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Spin Hall effect of Photons in a Static Gravitational Field

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Starting from an Hamiltonian description of the photon through the set of Bargmann-Wigner equations, we derive new semiclassical equations of motion for the photon propagating in static gravitational field. These equations which are obtained in the representation where we could diagonalize the Hamiltonian at the order \hbar , present the first order corrections to the geometrical optic equations. This Hamiltonian shows two new kinds of magneto-torsion couplings which can be also interpreted in terms of Berry curvatures. But the most important result is that even in the absence of torsion the photon does not follow the geodesic as a consequence of an anomalous velocity term responsible for the spin Hall effect of light. Besides the velocity of light is not changed at this order of the approximation.

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In the last few years many studies focused on the transport of quantum particles with spin. Indeed, manipulating spin polarization of electrons is a challenging goal in semiconductor spintronics. Achieving this goal requires the understanding of the spin transport mechanism in systems with spin orbit (SO) interaction. It was found that in such a system, a Berry phase in momentum space plays an important role by affecting both particle phase and its transport properties [1]. It is well known since the seminal work of Berry [2], that when a quantum mechanical system has an adiabatic evolution, a wave function acquires a geometric phase. But it is only recently that the possible influence of the berry phase on transport properties (in particular on the semiclassical dynamics) of several physical systems has been investigated. In semiconductor, SO coupling being greatly enhanced with respect to the vacuum case, a theory of spin transport in system with SO interaction is necessary. But even in the vacuum, new fundamental results concerning the semiclassical equations of motion of electrons were recently derived. For instance in [3] and [4], considering the Dirac equation in an external potential, it was shown that the position operator acquires a spin-orbit contribution which turns out to be a Berry connection rendering the coordinates algebraic structure non-commutative. This drastically modifies the semiclassical equations of motion and implies a topological spin transport similar to the intrinsic spin Hall effect in semiconductor [1]. A similar non-commutative algebra has been also found in the context of electrons in magnetic Bloch bands [5], leading to an anomalous velocity term.

Despite its very different nature, the photon displays many similar behaviors with electronic phenomena such as energy bands in photonic crystal and localization. These similarities stem from the wave like nature of quantum particles. Because photon is also a spinning particle it is important to understand if SO interaction may influ-

ence the transport of light in a similar way as electrons in vacuum or in semiconductors. It has been long known that there is no position operator with commuting components and as a consequence photons are not localizable. Therefore, one of the main differences between photons and electrons also disappears as the coordinates of both particles have actually noncommuting components. It is precisely this property which is at the origin of the SO interaction leaving no doubt about its contribution to the propagation of photons. Therefore these recent studies of the SO coupling in different systems taught us that one can treat both kind of particles on an equal footing. It is thus even more legitimate to wonder whether electronic phenomena has photonic counterparts.

In addition, the localization of light rays is the essential ingredient of the construction of the Minkowski space-time. Also in the context of general relativity, it has been argued that the Riemannian metric is determined by the properties of light propagation [6]. A deeper understanding of the properties of light and in particular of its SO interaction which is at the origin of the non localisability, is thus necessary to build a quantum version of space-time.

It has already been observed that the SO coupling induces a rotation of the polarization plane of light propagating in a optical fiber with torsion [7]. This effect had been predicted long ago by Rytov and Vladimirskii [8] [9] and can be interpreted in terms of Berry phase [2] in momentum space. Recently, it was shown that the noncommutativity of the coordinates (the Berry phase) affects the trajectory or ray of light itself when it is propagating in an isotropic inhomogeneous medium [3, 10]. Besides, other approaches of spin-orbit contributions to the propagation of light in isotropic inhomogeneous media has been the focus of several other works [11] and has led to a generalization of geometric optics called geometric spinoptics [12].

In this letter we will investigate how the propagation of a photon in a static gravitation field is affected by its SO interaction. Since the gravitation field is not scalar it can be seen as an anisotropic inhomogeneous medium. Our photon description is based on an Dirac like Hamiltonian in a gravitational field. The Bargmann-Wigner equations allows us to build the wave function of a spin one particle and to build dynamical operators which satisfy unusual commutation relations in the representation where the Hamiltonian has been diagonalized at the order \hbar . Here we reveal that this semi-classical Hamiltonian shows two new kinds of magneto-torsion couplings interpreted in terms of Berry curvatures. But even in the absence of torsion we will show that the photon does not follow the geodesic as a consequence of an anomalous velocity term but the velocity of light is equal to c at this order of the approximation. This deviation to the geodesic may be not negligible close to very strong gravitation field like black holes or at the first moments of the formation of the universe. It is found moreover that the helicity is general not conserved.

We start with the description of the photon dynamics in the vacuum by considering the two Bargmann-Wigner equations of a massless spin 1 particle:

$$\gamma_{\mu}^{(i)} \partial_{\mu} \Psi_{(a_1 a_2)}(x) = 0 \quad (i = 1, 2) \quad (1)$$

where $\Psi_{(ab)} = \Psi_{(ba)}$ is the symmetrized Bargmann-Wigner amplitude and $\gamma^{(i)}$ is a γ -matrix acting on $a_i = 1..4$ [13]. This symmetrized direct product of two Dirac spinors assures that the positive energy subspace forms a 3-d space corresponding to the irreducible representation of angular momenta 1 deduced from the composition of two states with angular momentum 1/2. It can be proved that Eqs.1 are equivalent to the Maxwell equations. Next, we write an Hamiltonian associated to each Bargmann-Wigner equations $\hat{H}^{(i)} = \alpha^{(i)} \cdot \mathbf{P}$ which can be diagonalized by the product of usual Foldy Wouthuysen (FW) unitary transformations [14], $U^{(i)}(\mathbf{P}) = (E + c\beta \cdot \alpha \cdot \mathbf{P})/E\sqrt{2}$ such that

$$\left(\prod_{j=1}^2 U^{(j)}(\mathbf{P}) \right) \hat{H}^{(i)} \left(\prod_{j=1}^2 U^{(j)}(\mathbf{P})^+ \right) = E\beta^{(i)}, \quad (2)$$

with $E = Pc$ the energy of the photon in the vacuum. The wave function is also transformed and becomes $\phi_{(a_1 a_2)}(\mathbf{P}) = E_P^{-1} \left(\prod_{j=1}^2 U^{(j)}(\mathbf{P})^+ \right) \Psi_{(a_1 a_2)}(\mathbf{P})$ in the FW representation. Clearly $\Psi_{(a_1 a_2)}(\mathbf{P})$ vanishes when the index 1 or 2 coexists with 3 or 4. Whereas the quasi-momentum is invariant through the action of $U^{(i)}$, i.e., $\mathbf{p} = U^{(1)} U^{(2)} \mathbf{P} U^{(1)+} U^{(2)+} = \mathbf{P}$, the position operator becomes $\mathbf{D} = i\hbar \partial_{\mathbf{p}} + \sum_{i=1}^2 \mathcal{A}^{(i)}$ with the gauge potential $\mathcal{A}^{(i)} = i\hbar U^{(i)}(\mathbf{p}) \partial_{\mathbf{p}} U^{(i)}(\mathbf{p})^+$. By projecting the gauge potential on the positive energy subspace, we obtain a non trivial gauge connection allowing us to define a new position operator \mathbf{r} for this particle $\mathbf{r} = i\hbar \partial_{\mathbf{p}} + (\mathbf{p} \wedge \mathbf{S})/p^2$

with $\mathbf{S} = \hbar(\sigma^{(1)} + \sigma^{(2)})/2$. This definition of the position operator gives rise, through the commutation relations, to a monopole in momentum space $[x^i, x^j] = i\hbar \theta^{ij}(\mathbf{p}) = -i\hbar \varepsilon_{ijk} \lambda p^k / p^3$ where $\lambda = \pm \hbar$ is the helicity and $\theta^{ij}(\mathbf{p})$ is the Berry curvature [3]. We then built the Hamiltonian of the free photon of positive energies as $H = (E^{(1)} + E^{(2)})/2 = pc$ with $E^{(i)} = pc$ the positive eigenvalue of the operator $\hat{H}^{(i)}$. Therefore, for free particles, the dynamical equations of motion in the FW representation are trivially given by $d\mathbf{r}/dt = \frac{i}{\hbar} [\mathbf{r}, H] = \mathbf{p}c/p$ and $d\mathbf{p}/dt = \frac{i}{\hbar} [\mathbf{p}, H] = 0$ which in particular implies for the light velocity $v = c$.

We now extend our previous approach to the case of a photon propagating in a static gravitation field where $g_{0\alpha} = 0$, which implies $ds^2 = g_{00}(dx^0)^2 - g_{ij}dx^i dx^j$. Consider again two Bargmann-Wigner equations of motion with the following associate Hamiltonian of the Dirac form :

$$H^{(i)} = \alpha^{(i)} \cdot \tilde{\mathbf{P}} \quad \text{with } i = 1, 2 \quad (3)$$

and $\tilde{\mathbf{P}}$ given by $\tilde{P}_{\alpha} = g_{00}(\mathbf{R}) h_{\alpha}^i(\mathbf{R}) (P_i + \hbar \varepsilon_{\rho\beta\gamma} \Gamma_i^{\rho\beta}(\mathbf{R}) \sigma^{\gamma})$ with h_i^{α} the vierbein, $\Gamma_i^{\alpha\beta}$ the spin connection components and $\varepsilon_{\alpha\beta\gamma} \sigma^{\gamma} = \frac{i}{8} (\gamma^{\alpha} \gamma^{\beta} - \gamma^{\beta} \gamma^{\alpha})$. It is known ([15]) that for a static gravitational field (which is the case considered here), the Hamiltonian $H^{(i)}$ is hermitian. For reason of convenience, we will redefine the vierbeins in the sequel $h_i^{\alpha} \rightarrow g_{00} h_i^{\alpha}$. We now want to diagonalize $H^{(i)}$ through a unitary transformation $U^{(i)}(\tilde{\mathbf{P}})$. Because the components of $\tilde{\mathbf{P}}$ depend both on the operators \mathbf{P} and \mathbf{R} the diagonalization at order \hbar is performed by adapting the method detailed in ([16]) to block diagonal Hamiltonians. To do so, after writing $H^{(i)}$ in a symmetrical way in \mathbf{P} and \mathbf{R} , we consider a formal situation where \mathbf{R} is at first considered as a parameter commuting with \mathbf{P} (this is of course a formal consideration). The Hamiltonian written $H_0^{(i)}$ (we add the index 0 when \mathbf{R} is a parameter) can then be exactly diagonalized by the usual FW matrix $U_0^{(i)}(\tilde{\mathbf{P}}) = (E_0^{(i)} + c\beta \alpha^{(i)} \cdot \tilde{\mathbf{P}}) / \sqrt{2(E_0^{(i)})^2}$ with $E_0^{(i)} = \sqrt{[\alpha^{(i)} \cdot \tilde{\mathbf{P}}]^2} = \sqrt{[\tilde{\mathbf{P}} \cdot \alpha^{(i)}][\alpha^{(i)} \cdot \tilde{\mathbf{P}}]}$ owing to the fact that for each parameter \mathbf{R} the matrices h_i^{α} and $\Gamma_i^{\alpha\beta}$ being independent of both the momentum and position operators, β and $\alpha \cdot \tilde{\mathbf{P}}$ anticommute and in the Taylor expansion of $E^{(i)}$ all terms commute with β and $\alpha^{(i)} \cdot \tilde{\mathbf{P}}$. In this context the diagonalized Hamiltonian is equal to $U^{(i)} H_0^{(i)} U^{(i)+} = c\beta^{(i)} \sqrt{(\alpha^{(i)} \cdot \tilde{\mathbf{P}})^2}$ which reads

$$E_0^{(i)}(\mathbf{P}) = c\beta^{(i)} [(P_i + \hbar \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta} \sigma^{\gamma}) g^{jj} (P_j + \hbar \varepsilon_{\alpha\beta\gamma} \Gamma_j^{\alpha\beta} \sigma^{\gamma}) + \hbar (P_i + \hbar \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta} \sigma^{\gamma}) \varepsilon_{\alpha\beta\gamma} h_a^i \tilde{T}^{a\alpha\beta} \sigma^{\gamma}]^{1/2} \quad (4)$$

where $\tilde{T}^{a\alpha\beta} = h_i^{\alpha} \Gamma^{ia\beta} - h_i^{\beta} \Gamma^{ia\alpha}$ is a torsion term without the gradient of the vierbein. Now, to diagonalize $H^{(i)}$ at the first order in \hbar it is shown in [16] that is enough to

apply the following FW transformation

$$U^{(i)}(\tilde{\mathbf{P}}) = \frac{E^{(i)} + c\beta\alpha^{(i)} \cdot \tilde{\mathbf{P}}}{\sqrt{2E^{(i)2}}} + X^{(i)} \quad (5)$$

and then to project the transformed Hamiltonian on the positive energy states. The energy in Eq.5 is given by expression Eq.4 and \mathbf{R} is no more considered as a parameter. All expressions in $U^{(i)}(\tilde{\mathbf{P}})$ are implicitly assumed to be symmetrized in \mathbf{P} and \mathbf{R} and the corrective term $X^{(i)}$ must be added to restore the unitarity of $U^{(i)}(\tilde{\mathbf{P}})$ which is destroyed by the symmetrization. It is shown in [16] that this term is expressed as

$$X^{(i)} = \frac{i}{4\hbar} \left[\mathcal{A}_{P^i}^{(i)}, \mathcal{A}_{R_i}^{(i)} \right] U^{(i)}(\tilde{\mathbf{P}}) \quad (6)$$

where we have defined the position and momentum (non projected) Berry phases $\mathcal{A}_R^{(i)} = i\hbar U^{(i)} \nabla_P U^{(i)+}$ and $\mathcal{A}_P^{(i)} = -i\hbar U^{(i)} \nabla_R U^{(i)+}$. After projection on the positive energy subspace needed to realize the diagonalization [16] the resulting position and momentum operators can thus be written $\mathbf{r}^{(i)} = i\hbar \partial_{\mathbf{P}} + \mathcal{A}_R^{(i)}$ and $\mathbf{p}^{(i)} = \mathbf{P} + \mathcal{A}_P^{(i)}$ where the explicit computation gives for the components

$$\mathcal{A}_{P^k}^{(i)} = -\hbar c^2 \frac{\varepsilon^{abc} \tilde{P}_a \sigma_b^{(i)} (\nabla_{R_k} \tilde{P}_c)}{2E^{(i)2}} + O(\hbar^2) \quad (7)$$

$$\mathcal{A}_{R_k}^{(i)} = \hbar c^2 \frac{\varepsilon^{abc} h_c^k \tilde{P}_a \sigma_b^{(i)}}{2E^{(i)2}} + O(\hbar^2) \quad (8)$$

Performing our diagonalization process leads us ultimately, after some computations, to the following expression for the energy operator

$$\begin{aligned} \tilde{\varepsilon}^{(i)} \simeq & \varepsilon^{(i)} + \frac{\hbar}{2\varepsilon^{(i)}} \left(P_i + \hbar \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta} \sigma^\gamma \right) \varepsilon_{\alpha\beta\gamma} h_a^i \tilde{T}_a^{\alpha\beta} \sigma^\gamma \\ & + \frac{i\hbar}{4\varepsilon^{(i)}} \left[\nabla_{R_i} (\alpha^{(i)} \cdot \tilde{\mathbf{P}}), \nabla_{P^i} (\alpha^{(i)} \cdot \tilde{\mathbf{P}}) \right] \\ & - (\nabla_{R_i} \varepsilon) \mathcal{A}_{R_i}^{(i)} - (\nabla_{P^i} \varepsilon) \mathcal{A}_{P^i}^{(i)} \end{aligned} \quad (9)$$

where $\varepsilon^{(i)} = c\sqrt{(p_i + \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta} \sigma^\gamma) g^{ij} (p_j + \varepsilon_{\alpha\beta\gamma} \Gamma_j^{\alpha\beta} \sigma^\gamma)}$. In this last expression we have neglected contributions of the curvature of order \hbar^2 . The second and third terms in Eq. 9 are kinds of gravitational spin-orbit terms and have the same form as the spin-orbit interaction in the diagonalization of the Dirac Hamiltonian in the presence of electromagnetic perturbations [4] (to compare we just have to do the replacement $\tilde{\mathbf{P}} \rightarrow \mathbf{P} - e\mathbf{A}(\mathbf{R})$). These two terms recombine to give $\frac{\hbar}{2\varepsilon^{(i)}} P_k T_{ab}^k \varepsilon^{abc} \sigma_c = \hbar \mathbf{B} \cdot \sigma / 2\varepsilon^i$ where we have introduced a "magneto-torsion" field $B_c = \frac{1}{2} P_k T_{ab}^k \varepsilon^{abc}$ with T_{ab}^k the torsion defined as $T_{ab}^k = h_b^l \partial_l h_a^k - h_a^l \partial_l h_b^k + \tilde{T}_{ab}^k$ and $\tilde{T}_{ab}^k = h_c^k \tilde{T}_{ab}^c$. By the same token, we can also recombine the two last terms in Eq. 9 and write $(\nabla_{R_i} \varepsilon) \mathcal{A}_{R_i}^{(i)} + (\nabla_{P^i} \varepsilon) \mathcal{A}_{P^i}^{(i)} = -(\mathcal{A}_R^{(i)} \times \mathbf{p}) \cdot \mathbf{B}' / \varepsilon^{(i)}$

with another "magneto-torsion" field which is vierbein independent $B'_c = \frac{1}{2} P_k \left(T_{ab}^k - \tilde{T}_{ab}^k \right) \varepsilon^{abc} = \frac{1}{2} P_k \left(h_b^l \partial_l h_a^k - h_a^l \partial_l h_b^k \right) \varepsilon^{abc}$. Therefore the Hamiltonian can be written $\tilde{\varepsilon}^{(i)} = \varepsilon^{(i)} + \hbar \mathbf{B} \cdot \sigma / 2\varepsilon - (\mathcal{A}_R^{(i)} \times \mathbf{p}) \cdot \mathbf{B}' / \varepsilon^i$.

We can now build (as in the free photon case) the Hamiltonian of the photon as the sum of the two Hamiltonians for one-half massless spinning particle $\tilde{\varepsilon} = (\tilde{\varepsilon}^{(1)} + \tilde{\varepsilon}^{(2)})/2$. By Taylor expanding the expression of $\varepsilon^{(i)}$ we see that at the leading order in \hbar the sum may be recombined to give

$$\tilde{\varepsilon} \simeq \varepsilon + \hbar \mathbf{B} \cdot \Sigma / 2\varepsilon - (\mathcal{A}_R \times \mathbf{p}) \cdot \mathbf{B}' / \varepsilon \quad (10)$$

where $\varepsilon = c\sqrt{(p_i + \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta} \Sigma^\gamma) g^{ij} (p_j + \varepsilon_{\alpha\beta\gamma} \Gamma_j^{\alpha\beta} \Sigma^\gamma)}$ with $\Sigma = (\sigma^{(1)} + \sigma^{(2)})/2$ the spin one connection and $\mathcal{A}_R = \sum_{i=1}^2 \mathcal{A}_R^{(i)}$ the Berry connection in position. Define also $\mathcal{A}_P = \sum_{i=1}^2 \mathcal{A}_P^{(i)}$, the projected position and momentum are defined through the sum of the two Berry connections associated to one-half spinning particles and can therefore be approximated at leading order in \hbar by the expression

$$\mathbf{r} = i\hbar \partial_{\mathbf{P}} + \hbar c^2 \frac{\mathbf{P} \times \Sigma}{\varepsilon^2} \quad (11)$$

$$\mathbf{p} = \mathbf{P} + \hbar c^2 \left(\frac{\mathbf{P} \times \Sigma}{\varepsilon^2} \right) \nabla_{\mathbf{R}} \tilde{\mathbf{P}} \quad (12)$$

where $\varepsilon^2 = c^2 P_i g^{ij} P_j$ at the approximation considered. Eq.10 shows that the spin couples to the gravitational field through the magneto-torsion field \mathbf{B} which therefore contains the spin connection. On the other hand, the interaction involving the orbital part $(\mathcal{A}_R \times \mathbf{p})$ should only interact with the spatial torsion so that \mathbf{B}' is present in this case. As a consequence, an hypothetical torsion of space may be revealed through the presence of these interactions. From Eqs. 11 and 12 we deduce the new commutations rules

$$[r^i, r^j] = i\hbar \Theta_{rr}^{ij} \quad (13)$$

$$[p^i, p^j] = i\hbar \Theta_{pp}^{ij} \quad (14)$$

$$[p^i, r^j] = -i\hbar g^{ij} + i\hbar \Theta_{pr}^{ij} \quad (15)$$

where $\Theta_{\alpha\beta}^{ij} = \partial_{\alpha^i} \mathcal{A}_{\beta^j} - \partial_{\beta^i} \mathcal{A}_{\alpha^j} + [\mathcal{A}_{\alpha^i}, \mathcal{A}_{\beta^j}]$. An explicit computation shows that

$$\begin{aligned} \Theta_{rr}^{ij} &= -\hbar \frac{(\Sigma^a \tilde{P}_a) \tilde{P}_c}{2E^4} \varepsilon^{abc} h_a^i h_b^j \\ \Theta_{pp}^{ij} &= -\hbar \frac{(\Sigma^a \tilde{P}_a) \tilde{P}_c}{2E^4} \nabla_{R_i} \tilde{P}_a \nabla_{R_j} \tilde{P}_b \varepsilon^{abc} \\ \Theta_{pr}^{ij} &= \hbar \frac{(\Sigma^a \tilde{P}_a) \tilde{P}_c}{2E^4} \nabla_{R_i} \tilde{P}_a h_b^j \varepsilon^{abc} \end{aligned} \quad (16)$$

This semi-classical Hamiltonian is one of the main result of this paper. It contains, in addition to the usual

term ε , new contributions coming from Berry connections corresponding to spin-orbit interactions. To deduce the equations of motion we need also $\Theta_{r\Sigma} = [r_i, \Sigma_j] = i\hbar c^2(-P_j \Sigma_i + \mathbf{P} \cdot \Sigma \delta_{ij})/\varepsilon^2$ and $\Theta_{r\Sigma} = [p_i, \Sigma_j] = -i\hbar c^2(-P_j \Sigma_i + \mathbf{P} \cdot \Sigma \delta_{ij})h_i^c \nabla_{R_i} \tilde{P}_c/\varepsilon^2$. The resulting equations of motion are then given by

$$\dot{\mathbf{r}} = \nabla_{\mathbf{p}} \tilde{\varepsilon} + \dot{\mathbf{p}} \times \Theta_{rr} + \frac{i}{\hbar} \nabla_{\Sigma} \tilde{\varepsilon} \Theta_{r\Sigma} \quad (17)$$

$$\dot{\mathbf{p}} = -\nabla_{\mathbf{r}} \tilde{\varepsilon} + (\dot{\mathbf{r}} \times \mathbf{B}') \cdot \Theta_{pr} + \frac{i}{\hbar} \nabla_{\Sigma} \tilde{\varepsilon} \Theta_{r\Sigma} \quad (18)$$

where $(\mathbf{f} \cdot \Theta)_i = \mathbf{f}_j \Theta_{ij}$. These are our new semiclassical equations of motion for a photon in a static gravitational field. The velocity equation contains the by now well known anomalous contribution $\dot{\mathbf{p}} \times \Theta^{rr}$ which is at the origin of the spin Hall effect of the photon in an isotropic inhomogeneous medium of indice $n(r)$ [3, 10, 11, 12]. This medium is characterized by a metric $g_{ij} = \delta_{ij} n^{-1}(r)$ and our results are thus a generalization to an anisotropic medium of indice $n_{ij}^{-1} = g_{ij}$. The momentum equation shows a new contribution (equal to zero in the absence of torsion) which being of order \hbar does not influence the velocity equation. These equations show that due to the Berry phases which correspond to spin-orbit interactions a photon cannot follow the geodesic. This is an important result because the propagation of light is at the origin of the construction of general relativity. This Hall effect of light is obviously a small effect (which nevertheless has been observed in a inhomogeneous media [18]) but can be greatly enhanced in a very strong gravitational field like a black hole. In addition one can check that the velocity of light get a correction of order \hbar^2 which lies beyond our approximation. At the leading order the helicity λ is not changed by the unitary transformation which diagonalizes the Hamiltonian so that it can be written $\lambda = \hbar \mathbf{p} \cdot \Sigma / p$. At the lowest order we get the equation

$$\hbar \frac{d}{dt} \left(\frac{\mathbf{p} \cdot \Sigma}{p} \right) = \tilde{P}_j \Gamma_j \cdot \mathcal{A}_R \quad (19)$$

where the vector Γ_j is defined by $\Gamma_{i,\gamma} = \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta}(\mathbf{r})$. Eq.19 shows that the helicity is not conserved and its variation is Berry phase dependent.

As a simple application let consider the symmetric case $g^{ij}(\mathbf{R}) = \delta^{ij} F(\mathbf{R})$. We start with the following Hamiltonian, as defined in [17], $H_0 = \alpha \tilde{\mathbf{P}}$ with $\tilde{\mathbf{P}} = (\mathbf{P}F(\mathbf{R}) + \mathbf{P}F(\mathbf{R}))/2$. Specializing our diagonalization to this case, leads us to define the FW transformation $U(\tilde{\mathbf{P}}) = (E + c\beta\alpha \tilde{P})/\sqrt{2E^2} + X$ where now $E = c(F\mathbf{P}^2F)^{1/2}$. The computation of the (projected) Berry phases at the first order in \hbar gives $\mathcal{A}_R = i\hbar \mathbf{P} \times \Sigma / 2\mathbf{P}^2$ and $\mathcal{A}_P = 0$ so that the dynamical variables turn out to be $\mathbf{r} = \mathbf{R} + \hbar \frac{\mathbf{P} \times \Sigma}{2\mathbf{P}^2}$ and $\mathbf{p} = \mathbf{P}$. As a consequence of $\mathcal{A}_P = 0$ we have $X = 0$ and the unitary matrix is simply given by the usual FW transformation. This leads us to the semiclassical diagonal Hamiltonian $\varepsilon =$

$c\sqrt{[\alpha \tilde{\mathbf{P}}(\mathbf{r}, \mathbf{p})]^2} = c(\mathbf{P}F(\mathbf{r}) + \mathbf{P}F(\mathbf{r}))/2$. We deduce the following the Berry curvatures from the dynamical variables commutators : $[r_i, r_j] = \hbar^2(\Sigma \cdot \mathbf{P})\mathbf{P}/P^4$, $[p_i, p_j] = 0$ and $[p_i, r_j] = -i\hbar g_{ij}$. As a consequence, we derive the following equations of motion $\dot{\mathbf{r}} = \nabla_{\mathbf{p}} \tilde{\varepsilon} + \dot{\mathbf{p}} \times \Theta_{rr}$ and $\dot{\mathbf{p}} = -\nabla_{\mathbf{r}} \tilde{\varepsilon}$ which explain the Magnus effect observed in [18]. In this case one check that the helicity is conserved as $\frac{d}{dt}(\mathbf{p} \cdot \Sigma / p) = 0$ in agreement with ([17]).

In this paper, we have diagonalized at the first order in \hbar the photon Hamiltonian in a static gravitational field. This diagonal Hamiltonian displays new interactions between two magneto-torsion fields and respectively spin and angular momentum. As a consequence the torsion of space, if any, could, be determined through these couplings. However, even in the absence of torsion, we found two new semiclassical equations of motion including Berry phases contributions for both dynamical variables and predicting that the photon does not follow the geodesic due to its spinning nature. The reason is an anomalous velocity, responsible for the spin Hall effect of light. This last result is in agreement with the modern point of view about the spinning particles evolution. Our results are actually not restricted to the gravitational field but also apply to systems with anisotropic indices.

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