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A Mathematical Programming Problem with Singular Mixed Pointwise-integral Constraints

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Abstract

We investigate an infinite dimensional optimization problem which constraints are singular integral-pointwise ones. We give some partial results of existence for a solution in some particular cases. However, the lack of compactness, even in L^1 prevents to conclude in the general case. We give an existence result for a weak solution (as a measure) that we are able to describe. The regularity of such a solution is still an open problem.

Key words: compactness, measure, minimization .

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1 Introduction

The generalized principal-agent model in the economic theory of delegation as well as the principal's optimization problem procedure ([1]) leads to the

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optimal control problem described below :

$$(\mathcal{P}_1) \quad \begin{cases} \min \mathcal{J}(h) \stackrel{def}{=} \int_0^1 \left(\left(1 - \frac{2t}{3}\right)h(t) - t^2 \sqrt{\frac{h(t)}{3}} \right) dt \\ 0 \leq th(t) \leq \alpha \int_0^t h(s) ds \quad \forall t \in]0, 1] , h \in L^1(0, 1) , \end{cases}$$

where $\alpha = 3/2$. It has been shown in [1] that this problem has a unique solution if $\alpha \in [1, 5]$ and we obtained an analytical form. Anyway, the question remains to prove the existence of the solution of this problem for any α and give some regularity properties of this solution if possible.

The functional \mathcal{J} is strictly convex with respect to h and the constraints are linear. Therefore the solution of this problem (if it exists) is unique. Though this problem seems quite simple, we are not able to prove any existence result with classical minimization techniques since the function \mathcal{J} is not coercive and the feasible set is not bounded. In what follows, we set

$$\mathcal{C}_\alpha = \left\{ h : [0, 1] \rightarrow \mathbb{R} \mid 0 \leq h(t) \leq \frac{\alpha}{t} \int_0^t h(s) ds , a.e.t \in [0, 1] \right\} .$$

Let us briefly recall the main results of [1] that have been established using the Karush-Kuhn-Tucker type optimality system when $\alpha \in [1, 5]$. Indeed in this case we may exhibit the solution h^* and the Lagrange multiplier λ^* associated to the constraint as :

$$h^*(t) = 3 \left(\frac{\alpha(\alpha + 1)}{(\alpha + 3)(\alpha + 5)} \right)^2 t^{\alpha-1}, \quad \lambda^*(t) = \frac{2}{3(\alpha + 1)} + \frac{(5 - \alpha)(\alpha + 3)}{6\alpha(\alpha + 1)} t^{\frac{3-\alpha}{2}} \quad (1.1)$$

The main result of [1] was the following :

Theorem 1.1 *If $1 \leq \alpha \leq 5$, the problem*

$$(\mathcal{P}_\infty) \quad \min \mathcal{J}(h), \quad h \in \mathcal{C}_\alpha \cap L^\infty(0, 1) .$$

has a unique solution h^* given by (1.1). If $1 \leq \alpha \leq 3$, problem (\mathcal{P}_1) has a unique solution h^* given by (1.1) and $\mathcal{J}(h^*) = -\frac{\alpha(\alpha+1)}{(\alpha+3)(\alpha+5)^2}$.

If $\alpha > 5$, then λ^* does not belong to $L^1(0,1)$ and its sign is not constant: we cannot conclude in this case.

The goal of this paper is to investigate all the values for α in a more general setting to give a complete study of this problem. Next section is devoted to the general formulation of the problem. Here we consider particular cases as well. In section 3, we give a partial existence result when the problem is set in the space $L^1(0,1)$ and we present a general “weak” framework in section 4. We give a counter-example where the solution does not exist in the last section.

2 General formulation of the problem and particular cases

To be more complete we now consider the problem

$$(\mathcal{P}_p) \quad \begin{cases} \min J(h) \stackrel{\text{def}}{=} \int_0^1 \left(\omega_1(t)h(t) - \omega_2(t)\sqrt{h(t)} \right) dt \\ h \in \mathcal{C}_\alpha \cap L^p(0,1) . \end{cases}$$

where $p \in [1, +\infty]$ and we assume :

$$\omega_1, \omega_2 \in L^\infty(0,1), \omega_2 \geq 0, \omega_2 \not\equiv 0 \text{ and } \exists \sigma_o > 0 \text{ s.t. } \forall t \in [0,1] \omega_1(t) \geq \sigma_o. \quad (2.1)$$

We note that non-increasing nonnegative functions in L^1 always belong to \mathcal{C}_α for any $\alpha \geq 1$. Note also that constant functions are not elements of \mathcal{C}_α if $0 < \alpha < 1$.

2.1 Case where $\alpha < 1$

It is clear that if $\alpha \leq 0$ then $\mathcal{C}_\alpha \cap L^1(0, 1) = \{0\}$ since

$$0 \leq h(t) \leq \frac{\alpha}{t} \int_0^t h(s) ds \leq 0, \forall t \in]0, 1].$$

Now we focus on the case $\alpha \in]0, 1[$.

Proposition 2.1 *If $\alpha \in]0, 1[$ and $\frac{1}{1-\alpha} < p \leq \infty$ then $\mathcal{C}_\alpha \cap L^p(0, 1) = \{0\}$*

Proof - We use a real analysis argument: Hardy's inequality on $L^p((0, \infty[)$ (see [7] ex 14 p.69). Let $F(t) = \frac{1}{t} \int_0^t f(s) ds$; then, for every $f \in L^p((0, \infty[)$ we have $\|F\|_p \leq \frac{p}{p-1} \|f\|_p$. Now, if $f \in \mathcal{C}_\alpha \cap L^p(0, 1)$ and non-zero (we extend it by 0 on $[1, \infty[$) then

$$\|f\|_p \leq \alpha \cdot \frac{p}{p-1} \|f\|_p$$

Hence this inequality can only hold with $1 \leq \frac{\alpha p}{p-1}$ and we get a contradiction.

This achieves the proof. □

Note that the Hardy's inequality is trivial for $p = \infty$.

Example 2.1 *In the case where $1 < p \leq \frac{1}{1-\alpha}$ one can verify that $t \mapsto t^{-1/p}$ belongs to $\mathcal{C}_\alpha \cap L^q(0, 1)$ for any $q \in]1, p[$. This is true if we choose $p = \frac{1}{1-\alpha}$. So $\forall p \in]1, \frac{1}{1-\alpha}[$ $\mathcal{C}_\alpha \cap L^p(0, 1) \neq \{0\}$.*

If $\alpha < 1$, we conclude that the minimization problem has to be studied in $L^p(0, 1)$ for $1 \leq p < \frac{1}{1-\alpha}$.

2.2 The unconstrained case

As J is convex and Gâteaux-differentiable at any h that does not vanish, a necessary and sufficient condition for h^c to be the unconstrained minimizer of J is

$$\forall h \in L^1(0, 1) \quad \nabla J(h^c) \cdot h = \int_0^1 \left(\omega_1(t) - \frac{\omega_2(t)}{2\sqrt{h^c(t)}} \right) h(t) dt = 0 .$$

A small computation gives (see also the direct computation at the beginning of section 3),

$$h^c(t) = \left(\frac{\omega_2(t)}{2\omega_1(t)} \right)^2 . \quad (2.2)$$

This L^∞ - function is nonnegative. It is the solution to (\mathcal{P}_1) if the constraint is satisfied that is when α satisfies

$$\alpha_c \stackrel{def}{=} \sup_{t \in [0,1]} \frac{th^c(t)}{\int_0^t h^c(s) ds} \leq \alpha .$$

More precisely

Proposition 2.2 *Assume that $\alpha_c < +\infty$. If $\alpha \geq \alpha_c$, then h^c is the unique solution to (\mathcal{P}_p) for every $p \in [1, +\infty]$. The optimal value is $I_\alpha = -\frac{1}{4} \int_0^1 \frac{\omega_2^2(t)}{\omega_1(t)} dt$.*

Proof - We note that $h^c \in L^\infty(0, 1)$ with assumption (2.1). If $\alpha \geq \alpha_c$ then the unconstrained solution belongs to $\mathcal{C}_\alpha \cap L^p(0, 1)$ for every $p \in [1, +\infty]$. \square

It may happen that $\alpha_c = +\infty$. However, for instance, if ω_2 and ω_1 are proportional then h^c is the (constant) solution of the problem for all $\alpha \geq \alpha_c = 1$ (and every $p \in [1, +\infty]$) if we minimize over $\mathcal{C}_\alpha \cap L^p(0, 1)$). If h^c is **non-increasing** then it belongs to \mathcal{C}_α for any α ; therefore, as h^c is a $L^p(0, 1)$ - function, then it is the solution of the L^p problem for every $p \in [1, +\infty]$.

Example 2.2 For the explicit case of [1] where $\omega_1(t) \equiv 1 - \frac{2t}{3}$ and $\omega_2(t) \equiv \frac{t^2}{\sqrt{3}}$ we obtain $h^c(t) = \frac{3t^4}{4(3-2t)^2}$, and a computation (with a formal computation software for example) shows that $t \mapsto \frac{th^c(t)}{\int_0^t h^c(s)ds}$ is increasing so that the “critical” value for α is $\alpha_c = \frac{h^c(1)}{\int_0^1 h^c(s)ds} \simeq 7.9671$; moreover $I_\alpha = -\frac{10}{507}$.

It remains to study the case where $1 \leq \alpha \leq \alpha_c$.

2.3 Case where the solution saturates the (upper) constraint

The trick in [1] was to assume a priori (with the help of numerical computation) that the solution h^* to the problem (\mathcal{P}_p) (for some $p \in [1, +\infty]$) was such that

$$\forall a.e. t \in [0, 1] \quad th^*(t) = \int_0^t h^*(s) ds \quad \text{and} \quad h^*(t) \geq 0. \quad (2.3)$$

It is easy to see that functions that satisfy (2.3) are the following

$$h(t) = Ct^{\alpha-1}, C \in \mathbb{R}^+.$$

We first note that $h \in L^p(0, 1)$ if and only if $1 + p(\alpha - 1) > 0$, that is, for any $p \in [1, +\infty]$ if $\alpha \geq 1$ and $p \leq \frac{1}{1-\alpha}$ else. The tool to prove that a “saturating” function (i.e. verifying (2.3)) is the solution consists of finding an appropriate Lagrange multiplier $\lambda^* \in L^{p'}(0, 1)$ (where $\frac{1}{p'} + \frac{1}{p} = 1$) such that $\lambda^* \geq 0$. As the problem is convex, this provides a sufficient condition to obtain the solution. In what follows, we set

$$\Psi(h) = \omega_1 h - \omega_2 \sqrt{h} \quad \text{and} \quad L_\alpha(h)(t) = th(t) - \alpha \int_0^t h(s)ds.$$

The operator L_α is linear and continuous from $L^p(0, 1)$ to $L^p(0, 1)$.

Step 1. Computation of the “saturating” function that could be the solution

of (\mathcal{P}_p)

We have seen that such a function can be written as $h(t) = Ct^{\alpha-1}$. We set

$$f(C) \stackrel{\text{def}}{=} J(Ct^{\alpha-1}) = C \int_0^1 \omega_1(t) t^{\alpha-1} dt - \sqrt{C} \int_0^1 \omega_2(t) t^{\frac{\alpha-1}{2}} dt .$$

The infimum of f is attained at C_α such that $f'(C_\alpha) = 0$. This gives

$$h^*(t) = C_\alpha t^{\alpha-1} \text{ with } C_\alpha = \left[\frac{1 \int_0^1 \omega_2(t) t^{\frac{\alpha-1}{2}} dt}{2 \int_0^1 \omega_1(t) t^{\alpha-1} dt} \right]^2 . \quad (2.4)$$

Note that $\omega_1(\cdot) t^{\alpha-1}$ and $\omega_2(\cdot) t^{\frac{\alpha-1}{2}}$ are L^1 -functions since ω_1 and ω_2 are L^∞ -functions and $\alpha > 0$. Moreover, we get a rough estimate for C_α :

$$0 \leq C_\alpha \leq \left(\frac{\alpha}{2\sigma_o} \int_0^1 \omega_2(t) t^{\frac{\alpha-1}{2}} dt \right)^2 \leq \left(\frac{\alpha}{\alpha+1} \frac{\|\omega_2\|_\infty}{\sigma_o} \right)^2 .$$

Step 2. Adjoint equation

The Lagrangian function is defined on $L^p(0, 1) \times L^{p'}(0, 1)$ as

$$\mathcal{L}(h, \lambda) = \int_0^1 \Psi(h)(t) dt + \int_0^1 \lambda(t) L_\alpha(h)(t) dt . \quad (2.5)$$

We set $V(t) = \int_0^t h(s) ds$; it is a L^∞ -function since h is at least a L^1 -function and $\|V\|_\infty \leq \|h\|_{L^1}$. Performing an integration by parts yields

$$\begin{aligned} & \int_0^1 \lambda(t) \left(\int_0^t h(s) ds \right) dt = \int_0^1 \lambda(t) V(t) dt \\ & = \left[\left(\int_1^t \lambda(s) ds \right) V(t) \right]_0^1 - \int_0^1 h(t) \left(\int_1^t \lambda(s) ds \right) dt = - \int_0^1 \left(\int_1^t \lambda(s) ds \right) h(t) dt . \end{aligned}$$

Therefore

$$\begin{aligned} & \int_0^1 \lambda(t) L_\alpha(h)(t) dt = \int_0^1 t \lambda(t) h(t) dt - \alpha \int_0^1 \lambda(t) \left(\int_0^t h(s) ds \right) dt \\ & = \int_0^1 t \lambda(t) h(t) dt + \alpha \int_0^1 \left(\int_1^t \lambda(s) ds \right) h(t) dt = \int_0^1 \left[t \lambda(t) + \alpha \left(\int_1^t \lambda(s) ds \right) \right] h(t) dt . \end{aligned}$$

A formal computation of the derivative of \mathcal{L} with respect to h gives

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial h}(h^*, \lambda^*) \cdot h &= \int_0^1 \left(\frac{\partial \Psi}{\partial h}(h^*) \cdot h + \lambda^*(t) L_\alpha(h)(t) \right) dt \\ &= \int_0^1 \left[\frac{\partial \Psi}{\partial h}(h^*) + t \lambda^*(t) + \alpha \left(\int_1^t \lambda^*(s) ds \right) \right] h(t) dt . \end{aligned}$$

Therefore λ^* must verify the so called adjoint equation :

$$\frac{\partial \Psi}{\partial h}(h^*) + t \lambda^*(t) + \alpha \left(\int_1^t \lambda^*(s) ds \right) = 0 . \quad (2.6)$$

If the solution λ^* of (2.6) is nonnegative and belongs to $L^{p'}(0, 1)$ (where p has to be chosen), then the following optimality system is satisfied by the pair $(h^*, \lambda^*) \in L^p(0, 1) \times L^{p'}(0, 1)$:

$$\left(\frac{\partial \mathcal{L}(h^*, \lambda^*)}{\partial h}, h - h^* \right) \geq 0 \quad \text{for all } h \geq 0, \quad (2.7a)$$

$$\lambda^* \geq 0 \quad \text{and} \quad \lambda^*(t) L_\alpha(h^*)(t) = 0 \quad \text{a.e. } t \in [0, 1] , \quad (2.7b)$$

$$L_\alpha(h^*)(t) \leq 0 \quad \text{and} \quad h^*(t) \geq 0 \quad \text{a.e. } t \in [0, 1] , \quad (2.7c)$$

Therefore h^* is the solution to (\mathcal{P}_p) .

Conversely, if λ^* happens to be negative on a measurable set with non zero measure, it proves that the saturating function **cannot** be the solution : indeed if it were the solution, λ^* should be nonnegative since the above optimality system is necessary and sufficient and λ^* is unique since it is given by (2.6).

Step 3. Resolution of the adjoint equation (2.6)

Setting $\Lambda(t) = \int_1^t \lambda^*(s) ds$ and using the computation of $\frac{\partial \Psi}{\partial h}(h)$, the adjoint equation is equivalent to

$$t \Lambda'(t) + \alpha \Lambda(t) + \omega_1(t) h^*(t) - \frac{\omega_2(t)}{2\sqrt{h^*(t)}} = 0, \quad \forall t \in]0, 1], \quad \Lambda(1) = 0 .$$

A standard computation gives $\Lambda(t) = t^{-\alpha} \int_1^t \Phi_\alpha(s) ds$, with

$$\Phi_\alpha(t) = \frac{\omega_2(t)}{2\sqrt{C_\alpha}} t^{\frac{\alpha-1}{2}} - \omega_1(t) C_\alpha t^{2(\alpha-1)}$$

and C_α is given by (2.4). Note that $\Phi_\alpha \in L^\infty(0,1)$ if $\alpha \geq 1$ (because of assumption (2.1)) and $\Phi_\alpha \in L^1(0,1)$ if $\alpha \geq 1/2$.

As $\lambda^*(t) = \Lambda'(t)$ we finally obtain $\lambda^*(t) = t^{-\alpha} \left[\Phi_\alpha(t) + \frac{\alpha}{t} \int_t^1 \Phi_\alpha(s) ds \right]$.

Step 4. Sufficient conditions to get $\lambda^* \geq 0$

We are not able to give precise results since ω_1 and ω_2 are general functions. Anyway, we may give sufficient conditions dealing with ω_1 , ω_2 and α , that have to be detailed once ω_1 and ω_2 are given.

An obvious necessary and sufficient condition to get $\lambda^* \geq 0$ is

$$\forall t \in]0, 1] \quad \Phi_\alpha(t) + \frac{\alpha}{t} \int_t^1 \Phi_\alpha(s) ds \geq 0.$$

A simple sufficient condition is : $\forall t \in]0, 1] \quad \Phi_\alpha(t) \geq 0$; that is

$$\forall t \in [0, 1] \quad t^{3(\frac{\alpha-1}{2})} \omega_1(t) \leq \frac{C_\alpha^{-3/2}}{2} \omega_2(t) .$$

This relation involves ω_1 , ω_2 and gives “good values” of α .

Step 5. Sufficient conditions to get $\lambda^* \in L^{p'}(0,1)$

First we note that if $\alpha \geq 1$ then the “ solution ” $h^* \in L^\infty(0,1)$. Therefore , we must find some $q = p' \in [1, +\infty]$ such that $\lambda^* \in L^q(0,1)$. The expression of λ^* shows that it belongs to $L^\infty(]0, 1])$: we must check its behaviour in a neighborhood of $t = 0$. Once again, as we do not know ω_1 and ω_2 explicitly, we only present the method since we cannot perform a complete study. If we know the explicit expressions of ω_1 and ω_2 , we know their behaviour in a neighborhood of $t = 0$ (in fact only ω_2 is needed since we assumed $\omega_1(0) \neq 0$). Then, it is easy to describe the behaviour of Φ_α and λ^* (see [1]) ; we may then

deduce suitable values for α to get some $q \in [1, +\infty]$ such that $\lambda^* \in L^q(0, 1)$.

3 Case where the function space is L^1

In what follows, we denote by λ the Lebesgue measure on $[0, 1]$ also denoted by dt or ds in the integrals. We use the notations $\int_0^1 f$ and $\int_0^1 f d\nu$ instead of $\int_0^1 f(t) dt$ and $\int_0^1 f(t) d\nu(t)$ to simplify the expressions.

3.1 Preliminary comments

We now denote $\mathcal{C}_\alpha = \mathcal{C}_\alpha \cap L^1(0, 1)$ since there is no ambiguity on the functional space. It is a convex cone and is weakly closed since it is closed for the strong topology in L^1 . Note that the family of sets \mathcal{C}_α is non-decreasing.

We denote by $I_\alpha = \inf\{J(h), h \in \mathcal{C}_\alpha\}$. Under the assumption (2.1) on ω_1 and ω_2 and from the following formula

$$J(h) = \int_0^1 \omega_1(t) \left[\sqrt{h(t)} - \frac{\omega_2(t)}{2\omega_1(t)} \right]^2 dt - \int_0^1 \frac{\omega_2^2(t)}{4\omega_1(t)} dt. \quad (3.1)$$

we get $J(h) \geq - \int_0^1 h^c(t) \omega_1(t) dt$ with h^c given by (2.2). We deduce that the infimum I_α is finite since $0 \leq h^c \omega_1 \in L^1(0, 1)$.

First of all, we remark that we cannot apply the Dunford-Pettis criterion to a bounded set of \mathcal{C}_α in L^1 in order to get weak compactness (see for example [2]). It is due to the fact that the best bound we can get is $0 \leq h(t) \leq \alpha M/t$ for $h \in \mathcal{C}_\alpha, \|h\|_1 \leq M$ for a finite M (take $h_\varepsilon(t) = t^{-\varepsilon}$ for optimality). To overcome this difficulty we performed a change of function. Roughly speaking

we consider $th(t)$ instead of $h(t)$. Unfortunately it was impossible to make the Dunford- Pettis theorem work: the singularity was just moved from 0 to $+\infty$.

So, we have to check for “weak” solutions that are not L^1 -functions but measures. First we give a useful property of the feasible set \mathcal{C}_α ..

Lemma 3.1 *For all $h \in \mathcal{C}_\alpha \cap L^1$ with h non identically zero, there exists $\gamma_0 > 0$ such that, if $\tilde{h} := \gamma_0 h$: for all $\gamma > 0$,*

$$J(\tilde{h}) \leq J(\gamma\tilde{h}) \tag{3.2}$$

or equivalently

$$\int_0^1 \omega_1 \tilde{h} = \frac{1}{2} \int_0^1 \omega_2 \sqrt{\tilde{h}}. \tag{3.3}$$

In particular $\tilde{h} \in \mathcal{C}_\alpha$ and (3.2) implies

$$J(\tilde{h}) \leq J(h) \text{ and } J(\tilde{h}) = - \int_0^1 \omega_1 \tilde{h} = -\frac{1}{2} \int_0^1 \omega_2 \sqrt{\tilde{h}}.$$

Proof - Let $f(\gamma) = J(\gamma h)$. The function f has a minimum at $\gamma_0 = \frac{1}{4} \frac{\left(\int_0^1 \omega_2 \sqrt{h}\right)^2}{\left(\int_0^1 \omega_1 h\right)}$. Let $\tilde{h} = \gamma_0 h$ then $J(\tilde{h}) \leq J(\gamma h)$ for all $\gamma > 0$. In particular, $J(\tilde{h}) \leq J(h)$ and $J(\tilde{h}) \leq J(\gamma\tilde{h})$ for all $\gamma > 0$. This last inequality is equivalent to $v'(1) = 0$ with $v(\gamma) = J(\gamma\tilde{h})$ i.e

$$2 \int_0^1 \omega_1 \tilde{h} = \int_0^1 \omega_2 \sqrt{\tilde{h}}.$$

The last inequality of the lemma is then obvious. This achieves the proof. \square

Now we may define

$$\mathcal{K}_\alpha = \{h \in \mathcal{C}_\alpha \mid h \text{ satisfies (3.3)} \}. \tag{3.4}$$

Note that \mathcal{K}_α is not convex; anyway we get the following result:

Proposition 3.1 *The following equality holds true*

$$I_\alpha = \inf_{h \in \mathcal{C}_\alpha \cap L^1} J(h) = \inf_{h \in \mathcal{K}_\alpha \cap L^1} J(h).$$

Proof - Let h_j be a minimizing sequence in $\mathcal{C}_\alpha \cap L^1$ of I_α . By lemma 3.1, there exists $\gamma_j > 0$ such that $\tilde{h}_j = \gamma_j h_j$ and satisfying $J(\tilde{h}_j) \leq J(\gamma \tilde{h}_j)$ for all $\gamma > 0$. In particular $J(\tilde{h}_j) \leq J(h_j)$. Then, we also have $I_\alpha = \lim_j J(\tilde{h}_j)$ with $\tilde{h}_j \in \mathcal{K}_\alpha$ since $J(\tilde{h}_j)$ is a better approximation of I_α . This proves the relation. \square

3.2 A partial existence result in L^1 .

In this subsection, we deduce a conditional result about the solution of the problem in the L^1 -setting. Such a result is not completely satisfactory since the solution may not be in the feasible set. But this first partial result is important to understand the problem under consideration. A decoupling argument is used for the function h which is in L^1 and its square root \sqrt{h} which is in L^2 . Difficulties appear when we consider a minimizing sequence since L^1 is not a reflexive space but L^2 is a reflexive space.

Any minimizing sequence $h_j \in \mathcal{C}_\alpha$ satisfies $h_j(t) \leq \alpha M/t$. We cannot apply Dunford-Pettis theorem to get a weak limit in $L^1([0, 1])$. Indeed, the unit approximation of δ_0 defined by $\omega_n = n\chi_{[0, 1/n]}$ is in \mathcal{C}_α , $\alpha \geq 1$ since ω_n is non-increasing and nonnegative. Then a weak limit of h_j may not be a function. We deal with this difficulty all along this paper.

In addition, the condition $h \in \mathcal{C}_\alpha$ leads to the following weak formulation.

$$\forall 0 \leq \varphi \in \mathcal{C}([0, 1]) \quad \int_0^1 t\varphi(t) d\mu(t) \leq \alpha \int_0^1 \varphi(t) \left[\int_0^t d\mu(s) \right] dt. \quad (3.5)$$

with $d\mu = h d\lambda$ i.e h is seen as a density of measure with respect to the Lebesgue measure. We shall make such a weak formulation precise in next section. Next lemma will be useful in what follows when we shall deal with a bounded measures sequence.

Lemma 3.2 *Let μ be a nonnegative finite measure on $[0, 1]$ satisfying (3.5). Then for any $0 < \delta \leq 1$ and for all $A \in \mathcal{B}([\delta, 1])$, where $\mathcal{B}([\delta, 1])$ denotes the set of borelian sets of $[\delta, 1]$.*

$$\mu(A) \leq \frac{\alpha\mu([0, 1])}{\delta} \lambda(A). \quad (3.6)$$

Therefore the measure μ restricted to $[\delta, 1]$ is absolutely continuous with respect to the Lebesgue measure λ .

Proof - Fix $0 < \delta < 1$ and let $[a, b] \subset [\delta, 1]$. We assume that $\delta < a$ and $b < 1$. We treat the other intervals of $[0, 1]$ similarly. We define the sequence (φ_n) of continuous functions such that $\varphi_n(t) = 1$ if $t \in [a, b]$, zero outside $[a - 1/n, b + 1/n]$ and φ_n is linear on the set $[a - 1/n, a] \cup [b + 1/n, b]$. In particular $0 \leq \varphi_n \leq 1$ and the support of φ_n is included in $[\delta, 1]$ for n large enough. We have the following inequalities, for n large enough,

$$\delta\mu([a, b]) \leq \int_0^1 t\varphi_n(t) d\mu(t) \leq \alpha\mu([0, 1]) \int_0^1 \varphi_n(t) dt \leq \alpha\mu([0, 1])\lambda([a-1/n, b+1/n]).$$

Taking the limit over n , we get: $\mu([a, b]) \leq \frac{\alpha\mu([0, 1])}{\delta} \lambda([a, b])$

and we also deduce : $\mu(]a, b]) \leq \frac{\alpha\mu([0, 1])}{\delta} \lambda(]a, b])$.

Now since the lebesgue measure λ is regular measure:

$$\forall A \in \mathcal{B} \quad \lambda(A) = \inf\{\lambda(\Omega), A \subset \Omega \text{ open in } [0, 1]\}.$$

Let Ω be an open set of \mathbb{R} (with $A \subset \Omega$), then Ω is a countable union of disjoint intervals I_n of $[0, 1]$: $\Omega = \cup_n I_n$. Hence

$$0 \leq \mu(A) \leq \mu(\Omega) \leq \sum_n \mu(I_n) \leq \frac{\alpha\mu([0, 1])}{\delta} \sum_n \lambda(I_n) = \frac{\alpha\mu([0, 1])}{\delta} \lambda(\Omega).$$

Taking the infimum over Ω , we get the inequality (3.6). □

We also need the following lemma.

Lemma 3.3 *Let μ be a nonnegative finite measure on $[0, 1]$ satisfying (3.5).*

Then $\mu = \beta\delta_0 + g\lambda$ with $0 \leq \beta < \infty$ and $0 \leq g \in L^1$.

Proof - Fix $0 < \delta \leq 1$, By lemma 3.2, the measure μ restricted to $[\delta, 1]$ is absolutely continuous with respect to λ the Lebesgue measure. By Radon-Nikodym theorem, there exists a unique nonnegative density $g_\delta \in L^1([\delta, 1])$ such that $\mu|_{[\delta, 1]} = g_\delta\lambda$. We define for all $t \in]0, 1]$, $g(t) = g_\delta(t)$ for $\delta \leq t \leq 1$ and $g(0) = 0$. By uniqueness of g_δ , g is well-defined, nonnegative and g is integrable. Indeed, for all $0 \leq \varphi \in \mathcal{C}([0, 1])$

$$\int_0^1 \varphi g = \lim_{\delta \rightarrow 0} \int_\delta^1 \varphi g_\delta = \lim_{\delta \rightarrow 0} \int_\delta^1 \varphi d\mu ;$$

with $\varphi \equiv 1$, we get the integrability of g . So, for any $\varphi \in \mathcal{C}([0, 1])$, we get

$$\int_0^1 \varphi d\mu = \varphi(0)\mu(\{0\}) + \int_0^1 \varphi g,$$

that is : $\mu = \beta\delta_0 + g\lambda$ with $\beta = \mu(\{0\})$. □

Remark 1 *The above result is coherent with the compactness-concentration principle (see [3-6]).*

Now, we may give the main result of this section

Theorem 3.1 *Assume (2.1) and $\omega_1 \in \mathcal{C}([0, 1])$, $\omega_2 \in L^2([0, 1])$. Then, there exists a positive finite measure ν on $[0, 1]$ such that $\nu = \beta\delta_0 + g\lambda$ with $0 \leq \beta < \infty$ and $0 \leq g \in L^1$ with g non-identically zero and a L^2 -function $\sqrt{H} \geq 0$ satisfying*

$$\forall 0 \leq \varphi \in \mathcal{C}([0, 1]) \quad \int_0^1 t\varphi(t) d\nu(t) \leq \alpha \int_0^1 \varphi(t) \left[\int_0^t d\nu \right] dt. \quad (3.7)$$

The measure ν and the function \sqrt{H} satisfy the following conditions

$$\forall 0 \leq \Psi \in L^2([0, 1]) \quad \int_0^1 \sqrt{t}\Psi(t)\sqrt{H(t)} dt \leq \sqrt{\alpha} \int_0^1 \Psi(t) \left[\int_0^t d\nu(s) \right]^{1/2} dt. \quad (3.8)$$

$$\forall 0 \leq \varphi \in \mathcal{C}([0, 1]) \quad \int_0^1 \varphi H \leq \int_0^1 \varphi d\nu \quad (3.9)$$

$$\int_0^1 \omega_1 d\nu = \frac{1}{2} \int_0^1 \omega_2 \sqrt{H}. \quad (3.10)$$

Moreover, the constant β and the density g satisfy:

$$\beta\omega_1(0) + \int_0^1 \omega_1 g \leq \frac{1}{4} \int_0^1 \frac{\omega_2^2}{\omega_1} < \infty.$$

Finally the infimum I_α is given by

$$I_\alpha = \int_0^1 \omega_1 d\nu - \int_0^1 \omega_2 \sqrt{H} = -\frac{1}{2} \int_0^1 \omega_2 \sqrt{H} = -\int_0^1 \omega_1 d\nu. \quad (3.11)$$

Remark 2 *The couple (ν, \sqrt{H}) a priori depends on α and the minimizing sequence.*

An important corollary of this theorem is the proposition below :

Proposition 3.2 *Under the hypothesis of Theorem 3.1 and if we assume $\beta =$*

0 then a (unique) solution h^* to $(\mathcal{P}_1,)$ exists : $h^* = g \in \mathcal{C}_\alpha \cap L^1$ is the density of the measure ν .

Proof - We assume $\beta = 0$, we have from (3.9),

$$\forall 0 \leq \varphi \in \mathcal{C}([0, 1]) \quad \int_0^1 \varphi g \geq \int_0^1 \varphi H. \quad (3.12)$$

we deduce that $g \geq H(\geq 0)$ almost everywhere and that $\sqrt{g} \geq \sqrt{H}$. This is the crucial point of the proof. As a consequence, we derive the following inequality

$$-\int_0^1 \omega_2 \sqrt{H} \geq -\int_0^1 \omega_2 \sqrt{g}.$$

With (3.11), this gives $I_\alpha \geq \int_0^1 \omega_1 H - \int_0^1 \omega_2 \sqrt{H} = J(H)$ one one hand, and on the other hand, $I_\alpha \geq \int_0^1 \omega_1 g - \int_0^1 \omega_2 \sqrt{g} = J(g)$.

Now, we prove that $g \in \mathcal{C}_\alpha$. By (3.7), for all $0 \leq \varphi \in \mathcal{C}([0, 1])$,

$$\int_0^1 t\varphi(t)g(t) dt \leq \alpha \int_0^1 \varphi(t) \left[\int_0^t g(s) ds \right] dt.$$

This implies

$$0 \leq tg(t) \leq \alpha \int_0^t g(s) ds \quad a.e.,$$

that is $g \in \mathcal{C}_\alpha \cap L^1$. Consequently, $I_\alpha = \inf_{h \in \mathcal{C}_\alpha \cap L^1} J(h) = J(g)$ with $g \in \mathcal{C}_\alpha \cap L^1$.

Uniqueness of the solution follows from the facts that the functional J and the set \mathcal{C}_α are convex. This achieves the proof. \square

Remark 3 *We do not know if $H \in \mathcal{C}_\alpha \cap L^1$ and $g = H$.*

3.3 Proof of Theorem 3.1

Step 1: Properties of some minimizing sequences and existence of I_α .

Let \bar{h}_k be a minimizing sequence in $\mathcal{C}_\alpha \cap L^1$ of I_α . By lemma 3.1, there exists

$h_k \in \mathcal{K}_\alpha$ such that $I_\alpha = \lim_k J(h_k)$ since $J(h_k)$ is a better approximation of I_α . We replace the sequence (\bar{h}_k) by the sequence (h_k) . We have in particular

$$\int_0^1 \omega_1 h_k = \frac{1}{2} \int_0^1 \omega_2 \sqrt{h_k}.$$

Let us show that (h_k) is a bounded sequence in L^1 .

Since, $\int_0^1 \omega_1 h_k = \frac{1}{2} \int_0^1 \frac{\omega_2}{\sqrt{\omega_1}} \sqrt{\omega_1 h_k}$, then by Hölder inequality,

$$\int_0^1 \omega_1 h_k \leq \frac{1}{2} \left(\int_0^1 \frac{\omega_2^2}{\omega_1} \right)^{1/2} \left(\int_0^1 \omega_1 h_k \right)^{1/2}.$$

So $\int_0^1 \omega_1 h_k \leq \frac{1}{4} \int_0^1 \frac{\omega_2^2}{\omega_1}$. On one hand, this yields that (h_k) is a bounded sequence

in L^1 since $\omega_1 \geq \sigma_0$, $h_k \geq 0$ and $\|h_k\|_1 = \int_0^1 h_k \leq \frac{1}{\sigma_0} \int_0^1 \frac{\omega_2^2}{\omega_1}$. The last term is finite since $\omega_2 \in L^2$ and $\omega_1 \geq \sigma_0$. On the other hand, as already obtained in a previous section I_α is finite and a lower bound in terms of ω_1 and ω_2 is given by:

$$I_\alpha = -\lim_k \int_0^1 \omega_1 h_k \geq -\frac{1}{4} \int_0^1 \frac{\omega_2^2}{\omega_1}. \quad (3.13)$$

In what follows, we denote by $M := \frac{1}{\sigma_0} \int_0^1 \frac{\omega_2^2}{\omega_1}$.

Step 2: Existence and properties of ν and \sqrt{H} .

By step 1, the sequence (h_k) is bounded in L^1 . We embed $L^1_+([0, 1])$ of non-negative integrable functions in the space $\mathcal{M}^+([0, 1])$ of nonnegative finite measure on the set $[0, 1]$. Then we set $\nu_k = h_k \lambda$ (with λ the Lebesgue measure). We have $\|\nu_k\| = \|h_k\|_1$ where $\|\nu_k\|$ denote the measure ν_k total variation norm. Since $\mathcal{M}([0, 1])$ is the dual space of $\mathcal{C}([0, 1])$, we can extract from (h_k) a subsequence (still denoted similarly) that converges in the $*$ -weak sense to

a measure ν i.e

$$\forall \varphi \in \mathcal{C}([0, 1]) \quad \lim_k \int_0^1 \varphi h_k = \int_0^1 \varphi d\nu.$$

In addition, ν is a finite nonnegative measure on $[0, 1]$: $\|\nu\| \leq M$ since $\|h_k\| \leq M$ and ν is nonnegative because $h_k \geq 0$. We may deduce some properties of ν : as

$$0 \leq t h_k(t) \leq \alpha \int_0^t h_k(s) ds \quad a.e.$$

$$\text{then } \forall 0 \leq \varphi \in \mathcal{C}([0, 1]) \quad \int_0^1 t \varphi(t) h_k(t) dt \leq \alpha \int_0^1 \varphi(t) \left[\int_0^t h_k(s) ds \right] dt.$$

We need the following lemma:

Lemma 3.4 *Assume that ν_k is \star -weak convergent to ν . Then,*

$$\limsup_k \int_0^t d\nu_k \leq \int_0^t d\nu.$$

We apply the above lemma to $\nu_k = h_k \lambda$. By the monotone convergence theorem, the following inequality holds:

$$\int_0^1 t \varphi(t) d\nu(t) \leq \alpha \int_0^1 \varphi(t) \left[\int_0^t d\nu(s) \right] dt. \quad (3.14)$$

The inequality (3.14) is a weak formulation of $h \in \mathcal{C}_\alpha$. In fact, this formulation (3.14) for the measure $\nu = h \lambda$ is equivalent to $h \in \mathcal{C}_\alpha$: this motivates the introduction of assumption (\mathcal{H}_1) of next section.

Proof of lemma 3.4.- We fix $t \in [0, 1]$. If $t = 1$, we have $\lim_k \int_0^1 d\nu_k = \int_0^1 d\nu$. We can assume $t < 1$. Let $\varphi_n(t) = 1$ on the set $[0, t]$, $\varphi_n(t) = 0$ on the set $[t+1/n, 1]$ (for n large enough) and linear on the set $[t, t+1/n]$ then $\varphi_n \in \mathcal{C}([0, 1])$. By monotone convergence theorem, $\lim_n \int_0^1 \varphi_n d\nu = \int_0^t d\nu$. Let $\varepsilon > 0$ and N such that for all $n \geq N$,

$$\int_0^t d\nu + \varepsilon \geq \int_0^1 \varphi_n d\nu \geq \limsup_k \int_0^1 \varphi_n d\nu_k \geq \limsup_k \int_0^t d\nu_k.$$

Let $\varepsilon \rightarrow 0$: we have proved the lemma. \square

We just proved that ν satisfies (3.5). With lemma 3.3 we conclude that $\nu = \beta\delta_0 + g\lambda$.

Since $\sqrt{h_k}$ is bounded in the Hilbert space L^2 , we can extract a new subsequence (still denoted similarly) that weakly converges to $\tilde{h} \in L^2$. Since h_k is nonnegative, the \tilde{h} is nonnegative as well and it can be named \sqrt{H} .

As $\omega_1 \in \mathcal{C}([0, 1])$ and $\omega_2 \in L^2([0, 1])$, we take the limit as $k \rightarrow +\infty$; we obtain

$$I_\alpha = \int_0^1 \omega_1 d\nu - \int_0^1 \omega_2 \sqrt{H}.$$

Step 3: Coupling conditions on (ν, \sqrt{H}) .

We have: $0 \leq \sqrt{t}\sqrt{h_k}(t) \leq \sqrt{\alpha} \left[\int_0^t h_k(s) ds \right]^{1/2}$ a.e .

Thus : $\forall 0 \leq \psi \in L^2 \quad 0 \leq \int_0^1 \sqrt{t}\psi(t)\sqrt{h_k}(t) dt \leq \int_0^1 \psi(t) \left[\int_0^t h_k(s) ds \right]^{1/2} dt$.

By lemma 3.4, we deduce

$$0 \leq \int_0^1 \sqrt{t}\psi(t)\sqrt{H}(t) dt \leq \int_0^1 \psi(t) \left[\int_0^t d\nu(s) \right]^{1/2} dt.$$

We also have $\int_0^1 \omega_2 \sqrt{H} = \frac{1}{2} \int_0^1 \omega_1 d\nu$.

At last, we have

$$\forall 0 \leq \varphi \in \mathcal{C}([0, 1]) \quad \lim_k \int_0^1 \varphi \left(\sqrt{h_k} - \sqrt{H} \right)^2 \geq 0,$$

thus $\lim_k \left(\int_0^1 \varphi h_k - 2 \int_0^1 \varphi \sqrt{h_k} \sqrt{H} + \int_0^1 \varphi H \right) \geq 0$ and we conclude that

$$\int_0^1 \varphi d\nu \geq \int_0^1 \varphi H. \tag{3.15}$$

Step 4: The density g is non-zero a.e .

Let $\alpha > 0$. Assume that $g = 0$ a.e so that $\nu = \beta\delta_0$. We show that $H = 0$ a.e.

as well. By (3.15), $0 \leq \int_0^1 \varphi H \leq \beta \varphi(0)$ for every $0 \leq \varphi \in \mathcal{C}([0, 1])$.

Let $n \geq 1$ and $\varphi_n(t) = 1$ if $t \in [1/n, 1]$ and $\varphi_n(t) = nt$ if $t \in [0, 1/n]$ then, by monotone convergence $0 \leq \int_0^1 H \leq \lim_n \int_0^1 \varphi_n(t) H \leq 0$. Since H is nonnegative, $H = 0$ a.e. We deduce that $I_\alpha = \beta \omega_1(0) \geq 0$ and we get a contradiction.

Indeed, when $\alpha > 0$ and $h_0(t) := t^{\alpha-1}$ then $h_0 \in \mathcal{C}_\alpha$ and

$$I_\alpha \leq \inf_{\gamma \geq 0} J(\gamma h_0) = - \int_0^1 \omega_1 \gamma_o h_0 < 0 \quad (3.16)$$

for some $\gamma_o > 0$. This achieves the proof. \square

Remark 4 1. *With relation (3.16) it is easy to see that*

$$\int \omega_1 d\nu \leq \frac{1}{4} \int \frac{\omega_2^2}{\omega_1};$$

moreover, if we choose constant test functions we obtain

$$\int \omega_1 d\nu \geq \frac{1}{4} \frac{\int \omega_2^2}{\int \omega_1};$$

the equality holds if ω_1 and ω_2 are proportional.

2. In step 1, we can consider a minimizing sequence satisfying a weaker condition than $\int_0^1 \omega_1 h_k = \frac{1}{2} \int_0^1 \omega_2 \sqrt{h_k}$ namely $J(h_k) \leq 0$.

This implies that $\int_0^1 \omega_1 h_k \leq \int_0^1 \omega_2 \sqrt{h_k}$. This gives also a uniform bound on the L^1 -norm of h_k larger than the one given in step 1 but which is enough to conclude that part.

The main challenge now, is to check when $\beta = 0$ so that the dirac measure at 0 disappears and we get a solution for problem (\mathcal{P}_1) .

4 A weak formulation : consideration in $\mathcal{M}^+([0, 1])$.

In this section, we consider the problem in the space of nonnegative finite measures space $\mathcal{M}^+([0, 1])$ with solutions in the so-called “weak” feasible set. With this formulation, we find a solution in the feasible set but we have lost (a priori) the uniqueness of the solution.

We have seen in the previous section that a “solution” exists : it is a measure and does not belong to the feasible set (except in the case $\beta = 0$). We now define an extension of the problem in order to get a solution in a “weak”-feasible set. Let $\mathcal{M}^+([0, 1])$ be the set of nonnegative finite measures on $[0, 1]$. We have to set some hypothesis on the measure $\mu \in \mathcal{M}^+([0, 1])$ and $0 \leq h \in L^1$ (i.e $h \in L^1_+$) to get a weak formulation for the feasible set. These assumptions are motivated by relations (3.7)-(3.8)-(3.9) of Theorem 3.1.

$$(\mathcal{H}_1) \quad \forall 0 \leq \varphi \in \mathcal{C}([0, 1]), \quad \int_0^1 t\varphi(t) d\mu(t) \leq \alpha \int_0^1 \varphi(t) \left[\int_0^t d\mu(s) \right] dt.$$

$$(\mathcal{H}_2) \quad \forall 0 \leq \Psi \in L^2([0, 1]), \quad \int_0^1 \sqrt{t}\Psi(t)\sqrt{h(t)} dt \leq \sqrt{\alpha} \int_0^1 \Psi(t) \left[\int_0^t d\mu(s) \right]^{1/2} dt.$$

$$(\mathcal{H}_3) \quad \forall 0 \leq \varphi \in \mathcal{C}([0, 1]), \quad \int_0^1 \varphi h \leq \int_0^1 \varphi d\mu$$

Lemma 4.1 (1) *The condition (\mathcal{H}_2) is equivalent to the pointwise inequality*

$$\sqrt{t}\sqrt{h(t)} \leq \sqrt{\alpha} \left(\int_0^t d\nu(s) \right)^{1/2} \quad \text{a.e}$$

that is (2.4) if $\nu = h \lambda$.

(2) *If $\mu = h \lambda$ then (\mathcal{H}_1) is equivalent to the pointwise inequality*

$$th(t) \leq \alpha \int_0^t h(s) ds \quad (4.1)$$

almost everywhere on $[0, 1]$. In that case (\mathcal{H}_1) and (\mathcal{H}_2) are equivalent to $h \in C_\alpha$. The inequality (\mathcal{H}_3) is an equality and is obvious.

Proof - We only sketch the proof. The first assertion follows by a well- known argument: we take a characteristic function of an appropriate set. This function is in L^1 since it contains bounded functions. The second assertion is deduced by the density of continuous function in the space L^1 and the same argument of the first assertion. \square

Since $(\mathcal{H}_1) - (\mathcal{H}_3)$ are natural generalizations of properties of (3.7) -(3.9) for a couple (μ, h) , we define the *weak* (or *generalized*) feasible set as follows

$$\tilde{\mathcal{C}}_\alpha = \{(\mu, h) \in \mathcal{M}^+([0, 1]) \times L_+^1([0, 1]) \mid (\mathcal{H}_1) - (\mathcal{H}_3) \text{ are verified.}\} \quad (4.2)$$

It is a cone (which is not necessarily convex) and the set \mathcal{C}_α can be identified as a subset of $\tilde{\mathcal{C}}_\alpha$. The generalized functional is defined by

$$\tilde{J}(\mu, h) = \int_0^1 \omega_1(t) d\mu(t) - \int_0^1 \omega_2(t) \sqrt{h(t)} dt \quad (4.3)$$

and the infimum of \tilde{J} on $\tilde{\mathcal{C}}_\alpha$ is denoted by

$$\tilde{I}_\alpha = \inf\{\tilde{J}(\mu, h) : (\mu, h) \in \tilde{\mathcal{C}}_\alpha\}. \quad (4.4)$$

We have the following relation between the sets \mathcal{C}_α and $\tilde{\mathcal{C}}_\alpha$ and the functionals J and \tilde{J} .

Lemma 4.2 *For $h \in \mathcal{C}_\alpha$ then $(h\lambda, h) \in \tilde{\mathcal{C}}_\alpha$ and $\tilde{J}(h\lambda, h) = J(h)$ where λ is the Lebesgue measure.*

The proof is straight forward. We have the analogue of lemma 3.1.

Lemma 4.3 *For all $(\nu, h) \in \tilde{\mathcal{C}}_\alpha$ with h and ν non identically zero, there exists $\gamma_1 > 0$ such that, if $(\nu_1, h_1) = (\gamma_1\nu, \sqrt{\gamma_1 h})$*

$$\forall \gamma > 0 \quad \tilde{J}(\nu_1, h_1) \leq \tilde{J}(\gamma\nu, \gamma h) \quad (4.5)$$

or equivalently

$$\int_0^1 \omega_1 d\nu_1 = \frac{1}{2} \int_0^1 \omega_2 \sqrt{h_1}. \quad (4.6)$$

In particular $(\nu_1, h_1) \in \tilde{\mathcal{C}}_\alpha$, relation (4.5) implies $\tilde{J}(\nu_1, h_1) \leq \tilde{J}(\nu, h)$ and (4.6) implies $\tilde{J}(\nu_1, h_1) = -\int_0^1 \omega_1 d\nu_1 = -\frac{1}{2} \int_0^1 \omega_2 \sqrt{h_1}$.

The proof is analogue to the proof of lemma 3.1.

The previous lemma indicates that we can use some specific minimizing sequence to solve the minimization problem. Indeed, we can add the following assumption on (ν, h) (similar to (3.10)) :

$$(\mathcal{H}_4) \quad \int_0^1 \omega_1 d\nu = \frac{1}{2} \int_0^1 \omega_2 \sqrt{h}.$$

Then we may define the set $\tilde{\mathcal{K}}_\alpha$ as the set of $(\nu, h) \in \tilde{\mathcal{C}}_\alpha$ satisfying (\mathcal{H}_4) .

This set is the analogue of \mathcal{K}_α defined in section 3.2 and we have a similar proposition to proposition 3.1 :

Proposition 4.1 *The following equality holds true*

$$\inf_{(\nu, h) \in \tilde{\mathcal{C}}_\alpha} \tilde{J}(\nu, h) = \inf_{(\nu, h) \in \tilde{\mathcal{K}}_\alpha} \tilde{J}(\nu, h).$$

We now formulate the theorem that gives a solution in the feasible set $\tilde{\mathcal{C}}_\alpha$ (or

$\tilde{\mathcal{K}}_\alpha$).

Theorem 4.1 *There exists a minimizer $(\mu^*, H^*) \in \tilde{\mathcal{C}}_\alpha$ of \tilde{J} on $\tilde{\mathcal{C}}_\alpha$ satisfying (\mathcal{H}_4) as well. In addition*

(1) *The infimum \tilde{I}_α is given by*

$$\tilde{I}_\alpha = \int_0^1 \omega_1 d\mu^* - \int_0^1 \omega_2 \sqrt{H^*} = \tilde{J}(\mu^*, H^*). \quad (4.7)$$

(2) *The measure μ^* has the following form $\mu^* = \beta^* \delta_0 + g^* \lambda$ with $0 \leq \beta^* < \infty$ and $0 \leq g^* \in L^1$ with g^* non-identically zero.*

(3) *The constant β^* and the density g^* satisfy:*

$$\beta^* \omega_1(0) + \int_0^1 \omega_1 g^* \leq \frac{1}{4} \int_0^1 \frac{\omega_2^2}{\omega_1} < \infty.$$

(4) *We have the relation*

$$\tilde{I}_\alpha = -\frac{1}{2} \int_0^1 \omega_2 \sqrt{H^*} = -\int_0^1 \omega_1 d\mu^* \quad (4.8)$$

Remark 5 *In particular, the minimizer has the specific form given by Theorem 3.1. The proposition 3.2 is valid for (μ^*, H^*) as well. The proof is similar.*

Proof - Step 1: A priori estimate of minimizing sequences in $\mathcal{M}^+([0, 1]) \times L^1([0, 1])$.

By lemma 4.3 we may choose a minimizing sequence (μ_k, h_k) in $\tilde{\mathcal{C}}_\alpha$ satisfying (\mathcal{H}_4) as well :

$$\int_0^1 \omega_1 d\mu_k = \frac{1}{2} \int_0^1 \omega_2 \sqrt{h_k} = \frac{1}{2} \int_0^1 \frac{\omega_2}{\sqrt{\omega_1}} \sqrt{\omega_1 h_k} \leq \frac{1}{2} \left(\int_0^1 \frac{\omega_2^2}{\omega_1} \right)^{1/2} \left(\int_0^1 \omega_1 h_k \right)^{1/2}.$$

We use (\mathcal{H}_3) with $\varphi = \omega_1 \in \mathcal{C}([0, 1])$,

$$\int_0^1 \omega_1 d\mu_k \leq \frac{1}{2} \left(\int_0^1 \frac{\omega_2^2}{\omega_1} \right)^{1/2} \left(\int_0^1 \omega_1 d\mu_k \right)^{1/2}.$$

It yields

$$\sigma_o \|\mu_k\| \leq \int_0^1 \omega_1 d\mu_k \leq \frac{1}{4} \left(\int_0^1 \frac{\omega_2^2}{\omega_1} \right)$$

Then $\|\mu_k\|$ is uniformly bounded by $M := \frac{1}{4\sigma_o} \left(\int_0^1 \frac{\omega_2^2}{\omega_1} \right)$.

Again by (\mathcal{H}_3) , with $\varphi = \omega_1 \in \mathcal{C}([0, 1])$, we deduce

$$0 \leq \int_0^1 \omega_1 h_k \leq \int_0^1 \omega_1 d\mu_k$$

Hence $\|h_k\|_1 \leq M$. Then (h_k) remains in a bounded set of L^1 and $(\sqrt{h_k})$ in a bounded set of L^2 .

Step 2: Existence of a \star -weak cluster point.

The sequence (μ_k) is \star -weak compact: it exists a subsequence still denoted similarly which converges \star -weak to μ^* in $\mathcal{M}^+([0, 1])$.

Similarly, as $(\sqrt{h_k})$ is bounded in L^2 , there exists a subsequence (still denoted similarly) which converges to $\sqrt{H^*}$ weakly in L^2 . Since $\omega_2 \in L^2$ and $\omega_1 \in \mathcal{C}([0, 1])$ we get

$$\tilde{I}_\alpha = \int_0^1 \omega_1 d\mu^* - \int_0^1 \omega_2 \sqrt{H^*}.$$

Step 3: Conditions $(\mathcal{H}_1) - (\mathcal{H}_4)$ for (μ^*, H^*) .

• Condition (\mathcal{H}_1) : Let $0 \leq \varphi \in \mathcal{C}([0, 1])$: then

$$\lim_k \int_0^1 t\varphi(t) d\mu_k(t) \leq \alpha \lim_k \int_0^1 \varphi(t) \left[\int_0^t d\mu_k(s) \right] dt.$$

The left-hand side has the following limit $\int_0^1 t\varphi(t) d\mu^*$ since $t\varphi(t)$ is continuous. We treat the right-hand side of the inequality above by lemma 3.4 and monotone convergence theorem : $\limsup_k \int_0^t d\mu_k \leq \int_0^t d\mu^*$.

• Condition (\mathcal{H}_2) : Let $0 \leq \Psi \in L^2([0, 1])$; then

$$\lim_k \int_0^1 \sqrt{t}\Psi(t)\sqrt{h_k(t)} dt \leq \sqrt{\alpha} \lim_k \int_0^1 \Psi(t) \left[\int_0^t d\mu_k(s) \right]^{1/2} dt.$$

The left-hand side is $\int_0^1 \sqrt{t} \Psi(t) \sqrt{H^*(t)} dt$. With lemma 3.4 and monotone convergence theorem, the left-hand side is $\sqrt{\alpha} \int_0^1 \Psi(t) \left[\int_0^t d\mu^*(s) \right]^{1/2} dt$.

• Condition (\mathcal{H}_3) : A priori we cannot take the limit in the left-hand side of (\mathcal{H}_3) for the subsequence (h_k) since the weak limit is known to exist only for $(\sqrt{h_k})$ not for (h_k) . Let $0 \leq \varphi \in \mathcal{C}([0, 1])$ and consider $\int_0^1 \varphi \left(\sqrt{h_k} - \sqrt{H^*} \right)^2 \geq 0$; then, with (\mathcal{H}_3) for (μ_k, H_k)

$$\lim_k \int_0^1 \varphi d\mu_k \geq \lim_k \int_0^1 \varphi h_k \geq \lim_k 2 \int_0^1 \varphi \sqrt{h_k} \sqrt{H^*} - \int_0^1 \varphi H^* = \int_0^1 \varphi H^* ;$$

this proves (\mathcal{H}_3) for (μ^*, H^*) .

• Condition (\mathcal{H}_4) :

$$\int_0^1 \omega_1 d\mu^* = \lim_k \int_0^1 \omega_1 d\mu_k = \lim_k \frac{1}{2} \int_0^1 \omega_2 \sqrt{h_k} = \frac{1}{2} \int_0^1 \omega_2 \sqrt{H^*}.$$

We deduce that $(\mu^*, H^*) \in \tilde{\mathcal{C}}_\alpha$ and it is a solution of the minimization problem.

Step 4: Decomposition of μ^* .

By lemma 3.3, condition (\mathcal{H}_1) is satisfied by μ^* ; so $\mu^* = \beta^* \delta_0 + g^* \lambda$ with $0 \leq \beta^* < \infty$ and $0 \leq g^* \in L^1$. To show that g^* is not identically zero we follow the same lines of proof of last step of Theorem 3.1. \square

We just proved that the “weak” problem (\mathcal{Q}) has a solution which has the same form as the measure found in Theorem 3.1. Therefore, the feasible domain of (\mathcal{Q}) can be reduced to elements $(\mu, h) \in \tilde{\mathcal{C}}_\alpha$ such that the measure μ has the specific form $\mu = \beta \delta_0 + g$ with $\beta \in \mathbb{R}^+$ and $g \in L^1(\Omega)_+$. It is also obvious that

$$\inf \mathcal{Q} \leq \inf \mathcal{P}_1 .$$

Unfortunately, we are not able to prove for the moment that the equality holds, that is the weak formulation (\mathcal{Q}) (which seems the most natural) is the

appropriate relaxed problem for (\mathcal{P}_1) .

5 A counter -example

We give with a negative result which shows that the condition $\omega_2 \in L^\infty(0, 1)$ is necessary to get a general result in $L^p(0, 1)$. We show that for each $1 < p \leq +\infty$, there exists functions ω_1 and ω_2 such that the problem of minimization has a solution in \mathcal{C}_α but **not** in L^p .

Proposition 5.1 *Let $\alpha \geq 1$. For every $1 < p \leq +\infty$, there exists $q \neq \infty$ and ω_2 that verify*

$$0 \leq \omega_2 \notin L^\infty(0, 1) \text{ and } \omega_2 \in L^q(0, 1)$$

such that for any $\omega_1 = \sigma_o > 0$ (constant function) the minimization problem (\mathcal{P}_p) has no solution in $L^p(0, 1)$. More precisely, (\mathcal{P}_p) has a solution in \mathcal{C}_α but not in $L^p(0, 1)$.

Proof - The idea is to construct an explicit solution satisfying the conditions of the proposition above which is also a solution of the unconstrained problem.

Let be $\alpha \geq 1$, $1 < p \leq +\infty$ and $0 < \varepsilon < 1$ such that $p \geq 1/\varepsilon$. We set $\omega_2(t) = t^{-\varepsilon/2}$: then $\omega_2 \in L^q(0, 1)$ if and only if $q < 2/\varepsilon$. In particular $\omega_2 \notin L^\infty(0, 1)$.

We choose $\omega_1(t) = \sigma_o = 1/2$. Then the unconstrained solution is

$$h^c(t) = \frac{\omega_2^2(t)}{4\omega_1^2(t)} = t^{-\varepsilon}.$$

We see that $h^c \in \mathcal{C}_\alpha$ for any $\alpha \geq 1$ since it is nonnegative, non-increasing and $\varepsilon < 1$. However, $h^c \notin L^p(0, 1)$ since $p \geq 1/\varepsilon$.

Let be

$$h_n(t) = \begin{cases} n^\varepsilon & \text{for } 0 \leq t \leq \frac{1}{n} \\ t^{-\varepsilon} & \text{for } \frac{1}{n} \leq t \leq 1 ; \end{cases}$$

therefore $h_n \in \mathcal{C}_\alpha \cap \mathcal{C}[0, 1]$ since h_n is a nonnegative, non-increasing and continuous function. We easily check that $\lim_{n \rightarrow +\infty} J(h_n) = J(h^c)$. We deduce that

$$J(h^c) \leq \inf_{\mathcal{C}_\alpha \cap L^p} J(h) \leq \inf_{\mathcal{C}_\alpha \cap \mathcal{C}[0,1]} J(h) \leq \lim_{n \rightarrow +\infty} J(h_n) = J(h^c) .$$

This completes the proof by uniqueness of the solution (when it exists). \square

Remark 6 (1) For ω_1, ω_2 as in the proof above, $h^c = t^{-\varepsilon}$ is the minimizer in $L^1(0, 1)$. It is also the solution under the constraint $h \in \mathcal{C}_\alpha$. Since h^c belongs to $L^p(0, 1)$ for $1 \leq p < 1/\varepsilon$, it is the minimizer of J under the constraint $h \in \mathcal{C}_\alpha \cap L^p(0, 1)$ for such a p (by uniqueness). In particular for this example, we have proved that for any $\alpha \geq 1$ and any $1 \leq p \leq \infty$,

$$\inf\{J(h), h \in \mathcal{C}_\alpha \cap \mathcal{C}[0, 1]\} = \inf\{J(h), h \in \mathcal{C}_\alpha \cap L^p\} = J(h^c)$$

(2) For $1 < p < 2$ with $0 < \varepsilon < 1$ such that $1/\varepsilon \leq p < 2/\varepsilon$ then $h^c \notin L^p(0, 1)$ but $\omega_2 \in L^p(0, 1)$.

(3) The choice of ω_1 is independent of p and has all the regularity we can expect since it is constant.

(4) The function ω_2 is always in $L^q(0, 1)$ with $1 \leq q \leq 2$.

(5) The solution h^c belongs to \mathcal{C}_α for $\varepsilon \in [1 - \alpha, 1[$ if $\alpha \in]0, 1[$.

6 Conclusion

Though this problem seems “simple”, it is still widely open. With results of section 2 , we may assert there exists $0 < \alpha_o < \alpha_c$ such that we get existence of a (more or less regular) solution if $\alpha \notin [\alpha_o, \alpha_c]$. The main challenge now, is to give conditions on α and ω_i to ensure that β (section 3) is equal to 0, or a contrario, provide some counter-examples. We conjecture that under the condition $\omega_2 \in L^\infty$ and appropriate conditions on α , there exists a solution of the problem in $L^p(0, 1)$ for some (any) $1 < p \leq \infty$. This is true for the unconstrained problem under the condition

$$\omega_2 \in L^\infty(0, 1) \text{ and } \omega_1 \geq \sigma_o,$$

since the solution $h^c = \frac{\omega_2^2}{4\omega_1^2}$ is then bounded. It remains two points to clarify :

- Find the cases and the conditions on ω_1 and ω_2 to ensure $\beta = 0$.
- Define an appropriate relaxed problem in the space of measures such that the minimum of the relaxed problem is equal to the infimum of the original problem (in L^1).

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