frac{1}{2}[\phi^5,\phi]^2+\frac{1}{2}\big(\overline{\lambda}\gamma_5[\phi^5,\lambda]\big)-\frac{1}{2}\big(\overline{\lambda}[\phi,\lambda]\big)\right) \tag{84}$$

and the supersymmetric algebra is :

$$\begin{aligned}
\delta^{Susy} A_\mu &= -i\big(\overline{\epsilon}\gamma_\mu\lambda\big) \\
\delta^{Susy}\phi^5 &= -\big(\overline{\epsilon}\gamma_5\lambda\big) \qquad \delta^{Susy}\phi = -\big(\overline{\epsilon}\lambda\big) \\
\delta^{Susy}\lambda &= \not{F}\epsilon + \gamma_5[\phi^5,\phi]\epsilon - i\big(\gamma_5\not{D}\phi^5 + \not{D}\phi\big)\epsilon + H_i\tau^i\epsilon \\
\delta^{Susy}H^i &= \Big(\overline{\epsilon}\tau^i\big(-i\not{D}\lambda + \gamma_5[\phi^5,\lambda] - [\phi,\lambda]\big)\Big)
\end{aligned} \tag{85}$$

It closes as follows

$$(\delta^{Susy})^2 = \delta^{\text{gauge}}\big((\overline{\epsilon}[\gamma_5\phi^5 - \phi]\epsilon) - i(\overline{\epsilon}\gamma^\mu\epsilon)A_\mu\big) - i(\overline{\epsilon}\gamma^\mu\epsilon)\partial_\mu \tag{86}$$

The supersymmetric gauge fixing-term is :

$$\begin{aligned}
-\ s\,Q\int d^4x\mathrm{Tr}\ \bar{\mu}\big(\partial^\mu A_\mu + \frac{\alpha}{2}b\big) = \\
\int d^4x\mathrm{Tr}\ \Big(-b\partial^\mu A_\mu - \frac{\alpha}{2}b^2 + \bar{\Omega}\partial^\mu D_\mu\Omega - \bar{c}\partial^\mu D_\mu c - \bar{\mu}\partial^\mu D_\mu\mu \\
- i\frac{\alpha}{2}(\overline{\epsilon}\gamma^\mu\epsilon)\bar{c}\partial_\mu\bar{c} + i\bar{c}(\overline{\epsilon}\not{\partial}\lambda) + \partial_\mu\bar{\mu}\big([D^\mu\Omega, c] + i(\overline{\epsilon}\gamma^\mu[\Omega,\lambda])\big)\Big)
\end{aligned} \tag{87}$$

We notice that this gauge-fixing term gives propagators between the shadow and the fermion fields, which depends on the supersymmetric parameters, as follows :

$$\langle c\ \lambda\rangle_0 = -\frac{\epsilon}{p^2} \qquad \langle c\ c\rangle_0 = (1-\alpha)\frac{(\overline{\epsilon}\not{p}\epsilon)}{(p^2)^2} \tag{88}$$

We checked that, no infra-red problem , which cannot be cured, is implied by the apparently infra-red dangerous propagator $\langle c\ c\rangle_0$. The dependence on the supersymmetric parameters ϵ of the propagators has no physical consequences for the observables, since the latter can be considered as gauge parameters. Their appearance in the propagators can be handled as that of the parameter α in the gauge propagator:

$$\langle A_\mu\ A_\nu\rangle_0 = -\frac{\delta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}}{p^2} \tag{89}$$

9 Conclusion

An important component of this paper is the introduction of a new sector in super-Yang–Mills theories, namely that of shadow fields. It defines new gauge-fixing terms

that depend on a more general family of gauge parameters. Some of them can be identified as the "constant ghosts" for supersymmetry transformations. The preserved BRST symmetry allows one to define supersymmetric covariant and gauge-invariant observables. Moreover, the Slavnov–Taylor identities imply that the correlation functions of the observables do not depend on these parameters. The ordinary Faddeev–Popov gauge-fixing, which is not supersymmetric, is obtained for a given choice of these parameters. Other choices give an explicitly supersymmetric gauge-fixing. This provides a proof that supersymmetry and gauge invariance of physical observables can be safely maintained, even in the simplest non-supersymmetric gauges.

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