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Bounded Eigenfunctions in the Real Hyperbolic Space

Sandrine Grellier and Jean-Pierre Otal

1 Introduction

Let \mathbb{H}^n be the real hyperbolic space of dimension n , that is, the complete and simply connected Riemannian manifold of constant curvature -1 . Using the Poincaré model, we identify \mathbb{H}^n with the unit ball of \mathbb{R}^n and its boundary at infinity $\partial\mathbb{H}^n$ with the unit sphere S^{n-1} of \mathbb{R}^n .

We denote by Δ the Laplace-Beltrami operator acting on functions on \mathbb{H}^n .

Definition 1.1. For $\lambda \in \mathbb{C}$, say that $f : \mathbb{H}^n \rightarrow \mathbb{C}$ is a λ -eigenfunction when f is C^2 and satisfies $\Delta f + \lambda f = 0$. Denote by \mathcal{E}_λ the space of the λ -eigenfunctions which are bounded.

The purpose of this paper is to describe the space \mathcal{E}_λ . The Harnack inequality (see, e.g., [6, page 199]) implies that the gradient of a bounded λ -eigenfunction f is bounded by a constant which only depends on λ and on the sup norm of f . This implies, using the Ascoli theorem, that the space \mathcal{E}_λ with the sup norm is a Banach space.

For $s \in \mathbb{C}$, we set $\lambda_s = s(n-1-s)$. The basic example of a λ_s -eigenfunction is provided by $x \rightarrow P^s(x, \xi)$, where $P(x, \xi)$ is the Poisson kernel of \mathbb{H}^n and ξ is any point of the boundary at infinity $\partial\mathbb{H}^n$ of \mathbb{H}^n . Any λ_s -eigenfunctions can be described in terms of *hyperfunctions* on $\partial\mathbb{H}^n$. Namely, for any λ_s -eigenfunction, there is a hyperfunction on $\partial\mathbb{H}^n$ such that

$$f(x) = \mathcal{P}^s(T)(x) = \langle T, P^s(x, \cdot) \rangle; \tag{1.1}$$

this is a theorem of Helgason and Minemura (cf. [7, 9]), which was also proven later by Agmon [1].

Definition 1.2. Call the function $x \rightarrow \langle T, \mathcal{P}^s(x, \cdot) \rangle$ *the Poisson-Helgason transform of T*. Denote it by $\mathcal{P}^s(T)$.

Furthermore, one knows that the hyperfunction T is unique, except for the cases when $s = -1, -2, \dots, -k, \dots$ (see [9, Proposition 2.1]). Moreover, if f grows slowly, that is, at most exponentially with the hyperbolic distance, then T is a *distribution*. This result has been proven initially by Ōshima and Sekiguchi [10] using microlocal techniques. In [15], the authors give a new proof within the framework of asymptotic expansions (see [15, Theorem 2-2]). In particular, the hyperfunction which represents a bounded eigenfunction is indeed a distribution. Thus, to describe the space of bounded eigenfunctions is equivalent to describe the distributions T on \mathbb{S}^{n-1} such that $\mathcal{P}^s(T)$ is bounded.

When T is equal to the normalized Lebesgue measure $d\sigma$ on $\partial\mathbb{S}^{n-1}$, then the eigenfunction $\mathcal{P}^s(d\sigma)$ is the *spherical* λ_s -eigenfunction: this is the unique eigenfunction h_s such that $h_s(o) = 1$, and which is invariant under the subgroup ($\simeq O(n)$) of the isometries of \mathbb{H}^n which fix the origin o .

First of all, it is easy to see that the only values of s such that $\mathcal{E}_{\lambda_s} \neq \{0\}$ are contained in the strip $\{s \in \mathbb{C}; \Re(s) \in [0, n-1]\}$. Indeed, let $f \in \mathcal{E}_{\lambda_s}$ be a nonzero function and let $x \in \mathbb{H}^n$ be a point where $f(x) \neq 0$. Up to precomposing f with an isometry which sends o to x , we can assume that $f(o) \neq 0$. Then, by averaging f over the subgroup ($\simeq O(n)$) of all rotations which fix o , we obtain a λ_s -eigenfunction which is bounded, nonzero, and invariant under the orthogonal group $O(n)$. But the unicity of the spherical function h_s implies that this function is a multiple of h_s ; therefore the spherical function h_s has to be bounded. It is now easy to see that the only values of s for which this can occur are exactly those such that $\Re s \in [0, n-1]$.

We denote by \mathcal{H} the vertical strip $\{s \in \mathbb{C} \mid \Re s \in]0, n-1[\}$. The method we use in this paper does not allow to deal with the values of s such that $\Re s = 0$ or $n-1$. Therefore, we will focus on the values of $s \in \mathcal{H}$.

When $n = 2$, there is a characterization of the distributions T such that $\mathcal{P}^s(T)$ is bounded [11]: for $\Re s \neq 0$, they are exactly the distributional derivatives of $\Re(s)$ -Hölder functions on $\partial\mathbb{H}^2$. In higher dimension, the notion of derivatives depends on the data of a vector field, and so does not have easily an intrinsic meaning. One way to avoid this difficulty is to use the spherical Laplacian Δ_σ acting on functions on \mathbb{S}^{n-1} . If a distribution T satisfies $\langle T, 1 \rangle = 0$, there is an integer k and a continuous function H such that the distribution T can be written as $\Delta_\sigma^k(H)$. Then the regularity of T can be measured in terms of the integer k and of the regularity of H . We use *the Lipschitz spaces of order α* for $\alpha > 0$

in order to measure the regularity of a function $H : \mathbb{S}^{n-1} \rightarrow \mathbb{C}$. Let us recall the definition and the basic properties of Lipschitz spaces.

Let $\theta \in [-\pi, \pi]$ and R_θ be a rotation of angle θ acting on \mathbb{S}^{n-1} . Let H be a function on \mathbb{S}^{n-1} . We denote by $D_\theta(H) = D_\theta^1(H)$ the function $H \circ R_\theta - H$, and for $n \geq 1$, we define inductively $D_\theta^n(H) = D_\theta \circ D_\theta^{n-1}(H)$.

Let $\alpha > 0$ and let $[\alpha]$ be its integer part. Denote by $\Lambda_\alpha(\mathbb{S}^{n-1})$ the space of (equivalence classes of) bounded measurable functions on \mathbb{S}^{n-1} for which there exists a constant $C > 0$, such that for any rotation R_θ of angle θ , one has

$$\|D_\theta^{[\alpha]+1}(H)\|_\infty \leq C|\theta|^\alpha. \quad (1.2)$$

One can define a norm on the space $\Lambda_\alpha(\mathbb{S}^{n-1})$, by setting $\|H\|_\alpha$ equal to the infimum of the L^∞ -norm of H and of the constants C with this property. With this norm, the space $\Lambda_\alpha(\mathbb{S}^{n-1})$ becomes a Banach space: it is called *the Lipschitz space of order α* .

If $\alpha \in]0, 1[$, the space $\Lambda_\alpha(\mathbb{S}^{n-1})$ is the classical Hölder space of order α , and the norm is equivalent to the classical Hölder norm. In fact, a smoothing argument shows that any function in $\Lambda_\alpha(\mathbb{S}^{n-1})$ is almost everywhere equal to a Hölder-continuous function of order α . Any function in $\Lambda_1(\mathbb{S}^{n-1})$ admits a continuous representative as well (see [14, page 331]).

If $\alpha > 1$, the following holds. Let X be any vector field on \mathbb{S}^{n-1} . Then, for $H \in \Lambda_\alpha(\mathbb{S}^{n-1})$, the Lie derivative $\mathcal{L}_X H$ is contained in $\Lambda_{\alpha-1}(\mathbb{S}^{n-1})$, with a norm bounded in terms of $\|H\|_\alpha$ and the norm of the vector field X . By induction, one can characterize, for $\alpha > 1$, $\alpha \notin \mathbb{N}$, the space $\Lambda_\alpha(\mathbb{S}^{n-1})$ as the space of functions which are $[\alpha]$ -times differentiable and which have all mixed derivatives of order $[\alpha] - 1$ in $\Lambda_{\alpha-[\alpha]+1}(\mathbb{S}^{n-1})$.

We will use the following important property of the Lipschitz spaces. For any integer $m \geq \alpha$, one can replace in the definition of $\Lambda_\alpha(\mathbb{S}^{n-1})$ the quantity $\|D_\theta^{[\alpha]+1}(H)\|_\infty$ by $\|D_\theta^{m+1}(H)\|_\infty$. The space defined in that way is equal to $\Lambda_\alpha(\mathbb{S}^{n-1})$ and the two norms are equivalent (see, e.g., [14, page 331]). In particular, for $\alpha \in]0, n-1[$, one can define the space $\Lambda_\alpha(\mathbb{S}^{n-1})$ as the space of functions $H \in L^\infty(\mathbb{S}^{n-1})$ such that $\|D_\theta^n(H)\|_\infty \leq C|\theta|^\alpha$, for some constant C independent of θ .

In the following, we denote by $\Lambda_\alpha^0(\mathbb{S}^{n-1})$ the space of functions $H \in \Lambda_\alpha(\mathbb{S}^{n-1})$ such that $\int_{\mathbb{S}^{n-1}} H \, d\sigma = 0$.

Given an integer $j \geq 0$, we denote by $\Delta_\sigma^j \Lambda_\alpha^0(\mathbb{S}^{n-1})$ the space of those distributions on \mathbb{S}^{n-1} which can be written as $\Delta_\sigma^j(H)$ for a function $H \in \Lambda_\alpha^0(\mathbb{S}^{n-1})$.

The paper is devoted to prove the following theorem, which describes the spaces \mathcal{E}_{λ_s} for $s \in \mathcal{H}$.

Theorem 1.3. Let $s = \delta + it \in \mathcal{H}$.

If n is odd, then \mathcal{P}_s induces an isomorphism between the Banach spaces $\Delta_\sigma^{(n-1)/2} \Lambda_\delta^0(\mathbb{S}^{n-1}) \oplus \mathbb{C} d\sigma$ and \mathcal{E}_{λ_s} ;

If n is even, then \mathcal{P}_s induces an isomorphism between the Banach spaces $\Delta_\sigma^{n/2} \Lambda_{\delta+1}^0(\mathbb{S}^{n-1}) \oplus \mathbb{C} d\sigma$ and \mathcal{E}_{λ_s} . \square

In this theorem, the λ_s -eigenfunctions which vanish at the origin are exactly the image of $\Delta_\sigma^{(n-1)/2} \Lambda_\delta^0(\mathbb{S}^{n-1})$ by the Poisson-Helgason transform. The image of the complex line $\mathbb{C} d\sigma$ is equal to the multiples of the spherical function h_s which equals 1 at the origin o .

Note that Theorem 1.3 does not include the bounded *harmonic* functions on \mathbb{H}^n ; they correspond to the case $s = n - 1$. The boundary values of those are characterized by the Fatou theorem: they are exactly the functions in $L^\infty(\mathbb{S}^{n-1})$. However, derivatives of order $n - 1$ of functions in $\Lambda^{n-1}(\mathbb{S}^{n-1})$ are not in $L^\infty(\mathbb{S}^{n-1})$, in general. Therefore, Theorem 1.3 cannot hold in the case $s = n - 1$.

The proof of Theorem 1.3 contains two parts. The first one amounts to show that \mathcal{P}^s restricted to $\Delta_\sigma^{(n-1)/2} \Lambda_\delta^0(\mathbb{S}^{n-1}) \oplus \mathbb{R} d\sigma$ or to $\Delta_\sigma^{n/2} \Lambda_{\delta+1}^0(\mathbb{S}^{n-1}) \oplus \mathbb{R} d\sigma$ according to the parity of n takes range in \mathcal{E}_{λ_s} and that it is a continuous operator. This is done in Section 2. If the real part of s is not of the form $n - 1 - 2j$ with $j \in \mathbb{N}$, the proof is a direct computation using easy estimates on the Poisson kernel. The other cases are obtained by interpolation of analytic families of operators.

The second part amounts to show that the restriction of \mathcal{P}^s is surjective. This is proven in two steps. First, we consider the case when $\delta = \Re s \geq (n - 1)/2$. Under this assumption, we assign in Section 4 to any function $f \in \mathcal{E}_{\lambda_s}$ a “boundary value”: it is a distribution T on \mathbb{S}^{n-1} which is contained in the expected space, that is, it satisfies $T = \Delta_\sigma^{(n-1)/2} H$ for some Lipschitz function H in $\Lambda_\delta(\mathbb{S}^{n-1})$ if n is odd or $T = \Delta_\sigma^{n/2} H$ for some Lipschitz function H in $\Lambda_{\delta+1}(\mathbb{S}^{n-1})$ if n is even. The proof relies on estimates of a certain differential operator \mathcal{D}_λ acting on functions on the real line; this operator is the radial part of the Laplace operator on \mathbb{H}^n , written in polar coordinates. These estimates are obtained in Section 3; we use in an essential way the assumption that $\delta \geq (n - 1)/2$.

In Section 5, one proves, under the assumption $\delta = \Re s \geq (n - 1)/2$, that the distribution T reproduces f , that is, that $\mathcal{P}^s(T) = f$. This shows that \mathcal{P}^s is onto under the assumptions of Theorem 1.3 and $\Re s \geq (n - 1)/2$.

In Section 6, we extend this to the case $\delta < (n - 1)/2$ by using a relation between the $(n - 1 - s)$ th power of the Poisson kernel and its s th power: they are related by a fractional integration (resp., derivation) operator which is studied in Proposition 6.1. We study some continuity properties of this operator acting on Lipschitz spaces

(Proposition 6.1(4)). This result is similar to the one for fractional integration (resp., derivation) operator acting on functions on \mathbb{R}^n , but as we did not find any reference, we propose an elementary proof of it.

Finally, since \mathcal{P}^s is continuous, onto, and injective, the isomorphism statement in Theorem 1.3 follows from the Banach closed graph theorem.

The computations in Section 3 to Section 5 use heavily estimates on the Green function and on the spherical function of the operator $\Delta + \lambda_s I$. These estimates are well known, but in order to make the paper self-contained, we give the proofs in an appendix.

In all the sequel, we assume that $n \geq 3$.

2 Continuity properties of the Poisson-Helgason transform \mathcal{P}^s

In order to prove the direct part of Theorem 1.3, we use the following estimate on the derivatives of the Poisson kernel.

Claim 2.1. Let $j \in \mathbb{N}$ and $s = \delta + it$ in \mathbb{C} . There exists a polynomial Q in one complex variable so that for any ζ and ξ in \mathbb{S}^{n-1} , for any point $x \in \mathbb{H}^n$ at distance r from o on the ray $[o\zeta[$, one has

$$|\Delta_\sigma^j \mathcal{P}^s(x, \xi)| \leq C(s) \frac{e^{-\delta r}}{(e^{-r} + |\zeta - \xi|)^{2\delta + 2j}}, \quad (2.1)$$

where $C(s) = |Q(s)|$. □

Proof. In the ball model, the Poisson kernel can be written as

$$\mathcal{P}(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^2}. \quad (2.2)$$

In Euclidean polar coordinates, one has $x = (\tanh(r/2))\zeta$. Therefore,

$$\mathcal{P}(x, \xi) = \frac{1 - \left(\tanh\left(\frac{r}{2}\right)\right)^2}{\left(\tanh\left(\frac{r}{2}\right)\right)^2 + 1 - 2 \tanh\left(\frac{r}{2}\right) \langle \zeta, \xi \rangle}. \quad (2.3)$$

The statement follows after differentiating this formula. ■

Proposition 2.2. Let $n \geq 3$. Let $s \in \mathcal{H}$.

If n is odd, then \mathcal{P}^s maps continuously $\Delta_\sigma^{(n-1)/2} \Lambda_\delta(\mathbb{S}^{n-1})$ to $L^\infty(\mathbb{H}^n)$. If n is even, then \mathcal{P}^s maps continuously $\Delta_\sigma^{n/2} \Lambda_{\delta+1}(\mathbb{S}^{n-1})$ to $L^\infty(\mathbb{H}^n)$. □

As it was said in Section 1, this result is false for $s = n - 1$ since, by the Fatou theorem, the Poisson-Helgason transform \mathcal{P}^{n-1} is an isomorphism from $L^\infty(\mathbb{S}^{n-1})$ to \mathcal{E}_0 .

Proof. Assume first that n is odd. Let $f \in \Lambda_\delta(\mathbb{S}^{n-1})$. We choose the integer $j \geq 1$ such that $n - 1 - \delta = 2j - \alpha$ with $\alpha \in]0, 2]$. Suppose also that $n - 1 - \delta$ is not an even integer, that is, that $\alpha \neq 2$.

The Green formula implies that for any $x \in \mathbb{H}^n$, one has

$$I := \mathcal{P}^s(\Delta_\sigma^{(n-1)/2} f)(x) = \langle \Delta_\sigma^j \mathcal{P}^s(x, \cdot), \Delta_\sigma^{(n-1)/2-j} f \rangle. \quad (2.4)$$

But the function $h = \Delta_\sigma^{(n-1)/2-j} f$ belongs to $\Lambda_\alpha(\mathbb{S}^{n-1})$ with norm bounded by $C\|f\|_\delta$.

As $j \geq 1$, for any $\zeta \in \mathbb{S}^{n-1}$, one has

$$I = \int_{\mathbb{S}^{n-1}} \Delta_\sigma^j \mathcal{P}^s(x, \cdot) h(\cdot) d\sigma = \int_{\mathbb{S}^{n-1}} \Delta_\sigma^j \mathcal{P}^s(x, \cdot) (h(\cdot) - h(\zeta)) d\sigma. \quad (2.5)$$

Let now ζ be the end of the ray ox on the sphere and denote by r the hyperbolic distance from x to o . For $\theta \in [0, \pi]$, denote by $B(\zeta, \theta) \subset \mathbb{S}^{n-1}$ the spherical ball of radius θ around ζ and set $\mathcal{P}(x, \theta) = \mathcal{P}(x, \xi)$ for any $\xi \in \partial B(\zeta, \theta)$.

Using geodesic polar coordinates on \mathbb{S}^{n-1} around the point ζ , this last integral also equals

$$\int_0^\pi \Delta_\sigma^j \mathcal{P}^s(x, \theta) \left(\int_{\partial B(\zeta, \theta)} (h(\cdot) - h(\zeta)) d\sigma_\theta \right) d\theta, \quad (2.6)$$

where $d\sigma_\theta$ is the Riemannian measure on the sphere $\partial B(\zeta, \theta)$ for the induced Riemannian metric. Denoting by ξ' the symmetric of $\xi \in \partial B(\zeta, \theta)$ with respect to the center ζ , one has

$$\left| \int_{\partial B(\zeta, \theta)} (h(\cdot) - h(\zeta)) d\sigma_\theta \right| = \frac{1}{2} \left| \int_{\partial B(\zeta, \theta)} (h(\xi) + h(\xi') - 2h(\zeta)) d\sigma_\theta(\xi) \right|. \quad (2.7)$$

Since $h \in \Lambda_\alpha(\mathbb{S}^{n-1})$ with $\alpha \in]0, 2]$, $|h(\xi) + h(\xi') - 2h(\zeta)| \leq \|h\|_\alpha |\xi - \zeta|^\alpha$. So, the left integral above is bounded by $C\|h\|_\alpha \theta^{n-2+\alpha}$.

Since $n - 2 + \alpha = \delta + 2j - 1$, it follows from Claim 2.1 that for any $x \in \mathbb{H}^n$,

$$|I| \leq \|h\|_\alpha \left(c_1 \int_0^{e^{-r}} \frac{e^{-\delta r}}{e^{-(2\delta+2j)r}} \theta^{\delta+2j-1} d\theta + c_2 \int_{e^{-r}}^\pi e^{-\delta r} \frac{\theta^{\delta+2j-1}}{\theta^{2\delta+2j}} d\theta \right). \quad (2.8)$$

Therefore, $|I| \leq C\|h\|_\alpha$ for a constant C which does not depend on h . So, $\|\mathcal{P}^s(\Delta_\sigma^{(n-1)/2} f)\|_\infty \leq C\|\Delta_\sigma^{(n-1)/2-j} f\|_\alpha \leq C\|f\|_\delta$. The result of the proposition is proved except when $n - 1 - \delta$ is an even integer.

To deal with the case $\alpha = 2$, that is, when $(n - 1) - \delta$ is even, we use interpolation of analytic families of operators. We give first a complement to the last proof. Let $\delta_0 = n - 1 - 2j$. Choose δ_2 in $]0, n - 1[$ with $\delta_0 < \delta_2 < \delta_0 + 2$. Let $s = \delta + it$, with $\delta \in]0, \delta_2]$.

Then, for $f \in \Lambda_{\delta_2}$, one has $\|\mathcal{P}^s(\Delta_\sigma^{(n-1)/2}f)\|_\infty \leq C'(s)\|f\|_{\delta_2}$ for a constant $C'(s)$ which is independent of f and is the absolute value of some polynomial R at point s . Indeed, writing $\delta_2 = n - 1 - 2j + \alpha_2$, the argument above gives, for $f \in \Lambda_{\delta_2}(\mathbb{S}^{n-1})$ and $h = \Delta_\sigma^{(n-1)/2-j}f$,

$$\begin{aligned} & |\mathcal{P}^s(\Delta_\sigma^{(n-1)/2}f)(x)| \\ & \leq C(s)\|h\|_{\alpha_2} \left(c_1 \int_0^{e^{-r}} \frac{e^{-\delta r}}{e^{-(2\delta+2j)r}} \theta^{\delta_2+2j-1} d\theta + c_2 \int_{e^{-r}}^\pi e^{-\delta r} \frac{\theta^{\delta_2+2j-1}}{\theta^{2\delta+2j}} d\theta \right), \end{aligned} \quad (2.9)$$

for $r = d(o, x)$. Therefore, if δ and δ_2 satisfy $\delta_2 < 2\delta$, one has

$$|\mathcal{P}^s(\Delta_\sigma^{(n-1)/2}f)(x)| \leq C'(s)\|f\|_{\delta_2} e^{-(\delta_2-\delta)r}, \quad (2.10)$$

where $C'(s)$ can be chosen to be the modulus of a polynomial R which does not vanish in the strip $\Re s \geq 0$.

Choose now δ_1 and δ_2 with $\delta_1 < \delta_0 < \delta_2$ and sufficiently close so that $\delta_2 < 2\delta_1$. Consider the Lipschitz spaces $\Lambda_{\delta_1}(\mathbb{S}^{n-1})$ and $\Lambda_{\delta_2}(\mathbb{S}^{n-1})$; it is known that the spaces $\Lambda_\delta(\mathbb{S}^{n-1})$ for $\delta \in [\delta_1, \delta_2]$ form an interpolating family [2]. On the closed strip $\mathcal{B} = \{s \mid \Re s \in [\delta_1, \delta_2]\}$, consider the analytic family of operators $T_s = (1/R(s))\mathcal{P}^s \circ \Delta_\sigma^{(n-1)/2}$. Those operators are defined on $\Lambda_{\delta_2}(\mathbb{S}^{n-1})$ and satisfy the following properties:

- (1) each operator T_s maps continuously $\Lambda_{\delta_2}(\mathbb{S}^{n-1})$ to the Banach (separable) space $\mathcal{C}^b(\mathbb{H}^n)$ of bounded continuous functions on \mathbb{H}^n (with the uniform norm);
- (2) for all $f \in \Lambda_{\delta_2}(\mathbb{S}^{n-1})$, for all $x \in \mathbb{H}^n$, the function $s \rightarrow T_s(f)(x)$ is continuous, analytic in the interior of \mathcal{B} and bounded over \mathcal{B} ;
- (3) for $f \in \Lambda_{\delta_2}(\mathbb{S}^{n-1})$, one has $\|T_{\delta_1+it}f\| \leq \|f\|_{\delta_1}$ and $\|T_{\delta_2+it}f\| \leq \|f\|_{\delta_2}$.

Then the hypotheses of the interpolation theorem (see, e.g., [3, Theorem 1]) are satisfied. This theorem gives that for $\Re s = \delta \in \mathcal{B}$, T_s is continuous from $\Lambda_\delta(\mathbb{S}^{n-1})$ to $\mathcal{C}^b(\mathbb{H}^n)$. Making $\delta = \delta_0$ finishes the proof of Proposition 2.2 for odd values of n .

The proof for even values of n is the same. ■

3 Estimates of the solutions of $\mathcal{D}_\lambda f = g$

In this section, we study the differential operator \mathcal{D}_λ which is defined as acting on functions on $]0, \infty[$ by

$$\mathcal{D}_\lambda = (\sinh r)^2 \left(\frac{\partial^2}{\partial r^2} + (n-1) \coth r \frac{\partial}{\partial r} + \lambda \text{Id} \right). \quad (3.1)$$

Recall that the hyperbolic Laplacian has the following form in polar hyperbolic coordinates (r, ζ) around \mathfrak{o} . For a C^2 function $u = u(r, \zeta) : \mathbb{H}^n \setminus \{\mathfrak{o}\} \rightarrow \mathbb{C}$, one has

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + (n-1) \coth r \frac{\partial u}{\partial r} + \frac{1}{(\sinh r)^2} \Delta_\sigma u(r, \cdot). \quad (3.2)$$

From this expression, the solutions of $\mathcal{D}_\lambda u = 0$ are precisely the radial λ -eigenfunctions. In Propositions 7.1 and 7.2, we will recall the behaviour near ∞ of the two basic solutions of the eigenfunction equations, that is, the spherical function h_s and the Green function g_s . The next result studies similarly the behaviour of a solution u of $\mathcal{D}_\lambda u = v$ in terms of the behaviour of v .

To simplify notations, we write $\lambda = \lambda_s = s(n-1-s)$.

Throughout this section, we make the assumption that $\delta = \Re s \geq (n-1)/2$.

Proposition 3.1. Let $r_0 > 0$ and let l be an integer such that $l \leq n-3-\delta$. Let v be a smooth function on $[r_0, \infty[$ such that $|v(r)| \leq Ce^{-lr}$ for $r \geq r_0$. Let $u : [r_0, \infty[\rightarrow \mathbb{C}$ be a smooth function such that $\mathcal{D}_\lambda u = v$. Then, if $l < n-3-\delta$, one has

$$|u(r)| \leq \kappa(C + |u(r_0)| + |u'(r_0)|)e^{-(l+2)r}. \quad (3.3)$$

If $l = n-3-\delta$, one has

$$|u(r)| \leq \kappa(C + |u(r_0)| + |u'(r_0)|)re^{-(n-1-\delta)r}. \quad (3.4)$$

In both cases, κ is a constant which depends only on r_0 and on s . □

Proof. The solution u of $\mathcal{D}_\lambda u = v$ is uniquely determined by its behaviour up to first order at r_0 . Let a and b in \mathbb{C} be such that $ag_s(r_0) + bh_s(r_0) = 0$ and $ag'_s(r_0) + bh'_s(r_0) = 1$. This solution (a, b) exists and is unique, since the determinant of this system equals the Wronskian of g_s and h_s at r_0 which does not vanish. Then, by subtracting from u the combination $ag_s + bh_s$, one reduces to the case when $u(r_0) = u'(r_0) = 0$. This linear combination grows less than $\kappa(|u(r_0)| + |u'(r_0)|)e^{-(n-1-\delta)r}$, where κ is a constant depending only on r_0 . Therefore, to prove Proposition 3.1, we can assume that $u(r_0) = u'(r_0) = 0$.

Let $U(r)$ be the vector with coordinates $u'(r)$ and $u(r)$ and let $V(r)$ be the vector with coordinates $v(r)/(\sinh r)^2$ and 0. Then

$$U'(r) = A(r)U(r) + V(r) \quad \text{on } [r_0, \infty[, \quad (3.5)$$

where

$$A(r) = \begin{pmatrix} -(n-1) \coth r & -\lambda \\ 1 & 0 \end{pmatrix}. \quad (3.6)$$

Let $M(r)$ be the solution of $M'(r) = -M(r)A(r)$ which vanishes at r_0 ; then

$$U(r) = \int_{r_0}^r M^{-1}(r)M(t)V(t)dt. \quad (3.7)$$

For any solution $Y(r)$ of the equation $Y'(r) = A(r)Y(r)$, the vector $M(r)Y(r)$ is constant. This holds in particular for the *spherical vector* $H_s(r)$ with coordinates $(h'_s(r), h_s(r))$ and for the *Green vector* $G_s(r) = (g'_s(r), g_s(r))$. Now, since the Wronskian $W(t)$ does not vanish, we may write $V(t)$ as a linear combination of these two vectors: $V(t) = \alpha(t)H_s(t) + \beta(t)G_s(t)$. It comes out that

$$U(r) = \left(\int_{r_0}^r \alpha(t)dt \right) H_s(r) + \left(\int_{r_0}^r \beta(t)dt \right) G_s(r), \quad (3.8)$$

where $\alpha(t)$ and $\beta(t)$ are given, respectively, by

$$\alpha(t) = c_{n-1}v(t)g_s(t)(\sinh t)^{n-3}, \quad \beta(t) = -c_{n-1}v(t)h_s(t)(\sinh t)^{n-3}. \quad (3.9)$$

Using the classical estimates on h_s and g_s recalled in the appendix, one obtains easily the required estimates of Proposition 3.1. \blacksquare

Corollary 3.2. Let $r_0 > 0$. Let $j \in \mathbb{N}, l \in \mathbb{Z}$ be such that $l + 2j \leq n - 1 - \delta$. Let v be a smooth function on $[r_0, \infty[$ such that $|v(r)| \leq Ce^{-lr}$. Assume that u is a smooth function on $[r_0, \infty[$ which solves the differential equation $\mathcal{D}_\lambda^j u = v$. Then if $l + 2j < n - 1 - \delta$, one has

$$|u(r)| \leq \kappa \left(C + \sum_{p=0}^{2j-1} |u^{(p)}(r_0)| \right) e^{-(l+2j)r}. \quad (3.10)$$

If $l + 2j = n - 1 - \delta$, one has

$$|u(r)| \leq \kappa \left(C + \sum_{p=0}^{2j-1} |u^{(p)}(r_0)| \right) re^{-(n-1-\delta)r}, \quad (3.11)$$

where the constant κ depends only on r_0 and on s . \square

Proof. By decreasing induction on m , $1 \leq m \leq j-1$, one proves, using Proposition 3.1, that

$$|\mathcal{D}^m u(r)| \leq \kappa_m \left(C + \sum_0^{2(j-m)-1} |u^{(p)}(r_0)| \right) e^{-(l+2(j-m))r} \quad (3.12)$$

for $r \geq r_0$, where the constant κ_m depends only on r_0 . For $m = 0$, it gives the result. Notice that one needs to use the degenerate case of Proposition 3.1 only at the last step of the induction and when $l + 2j = n - 1 - \delta$. \blacksquare

4 Construction of a distribution on \mathbb{S}^{n-1} associated to a bounded eigenfunction

Now, we begin the proof of the fact that \mathcal{P}^s is onto. Let $f \in \mathcal{E}_{\lambda_s}$. We are going to construct an explicit distribution T such that $f = \mathcal{P}^s(T)$. Notice first that we may assume that $f(o) = 0$, since up to replacing f by $f - f(o)h_s$, we get a bounded eigenfunction in \mathcal{E}_{λ_s} which has L^∞ -norm less than $2\|f\|_\infty$. In this section, we associate to f a distribution T_s on \mathbb{S}^{n-1} and study its Lipschitz regularity. In the next section, we will prove that it satisfies $\mathcal{P}^s(T_s) = f$.

As in the preceding section, we assume that $\mathfrak{A}s = \delta \in [(n-1)/2, n-1[$ and we write $\lambda = \lambda_s = s(n-1-s)$ to simplify notations. Consider for $r > 0$ the function

$$T(r, \xi) = (\sinh r)^{n-1} \left(f \frac{\partial g_s}{\partial r} - \frac{\partial f}{\partial r} g_s \right) (r, \xi), \quad (4.1)$$

where g_s is the Green function: notice that the Green function depends on s and not only on λ_s (cf. Section 6).

Proposition 4.1. Let $s = \delta + it$ with $\delta \in [(n-1)/2, n-1[$. Then the distributions $T(r, \cdot)$ converge to a distribution $T = T_s$ on \mathbb{S}^{n-1} . Furthermore, if $\delta = n-1-2j+\alpha$, with $\alpha \in]0, 2]$, there is a function $H \in \Lambda_\alpha^0(\mathbb{S}^{n-1})$ such that $T = \Delta_\sigma^j(H)$. This function H satisfies $\|H\|_\alpha \leq C\|f\|_\infty$ for a constant C which depends only on s . \square

Remark 4.2. As a corollary, we obtain that the limit distribution T can always be written as $T = \Delta_\sigma^{(n-1)/2} h$ with $h \in \Lambda_\delta^0(\mathbb{S}^{n-1})$ if n is odd or that $T = \Delta_\sigma^{n/2} h$ with $h \in \Lambda_{\delta+1}^0(\mathbb{S}^{n-1})$ if n is even. Indeed, since H has mean-value zero over \mathbb{S}^{n-1} , one may write it as $H = \Delta_\sigma^{(n-1)/2-j} h$ if n is odd and $H = \Delta_\sigma^{n/2-j} h$ if n is even. As powers of Laplacians give isomorphisms between Lipschitz spaces of different parameters (see [12]), we get that $h \in \Lambda_\delta^0(\mathbb{S}^{n-1})$ if n is odd and that $h \in \Lambda_{\delta+1}^0(\mathbb{S}^{n-1})$ if n is even since $H \in \Lambda_\alpha^0(\mathbb{S}^{n-1})$.

Proof. The following properties of $T(r, \cdot)$ are easy to establish.

- (1) $\int_{\mathbb{S}^{n-1}} T(r, \cdot) d\sigma = 0$: this comes from the Green formula since $f(o) = 0$ and since in the distributional sense, $\Delta g_s + \lambda_s g_s$ equals the Dirac mass at the origin δ_o .
- (2) $(\partial T / \partial r)(r, \xi) = -(\sinh r)^{n-3} g_s(r) \Delta_\sigma f(r, \xi)$: this comes by differentiating the expression of $T(r, \xi)$ and using the expression for the Laplacian in polar coordinates.

In order to illustrate the method, we consider first the easiest case $\delta = \mathfrak{R}s \in]n-3, n-2[$. By (1), there is a unique smooth function $\xi \rightarrow H(r, \xi)$ which has zero mean and such that $T(r, \cdot) = \Delta_\sigma H(r, \cdot)$. By (2), we have

$$\frac{\partial}{\partial r} H(r, \cdot) = -(\sinh r)^{n-3} g_s(r) f(r, \cdot). \quad (4.2)$$

Fix an $r_0 > 0$. Using the estimates on g_s and the boundedness of f , one obtains that there exists $C > 0$ which depends only on s and such that for any $r \geq r_0$,

$$|(\sinh r)^{n-3} g_s(r) f(r, \cdot)| \leq C \|f\|_\infty e^{(n-3-\delta)r}. \quad (4.3)$$

Therefore,

$$H(\cdot) = H(r_0, \cdot) - \int_{r_0}^{\infty} (\sinh t)^{n-3} g_s(t) f(t, \cdot) dt \quad (4.4)$$

defines a continuous function on \mathbb{S}^{n-1} . In the distributional sense, we have

$$\Delta_\sigma H(\cdot) = \lim_{r \rightarrow \infty} T(r, \cdot) =: T(\cdot). \quad (4.5)$$

We are going to prove that $H \in \Lambda_{\delta-(n-3)}^0(\mathbb{S}^{n-1})$. By assumption $\delta - (n-3) \in]0, 1[$ so it suffices to show that $\|D_\theta H\|_\infty \leq C \|f\|_\infty |\theta|^{\delta-(n-3)}$, for a constant C which only depends on s . Recall that D_θ is the first-order difference operator D_{R_θ} associated to the rotation $R_\theta \in O(n)$, that is, the operator which assigns to a function F the function $D_{R_\theta} F = F \circ R_\theta - F$.

We have

$$\begin{aligned} D_\theta H(\cdot) &= D_\theta H(r_0, \cdot) - \int_{r_0}^{\infty} (\sinh t)^{n-3} g_s(t) D_\theta f(t, \cdot) dt = I + II, \\ |II| &\leq \left| \int_{r_0}^{\infty} (\sinh t)^{n-3} g_s(t) D_\theta f(t, \cdot) dt \right| \\ &\leq \left| \int_{r_0}^{-\log|\theta|} (\sinh t)^{n-3} g_s(t) D_\theta f(t, \cdot) dt \right| + \left| \int_{-\log|\theta|}^{\infty} (\sinh t)^{n-3} g_s(t) D_\theta f(t, \cdot) dt \right| \\ &= (1) + (2). \end{aligned} \quad (4.6)$$

By the Harnack inequalities, since f is bounded, the gradient of f is bounded in terms of $\|f\|_\infty$ and s ; it follows that

$$|D_\theta f(r, \xi)| \leq C \|f\|_\infty |\theta| e^r \quad (4.7)$$

for some constant C which only depends on s . This gives

$$(1) \leq \|f\|_\infty \int_{r_0}^{-\log|\theta|} e^{t(n-2-\delta)} |\theta| dt \leq C \|f\|_\infty |\theta|^{\delta-(n-3)}. \quad (4.8)$$

Clearly, one has also, for any r , $\|D_\theta f(r, \cdot)\|_\infty \leq 2\|f\|_\infty$, so that one gets

$$(2) = \left| \int_{-\log|\theta|}^{\infty} (\sinh t)^{n-3} g_s(t) D_\theta f(t, \cdot) dt \right| \leq C \|f\|_\infty \int_{-\log|\theta|}^{\infty} e^{t(n-3-\delta)} dt \leq C \|f\|_\infty |\theta|^{\delta-(n-3)}. \quad (4.9)$$

So $|II| \leq C \|f\|_\infty |\theta|^{\delta-(n-3)}$. It remains to estimate the term $I = D_\theta H(r_0, \cdot)$. We have $\Delta_\sigma D_\theta H(r_0, \cdot) = D_\theta f(r_0, \cdot)$. By the Harnack inequality, and due to the compacity of \mathbb{S}^{n-1} , Δ_σ^{-1} is a continuous operator for the L^∞ -norm, so one has

$$|I| \leq \|D_\theta H(r_0, \cdot)\|_\infty \leq C \|f\|_\infty |\theta| \leq C \|f\|_\infty |\theta|^{\delta-(n-3)}, \quad (4.10)$$

for a constant C which only depends on s . It follows that $H \in \Lambda_{\delta-(n-3)}^0(\mathbb{S}^{n-1})$, with a norm smaller than $C \|f\|_\infty$.

We now show how this method can be extended to handle the general case. We are going to use the results of Section 3 on the operator \mathcal{D}_λ ; this is why we need to assume that $\Re s = \delta \in [(n-1)/2, n-1[$. Let us denote by j the integer such that $\delta = n-1-2j+\alpha$ with $\alpha \in]0, 2]$.

Since f is a λ -eigenfunction, it satisfies

$$\mathcal{D}_\lambda f(r, \xi) = -\Delta_\sigma f(r, \xi). \quad (4.11)$$

From the study of the Green function g_s in the appendix, there is an r_0 depending only on s such that $g_s(r)$ does not vanish for $r \geq r_0$. Therefore, by iterating the differential equation satisfied by $T(r, \cdot)$, we see that for $r \geq r_0$, one has

$$\mathcal{D}_\lambda^{j-1} \left(\frac{(\sinh r)^{3-n}}{g_s(r)} \frac{\partial T}{\partial r}(r, \cdot) \right) = (-1)^j \Delta_\sigma^j f(r, \cdot). \quad (4.12)$$

Since $T(r, \cdot)$ has zero mean, there exists a unique smooth function $H(r, \cdot)$ on \mathbb{S}^{n-1} which has zero mean and such that $T(r, \cdot) = \Delta_\sigma^j H(r, \cdot)$. This uniqueness property of $H(r, \cdot)$ implies that

$$\mathcal{D}_\lambda^{j-1} \left(\frac{(\sinh r)^{3-n}}{g_s(r)} \frac{\partial H}{\partial r}(r, \cdot) \right) = (-1)^j f(r, \cdot). \quad (4.13)$$

Since f is bounded and since $2(j-1) \leq n-1-\delta$, Corollary 3.2 can be applied with $l=0$. This gives

$$\begin{aligned} \left| \frac{(\sinh r)^{3-n}}{g_s(r)} \frac{\partial H}{\partial r}(r, \cdot) \right| &\leq C \left(\|f\|_\infty + \sum_0^{2j-3} \left| \frac{\partial^p H}{\partial r^p}(r_0, \cdot) \right| \right) e^{-2(j-1)r} \quad \text{if } \alpha < 2, \\ \left| \frac{(\sinh r)^{3-n}}{g_s(r)} \frac{\partial H}{\partial r}(r, \cdot) \right| &\leq C \left(\|f\|_\infty + \sum_0^{2j-3} \left| \frac{\partial^p H}{\partial r^p}(r_0, \cdot) \right| \right) r e^{-2(j-1)r} \quad \text{if } \alpha = 2, \end{aligned} \quad (4.14)$$

for a constant C , which only depends on s and on r_0 . One also has

$$\Delta_\sigma^j \left(\frac{\partial^p H}{\partial r^p}(r_0, \cdot) \right) = \frac{\partial^p T}{\partial r^p}(r_0, \cdot). \quad (4.15)$$

Going back to the definition of $T(r, \cdot)$ and using the Harnack inequality, one finds

$$\left\| \frac{\partial^p T}{\partial r^p}(r_0, \cdot) \right\|_\infty \leq C \|f\|_\infty, \quad (4.16)$$

for $p \leq 2j-3$, the constant C only depending on r_0 and j . Therefore, we have

$$\begin{aligned} \left\| \frac{\partial^p H}{\partial r^p}(r_0, \cdot) \right\|_\infty &\leq C \|f\|_\infty, \\ \left\| \frac{\partial H}{\partial r}(r, \cdot) \right\|_\infty &\leq C (\sinh r)^{n-3} |g_s(r)| e^{-2(j-1)r} \|f\|_\infty \leq C \|f\|_\infty e^{-\alpha r} \quad \text{if } \alpha < 2, \\ &\leq C (\sinh r)^{n-3} |g_s(r)| r e^{-2(j-1)r} \|f\|_\infty \leq C \|f\|_\infty r e^{-\alpha r} \quad \text{if } \alpha = 2, \end{aligned} \quad (4.17)$$

for a constant C which only depends on r_0 and on s . In any cases, since the last term is integrable over $[r_0, \infty[$, uniformly on \mathbb{S}^{n-1} , we may define a continuous function H on \mathbb{S}^{n-1} by setting

$$H(\cdot) = \lim_{r \rightarrow \infty} H(r, \cdot) = H(r_0, \cdot) + \int_{r_0}^\infty \frac{\partial H}{\partial t}(t, \cdot) dt. \quad (4.18)$$

In the distributional sense, this function satisfies

$$\Delta_\sigma^j H(\cdot) = \lim_{r \rightarrow \infty} \Delta_\sigma^j H(r, \cdot) =: T(\cdot). \quad (4.19)$$

We are now going to prove that $H \in \Lambda_\alpha^0(\mathbb{S}^{n-1})$. Let $R_\theta \in O(n)$ be a rotation of angle $\theta \in]-\pi, \pi[$. Since $\alpha \in]0, 2]$, we just need to prove that $\|D_\theta^3 H\|_\infty \leq C\|f\|_\infty |\theta|^\alpha$ for a constant C which does not depend on θ nor on R_θ (see [14, page 331]). We have

$$D_\theta^3 H(\cdot) = D_\theta^3 H(r_0, \cdot) + \int_{r_0}^\infty \frac{\partial D_\theta^3 H}{\partial t}(t, \cdot) dt. \quad (4.20)$$

For the term $D_\theta^3 H(r_0, \cdot)$, we note that $\Delta_\sigma^{j-1} D_\theta^3 H(r_0, \cdot) = D_\theta^3 T(r_0, \cdot)$. By the Harnack inequality, $\|D_\theta^3 T(r_0, \cdot)\|_\infty$ is bounded by $C\|f\|_\infty |\theta|^2$, for a constant C which depends only on r_0 . Therefore, since Δ_σ^{-1} is continuous for the L^∞ -norm, a similar bound holds also for $D_\theta^3 H(r_0, \cdot)$: its L^∞ -norm is smaller than $C\|f\|_\infty |\theta|^2$ which is smaller than $C\|f\|_\infty |\theta|^\alpha$.

It remains to deal with the term corresponding to the integral. Arguing as in the case $\delta \in]n-3, n-2[$, we split this integral into the sum of $I = \int_{r_0}^{-\log|\theta|}$ and $II = \int_{-\log|\theta|}^\infty$. We have

$$\mathcal{D}_\lambda^{j-1} \left(\frac{(\sinh r)^{3-n}}{g_s(r)} \frac{\partial D_\theta^3 H}{\partial r}(r, \cdot) \right) = (-1)^j D_\theta^3 f(r, \cdot). \quad (4.21)$$

Since $\|D_\theta^3 f(r, \cdot)\|_\infty \leq C\|f\|_\infty |\theta|^3 e^{3r}$, for a constant C independent of r and θ , we may apply Corollary 3.2 with $l = -3$: it comes out that for $\alpha \in]0, 2]$,

$$\left\| \frac{\partial D_\theta^3 H}{\partial r}(r, \cdot) \right\|_\infty \leq C\|f\|_\infty |\theta|^3 e^{(3-\alpha)r}, \quad (4.22)$$

so that

$$|I| \leq C\|f\|_\infty |\theta|^3 \int_{r_0}^{-\log|\theta|} e^{(3-\alpha)t} dt \leq C\|f\|_\infty |\theta|^\alpha. \quad (4.23)$$

Now, if $\alpha < 2$, we already proved during the construction of H that $\|(\partial H / \partial r)(r, \cdot)\|_\infty \leq C\|f\|_\infty e^{-\alpha r}$; we therefore have also $\|(\partial D_\theta^3 H / \partial r)(r, \cdot)\|_\infty \leq C\|f\|_\infty e^{-\alpha r}$. This gives

$$|II| \leq C\|f\|_\infty |\theta|^\alpha. \quad (4.24)$$

Adding the estimates above, we deduce that $\|D_\theta^3 H\|_\infty \leq C\|f\|_\infty |\theta|^\alpha$. This finishes the proof in the case $\alpha < 2$.

It remains to consider the case $\alpha = 2$. There we are in the critical case for applying Corollary 3.2 with $l = 0$ and we argue differently. By the Harnack inequality, we have $\|D_\theta f(r, \cdot)\|_\infty \leq C\|f\|_\infty |\theta| e^r$ for a constant C independent of f and θ . Applying Corollary 3.2 with $l = -1$, we deduce that $\|(\partial D_\theta H / \partial r)(r, \cdot)\|_\infty \leq C\|f\|_\infty |\theta| e^{-r}$. But, we have

$$\left\| \frac{\partial D_\theta^3 H}{\partial r}(r, \cdot) \right\|_\infty \leq 4 \left\| \frac{\partial D_\theta H}{\partial r}(r, \cdot) \right\|_\infty \leq C\|f\|_\infty |\theta| e^{-r}, \quad (4.25)$$

for a constant C which is independent of R_θ and f . It follows that

$$\left| \int_{-\log|\theta|}^{\infty} \frac{\partial D_\theta^3 H}{\partial t}(t, \cdot) dt \right| \leq C \|f\|_\infty |\theta|^2. \quad (4.26)$$

It ends the proof of the case $\alpha = 2$ and of Proposition 4.1 also. \blacksquare

5 Proof of Theorem 1.3 when $\delta \geq (n-1)/2$

Let $f \in \mathcal{E}_{\lambda_s}$. By a theorem of Helgason and Minemura ([7, 9], see also [1]), any λ_s -eigenfunction can be written as $\mathcal{P}^s(D)$, where D is a hyperfunction on \mathbb{S}^{n-1} . Also, T is unique except when $s \in -\mathbb{N}^*$ [9].

To prove Theorem 1.3 when $\Re s \geq (n-1)/2$, we show that the distribution T we constructed in the previous section is equal to D , that is, that one has $f = \mathcal{P}^s(T)$. Theorem 1.3 will then follow from Proposition 4.1. We suppose first that $f = \mathcal{P}^s(\phi)$, for a C^1 -function ϕ on \mathbb{S}^{n-1} . By Proposition 2.2, f is bounded and we can consider the corresponding distribution T_s . We need to show that $T_s = D$. To simplify the notations, we denote T_s by T .

Recall that the distribution T is the limit as $r \rightarrow \infty$ of the distributions

$$T(r, \cdot) = (\sinh r)^{n-1} \left(\frac{\partial f(r, \cdot)}{\partial r} g_s - \frac{\partial g_s}{\partial r} f(r, \cdot) \right). \quad (5.1)$$

With these notations, we have the following result.

Proposition 5.1. Let ϕ be a C^1 function on \mathbb{S}^{n-1} and $f = \mathcal{P}^s(\phi)$. Then, one has

$$\lim_{r \rightarrow \infty} \langle T(r, \cdot), \mathcal{P}^s(z, \cdot) \rangle = f(z). \quad (5.2)$$

\square

Proof. We begin by studying the kernel function $K_r^{(s)}(\zeta, \xi)$ defined by

$$K_r^{(s)}(\zeta, \xi) = (\sinh r)^{n-1} \left[\frac{\partial \mathcal{P}^s}{\partial r}(r\zeta, \xi) g_s(r) - \mathcal{P}^s(r\zeta, \xi) \frac{\partial g_s}{\partial r}(r) \right]. \quad (5.3)$$

\blacksquare

Lemma 5.2. The kernel $K_r^{(s)}(\zeta, \xi)$ has the following two properties:

- (1) for any $\xi \in \mathbb{S}^{n-1}$, one has $\int_{\mathbb{S}^{n-1}} K_r^{(s)}(\zeta, \xi) d\sigma(\zeta) = 1$;
- (2) for $\Re(s) = \delta \geq (n-1)/2$, and $r_0 > 0$, there is a constant $C > 0$ such that for all $r \geq r_0$, for all ζ, ξ in \mathbb{S}^{n-1} ,

$$|K_r^{(s)}(\zeta, \xi)| \leq C \frac{e^{(n-1-2\delta)r}}{(e^{-r} + |\zeta - \xi|)^{2\delta}}, \quad |K_r^{(s)}(\zeta, \xi)| \leq C \frac{e^{r(n-3-2\delta)}}{|\zeta - \xi|^{2\delta+2}}. \quad (5.4)$$

\square

Proof. Property (1) follows from the Green formula applied to the functions $P^s(\cdot, \xi)$ and g_s and from the fact that $P^s(o, \xi) = 1$ for all $\xi \in \mathbb{S}^{n-1}$.

To prove (2), we assume first that $\delta > (n-1)/2$. For any $r \geq r_0 > 0$, we know from the appendix that $|g_s(r)| \leq Ce^{-\delta r}$ for a constant C which is uniform as s remains in a compact of the set $\{s \mid \mathfrak{R}(s) \in [(n-1)/2, n-1]\}$. By the Harnack inequality, it follows that $|\partial g_s(r)/\partial r| \leq Ce^{-\delta r}$, where the constant C is uniform as s remains in a compact set. This leads first to the inequality

$$|K_r^{(s)}(\zeta, \xi)| \leq C \frac{e^{(n-1-2\delta)r}}{(e^{-r} + |\zeta - \xi|)^{2\delta}}, \quad (5.5)$$

where the constant C is independent of $r \geq r_0$ and of s as long as it stays in a compact set. In particular, for all $\zeta \neq \xi$, $K_r^{(s)}(\zeta, \xi) \rightarrow 0$ as $r \rightarrow \infty$ when $\delta > (n-1)/2$. The derivative with respect to r of $K_r^{(s)}(\zeta, \xi)$ can be computed as we did in the previous section for the construction of the distribution T . This gives

$$\frac{\partial K_r^{(s)}(\zeta, \xi)}{\partial r} = (\sinh r)^{n-3} g_s(r) \Delta_\sigma P^s(r\zeta, \xi). \quad (5.6)$$

Using the same estimates as above on g_s and its derivative, and on $\Delta_\sigma P^s$ (cf. Claim 2.1), it comes out that

$$\left| \frac{\partial K_r^{(s)}(\zeta, \xi)}{\partial r} \right| \leq C \frac{e^{r(n-3-2\delta)}}{(e^{-r} + |\zeta - \xi|)^{2\delta+2}}. \quad (5.7)$$

Integrating with respect to r and using that $K_t(\zeta, \xi) \rightarrow 0$ as $t \rightarrow \infty$, it gives that

$$|K_r^{(s)}(\zeta, \xi)| \leq C \frac{e^{(n-3-2\delta)r}}{|\zeta - \xi|^{2\delta+2}}, \quad (5.8)$$

for a constant C which does not depend on s as long as it stays in a compact set of $\mathfrak{R}s \geq (n-1)/2$.

Since the kernel $K_r^{(s)}$ is continuous with respect to s , the last estimate holds also for any s with $\mathfrak{R}s = (n-1)/2$ (for a constant which depends on r_0 and on s): it suffices to approximate $s = (n-1)/2 + it$ by $s_k + it$ for a sequence s_k which tends to $(n-1)/2$ from above. \blacksquare

Proof of Proposition 5.1. As $f = \mathcal{P}^s(\phi)$, we have

$$T(r, \zeta) = \int_{\mathbb{S}^{n-1}} K_r^{(s)}(\zeta, \xi) \phi(\xi) d\sigma(\xi). \quad (5.9)$$

So, it suffices to show that if ϕ is a C^1 -function, one has for all ζ, ξ in \mathbb{S}^{n-1} that

$$\int_{\mathbb{S}^{n-1}} K_r^{(s)}(\zeta, \xi)(\phi(\xi) - \phi(\zeta)) d\sigma(\xi) \longrightarrow 0 \quad (5.10)$$

as $r \rightarrow \infty$. Since ϕ is smooth,

$$\left| \int_{\mathbb{S}^{n-1}} K_r^{(s)}(\zeta, \xi)(\phi(\xi) - \phi(\zeta)) d\sigma(\xi) \right| \leq \|\phi\|_{C^1} \int |K_r^{(s)}(\zeta, \xi)| |\zeta - \xi| d\sigma(\xi). \quad (5.11)$$

We split this integral as the sum of the integral over the domains $|\zeta - \xi| \leq e^{-r}$ and $|\zeta - \xi| \geq e^{-r}$. Using the first estimate in (2) of the preceding lemma on the domain $|\zeta - \xi| \leq e^{-r}$ and the second on the domain $|\zeta - \xi| \geq e^{-r}$, one finds

$$\int_{\mathbb{S}^{n-1}} |K_r^{(s)}(\zeta, \xi)| |\zeta - \xi| d\sigma(\xi) \leq C e^{-r}. \quad (5.12)$$

This ends the proof of Proposition 5.1. ■

It remains to deal with general f . Our aim is to show that $\mathcal{P}^s(\Gamma) = f$, where Γ is the distribution that was constructed in Section 3. To that purpose, we use the characterization of eigenfunctions f which are the Poisson-Helgason transforms of a distribution as those which grow at most exponentially with the hyperbolic distance [15]: there are constants a and b such that

$$|f(x)| \leq a e^{d(o, x)}. \quad (5.13)$$

In particular if $f \in \mathcal{E}_{\lambda_s}$, then $f = \mathcal{P}^s(D)$ for a distribution D on $C^k(\mathbb{S}^{n-1})$.

Now, let $\epsilon > 0$; choose a smooth positive function ϕ_ϵ supported in the ϵ -neighborhood of $\text{Id} \in O(n)$ and with $\int \phi_\epsilon(g) dg = 1$. For $g \in O(n)$, denote by $g^*(f)$ the function $f \circ g^{-1}$; for a distribution D , denote also by $g^*(D)$ the distribution which is defined by duality. Then the function $f_\epsilon = \int g^*(f) \phi_\epsilon(g) dg$ belongs to \mathcal{E}_{λ_s} with L^∞ -norm smaller than $\|f\|_\infty$. One has $f_\epsilon = \mathcal{P}^s(D_\epsilon)$, where $D_\epsilon = \int g^*(D) \phi_\epsilon(g) dg$. Since D is a distribution and ϕ_ϵ is smooth, the distribution D_ϵ is a function at least C^1 . Applying Proposition 5.1, we obtain that D_ϵ equals the distribution T_ϵ that we had associated in the previous section to f_ϵ . Therefore, if we assume for instance that n is odd, we may write $D_\epsilon = \Delta_\sigma^{(n-1)/2} h_\epsilon$, where $h_\epsilon \in \Lambda_\delta^0(\mathbb{S}^{n-1})$ has a norm bounded by $C\|f_\epsilon\|_\infty \leq C\|f\|_\infty$.

By the Ascoli theorem, one can assume (up to taking a subsequence) that the sequence (h_ϵ) converges to h in $\Lambda_\delta(\mathbb{S}^{n-1})$. One has in the distributional sense $D_\epsilon = \Delta_\sigma^{(n-1)/2} h_\epsilon \rightarrow \Delta_\sigma^{(n-1)/2} h$ as $\epsilon \rightarrow 0$. Since $f_\epsilon(z) \rightarrow f(z)$, one has, for all z , $\mathcal{P}^s(D_\epsilon)(z) \rightarrow \mathcal{P}^s(D)(z)$.

Therefore, $f = \mathcal{P}^s(\Delta_\sigma^{(n-1)/2} \mathfrak{h}) = \mathcal{P}^s(D)$. By the unicity part in the Helgason-Minemura theorem, $D = \Delta_\sigma^{(n-1)/2} \mathfrak{h}$. This proves that \mathcal{P}^s is onto, when restricted to the space $\Delta_\sigma^{(n-1)/2} \Lambda_\delta(\mathbb{S}^{n-1})$.

It follows now from the Banach closed graph theorem that \mathcal{P}^s is an isomorphism between $\Delta_\sigma^{(n-1)/2} \Lambda_\delta^0(\mathbb{S}^{n-1}) \oplus \mathbb{C} d\sigma$ and \mathcal{E}_{λ_s} . This proves Theorem 1.3 for $\Re s \geq (n-1)/2$, when n is odd.

The proof when n is even is exactly the same.

This ends the proof of Theorem 1.3 in the case $\Re s \geq (n-1)/2$.

6 Proof of Theorem 1.3 when $\Re s < (n-1)/2$

The proof of Theorem 1.3 in the case $\Re s < (n-1)/2$ is not direct; we did not succeed in constructing the distribution \mathbb{T} as we did in Section 4 for $\Re s \geq (n-1)/2$. To deal with that case, we use the result for $\Re s \geq (n-1)/2$, which we have just proven. We also use an important identity which connects the s -power $\mathcal{P}^s(z, \cdot)$ of the Poisson kernel to its $(n-1-s)$ -power $\mathcal{P}^{n-1-s}(z, \cdot)$. This relation is classical, but we explain it again here in order to make the paper self-contained.

Let $z \in \mathbb{H}^n$; let γ be an isometry of \mathbb{H}^n such that $z = \gamma^{-1}(0)$. Then $|\gamma'(\xi)| = P(z, \xi)$, where $|\gamma'(\xi)|$ is the conformal factor of the derivative of γ at the point $\xi \in \mathbb{S}^{n-1}$. For two points $\xi, \xi' \in \mathbb{S}^{n-1}$, denote their chordal distance by $|\xi' - \xi| = 2 \sin(\theta/2)$ for θ the spherical distance between ξ and ξ' . One has the “intermediate value formula” (cf. [13])

$$|\gamma(\xi') - \gamma(\xi)|^2 = |\gamma'(\xi')| |\gamma'(\xi)| |\xi' - \xi|^2. \quad (6.1)$$

Assume now that $s \in \mathbb{C}$ is such that $\Re s < (n-1)/2$. Then, one has

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} \frac{|\gamma'(\xi')|^{n-1-s}}{|\xi' - \xi|^{2s}} d\sigma(\xi') &= \int_{\mathbb{S}^{n-1}} \frac{|\gamma'(\xi')|^{n-1}}{|\gamma'(\xi')|^s |\xi' - \xi|^{2s}} d\sigma(\xi') \\ &= |\gamma'(\xi)|^s \int_{\mathbb{S}^{n-1}} \frac{|\gamma'(\xi')|^{n-1}}{|\gamma(\xi') - \gamma(\xi)|^{2s}} d\sigma(\xi') \\ &= \mathcal{C}(s) |\gamma'(\xi)|^s, \end{aligned} \quad (6.2)$$

where $\mathcal{C}(s) = \int_{\mathbb{S}^{n-1}} (1/|\xi' - \xi|^{2s}) d\sigma(\xi')$. An easy computation gives that \mathcal{C} satisfies $s(n-2-s)\mathcal{C}(s) + 2s(2s+3-n)\mathcal{C}(s+1) = 0$. It follows that \mathcal{C} extends analytically over \mathbb{C} to a meromorphic function. In fact, $\mathcal{C}(s) = 2^{n-1-2s} \pi^{(n-1)/2} (\Gamma((n-1)/2-s)/\Gamma(n-1-s))$, where $\Gamma(\cdot)$ stands for the Gamma function (analytic on $\mathbb{C} \setminus (-\mathbb{N})$). In particular, the function \mathcal{C} does not vanish in \mathcal{H} , and its poles in this strip are simples and of the form $(n-1)/2 + k$, $k \in \mathbb{N}$.

For $\Re s < (n-1)/2$, consider the operator \mathcal{J}_s which is defined as acting on functions on \mathbb{S}^{n-1} by

$$\mathcal{J}_s(f)(\xi) = \int_{\mathbb{S}^{n-1}} \frac{f(\xi')}{|\xi' - \xi|^{2s}} d\sigma(\xi'). \quad (6.3)$$

In this range of values of s , since the kernel $1/|\xi' - \xi|^{2s}$ is integrable, \mathcal{J}_s is a bounded operator from $L^\infty(\mathbb{S}^{n-1})$ to itself and it maps also $C^\infty(\mathbb{S}^{n-1})$ to itself. In order to give a meaning to this operator for other values of s , we show that it satisfies a certain functional equation. For this, we compute first the (spherical) Laplacian of $1/|\xi' - \xi|^{2s}$ with respect to the variable ξ' . Recall that the spherical Laplacian of a function $u = u(\xi')$ on \mathbb{S}^{n-1} which only depends on the distance $\theta = d(\xi, \xi')$ to a point ξ is

$$\Delta_\sigma u = u''(\theta) + (n-2) \cot \theta u'(\theta). \quad (6.4)$$

Thus,

$$\Delta_\sigma \left(\frac{1}{|\xi' - \xi|^{2s}} \right) = s(n-2-s) \frac{1}{|\xi' - \xi|^{2s}} + 2s(2s+3-n) \frac{1}{|\xi' - \xi|^{2s+2}}. \quad (6.5)$$

So assuming that $2\Re s + 2 < n-1$ and $f \in C^2(\mathbb{S}^{n-1})$, we find, by integrating with respect to ξ' over \mathbb{S}^{n-1} and using the Green formula,

$$\mathcal{J}_s(\Delta_\sigma f) = s(n-2-s)\mathcal{J}_s(f) + 2s(n-3-2s)\mathcal{J}_{s+1}(f). \quad (6.6)$$

This functional equation allows to extend analytically $s \rightarrow \mathcal{J}_s(f)$ for any C^∞ -function f to a meromorphic function. As for $\mathcal{C}(s)$, these poles are at most simple and are located in $\{(n-1)/2 + \mathbb{N}\}$. Therefore, if one defines for a C^∞ -function f , $\mathcal{K}_s(f) = (1/C(s))\mathcal{J}_s(f)$, then $s \rightarrow \mathcal{K}_s(f)$ is holomorphic over the half-plane $\Re s < n-1$. For $s+1$ in this half-plane, it satisfies also the functional equation

$$\mathcal{K}_s(\Delta_\sigma f) = s(n-2-s)[\mathcal{K}_s(f) + \mathcal{K}_{s+1}(f)]. \quad (6.7)$$

We summarize the properties of the operator \mathcal{K}_s that we need in the following proposition.

Proposition 6.1. There exists a family of operators (\mathcal{K}_s) for $s \in \mathcal{H}$ which depends holomorphically on the parameter $s \in \{\Re s < n-1\}$, which extends the family of operators (\mathcal{K}_s) for $\{s; \Re s < (n-1)/2\}$ and which has the following properties.

(1) Each operator \mathcal{K}_s maps continuously $C^\infty(\mathbb{S}^{n-1})$ to itself and commutes with the spherical Laplacian Δ_σ . Furthermore, one has, if $(s+1) \in \mathcal{H}$,

$$\Delta_\sigma(\mathcal{K}_s(f)) = \mathcal{K}_s(\Delta_\sigma(f)) = s(n-2-s)[\mathcal{K}_s(f) + \mathcal{K}_{s+1}(f)]. \quad (6.8)$$

(2) The operator \mathcal{K}_s connects the $(n-1-s)$ -power and the s -power of the Poisson kernel: for all $z \in \mathbb{H}^n$, one has

$$\mathcal{K}_s(P^{n-1-s}(z, \cdot)) = P^s(z, \cdot). \quad (6.9)$$

(3) For any $s \in \mathcal{H}$, the operators \mathcal{K}_s and \mathcal{K}_{n-1-s} are inverse to each other.

(4) Let $s \in \mathcal{H}$ with $\Re s \neq (n-1)/2$. Let $\beta > 0$ be such that $\beta + n - 1 - 2\Re s > 0$. Then \mathcal{K}_s maps continuously $\Lambda_\beta(\mathbb{S}^{n-1})$ to $\Lambda_{\beta+n-1-2\Re s}(\mathbb{S}^{n-1})$. \square

Proof. For $\Re s < (n-1)/2$, the operator \mathcal{K}_s commutes with the spherical Laplacian Δ_σ ; this comes from the Green formula and the fact that $\Delta_{\sigma, \xi}(1/|\xi' - \xi|^{2s}) = \Delta_{\sigma, \xi'}(1/|\xi' - \xi|^{2s})$. By analytic continuation, the same holds also for $\Re s \geq (n-1)/2$. The functional equation for \mathcal{K}_s is a consequence of the equation for \mathcal{J}_s .

It implies that the operator \mathcal{K}_s maps continuously the space of C^∞ -functions on \mathbb{S}^{n-1} to itself, since it does so for $\Re s \leq (n-1)/2$.

Statement (2) has already been proven for s with $2\Re s < n-1$. It extends by analytic continuation to any value of $s \in \mathcal{H}$.

We now prove part (3) of the proposition. First from (2), we get that $\mathcal{K}_s \circ \mathcal{K}_{n-1-s}(P^s(z, \cdot)) = P^s(z, \cdot)$. We noticed already that for $s \in \mathcal{H}$, the distribution representing an $s(n-1-s)$ -eigenfunction is unique. This uniqueness property implies in particular that for any $k \geq 0$, the functions $P^s(z, \cdot)$ generate a dense linear subspace of $C^k(\mathbb{S}^{n-1})$ as z describes \mathbb{H}^n . It follows that for $s \in \mathcal{H}$, the operator $\mathcal{K}_s \circ \mathcal{K}_{n-1-s}$ is the identity.

In order to prove (4), we begin with the following result. \blacksquare

Lemma 6.2. Let $s \in \mathcal{H}$ with $\Re s < (n-1)/2$ and let $\beta > 0$. Then \mathcal{J}_s maps continuously $\Lambda_\beta(\mathbb{S}^{n-1})$ to $\Lambda_{\beta+n-1-2\Re s}(\mathbb{S}^{n-1})$. \square

Proof. Choose integers p and q where $p > [\beta] + 1$, and where $q > n-1$. Let $f \in \Lambda_\beta(\mathbb{S}^{n-1})$. For any rotation of \mathbb{S}^{n-1} of angle $\theta \in [-\pi, \pi]$, one has $\|D_\theta^p(f)\|_\infty \leq C\|f\|_\beta|\theta|^\beta$, for a constant C which does not depend on f nor on the rotation. Set $m = p + q$. Since $m > \beta + n - 1 - 2\Re s$, it suffices to check that $\|D_\theta^m(\mathcal{J}_s(f))\|_\infty \leq C\|f\|_\beta|\theta|^{\beta+n-1-2\Re s}$, for a constant C which has the same properties. We observe that \mathcal{J}_s commutes with the difference operator D_θ .

Therefore, one has

$$D_\theta^m(\mathcal{J}_s(f))(\xi) = D_\theta^q(\mathcal{J}_s(D_\theta^p(f))) (\xi) = \int_{\mathbb{S}^{n-1}} D_\theta^p f(\xi') D_\theta^q \left(\frac{1}{|\xi' - \xi|^{2s}} \right) d\sigma(\xi'), \quad (6.10)$$

where in the last integrand, the difference operator D_θ^q acts on the variable ξ . We split the last integral as the sum of two terms, I and II, where I is the integral over the domain $|\xi' - \xi| \leq 2m|\theta|$ and where II is the integral over $|\xi' - \xi| \geq 2m|\theta|$.

First, we have

$$|I| \leq \|f\|_\beta |\theta|^\beta 2^m \int_{|\xi' - \xi| \leq 3m|\theta|} \frac{1}{|\xi' - \xi|^{2\Re s}} d\sigma(\xi'). \quad (6.11)$$

Therefore, since $2\Re s < n - 1$, $|I| \leq C\|f\|_\beta |\theta|^{\beta+n-1-2s}$, for a constant C which is independent of f .

By the Taylor formula, we have for $|\xi' - \xi| \geq 2m\theta$,

$$\left| D_\theta^q \left(\frac{1}{|\xi' - \xi|^{2s}} \right) \right| \leq C' |\theta|^q \frac{1}{|\xi' - \xi|^{2\Re s + q}}. \quad (6.12)$$

Therefore, we obtain

$$|II| \leq C|\theta|^{\beta+q} \int_{|\xi' - \xi| \geq 2m\theta} \frac{1}{|\xi' - \xi|^{2\Re s + q}} d\sigma(\xi'), \quad (6.13)$$

for a constant C which is independent of f . Since $2\Re s + q > n - 1$, the integral is $O(|\theta|^{n-1-2\Re s-q})$. This implies the lemma. \blacksquare

We apply now Lemma 6.2 to the proof of Proposition 6.1(4). As $\mathcal{C}(s)$ has no poles when $\Re s < (n-1)/2$, the continuity property of \mathcal{K}_s follows from the continuity property of \mathcal{J}_s . Let $s \in \mathcal{H}$ with $\Re s > (n-1)/2$. Then, since $n \geq 3$, there is an integer k such that $s - k \in \mathcal{H}$, with $\Re(s - k) < (n-1)/2$. From the functional equation satisfied by \mathcal{K}_s , and using that \mathcal{K}_s commutes with Δ_σ , we know that $\mathcal{K}_s(f)$ is a linear combination of $\mathcal{K}_{s-1}(f)$ and of $\Delta_\sigma \mathcal{K}_{s-1}(f)$. By induction, it comes out that $\mathcal{K}_s(f)$ is a linear combination of the functions $\Delta_\sigma^j(\mathcal{K}_{s-k}(f))$ for $j \in \{0, 1, \dots, k\}$; this is also a linear combination of the functions $\Delta_\sigma^j(\mathcal{J}_{s-k}(f))$. Suppose now that $f \in \Lambda_\beta(\mathbb{S}^{n-1})$, with $\beta + n - 1 - 2\Re s > 0$. Then by Lemma 6.2, $\mathcal{J}_{s-k}(f) \in \Lambda_{\beta+n-1-2(\Re s-k)}(\mathbb{S}^{n-1})$ with norm controlled by the Λ_β -norm of f . Since $\beta + n - 1 - 2\Re s > 0$, we deduce that $\Delta_\sigma^k(\mathcal{J}_{s-k}(f)) \in \Lambda_{\beta+n-1-2\Re s}(\mathbb{S}^{n-1})$; the same holds a fortiori for $\Delta_\sigma^j(\mathcal{J}_{s-k}(f))$ when $j = 0, \dots, k-1$. This gives that \mathcal{K}_s maps continuously $\Lambda_\beta(\mathbb{S}^{n-1})$ to $\Lambda_{\beta+n-1-2\Re s}(\mathbb{S}^{n-1})$.

We are now in position to complete the proof of Theorem 1.3 when $2\Re s < n - 1$. Let $f \in \mathcal{E}_{\lambda_s}$ with $f(o) = 0$. By [15], there exists a distribution T so that $f(z) = \langle T, P^s(z, \cdot) \rangle$;

since $f(o) = 0$ this function satisfies $\langle T, \mathbb{1} \rangle = 0$. By Proposition 6.1, we have

$$f(z) = \langle T, \mathcal{K}_s(P^{n-1-s}(z, \cdot)) \rangle = \langle \mathcal{K}_s(T), P^{n-1-s}(z, \cdot) \rangle. \quad (6.14)$$

Suppose that the dimension n is odd. As $\Re(n-1-s) > (n-1)/2$, we know from Section 4 that $\mathcal{K}_s(T) = \Delta_\sigma^{(n-1)/2}(H)$ with $H \in \Lambda_{n-1-\delta}^0(\mathbb{S}^{n-1})$. By Proposition 6.1(3), $T = \Delta_\sigma^{(n-1)/2}(\mathcal{K}_{n-1-s}(H))$. By Proposition 6.1(4) and since $\delta > 0$, $\mathcal{K}_{n-1-s}(H) \in \Lambda_\delta^0(\mathbb{S}^{n-1})$. This gives Theorem 1.3 when $2\Re s < n-1$ and n is odd. The proof when n is even is exactly the same.

7 Radial λ -eigenfunctions

In this section, we review some properties of the radial eigenfunctions of the Laplacian. We write $\lambda = \lambda_s = s(n-1-s)$. A function $h : \mathbb{H}^n \rightarrow \mathbb{C}$ is *radial*, if it depends only on the distance to $o : h(x) = \phi(d(o, x))$. A radial function u is a λ -eigenfunction if and only if

$$u''(r) + (n-1) \coth r \cdot u'(r) + \lambda u(r) = 0. \quad (7.1)$$

The change of variable $r = (-\sinh r)^2$ shows that u is a radial λ -eigenfunction if and only if the function f defined by $u(r) = f((-\sinh r)^2)$ satisfies

$$z(1-z)f''(z) + \left(\frac{n}{2} - \frac{n+1}{2}z\right)f'(z) - \frac{\lambda}{4}f(z) = 0, \quad (7.2)$$

(cf. [5, page 135]), that is, the *hypergeometric equation* $E(a, b; c)$:

$$z(1-z)f''(z) + (c - (a+b+1)z)f'(z) - abf(z) = 0, \quad (7.3)$$

for the parameters $c = n/2$, $a = (n-1-s)/2$, and $b = s/2$ (cf. [4, ?]). One particular solution of the equation $E(a, b; c)$ is the *hypergeometric function*

$$F(a, b; c; z) = \sum_{k=0}^{k=\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k. \quad (7.4)$$

In the above formula, $(a)_k$ is defined by setting $(a)_0 = 1$ and for $k \geq 1$, $(a)_k = a(a+1)\dots(a+k-1)$. For any a, b in \mathbb{C} , and for any $c \notin -\mathbb{N}$, this series converges uniformly on the unit disc. It can be extended to an analytic function on each simply connected open set which is disjoint from the poles of the equation, in particular on the complement of the half-line $[1, \infty[$.

We introduce two particular radial λ -eigenfunctions, the spherical function and the Green function.

The *spherical function*. The space of solutions of the equation for radial eigenfunction is two-dimensional. An inspection of the equation shows that up to scalar multiple, there is only one radial λ_s -eigenfunction which is defined on \mathbb{H}^n and smooth at the origin o . The spherical function h_s is defined as the solution which equals 1 at o . It can be expressed in terms of the hypergeometric functions. Indeed, one has

$$h_s(r) = F\left(\frac{n-1-s}{2}, \frac{s}{2}; \frac{n}{2}; -(\sinh r)^2\right), \quad (7.5)$$

since the right term is a radial λ_s -eigenfunction which takes the value 1 at the origin o .

As a function of the parameter, $s \rightarrow h_s(r)$ is a holomorphic function.

In order to find the behaviour of $h_s(r)$ for large values of r , we use the following relation which is valid for z in the half-plane $\Re(z) < 1/2$ (cf. [4], page 64):

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}\right). \quad (7.6)$$

It comes out that

$$h_s(r) = (\cosh r)^{n-1-s} F\left(\frac{n-1-s}{2}, \frac{n-s}{2}, \frac{n}{2}; (\tanh r)^2\right). \quad (7.7)$$

Now, when $\Re \gamma > \Re \alpha + \Re \beta$ (cf. [4], page 61), one has

$$\lim_{z \rightarrow 1^-} F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}. \quad (7.8)$$

Therefore, if $\Re(s) \in](n-1)/2, n-1[$, we have

$$h_s(r) = (\cosh r)^{n-1-s} \left(\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2s-n+1}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right)} + o(r) \right). \quad (7.9)$$

From its definition, and from the symmetry between α and β in the definition of the hypergeometric function $F(\alpha, \beta, \gamma; z)$, one has $h_s(r) = h_{n-1-s}(r)$. This property and the estimate above gives therefore an estimate for $h_s(r)$ for large values of r , once $\Re s \neq (n-1)/2$.

It remains to consider the case when s can be written as $(n-1)/2 - it$ for $t \in \mathbb{R}$. One can use the following relation, which holds for $z \in \mathbb{C} - \mathbb{R}^+$, and for $\alpha - \beta \notin \mathbb{Z}$ (cf. [4],

page 108):

$$\begin{aligned} F(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\beta)\Gamma(\gamma - \alpha)} (-z)^{-\alpha} F(\alpha, 1 - \gamma + \alpha, 1 - \beta + \alpha; z^{-1}) \\ &\quad + \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\alpha)\Gamma(\gamma - \beta)} (-z)^{-\beta} F(\beta, 1 - \gamma + \beta, 1 - \alpha + \beta; z^{-1}). \end{aligned} \quad (7.10)$$

We apply it for $\alpha = (n-1)/4 + it/2$, $\beta = (n-1)/4 - it/2$, and $\gamma = n/2$. It gives that for $t \neq 0$,

$$\begin{aligned} h_{(n-1)/2+it}(r) &= (\sinh r)^{-((n-1)/2)r} \\ &\quad \times (c(t)(\sinh r)^{-(it/2)r} F(\alpha, 1 - \gamma + \alpha; 1 - \beta + \alpha; -(\sinh r)^{-2}) \\ &\quad + \bar{c}(t)(\sinh r)^{(it/2)r} \bar{F}(\alpha, 1 - \gamma + \alpha; 1 - \beta + \alpha; -(\sinh r)^{-2})), \end{aligned} \quad (7.11)$$

with

$$c(t) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(-it)}{\Gamma\left(\frac{n-1}{4} - \frac{it}{2}\right)\Gamma\left(\frac{n+1}{4} - \frac{it}{2}\right)}. \quad (7.12)$$

To finish, we consider the case when $t = 0$, that is, $s = (n-1)/2$. We observed already that the spherical λ_s -eigenfunction h_s can be expressed using the Poisson-Helgason transform: if $d\sigma$ denotes the normalized spherical measure on \mathbb{S}^{n-1} , one has $h_s(r) = \int_{\mathbb{S}^{n-1}} \mathcal{P}^s(z, \xi) d\sigma(\xi)$ if $d(o, z) = r$. The formula of the spherical measure leads then to

$$h_s(r) = c(\cosh r)^{-s} \int_0^\pi (1 - \tanh r \cos \theta)^{-s} (\sin \theta)^{n-2} d\theta, \quad (7.13)$$

for some explicit constant c . For $s = (n-1)/2$, an easy calculation gives that this last integral is $O(r)$.

Proposition 7.1. Let $s = \delta + it$ with $\delta \in]0, n-1[$. The spherical function h_s depends holomorphically on s : for any r , $s \rightarrow h_s(r)$ is holomorphic. It satisfies the symmetry $h_s(r) = h_{n-1-s}(r)$. For $\Re s \geq (n-1)/2$,

$$\begin{aligned} |h_s(r)| &\leq C_s \exp(-\delta r) \quad \text{if } s \neq \frac{n-1}{2}, \\ |h_{(n-1)/2}(r)| &\leq C r \exp\left(-\frac{n-1}{2}r\right) \quad \text{if } s = \frac{n-1}{2}. \end{aligned} \quad (7.14) \quad \square$$

Recall that, from the Harnak estimate, one has a similar estimate for the gradients of h_s : $|h'_s(r)| \leq C'_s \exp(-\delta r)$ and $|h'_{(n-1)/2}(r)| \leq C'r \exp(-((n-1)/2)r)$.

The *Green function*. Other solutions to equation $E(a, b; c)$ can be expressed with hypergeometric functions (cf. [4], page 105). In particular, if $b - a + 1$ is not contained in $-\mathbb{N}$, then the function

$$G(a, b; c; z) = (-z)^{-b} F(b + 1 - c, b; b + 1 - a; z^{-1}) \quad (7.15)$$

is a solution on $\mathbb{C} - [0, \infty[$ (the sth power is defined using the standard branch of the logarithm).

Let \mathcal{E} be the exceptional set $\{(n - 3)/2 - \mathbb{N}\}$.

For $r = d(o, z)$, the function

$$z \longrightarrow G_s(r) = (\sinh r)^{-s} F\left(\frac{2-n+s}{2}, \frac{s}{2}, \frac{3-n+2s}{2}; -(\sinh r)^{-2}\right) \quad (7.16)$$

is a λ_s -eigenfunction.

For r near ∞ , it behaves like $G_s(r) = (\sinh r)^{-s}(1 + O(e^{-2r}))$, where the equivalent $O(\cdot)$ is uniform as s stays in a compact set of $\mathbb{C} - \mathcal{E}$.

In order to find the behaviour of $G_s(r)$ for small values of r , we argue as for the spherical function:

$$F(\alpha, \beta; \gamma; z) = (1 - z)^{-\alpha} F\left(\alpha, \gamma - \beta; \gamma; \frac{z}{z-1}\right), \quad (7.17)$$

for z in a neighborhood of $0 \in \mathbb{C}$. It comes out that

$$G_s(r) = (\sinh r)^{-s} (\cosh r)^{n-s-2} F\left(\frac{2-n+s}{2}, \frac{3-n+s}{2}, \frac{3-n+2s}{2}; (\cosh r)^{-2}\right). \quad (7.18)$$

Now, using as above, that if $\Re\gamma > \Re\alpha + \Re\beta$ and $\gamma \notin -\mathbb{N}$,

$$\lim_{z \rightarrow 1^-} F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}. \quad (7.19)$$

Since $n \geq 3$, this applies to the function G_s if $s \notin \mathcal{E}$.

One gets, for r near 0,

$$G_s(r) = (\sinh r)^{2-n} (\cosh r)^{n-s-2} \left(\frac{\Gamma\left(\frac{3-n+2s}{2}\right)\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right)} + o(r) \right). \quad (7.20)$$

One has also, after differentiating the formula for $G_s(r)$ and using the relation

$$\begin{aligned}
F'(\alpha, \beta; \gamma, z) &= \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z), \\
G'_s(r) &= 2(\sinh r)^{-s-3} \cosh r \frac{\left(\frac{2-n+s}{2}\right) \left(\frac{s}{2}\right)}{\frac{3-n+2s}{2}} \\
&\quad \times F\left(\frac{4-n+s}{2}, \frac{s}{2} + 1, \frac{5-n+2s}{2}; -(\sinh r)^{-2}\right) \\
&\quad - s(\sinh r)^{-s-1} \cosh r F\left(\frac{2-n+s}{2}, \frac{s}{2}, \frac{3-n+2s}{2}; -(\sinh r)^{-2}\right).
\end{aligned} \tag{7.21}$$

As before, we find that, if $s \in \mathbb{C} - \mathcal{E}$, and r is near to 0

$$G'_s(r) = (\sinh r)^{1-n} \left(\frac{-2\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3-n+2s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)} + o(r) \right). \tag{7.22}$$

For $s \in \mathbb{C} - \mathcal{E}$, define a function on \mathbb{H}^n by

$$\begin{aligned}
g_s(x) &= \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{-2c_{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3-n+2s}{2}\right)} G_s(d(o, x)) \\
&= \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{-2c_{n-1}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3-n+2s}{2}\right)} (\sinh r)^{-s} \\
&\quad \times F\left(\frac{2-n+s}{2}, \frac{s}{2}, \frac{3-n+2s}{2}; -(\sinh r)^{-2}\right),
\end{aligned} \tag{7.23}$$

c_{n-1} being the volume of the unit sphere in \mathbb{R}^n .

This function g_s is locally integrable on \mathbb{H}^n (since $n \geq 3$) and satisfies in the distributional sense $(\Delta + \lambda_s)g_s = \delta_o$, where δ_o is the Dirac mass at the origin o . We call it the *Green function*. A priori, it is defined only for the values of $s \in \mathbb{C}$ but since $(1/\Gamma(\gamma))F(\alpha, \beta, \gamma, z)$ is a holomorphic function of its parameters for $|z| < 1$ (cf. [8, page 245]), it does extend to a holomorphic function of s for $s \in \mathcal{H}$.

We summarize the above discussion in the following.

Proposition 7.2. Let $s \in \mathcal{H}$. There exists a function $g_s : \mathbb{H}^n - \{o\} \rightarrow \mathbb{C}$ which satisfies $\Delta g_s + \lambda_s g_s = \delta_o$ in the distributional sense. The function $s \rightarrow g_s(z)$ is holomorphic. For $s \in \mathcal{H} - \mathcal{E}$, one has

$$\lim_{x \rightarrow \infty} g_s(x) e^{sd(0,x)} = \frac{\Gamma\left(\frac{1+s}{2}\right) \Gamma\left(\frac{s}{2}\right)}{-2c_{n-1} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{3-n+2s}{2}\right)}, \quad (7.24)$$

the convergence being uniform for s in a compact subset of $\{\Re s \in]0, n-1[-\mathcal{E}\}$. \square

As for the spherical function, the Harnack estimate gives that the same estimate holds for the gradient of g_s .

We need also to consider the value of the Wronskian of g_s and h_s , whose expression is $W(r) = h'_s(r)g_s(r) - h_s(r)g'_s(r)$; therefore $W'(r) = -(n-1) \coth r W(r)$. Thus, it follows from our normalizations that $(\sinh r)^{n-1} W(r) = 1/c_{n-1}$.

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