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A RELATION BETWEEN THE PARABOLIC CHERN CHARACTERS OF THE DE RHAM BUNDLES

JAYA NN. IYER AND CARLOS T. SIMPSON

ABSTRACT. In this paper, we consider the weight i de Rham–Gauss–Manin bundles on a smooth variety arising from a smooth projective morphism $f : X_U \rightarrow U$ for $i \geq 0$. We associate to each weight i de Rham bundle, a certain parabolic bundle on S and consider their parabolic Chern characters in the rational Chow groups, for a good compactification S of U . We show the triviality of the alternating sum of these parabolic bundles in the (positive degree) rational Chow groups. This removes the hypothesis of semistable reduction in the original result of this kind due to Esnault and Viehweg.

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1. INTRODUCTION

Suppose X and S are irreducible projective varieties defined over the complex numbers and $\pi : X \rightarrow S$ is a morphism such that the restriction $X_U \rightarrow U$ over a nonsingular dense open set is smooth of relative dimension n . The following bundle on U , for $i \geq 0$,

$$\mathcal{H}^i := R^i \pi_* \Omega_{X_U/U}^\bullet$$

is equipped with a flat connection ∇ , called as the Gauss–Manin connection. We call the pair (\mathcal{H}^i, ∇) the de Rham bundle or the Gauss–Manin bundle of weight i .

Suppose S is a nonsingular compactification of U such that $D := S - U$ is a normal crossing divisor and the associated local system $\ker \nabla$ has unipotent monodromies along the components of D . The bundle $\overline{\mathcal{H}}^i$ is the *canonical extension* of \mathcal{H}^i ([De1]) and is

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equipped with a logarithmic flat connection $\overline{\nabla}$. It is characterised by the property that it has nilpotent residues.

By the Chern–Weil theory, the de Rham Chern classes

$$c_i^{dR}(\mathcal{H}^k) \in H_{dR}^{2i}(U)$$

vanish, and by a computation shown in [Es-Vi1, Appendix B], the de Rham classes

$$c_i^{dR}(\overline{\mathcal{H}}^k) \in H_{dR}^{2i}(S)$$

vanish too. The essential fact used is that the residues of $\overline{\nabla}$ are nilpotent.

The algebraic Chern–Simons theory initiated by S. Bloch and H. Esnault ([BE]) studies the Chern classes (denoted by c_i^{Ch}) of flat bundles in the rational Chow groups of S and S . It is conjectured by H. Esnault that the classes $c_i^{Ch}(\mathcal{H}^k)$ and $c_i^{Ch}(\overline{\mathcal{H}}^k)$ are trivial for all $i > 0$ and k ([Es1, p. 187–188], [Es2]).

The cases where it is known to be true are as follows. In [Mu], Mumford proved this for any family of stable curves. In [vdG], van der Geer proved that $c_i^{Ch}(\mathcal{H}^1)$ is trivial when $X \rightarrow S$ is a family of abelian varieties. Further, for any family of abelian varieties of dimension g , the rational Chow group elements $c_i^{Ch}(\overline{\mathcal{H}}^1)$, $i \geq 1$, were proved to be trivial by Iyer under the assumption that $g \leq 5$ ([Iy]) and by Esnault and Viehweg and for all $g > 0$ ([Es-Vi2]). Further, for some families of moduli spaces, Biswas and Iyer ([Bi-Iy]) have checked the triviality of the classes in the rational Chow groups.

In this paper we consider parabolic bundles associated to logarithmic connections (section 3) instead of canonical extensions. By Steenbrink’s theorem [St, Proposition 2.20], the monodromy of the VHS associated to families is quasi-unipotent and the residues have rational eigenvalues. It is natural to consider the parabolic bundles associated to such local systems. Further, these parabolic bundles are compatible with pullback morphisms (Lemma 2.5), unlike canonical extensions. One can also define the Chern character of parabolic bundles in the rational Chow groups (section 2.3). All this is possible by using a correspondence of these special parabolic bundles, termed as locally abelian parabolic bundles, with vector bundles on a particular DM-stack (Lemma 2.3).

In this framework, we show

Theorem 1.1. *Suppose $\pi : X_U \rightarrow U$ is a smooth projective morphism of relative dimension n between nonsingular varieties. Consider a nonsingular compactification $U \subset S$ such that $S - U$ is a normal crossing divisor. Then the Chern character of the alternating sum of the parabolic bundles $\overline{\mathcal{H}}^i(X_U/U)$ in each degree,*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(X_U/U))$$

lies in $CH^0(S)_{\mathbb{Q}}$ or equivalently the pieces in all of the positive-codimension Chow groups with rational coefficients vanish.

Here $\overline{\mathcal{H}}^i(X_U/U)$ denotes the parabolic bundle associated to the weight i de Rham bundle on U .

In fact we will prove the same thing when the morphism is not generically smooth, where U is the open set over which the map is topologically a fibration, see Theorem 5.1.

If $X_U \rightarrow U$ has is a semi-stable extension $X \rightarrow S$ (or a compactified family satisfying certain conditions) then the triviality of the Chern character of the alternating sum of de Rham bundles in the (positive degree) rational Chow groups is proved by Esnault and Viehweg in [Es-Vi2, Theorem 4.1]. This is termed as a logarithmic Grothendieck Riemann-Roch theorem (GRR) since GRR is applied to the logarithmic relative de Rham sheaves to obtain the relations. It might be possible to generalize the calculation of [Es-Vi2] to the case of a weak toroidal semistable reduction which always exists by [AK]. However, this seems like it would be difficult to set up.

We generalise the Esnault-Viehweg result and a good compactified family is not required over S . In particular, we do not use calculations with the Grothendieck Riemann-Roch formula, although we do use the general existence of such a formula. Further, Theorem 1.1 shows that the singularities in the fibres of extended families do not play any role, in higher dimensions. Instead, our inductive argument calls upon de Jong's semistable reduction for curves [dJ] at each inductive step.

As an application, we show

Corollary 1.2. *Suppose $X_U \rightarrow U$ is any family of projective surfaces and S is a good compactification of U . Then the parabolic Chern character satisfies*

$$ch(\overline{\mathcal{H}}^i(X_U/U)) \in CH^0(S)_{\mathbb{Q}}$$

for each $i \geq 0$.

This is proved in §5.5 Proposition 5.13. On a non-compact base S supporting a smooth family of surfaces, this was observed in [BE, Example 7.3]. We use the weight filtration on the cohomology of the singular surface, a resolution of singularities, the triviality of the classes of $\overline{\mathcal{H}}^1$ ([Es-Vi2]), and using Theorem 1.1 we deduce the proof.

The proof of Theorem 1.1 is by induction principle and using the Lefschetz theory. We induct on the relative dimension n . By the Lefschetz theory, the cohomology of an n -dimensional nonsingular projective variety is expressed in terms of the cohomology of a nonsingular hyperplane section and the cohomology of the (extension of) *variable local system* on \mathbb{P}^1 . This helps us to apply induction and conclude the relations between the Chern classes of the de Rham bundles in the rational Chow groups (Theorem 1.1). Our proof requires a certain amount of machinery such as the notion of parabolic bundle. The reason for this is that the local monodromy transformations of a Lefschetz pencil whose fiber dimension is even, are reflections of order two rather than unipotent transvections. In spite of this machinery we feel that the proof is basically pretty elementary, and in particular it doesn't require us to follow any complicated calculations with GRR.

A result of independent interest is Theorem 6.1 (in section 6, Appendix, due to the second author), which is an analogue of Steenbrink’s theorem [St, Theorem 2.18]. In this case, the relative dimension is one and the relative (logarithmic) de Rham complex has coefficients in an extension of a unipotent local system. It is proved that the associated cohomology sheaves are locally free and having a Gauss–Manin connection. Furthermore, the logarithmic connection has nilpotent residues along the divisor components (where it has poles).

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2. PARABOLIC BUNDLES

We treat some preliminaries on the notion of parabolic bundle [Se]. This takes a certain amount of space, and we are leaving without proof many details of the argument. The purpose of this discussion in our proof of the main theorem is to be able to treat the case of Lefschetz pencils of even fiber dimension, in which case the monodromy transformations are reflections of order two (rather than the more classical unipotent transformations in the case of odd fiber dimension). Thus we will at the end be considering parabolic structures with weights $0, \frac{1}{2}$ and the piece of weight $\frac{1}{2}$ will have rank one. Furthermore we will assume by semistable reduction that the components of the divisor of singularities don’t touch each other. Nonetheless, it seems better to give a sketch of the general theory so that the argument can be fit into a proper context.

Suppose X is a smooth variety and D is a normal crossings divisor. Write $D = \bigcup_{i=1}^k D_i$ as a union of irreducible components, and we assume that the D_i are themselves smooth meeting transversally.

We will define the notion of *locally abelian parabolic bundle on (X, D)* . We claim that this is the right definition of this notion; however intermediate definitions may or may not be useful (e.g. our notion of “parabolic sheaf” might not be the right one). The notion which we use here appeared for example in Mochizuki [Mo], and is slightly different from the one used by Maruyama and Yokogawa [Ma-Yo] in that we consider different filtrations for all the different components of the divisor.

Also we shall only consider parabolic structures with rational weights.

A *parabolic sheaf* on (X, D) is a collection of torsion-free sheaves F_α indexed by multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_i \in \mathbb{Q}$, together with inclusions of sheaves of \mathcal{O}_X -modules

$$F_\alpha \hookrightarrow F_\beta$$

whenever $\alpha_i \leq \beta_i$ (a condition which we write as $\alpha \leq \beta$ in what follows), subject to the following hypotheses:

- (normalization/support) let δ^i denote the multiindex $\delta_i^i = 1$, $\delta_j^i = 0$, $i \neq j$, then $F_{\alpha+\delta^i} = F_\alpha(D_i)$ (compatibly with the inclusion); and
- (semicontinuity) for any given α there exists $c > 0$ such that for any multiindex ε with $0 \leq \varepsilon_i < c$ we have $F_{\alpha+\varepsilon} = F_\alpha$.

It follows from the normalization/support condition that the quotient sheaves F_α/F_β for $\beta \leq \alpha$ are supported in a schematic neighborhood of the divisor D , and indeed if $\beta \leq \alpha \leq \beta + \sum n_i \delta^i$ then F_α/F_β is supported over the scheme $\sum_{i=1}^k n_i D_i$. Let $\delta := \sum_{i=1}^k \delta^i$. Then

$$F_{\alpha-\delta} = F_\alpha(-D)$$

and $F_\alpha/F_{\alpha-\delta} = F_\alpha|_D$.

The semicontinuity condition means that the structure is determined by the sheaves F_α for a finite collection of indices α with $0 \leq \alpha_i < 1$. We loosely call these indices the *weights*.

We can be a bit more precise about the notion of weight. For each component D_i of the divisor D , we have

$$F_\alpha|_{D_i} = F_\alpha/F_{\alpha-\delta^i}.$$

Thus for $\alpha = 0$ we have

$$F_0|_{D_i} = F_0/F_{-\delta^i}.$$

This sheaf over D_i has a filtration by subsheaves which are the images of the $F_{\beta_i \delta^i}$ for $-1 < \beta_i \leq 0$. The filtration stops at $\beta_i = -1$ where by definition the subsheaf is zero. Call the image subsheaf F_{D_i, β_i} and for $\beta_i > -1$ put

$$Gr_{D_i, \beta_i} := F_{D_i, \beta_i}/F_{D_i, \beta_i - \varepsilon}$$

with ε small. There are finitely many values of β_i such that the Gr is nonzero. These values are the *weights* of F along the component D_i . Any \mathbb{Z} -translate of one of these weights will also be called a weight. The global parabolic structure is determined by the sheaves F_α for multiindices α such that each α_i is a weight along the corresponding component D_i .

2.1. The locally abelian condition. A *parabolic line bundle* is a parabolic sheaf F such that all the F_α are line bundles. An important class of examples is obtained as follows: if α is a multiindex then we can define a parabolic line bundle denoted

$$F := \mathcal{O}_X\left(\sum_{i=1}^k \alpha_i D_i\right)$$

by setting

$$F_\beta := \mathcal{O}_X\left(\sum_{i=1}^k a_i D_i\right)$$

where each a_i is the largest integer such that $a_i \leq \alpha_i + \beta_i$.

If F is a parabolic sheaf, set F_∞ equal to the extension $j_*(j^*F_\alpha)$ for any α , where $j : X - D \hookrightarrow X$ is the inclusion. It is the sheaf of sections of F which are meromorphic along D , and it doesn't depend on α . Note that the F_α may all be considered as subsheaves of F_∞ .

We can define a tensor product of torsion-free parabolic sheaves: set

$$(F \otimes G)_\alpha$$

to be the subsheaf of $F_\infty \otimes_{\mathcal{O}_X} G_\infty$ generated by the $F_{\beta'} \otimes G_{\beta''}$ for $\beta' + \beta'' \leq \alpha$.

On the other hand, if E is a torsion-free sheaf on X then it may be considered as a parabolic sheaf (we say “with trivial parabolic structure”) by setting E_α to be $E(\sum a_i D_i)$ for a_i the greatest integer $\leq \alpha_i$.

With this notation we may define for any vector bundle E on X the parabolic bundle

$$E\left(\sum_{i=1}^k \alpha_i D_i\right) := E \otimes \mathcal{O}_X\left(\sum_{i=1}^k \alpha_i D_i\right).$$

Lemma 2.1. *Any parabolic line bundle has the form $L(\sum_{i=1}^k \alpha_i D_i)$ for L a line bundle on X . This may be viewed as $L(B)$ where B is a rational divisor on X (supported on D).*

□

Definition 2.2. *A parabolic sheaf F is a locally abelian parabolic bundle if, in a Zariski neighborhood of any point $x \in X$ there is an isomorphism between F and a direct sum of parabolic line bundles.*

In fact these are the only objects we will be interested in: the definitions of various structures above, such as tensor product, may or may not be good for objects other than these. We claim only that they are the right definitions for locally abelian parabolic bundles. Most questions about locally abelian parabolic bundles can be treated by reducing to a local question then looking at the case of line bundles and using the structure result of Lemma 2.1.

2.2. Relationship with bundles on DM-stacks. This material is the subject of a number of recent papers and preprints by Biswas [Bi2], Cadman [Cad], Matsuki and Olsson [Ma-Ol] and specially Niels Borne [Bo]. We thank A. Chiodo for useful discussions about this and for informing us of these references.

Suppose $f : (X', D') \rightarrow (X, D)$ is a morphism of smooth varieties with normal crossings divisors, such that $f^{-1}(D) \subset D'$. Then we would like to define the *pullback* f^*F of a locally abelian parabolic bundle F on (X, D) , as a locally abelian parabolic bundle on (X', D') . This is not entirely trivial to do. We propose two approaches. We first will pass through the notion of bundles on Deligne-Mumford stacks in order to give a precise definition. On

the other hand, we can use the locally abelian structure to reduce to the case of parabolic line bundles. In this case we require that

$$f^*(L(B)) = (f^*L)(f^*B)$$

using the pullback $f^*(B)$ for a rational divisor B (note that if B is supported on D then the pullback will be supported on D'). Finally, in order to give a global computation we will discuss a standard resolution of a locally abelian parabolic bundle by bundles of the form $E(B)$ for rational divisors B and vector bundles E .

Any construction using a local choice of frame as in Definition 2.2 or something else such as the resolution we will discuss later, leaves open the problem of seeing that the construction is independent of the choices which were made. In order to get around this kind of problem, we use the relationship between parabolic bundles and bundles on certain Deligne-Mumford stacks. This is only sketched here since we only need to refer to the existence of such a theory.

We define a DM-stack denoted $Z := X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$. Localizing in the etale topology over X we may assume that the D_i are defined by equations z_i . Only some of the components will appear in any local chart; renumber things so that these are $z_i = 0$ for $i = 1, \dots, k'$. Then define the local chart for Z to be given with coordinates u_i by the equations $z_i = u_i^{n_i}$ for $i = 1, \dots, k'$ and $z_i = u_i$ for the other i . Without repeating the general theory of DM-stacks, this defines a smooth DM-stack Z whose coarse moduli space is X and whose stabilizer groups along the components D_i are the groups $\mathbb{Z}/n_i\mathbb{Z}$.

This DM-stack may alternatively be considered as a ‘‘Q-variety’’ in the sense of Mumford [Mu, section 2].

On Z we have divisors which we may write as $\frac{D_i}{n_i}$, given in the local coordinate patch by $u_i = 0$. In particular, if B is a rational divisor supported along D such that $B = \sum b_i D_i$ and if $n_i b_i \in \mathbb{Z}$ (that is, the denominator of b_i divides n_i) then B becomes an actual divisor on Z by writing

$$B = \sum (n_i b_i) \frac{D_i}{n_i}.$$

Let $p : Z \rightarrow X$ denote the projection. If E is a vector bundle on Z then p_*E is a torsion-free sheaf on X , and in fact it is a bundle. We may define the *associated parabolic bundle* $\mathbf{a}(E)$ on (X, D) by the formula

$$\mathbf{a}(E)_\alpha := p_*(E(\sum_i \alpha'_i D_i))$$

where α'_i is the greatest rational number $\leq \alpha_i$ with denominator dividing n_i .

Lemma 2.3. *The above construction establishes an equivalence of categories between vector bundles on Z , and locally abelian parabolic bundles on (X, D) whose weights have denominators dividing the n_i .*

Proof. See [Bo, Theorem 5]. □

The local abelian condition on parabolic bundles corresponds to the condition that upon pullback to the appropriate DM stack we have an actual vector bundle. One could expect more general parabolic bundles to correspond to certain saturated sheaves on these DM stacks but that goes beyond our present requirements.

The inverse functor in Lemma 2.3 will be taken as the definition of the pullback p^* from certain locally abelian parabolic bundles on (X, D) (that is, those with appropriate denominators) to bundles considered as having their trivial parabolic structure on the stack Z via the map $p : Z \rightarrow X$.

Lemma 2.4. *The inverse functor p^* is compatible with morphisms of DM-stacks of the form*

$$Z_1 := X\left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}\right] \xrightarrow{q} Z_2 := X\left[\frac{D_1}{m_1}, \dots, \frac{D_k}{m_k}\right]$$

whenever m_i divides n_i . In other words, if F is a parabolic bundle whose denominators divide the m_i and if $E = p^*F$ is the associated bundle on Z_2 , then q^*E is the associated bundle on Z_1 (by a natural isomorphism satisfying a cocycle condition).

□

Lemma 2.5. *Suppose $f : (X', D') \rightarrow (X, D)$ is a morphism of smooth varieties with normal crossings divisors, such that $f^{-1}(D) \subset D'$. Then the pullback f^*F of a locally abelian parabolic bundle F on (X, D) , is defined as a locally abelian parabolic bundle on (X', D') .*

Proof. Using the equivalence of Lemma 2.3, we can define the pullback of a locally abelian parabolic bundle. Indeed, given a morphism $f : (X', D') \rightarrow (X, D)$ and any positive integers n_1, \dots, n_k for the components D_i of D , there exist integers $n'_1, \dots, n'_{k'}$ for the components D'_i of D' such that f extends to a morphism

$$f_{DM} : X'\left[\frac{D'_1}{n'_1}, \dots, \frac{D'_{k'}}{n'_{k'}}\right] \rightarrow X\left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}\right]$$

If F is a locally abelian parabolic bundle on X we can choose integers n_i divisible by the denominators of the weights of F , then choose n'_j as above. Thus F corresponds to a bundle E on the DM-stack $X\left[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}\right]$. The pullback $f_{DM}^*(E)$ is a bundle on the DM-stack $X'\left[\frac{D'_1}{n'_1}, \dots, \frac{D'_{k'}}{n'_{k'}}\right]$. Define $f^*(F)$ to be the locally abelian parabolic bundle on (X', D') corresponding (again by Lemma 2.3) to the bundle $f_{DM}^*(E)$.

Using Lemma 2.4, the parabolic bundle $f^*(F)$ is independent of the choice of n_i and n'_j . □

This definition can be extended, using local charts which are schemes, to the pullback to any DM-stack, and it is compatible with the previous notation in the sense that if $p : Z \rightarrow X$ is the projection from the DM-stack used in Lemma 2.3 to the original variety, then the bundle $E = p^*(F)$ which corresponds to F is indeed the pullback as defined

two paragraphs ago (which would be a parabolic bundle on the DM-stack Z but which is actually a regular bundle with trivial parabolic structure).

2.3. Chern character of parabolic bundles. Mumford, Gillet, Vistoli ([Mu],[Gi],[Vi]) have defined Chow groups for DM-stacks. Starting with (X, D) and choosing denominators n_i we obtain the stack $Z := X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$. The coarse moduli space of Z is the original X , so from [Gi, Theorem 6.8], the pullback and pushforward maps establish an isomorphism of rational Chow groups

$$CH^*(Z)_{\mathbb{Q}} \cong CH^*(X)_{\mathbb{Q}}.$$

We also have a Chern character for bundles on a DM-stack [Gi, section 8]. If F is a locally abelian parabolic bundle on (X, D) , choose n_i divisible by the denominators of the weights of F and define the Chern character of F by

$$\text{ch}^{\text{par}}(F) := p_*(\text{ch}(p^*F)) \in CH^*(X)_{\mathbb{Q}}.$$

This doesn't depend on the choice of n_i , by Lemma 2.4, and the compatibility of the Chern character with pullbacks on DM-stacks.

Using the fact that Chern character commutes with pullback for bundles on DM-stacks, we obtain:

Lemma 2.6. *Suppose $f : (X', D') \rightarrow (X, D)$ is a morphism of smooth varieties with normal-crossings divisors, such that $f^{-1}(D)$ is supported on D' . If F is a locally abelian parabolic bundle on (X, D) then the pullback f^*F is a locally abelian parabolic bundle on (X', D') and this construction commutes with Chern character:*

$$\text{ch}^{\text{par}}(f^*F) = f^*(\text{ch}^{\text{par}}(F)) \in CH^*(X')_{\mathbb{Q}}.$$

□

If F is a usual vector bundle considered with its trivial parabolic structure, then $\text{ch}(F)$ is the usual Chern character of F (this is the case $n_i = 1$ in the definition). Also, if f is a morphism then the pullback $f^*(F)$ is again a usual vector bundle with trivial parabolic structure.

We can easily describe the pullback and Chern characters for parabolic line bundles. If $B = \sum b_i D_i$ is a rational divisor on X , supported on D , and if $f : (X', D') \rightarrow (X, D)$ is a morphism as before, we have the formula

$$f^*(\mathcal{O}_X(B)) = \mathcal{O}_{X'}(B')$$

where $B' = \sum b_i (f^*D_i)$ with $f^*(D_i)$ the usual pullback of Cartier divisors. Similarly, if E is a vector bundle on X then we have

$$f^*(E(B)) = (f^*E)(B').$$

For the Chern character, recall that $\text{ch}(\mathcal{O}_X(B)) = e^B$ for a divisor with integer coefficients B . This formula extends to the case of a rational divisor, to give the formula for the

Chern character of a parabolic line bundle in the rational Chow group. More generally for a vector bundle twisted by a rational divisor we have

$$\mathrm{ch}^{\mathrm{par}}(E(B)) = \mathrm{ch}(E) \cdot e^B.$$

The Chern character is additive on the K_0 -group of vector bundles on a DM-stack, so (modulo saying something good about choosing appropriate denominators whenever we want to apply the formula) we get the same statement for the Chern character on the K_0 -group of locally abelian parabolic bundles. One should say what is an *exact sequence* of locally abelian parabolic bundles. Of course a morphism of parabolic sheaves $F \rightarrow G$ was just a collection of morphisms of sheaves $F_\alpha \rightarrow G_\alpha$. A short sequence

$$0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$$

of parabolic sheaves is *exact* if the resulting sequences

$$0 \rightarrow F_\alpha \rightarrow G_\alpha \rightarrow H_\alpha \rightarrow 0$$

are exact for all α . This notion is preserved by the equivalence of Lemma 2.3 and indeed more generally the pullback functor f^* preserves exactness. All objects in these sequences are bundles so there are no higher *Ext* sheaves and in particular there is no requirement of flatness when we say that f^* preserves exactness, although the p^* of Lemma 2.3 is flat.

Thus we can define the K_0 of the category of locally abelian parabolic bundles, and the Chern character is additive here. Also the Chern character is multiplicative on tensor products (this might need some further proof which we don't supply here). And the pullback f^* is defined on the K_0 group.

Caution: The Chern character doesn't provide an isomorphism between K_0 and the Chow group, on a DM-stack. For this one must consider an extended Chern character containing further information over the stacky locus, see [To] [Ch-Ru] [AGV]. Thus, the same holds for parabolic bundles: two parabolic bundles with the same Chern character are not necessarily the same in the K_0 -group of parabolic bundles.

We would like to obtain a formula for the Chern character of a parabolic bundle F in terms of the Chern characters of the constituent bundles F_α . This point seems to be somewhat delicate, for example D. Panov in his thesis [Pa] points out that some of the formulae obtained by Biswas [Bi] contain errors.

In order to obtain a formula, we will indicate a semi-canonical way to resolve a locally abelian parabolic bundle. We obtain a resolution whose components are direct sums of parabolic bundles of the form $E(B)$ for vector bundles E and \mathbb{Q} -divisors B . The resolution is semi-canonical because it depends on a choice of subsets of rational numbers containing at least the weights.

Start with a parabolic bundle F on (X, D) . Choose for each component D_i of D , a finite set of rational numbers $A_i \subset \mathbb{Q}$. Assume that A_i contains at least one \mathbb{Z} -translate of every weight of F along D_i . For any such A_i , let \overline{A}_i denote the corresponding one-dimensional

linear graph, the vertices being the rational numbers and the edges being the nonempty bounded intervals between them.

For any α let F_α denote the locally free sheaf which is a component of F , considered itself as a bundle.

We can construct a resolution of the parabolic bundle F whose terms are indexed by the cells in the product $\overline{A}_1 \times \cdots \times \overline{A}_k$ as follows. A cell c is a product of a certain number of intervals with a certain number of single weights. Let $\alpha_i(c)$ and $\beta_i(c)$ be the lower and upper weights (which are equal for terms corresponding to vertices, and different for terms corresponding to intervals). The dimension d of the cell is the number of intervals. For this cell, put a term in degree $-d$, equal to $F_{\alpha(c)} \otimes \mathcal{O}(-\beta(c))$. The maps in the complex are \pm the canonical inclusions. This gives a complex in degrees $-k, \dots, 0$.

Lemma 2.7. *The complex constructed above resolves the parabolic bundle F .*

Proof. The complex is well defined and maps to F . To see that it is a resolution, the question is local and by the locally abelian hypothesis we can assume that F is a rank one parabolic bundle. In particular, in this case there is only one nontrivial weight. One possibility is therefore that F is resolved by itself considered as $\mathcal{O}(-\alpha)$ for an appropriate α . Thus, the only task is to see that if we choose A_i to contain more indices than necessary, it is still a resolution. \square

Corollary 2.8. *If $F = \{F_\alpha\}$ is a locally abelian parabolic bundle. Let A_i be a set of weights for F along the divisor D_i . Then we have the following formula for the Chern character of F in the rational Chow groups:*

$$\mathrm{ch}^{\mathrm{par}}(F) = \sum_c (-1)^{\dim c} \mathrm{ch}(F_{\alpha(c)}) e^{-\sum_i \beta_i(c) D_i}$$

The sum is over the cells in the cubical complex $\overline{A}_1 \times \cdots \times \overline{A}_k$, and as above $\alpha(c)$ and $\beta(c)$ are the upper and lower bounds for c .

Corollary 2.9. *Suppose F is a parabolic bundle. Then the difference between F and any one of its F_α in the rational Chow ring of X or equivalently the rational K -theory of X , is an element which is concentrated over D (i.e. it may be represented by a rational combination of sheaves concentrated on D).*

Corollary 2.10. *If $\mathrm{ch}^{\mathrm{par}}(F) \in CH^0(X)_\mathbb{Q}$ for a parabolic bundle, then any F_α is equivalent to an element coming from $CH^*(D)$.*

3. THE PARABOLIC BUNDLE ASSOCIATED TO A LOGARITHMIC CONNECTION

Suppose as above that X is a smooth projective variety with normal crossings divisor D , and let $U := X - D$. Write as before $D = D_1 \cup \cdots \cup D_k$ the decomposition of D into smooth irreducible components.

Suppose (E, ∇) is a vector bundle with logarithmic connection ∇ on X such that the singularities of ∇ are concentrated over D . Fix a component D_i of the divisor. Define the *residue* of (E, ∇) along D_i to be the triple $(E|_{D_i}, \nabla_{D_i}, \eta_i)$ where $E|_{D_i}$ is the restriction of E to D_i ; and ∇_{D_i} is the restriction of ∇ to operations of vector fields tangent to D_i . The connection ∇_{D_i} is logarithmic with respect to the divisor in D_i given by the remaining components of D . Finally, the residual transformation η_i is the action of the vector field $z \frac{\partial}{\partial z}$ where z is a local coordinate of D_i . This is independent of the choice of local coordinate, and η_i is an automorphism of the bundle with connection $(E|_{D_i}, \nabla_{D_i})$, in particular the eigenvalues of η_i are locally constant functions along D_i (actually constant because we assume D_i irreducible).

Assume the following condition:

Definition 3.1. *We say that (E, ∇) has rational residues if the eigenvalues of the residual transformations of ∇ along components of D (that is, the η_i above) are rational numbers.*

In this case we will construct a parabolic vector bundle $F = \{F_\alpha\}$ on (X, D) together with isomorphisms

$$F_\alpha|_U \cong E|_U.$$

We require that ∇ extend to a logarithmic connection on each F_α . Finally, we require that the residue of ∇ on the piece $F_\alpha/F_{\alpha-\varepsilon\delta^i}$ which is concentrated over D_i , be an operator with eigenvalue $-\alpha_i$. A parabolic bundle with these data and properties is called a *parabolic bundle associated to (E, ∇)* .

The notion of a parabolic bundle associated to a connection was discussed in [In] and [IIS]. In those places, no restriction is placed on the residues, and parabolic structures with arbitrary complex numbers for weights are considered. They use full flags and treat the case of curves. It isn't immediately clear how the full flags would generalize for normal crossings divisors in higher dimensions.

Lemma 3.2. *For any vector bundle with logarithmic connection having rational residues, there exists a unique associated parabolic bundle, and furthermore it is locally abelian.*

Proof. This construction is basically the one discussed by Deligne in [De1] Proposition 5.4 and attributed to Manin [Man]. Suppose E is a vector bundle with a logarithmic connection on (X, D) . Consider the residue $\eta_i \in \text{End}(E|_{D_i})$ whose eigenvalues α_i^j are rational numbers. The generalized eigenspaces corresponds to a filtration $\{F_i^{\alpha_i^j}\}$ by vector bundles with logarithmic connection of $E|_{D_i}$. Furthermore, the filtration corresponds to the string of rational numbers $\alpha_i^1 < \alpha_i^2 < \cdots < \alpha_i^{n_i}$ so that α_i^j is the weight associated to $F_i^{\alpha_i^j}$ and $F_i^{\alpha_i^j} = F_i^{\alpha_i^{j-1}}/A_{\alpha_i^j}$. Here $A_{\alpha_i^j}$ is the eigenspace for α_i^j . Define the subbundles $\overline{F_i^{\alpha_i^j}}$ of E by the exact sequence:

$$0 \longrightarrow \overline{F_i^{\alpha_i^j}} \longrightarrow E \longrightarrow E|_{D_i}/F_i^{\alpha_i^j} \longrightarrow 0.$$

Given a set of rational weights $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, i.e, α_i is a weight along D_i , we can associate a subsheaf

$$F_\alpha = \sum_{i=1}^k \overline{F_i^{\alpha_i}}$$

of E . This defines a parabolic structure on (X, D) . Clearly ∇ restricts to a logarithmic connection on each $F_i^{\alpha_i^j}$. Hence it induces a logarithmic connection on each $\overline{F_i^{\alpha_i^j}}$ and thus on F_α . Moreover, each F_α restricts on U to $E|_U$.

Locally abelian property is easily checked using the sum decomposition of each F_α . \square

Another way of saying the previous lemma is to say that we choose n_i such that the residues of ∇ along D_i are integer multiples of $\frac{1}{n_i}$. Then the parabolic bundle associated to (E, ∇) corresponds to the Deligne canonical extension of (E, ∇) over the Deligne-Mumford stack $X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$, see §2.2. Note that the pullback of (E, ∇) to a logarithmic connection over this Deligne-Mumford stack, has integer residues so exactly the construction of [De1] can be carried out here. One has to say something about uniqueness of the construction implying descent from an etale covering of the stack to the stack itself.

Lemma 3.3. *Let F be the parabolic bundle associated to a logarithmic connection. On the bundle F_α the eigenvalues of the residue of ∇ along D_i are contained in the interval $[-\alpha_i, 1 - \alpha_i)$.*

Proof. The restriction to D_i may be expressed as

$$F_\alpha|_{D_i} = F_\alpha / F_{\alpha - \delta^i},$$

so the condition in the definition of associated parabolic bundle fixes the eigenvalues on the graded pieces here as being in the required interval. \square

The proof of 3.2 also shows that

Lemma 3.4. *Given an exact sequence of logarithmic connections with rational residues along D :*

$$0 \longrightarrow E'' \longrightarrow E \longrightarrow E' \longrightarrow 0$$

there is an exact sequence of parabolic bundles

$$0 \longrightarrow F_{E''} \longrightarrow F_E \longrightarrow F_{E'} \longrightarrow 0$$

such that $F_{E''}, F_E, F_{E'}$ are the parabolic bundles associated to E'', E and E' respectively. Similarly the construction is compatible with direct sums.

Lemma 3.5. *Suppose $Y \subset X$ is a subvariety intersecting D transversally. Let F be the parabolic bundle associated to a vector bundle with logarithmic connection (E, ∇) over X . Then $F|_Y$ is the parabolic bundle associated to the restriction of (E, ∇) to Y .*

Proof. Since the intersection of Y and D is transversal, D restricts to a normal crossing divisor D_Y on Y . The restriction of (E, ∇) on Y has rational residues $\{\alpha_i\}$ which is a subset of the rational residues of (E, ∇) and corresponds to those sheaves F_{α_i} which restricted to Y are nonzero. In other words, using the construction of parabolic structure in Lemma 3.2, it follows that $F|_Y$ is the parabolic bundle associated to the restriction of (E, ∇) to Y . \square

3.1. Computing cohomology using the associated parabolic bundle. Suppose X is a smooth projective variety with normal crossings divisor $D \subset X$ and let $U := X - D$. Let $j : U \rightarrow X$ denote the inclusion.

Suppose (E, ∇) is a vector bundle on X with logarithmic connection and rational residues along D , with associated parabolic bundle F . Let $DR(E, \nabla)$ denote the de Rham complex

$$E \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^1(\log D) \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^2(\log D) \rightarrow \dots$$

with d_{∇} as differential.

Define the *logarithmic de Rham cohomology*

$$\mathcal{H}_{DR}(X; E, \nabla) := \mathbb{H}(X, DR(E, \nabla)).$$

In this notation the fact that it is logarithmic along D is encoded in the fact that ∇ is a logarithmic connection along D . On the other hand, let E_U^{∇} denote the local system on U of flat sections for the connection ∇ .

Theorem 3.6. (Deligne) *Suppose that the residues of ∇ have no strictly positive integers as eigenvalues. Then*

$$\mathcal{H}_{DR}^i(X; E, \nabla) \cong H^i(U, E_U^{\nabla}).$$

Proof. This is [De1], Proposition 3.13 and Corollaire 3.14. \square

Corollary 3.7. *Suppose (E, ∇) is a logarithmic connection. Let F denote the associated parabolic bundle, so that for any α we obtain again a logarithmic connection (F_{α}, ∇) . If $\alpha = (\alpha_1, \dots, \alpha_k)$ with all $\alpha_i > 0$ then*

$$\mathcal{H}_{DR}^i(X; F_{\alpha}, \nabla) \cong H^i(U, E_U^{\nabla}).$$

Proof. By Lemma 3.3 the eigenvalues of the residue of ∇ on F_{α} are in $[-\alpha_i, 1 - \alpha_i)$. If $\alpha_i > 0$ this is guaranteed not to contain any positive integers. \square

3.2. The relative logarithmic de Rham complex. Suppose (E, ∇) is a vector bundle with logarithmic connection relative to a pair (X, D) where X is smooth and D is a normal crossings divisor. In practice, E will be of the form F_{α} as per Theorem 3.7.

Suppose we have a map $f : X \rightarrow Y$ such that components of D are either over the singular divisor J in Y , or have relative normal crossings over Y [De1]. We have the

logarithmic complex $\Omega_X(\log D)$ and the subcomplex $f^*\Omega_Y(\log J)$. Put

$$\Omega_{X/Y}(\log D) := \Omega_X(\log D) / f^*\Omega_Y(\log J).$$

We can form a differential d_∇ using ∇ on $E \otimes \Omega_{X/Y}(\log D)$. Call the resulting logarithmic de Rham complex $DR(X/Y, E, \nabla)$. In this notation the divisors D and J are implicit.

Define a complex of quasicoherent sheaves on Y as follows:

$$\mathcal{H}_{DR}(X/Y; E, \nabla) := \mathbb{R} f_* DR(X/Y, E, \nabla).$$

To define this precisely, choose an open covering of X by affine open sets X_β such that the map f is affine on the multiple intersections. Let $\check{C}DR(X/Y, E, \nabla)$ denote the simple complex associated to the double complex obtained by applying the Čech complex in each degree. The components are acyclic for f_* so we can set

$$\mathbb{R} f_* DR(X/Y, E, \nabla) := f_* \check{C}DR(X/Y, E, \nabla).$$

The component sheaves of the Čech complex on X are quasicoherent but the differentials are not \mathcal{O}_X -linear. However, the differentials are $f^{-1}\mathcal{O}_Y$ -linear, so the direct image complex consists of quasicoherent sheaves on Y .

The complex $DR(X/Y, E, \nabla)$ is filtered by the Hodge filtration which is the “filtration bête”. This presents $DR(X/Y, E, \nabla)$ as a successive extension of complexes $E \otimes \Omega_{X/Y}^m(\log D)[-m]$. The higher derived direct image becomes a successive extension of the higher derived direct images of these component complexes, for example this is exactly true if we use the Čech construction above.

The standard argument of Mumford [Mu] shows that each

$$\mathbb{R} f_* E \otimes \Omega_{X/Y}^m(\log D)[-m]$$

is a perfect complex over Y and formation of this commutes with base changes $b : Y' \rightarrow Y$. Therefore the same is true of the successive extensions. Thus formation of $\mathcal{H}_{DR}(X/Y; E, \nabla)$ commutes with base change.

At this point we are imprecise about the conditions on Y' and b . All objects should be defined after base change. For example if $X' := Y' \times_Y X$ is smooth and the pullbacks of D to X' and J to Y' are normal crossings, then everything is still defined. In §6 we will want to consider a more general situation in which the logarithmic de Rham complex is still defined. To be completely general, refer to Illusie [Il] for example for an intrinsic definition, but we could take as definition

$$DR(X'/Y', E', \nabla') := pr_2^* DR(X/Y, E, \nabla)$$

for $pr_2 : X' \rightarrow X$ the projection, and with that the base-change formula holds for any $b : Y' \rightarrow Y$.

This gives the following lemma which is well known.

Lemma 3.8. *Suppose (E, ∇) is a vector bundle with logarithmic connection on (X, D) , with nilpotent residues on the vertical components of D . The relative logarithmic de Rham*

cohomology $\mathcal{H}_{DR}(X/Y; E, \nabla)$ is a perfect complex on Y and if $b : Y' \rightarrow Y$ is a morphism then letting $X' := Y' \times_Y X$ and (E', ∇') be the pullbacks, we have a quasiisomorphism

$$\mathcal{H}_{DR}(X'/Y'; E', \nabla') \stackrel{qi}{\sim} b^* \mathcal{H}_{DR}(X/Y; E, \nabla).$$

The complex $\mathcal{H}_{DR}(X/Y; E, \nabla)$ has a logarithmic Gauss-Manin connection which induces the usual Gauss-Manin connection on the cohomology sheaves. This is well known from the theory of \mathcal{D} -modules. We discuss it in §6.

As Deligne says in [De1] 3.16, the calculation of cohomology also works in the relative case. Restrict over the open set U which is the complement of J . Then $X_U \rightarrow U$ is a smooth map and the divisor $D_U := D \cap X_U$ is a union of components all of whose multiple intersections are smooth over Y . Let $W := X_U - D_U$. Let $f_W : W \rightarrow U$ be the map restricted to our open set. With the current hypotheses, topologically it is a fibration. The relative version of Lemma 3.6 says the following.

Lemma 3.9. *In the above situation, suppose that the eigenvalues of the residue of ∇ along horizontal components, that is components of D_U , are never positive integers. Then the vector bundle with Gauss-Manin connection $\mathcal{H}_{DR}^i(X_U/U; E_U, \nabla)$ on U is the unique bundle with connection and regular singularities corresponding to the local system $R^i f_{W,*} \mathbb{C}_W$ on U .*

□

3.3. Steenbrink's result. Steenbrink investigates the relationship between the relative logarithmic de Rham cohomology which is defined on all of X , and topological data. This gives, among other things, a sort of base-change result for the relative logarithmic de Rham cohomology. An analogue of Steenbrink's theorem ([St, Theorem 2.18]) is the following statement, whose proof will be given in section 6.

Proposition 3.10. *Suppose that the family f is semistable of relative dimension one and the residues of the logarithmic connection on E are nilpotent on the vertical components of D . Then the cohomology sheaves $\mathcal{H}_{DR}^i(X/Y; E, \nabla)$ are locally free, and locally on Y the perfect complex splits as a direct sum of its cohomology sheaves. The residues of the Gauss-Manin connection on the cohomology sheaves are nilpotent, so the $\mathcal{H}_{DR}^i(X/Y; E, \nabla)$ are the canonical extensions of their restrictions to the open set $U := Y - J$.*

Notice that Lemma 3.2 say that if residues are nilpotent (i.e., the eigenvalues of the residues are zero) then the parabolic bundle is the same as the Deligne canonical extension. In other words, the Deligne canonical extension has a trivial parabolic structure.

3.4. Chern characters for higher direct images. We now introduce some notation for calculations in Chow groups. Suppose $f : X \rightarrow Y$ is a flat morphism. If E is a vector

bundle on X then its higher direct image complex $\mathbb{R}f_*(E)$ is a perfect complex on Y and we can define its Chern character by

$$\mathrm{ch}(\mathbb{R}f_*(E)) := \sum_i (-1)^i \mathrm{ch}(\mathbb{R}^i f_*(E)).$$

The Grothendieck-Riemann-Roch theorem implies the following observation which says that the Chern character of the higher direct image complex is a function of the Chern character of E .

Proposition 3.11. *Suppose $f : X \rightarrow Y$ is a flat morphism. Then there is a map on rational Chow groups*

$$\chi_{X/Y} : CH(X)_{\mathbb{Q}} \rightarrow CH(Y)_{\mathbb{Q}}$$

which represents the Euler characteristic for higher direct images in the sense that

$$\mathrm{ch}(\mathbb{R}f_*(E)) = \chi_{X/Y}(\mathrm{ch}(E))$$

for any vector bundle or coherent sheaf E on X .

Proof. The GRR formula gives an explicit formula for $\chi_{X/Y}$. Alternatively in the case where the varieties in question are smooth, one can note that the Chow groups calculate the rational K_0 and it is clear that $\mathrm{ch} \circ \mathbb{R}f_*$ is additive on exact sequences so it passes to a function on K_0 . \square

Esnault-Viehweg in [Es-Vi2] use the explicit formula for $\chi_{X/Y}$. We use less of GRR here in the sense that we use only existence of the function rather than calculating with the formula.

3.5. Modification of the base. Consider the situation of a family $f : X \rightarrow S$. Let \mathcal{O}_X denote the trivial bundle with its standard connection on X . We assume that S is irreducible. Let $U \subset S$ denote a nonempty open set over which f is smooth, and let $X_U := X \times_S U$.

The $\mathcal{H}_{DR}^i(X_U/U)$ are vector bundles on U . If the complementary divisor $J := S - U$ has normal crossings then the Gauss-Manin connections on the cohomology vector bundles have logarithmic singularities with rational residues (this last statement is the content of the well-known ‘‘monodromy theorem’’). Let

$$\overline{\mathcal{H}}_{DR}^i(X_U/U)$$

denote the parabolic bundle on S associated to this logarithmic connection. Our goal is to investigate the alternating sum of the Chern characters of these bundles on S . In view of this goal, we can make the following reduction.

Lemma 3.12. *In the above situation, suppose $p : S' \rightarrow S$ is a generically finite surjective map from an irreducible variety to S , and if $U' \subset S'$ is a nonempty open set mapping to U , let $X' := X \times_S S'$. Assume that $J' := S' - U'$ again has normal crossings. Then the parabolic bundle $\overline{\mathcal{H}}_{DR}^i(X_U/U)^{\mathrm{par}}$, which is defined as the one which corresponds to the*

Gauss-Manin connection on $\mathcal{H}_{DR}^i(X'_{U'}/U')$, is isomorphic to the pullback from S to S' of the parabolic bundle $\overline{\mathcal{H}}_{DR}^i(X_U/U)^{\text{par}}$ on S .

Proof. Notice that over U' the statement is true for the cohomology bundle, since it is a base-change theorem for smooth morphisms. Now by Lemma 2.5, we deduce the statement for the parabolic bundle as well. \square

Corollary 3.13. *Let $p : S' \rightarrow S$ and U' be as in the previous lemma. Then*

$$\sum_i (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}_{DR}^i(X_U/U)) = p^* \sum_i (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}_{DR}^i(X_U/U)).$$

In particular, $\sum_i (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}_{DR}^i(X_U/U))$ is in the degree zero piece of the rational Chow group of S , if and only if $\sum_i (-1)^i \text{ch}(\overline{\mathcal{H}}_{DR}^i(X_U/U)^{\text{par}})$ is in the degree zero piece of the rational Chow group of S' .

\square

Since our goal is to prove that the alternating sum in question at the end of the previous corollary is in the degree zero piece of the rational Chow group, we can safely replace $X \rightarrow S$ and the open set U by the base-change $X' \rightarrow S'$ whenever $S' \rightarrow S$ is a generically finite rational map, with any nonempty open subset $U' \subset S'$. In what follows we will often make this type of reduction. There are several different flavours, for example we could simply decrease the size of the open set U ; we could do any kind of birational transformation on S ; or we could take a finite covering of an open set of S and complete to a smooth variety.

In particular by using finite coverings we may assume that the monodromy transformations at infinity are unipotent rather than just quasi-unipotent. In this case, the associated parabolic extension is just a regular bundle and it is the same as the Deligne canonical extension.

In order to avoid overly heavy notation in what follows, we will keep the same letters when we replace S, X, U and speak of this as “making a generically finite modification of S ,” or “passing to a smaller open set U ” or some similar such phrase.

4. LEFSCHETZ FIBRATIONS

In this subsection, we recall the fixed and the variable cohomology for Lefschetz fibrations and basic lemmas without proof. We refer to [Kz], for the details.

Suppose $X \subset \mathbb{P}^N$ is a nonsingular projective variety of dimension n , defined over the complex numbers. Consider the dual variety $\hat{X} \subset \hat{\mathbb{P}}^N$. A Lefschetz pencil is a projective line $\mathbb{P}^1 \subset \hat{\mathbb{P}}^N$ such that its intersection with \hat{X} is a reduced zero-dimensional subscheme. Furthermore, these points correspond to hyperplane sections of X with only one ordinary double point as its singularity and are usually called the critical points.

Then there is a Lefschetz fibration

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & X \\ \downarrow \varphi & & \\ \mathbb{P}^1 & & \end{array}$$

Here $Z = \{(x, H) \in X \times \mathbb{P}^1 : x \in H \cap X\}$ is a nonsingular projective variety and is the blow-up of X along the base locus $B \subset X$ defined by the Lefschetz pencil $\mathbb{P}^1 \subset \hat{\mathbb{P}}^N$.

Lemma 4.1. *We have a decomposition of the cohomology*

$$H^i(Z, \mathbb{Q}) = H^i(X, \mathbb{Q}) + H^{i-2}(B, \mathbb{Q}).$$

Further, if Y is any nonsingular fibre of the fibration $Z \rightarrow \mathbb{P}^1$ then the restriction map

$$H^i(Z, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q})$$

is the same as the restriction

$$H^i(X, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q}).$$

Definition 4.2. *The image of the restriction map*

$$H^i(Z, \mathbb{Q}) \rightarrow H^i(Y, \mathbb{Q})$$

is called the fixed cohomology and is denoted by $H_{fix}^i(Y, \mathbb{Q})$. The orthogonal subspace to $H_{fix}^i(Y, \mathbb{Q})$ for the intersection form is called the variable cohomology.

Remark 4.3. *The Lefschetz hyperplane section theorem and Lemma 4.1 imply that*

$$H^i(X, \mathbb{Q}) \simeq H_{fix}^i(Y, \mathbb{Q}) = H^i(Y, \mathbb{Q}), \text{ for } i < n - 1$$

and

$$H^n(X, \mathbb{Q}) \simeq H_{fix}^n(Y, \mathbb{Q}).$$

Consider the fibration $\varphi : Z \rightarrow \mathbb{P}^1$ and the inclusion $j : Z \hookrightarrow X \times \mathbb{P}^1$ together with the second projection $\beta : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$. Let $K \hookrightarrow \mathbb{P}^1$ be the finite set of critical points and consider the open embedding $\mathbb{P}^1 - K \xrightarrow{t} \mathbb{P}^1$.

We have

Lemma 4.4.

$$R^i \beta_* \mathbb{Q} \simeq R^i \varphi_* \mathbb{Q}, \text{ for } i < n - 1$$

and

$$R^{n-1} \varphi_* \mathbb{Q} \simeq R_{fix}^{n-1} \varphi_* \mathbb{Q} \oplus t_* \mathcal{V}^\nabla.$$

Here \mathcal{V}^∇ (resp. $R_{fix}^{n-1} \varphi_* \mathbb{Q}$) is the local system associated to the variable (resp. fixed) cohomology on $\mathbb{P}^1 - K$ and

$$R^{n-1} \beta_* \mathbb{Q} \simeq R_{fix}^{n-1} \varphi_* \mathbb{Q}.$$

Proof. The inclusion j induces a morphism of local systems

$$j^* : R^i \beta_* \mathbb{Q} \longrightarrow R^i \varphi_* \mathbb{Q}.$$

By Remark 4.3, the morphism induced at the stalks is an isomorphism if $i < n - 1$ and is injective if $i = n - 1$. Further, the image $j^*(R^{n-1} \beta_* \mathbb{Q})$ is the local system $R_{fix}^{n-1} \varphi_* \mathbb{Q}$, corresponding to the fixed cohomology on the fibres. The variable cohomology corresponds to the local system \mathcal{V}^∇ on $\mathbb{P}^1 - K$ and is a direct summand of $R^i \varphi_* \mathbb{Q}$. Hence the extensions on \mathbb{P}^1 gives a decomposition of $R^i \varphi_* \mathbb{Q}$, as asserted. \square

Remark 4.5. *We remark that if n is odd, then the monodromy transformations of the Lefschetz fibration in the middle dimensional cohomology, are reflections (i.e., they have order 2 with one eigenvalue as -1 and the rest as $+1$). In particular, they are not unipotent transformations.*

Many of the facts we have recalled above will not actually be used in the argument in §4.2 and §5 below. We have included them here because they are relevant to the original goal of this project which was to try to use Lefschetz devissage to attack the triviality of Chern characters of the individual cohomology bundles. We were unable to make any progress on this question because of the problem that the local system $R^{n-1} \varphi_* \mathbb{Q}$ divides into two pieces, the fixed and variable parts, and these contribute to cohomology in different degrees. Thus we could only treat the alternating sum as in [Es-Vi2].

4.1. Semistable reduction for families of rational curves. We need the genus 0 version of semistable reduction as proven by de Jong [dJ]. We start with a morphism $P \rightarrow S$ whose fibers are \mathbb{P}^1 . Suppose given a divisor $K \subset P$ which over a general point of S decomposes into a disjoint union of sections. Then there is an open set $U \subset S$ over which we have a family of marked rational curves. This gives a morphism $U \rightarrow \mathbf{M}_{0,n}$ to the moduli stack of marked rational curves. Use the Deligne-Mumford compactification $\overline{\mathbf{M}}_{0,n}$. In fact there is a finite covering corresponding to some level structure, denoted $\overline{\mathbf{M}}_{0,n}^\gamma \rightarrow \overline{\mathbf{M}}_{0,n}$, such that $\overline{\mathbf{M}}_{0,n}^\gamma$ is a scheme and has a universal family over it.

Let S_1 be the closure in $S \times \overline{\mathbf{M}}_{0,n}$ of the graph of the morphism which was defined over U , and let S_2 be a resolution of singularities. Then let S_3 be a resolution of singularities of the fiber product $S_2 \times_{\overline{\mathbf{M}}_{0,n}} \overline{\mathbf{M}}_{0,n}^\gamma$. We have a generically finite map $S_3 \rightarrow S$, and over S_3 there is the pullback of the universal family. This means that there is a modification P_3 of $P \times_S S_3$ with divisor $K_3 \subset P_3$ such that $(P_3, K_3) \rightarrow S_3$ is a stable family of marked curves. This is a quick sketch of the proof of the following lemma. For full details refer to [dJ].

Lemma 4.6. (de Jong) *Suppose $P \rightarrow S$ is a morphism whose general fibers are projective lines, with a divisor $K \subset P$. Then there is a finite covering and modification of the base $S' \rightarrow S$ and a family $P' \rightarrow S'$ with divisor $K' \subset P'$ such that $(P', K') \rightarrow S'$ is a semistable family of marked rational curves, and such that the open set $U' \subset S'$ over which the family*

is a smooth family of marked projective lines, has complement $D' := S' - U'$ which is a divisor with normal crossings. Further, the variety P' can be assumed to be smooth .

Proof. This is [dJ, Theorem 4.1, Proposition 3.6]. □

This generically finite modification of S will fit as a reduction such as was considered in Lemma 3.12 and Corollary 3.13.

4.2. A Lefschetz fibration for families. Suppose S is a smooth projective variety and $J \subset S$ is a normal crossings divisor with smooth components. Let $U := S - J$. Suppose $f : X \rightarrow U$ is a smooth projective morphism over U , where X is smooth. Let n denote the relative dimension of X over U .

In the following discussion we allow modification of the base S as described in Lemma 3.12 and Corollary 3.13.

Suppose we are given an embedding $X \rightarrow S \times \mathbb{P}^N$. Choose a general line $\mathbb{P}^1 \subset \mathbb{P}^N$. After possibly restricting to a smaller open set we may assume that this line defines a Lefschetz pencil for the subvariety X_u for every $u \in U$.

Let $P_0 := U \times \mathbb{P}^1$, with projection

$$q : P_0 \longrightarrow S$$

so that we have a family of Lefschetz pencils

$$X_U \xleftarrow{\alpha} Z_0 \xrightarrow{\varphi} P_0$$

where $\alpha : Z_0 \rightarrow X_U$ is blowing up the smooth family of base loci $B_0 \rightarrow U = S - J$, and the map $Z_0 \rightarrow P_0$ is a Lefschetz pencil of the fiber X_s for $s \in S - J$.

Let $K_0 \subset P_0$ denote the family of points over which the Lefschetz pencils have singular fibers. This is a divisor which is smooth over the base open set U : the different double points stay distinct as $u \in U$ moves by the hypothesis that we have a Lefschetz pencil for every $u \in U$.

By going to a finite covering of U we may suppose that each irreducible component of the divisor $K_0 \subset P_0$ maps isomorphically onto U . Thus we can consider (P_0, K_0) as a family of marked rational curves.

Apply the semistable reduction result Lemma 4.6, recalled in the previous subsection. After further modification of the base, then completing the families X, Z and B we obtain the following situation.

Keep the notations that $J \subset S$ is a divisor with normal crossings and $U := S - J$ an open subset. We have a semistable marked rational curve $q : (P, K) \rightarrow S$ with divisor

$D := K \cup q^{-1}(J) \subset P$. We have a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & P \\ \downarrow \alpha & & \downarrow q \\ X & \xrightarrow{f} & S. \end{array}$$

We also have a subvariety $B \subset X$ corresponding to the base loci.

If we denote in general by a subscript U the restriction of objects over the open set U , then $P_U \cong U \times \mathbb{P}^1$ and

$$X_U \xleftarrow{\alpha_U} Z_U \xrightarrow{\varphi_U} P_U \cong U \times \mathbb{P}^1$$

is a family of Lefschetz pencils indexed by U . The divisor $K_U \subset P_U$ which is a union of components mapping isomorphically to U , is the divisor of singularities of φ_U . The subvariety $B_U \subset X_U$ is the family of base loci for the Lefschetz pencils.

Lemma 4.7. *There is a decomposition of local systems over U ,*

$$R^i \alpha_* \mathbb{Q} = R^0 q_* (R^i \varphi_* \mathbb{Q}) \oplus R^1 q_* (R^{i-1} \varphi_* \mathbb{Q}) \oplus R^2 q_* (R^{i-2} \varphi_* \mathbb{Q}).$$

Proof. See [De2, p.112]. □

We need a de Rham version of this statement. It would be good to have a direct analogue, however the singularities in the Lefschetz fibration are not normal crossings singularities. Thus, we consider an open subset W obtained by removing the singular fibers. In the start of the next section we will give a series of reductions which basically say that it suffices to look at this open set.

It should be possible to obtain a direct de Rham version of Lemma 4.7 by applying the general theory of \mathcal{D} -modules, see [CK].

Consider \mathcal{O}_{Z_U} the trivial vector bundle with standard connection d . This corresponds to the constant local system.

Taking the relative de Rham cohomology we obtain a vector bundle together with its Gauss-Manin connection

$$\mathcal{H}_{DR}^i(Z_U/U, (\mathcal{O}_{Z_U}, d)) := \mathbb{R}^i(f \circ \alpha)_* DR(Z_U/U, \mathcal{O}_{Z_U}, d)$$

on U . Let $W := P_U - K_U$ be the complement of the locus of singularities of the family of Lefschetz pencils. Let $Z_W := \varphi^{-1}(W) \subset Z$. The map φ is smooth over W .

Similarly using the morphism φ , define (E_W^i, ∇_W) to be the vector bundle with Gauss-Manin connection

$$E_W^i := \mathcal{H}_{DR}^i(Z_W/W, (\mathcal{O}_{Z_W}, d)) := \mathbb{R}^i \varphi_* DR(Z_W/W, \mathcal{O}_{Z_W}, d)$$

As usual this extends over P to a logarithmic connection with rational residues [St]. Let (E^i, ∇) denote the associated parabolic bundle on P , and let (E_U^i, ∇_U) denote the restriction to P_U which is also the associated parabolic bundle on P_U . Note that (E^i, ∇) has singularities along the divisor $D := K \cup q^{-1}(J)$ so (E_U^i, ∇_U) has singularities along K_U .

For any multi-index α for the divisor D we obtain a vector bundle with logarithmic connection (E_α^i, ∇) on P with singularities along D . Let $(E_{\alpha,U}^i, \nabla_U)$ be its restriction to U . This is the same as the corresponding bundle associated to the parabolic bundle E_U^i for multi-index α_U which contains only the indices for the components of the divisor K_U .

In what follows we will also consider the morphism $Z_W \rightarrow U$ which doesn't have projective fibers. We can still define the relative de Rham cohomology $\mathcal{H}_{DR}^m(Z_W/U, (\mathcal{O}_Z, d))$ which is again a vector bundle with integrable Gauss-Manin connection on U . This has regular singularities and corresponds to the local system $R^m(\varphi_W \circ q_W)_* \mathbb{C}_{Z_W}$ on U . For our present purposes this can be considered as a matter of notation: the $\mathcal{H}_{DR}^m(Z_W/U, (\mathcal{O}_Z, d))$ can be defined as the unique bundle with regular singular integrable connection on U corresponding to the local system $R^m(\varphi_W \circ q_W)_* \mathbb{C}_{Z_W}$, so we don't need to explain in detail how to construct it which would involve resolving the double point singularities in the Lefschetz fibration and then taking relative logarithmic de Rham cohomology.

Lemma 4.8. *Assume that $\alpha_i > 0$ for indices corresponding to components of the divisor K . The Leray spectral sequence for $Z_W \rightarrow W \rightarrow U$ is*

$$E_2^{i,j} = \mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla) \Rightarrow \mathcal{H}_{DR}^{i+j}(Z_W/U, (\mathcal{O}_Z, d)).$$

This is a spectral sequence in the category of vector bundles over U with integrable connection having regular singularities at the complementary divisor J .

Proof. The category of vector bundles over U with integrable connection having regular singularities at the complementary divisor J is equivalent to the category of local systems over U , by [De1]. Therefore it suffices to have the spectral sequence for the associated local systems.

The Leray spectral sequence for $Z_W \xrightarrow{\varphi_W} W \xrightarrow{q_W} U$ is

$$E_2^{i,j} = R^i q_{W,*} (R^j \varphi_{W,*} \mathbb{C}_W) \Rightarrow R^{i+j} (q_W \circ \varphi_W)_* \mathbb{C}_W.$$

By Corollary 3.7 in each fiber, we have that

$$\mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^k, \nabla)$$

is the vector bundle with regular singular integrable connection corresponding to the local system

$$R^i q_{W,*} (R^j \varphi_{W,*} \mathbb{C}_W).$$

Note that the Gauss-Manin connection on $\mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$ is known to have regular singularities, in fact we shall use its extension as a logarithmic connection given by the relative logarithmic de Rham cohomology over S .

On the other hand,

$$\mathcal{H}_{DR}^{i+j}(Z_W/U, (\mathcal{O}_Z, d))$$

is the vector bundle with integrable connection corresponding to the local system $R^{i+j} (q_W \circ \varphi_W)_* \mathbb{C}_W$. If we use the paragraph before the statement of the present lemma then this is by definition, even.

In view of the equivalence of categories, we get the desired spectral sequence. \square

Let

$$\overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$$

and

$$\overline{\mathcal{H}}_{DR}^{i+j}(Z_W/U, (\mathcal{O}_Z, d))$$

denote the parabolic bundles on S associated to these vector bundles with logarithmic connection on U .

Corollary 4.9. *Assume that $\alpha_i > 0$ for indices corresponding to components of the divisor K . In the Chow group of S tensored with \mathbb{Q} we have*

$$\sum_m (-1)^m \text{ch}^{\text{par}} \overline{\mathcal{H}}_{DR}^m(Z_W/U, (\mathcal{O}_Z, d)) = \sum_{i,j} (-1)^{i+j} \text{ch}^{\text{par}} \overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla).$$

Proof. Formation of the associated parabolic bundle commutes with exact sequences by Lemma 3.4. The spectral sequence of Lemma 4.8 thus gives the result. \square

5. MAIN THEOREM

5.1. The main statement. We now give the statement of our main theorem in the form suitable for our inductive argument.

Suppose S is a smooth projective variety and $A \subset S$ is a normal crossings divisor with smooth components. Let $U := S - A$. Suppose $f : X \rightarrow S$ is a projective morphism which is smooth over U , and let $X_U := X \times_S U$. Let

$$\mathcal{H}_{DR}^i(X_U/U) := \mathbb{R}^i f_*(\Omega_{X_U/U}, d)$$

denote the bundle of i -th relative de Rham cohomology bundle on U . It has a Gauss-Manin connection ∇ which is well-known to have regular singularities and rational residues ([St, Proposition 2.20]). Let $\overline{\mathcal{H}}_{DR}^i(X_U/U)$ denote the parabolic bundle associated to $(\mathcal{H}_{DR}^i(X_U/U), \nabla)$ on U .

We extend this notation to the case when $f : X \rightarrow S$ is not smooth. There is still an open set $U \subset S$ over which the map is topologically a fibration for the usual topology, and we assume that the complement has normal crossings. In this case, define $\mathcal{H}_{DR}^i(X_U/U)$ to be the unique vector bundle with regular singular connection still denoted ∇ which corresponds to the local system $R^i f_{U,*} \mathbb{C}_{X_U}$ over U . Again let $\overline{\mathcal{H}}_{DR}^i(X_U/U)$ denote the associated parabolic bundle on S .

We state a general form of our main theorem. It will be useful to have this generality in the inductive proof.

Theorem 5.1. *Fix n . Suppose X, A, U, Y, f are as above, including the case where f is projective but not generically smooth. Suppose that $f : X \rightarrow S$ has relative dimension*

$\leq n$. Then the alternating sum of Chern characters of the parabolic extensions to S

$$\sum_{i=0}^{2n} (-1)^i \text{ch}^{\text{par}}(\overline{\mathcal{H}}^i(X_U/U))$$

lies in $CH^0(S)_{\mathbb{Q}}$ or equivalently the pieces in all of the positive-codimension Chow groups with rational coefficients vanish.

The proof will be by induction on n and using Lefschetz pencils.

5.2. Preliminary reductions. We start with some reductions. The first one was discussed in §3.5, Lemma 3.12 and Corollary 3.13.

Lemma 5.2. *In proving the theorem for $f : X \rightarrow S$, we can modify S by a generically finite morphism $S' \rightarrow S$ and it suffices to prove the theorem for $X' \rightarrow S'$.*

□

Lemma 5.3. *In order to prove the theorem for n and for morphisms $X \rightarrow S$ which may not be smooth, it suffices to prove the theorem for any $Y_U \rightarrow U$ a smooth projective family of relative dimension $\leq n$.*

Proof. We prove the present lemma also by induction on n . Therefore, we may consider a family $f : X \rightarrow S$ of relative dimension n and assume that the theorem is known for arbitrary (not necessarily smooth) projective families of relative dimension $< n$.

After possibly making a modification of the base S as in Lemma 5.2, specially to decrease the size of the open set U and resolve singularities of the complementary divisor, we can assume that we have an open set $W \subset X$ such that $f : W \rightarrow U$ is smooth and topologically a fibration. Let $T := X - W$ and assume that $(X_U, T_U) \rightarrow U$ is topologically a fibration.

Choose another compactification $W \subset Y$ with a morphism $g : Y \rightarrow U$ such that g is smooth and projective. Let $V := Y - W$ be the complementary divisor which we assume to have relative normal crossings over U .

Fix a point $u \in U$. We have pairs (X_u, T_u) and (Y_u, V_u) . In both of these, the smooth complementary open set is W_u . The cohomology of either of these pairs is equal to the compactly supported cohomology of W_u . Poincaré duality says

$$H^i((X_u, T_u), \mathbb{C}) \cong H^{2n-i}(W_u, \mathbb{C}) \cong H^i((Y_u, V_u), \mathbb{C}).$$

On the other hand, we have a long exact sequence relating $H^*(X_u, \mathbb{C})$, $H^*(T_u, \mathbb{C})$, and $H^i((X_u, T_u), \mathbb{C})$. Similarly we have a long exact sequence relating $H^*(Y_u, \mathbb{C})$, $H^*(V_u, \mathbb{C})$, and $H^i((Y_u, V_u), \mathbb{C})$. As u varies in U , the Poincaré duality isomorphisms are isomorphisms of local systems over U , and the long exact sequences are long exact sequences in the category of local systems over U . In view of the fact that the associated parabolic bundle

commutes with isomorphisms and with taking exact sequences, we obtain a long exact sequence relating

$$\overline{\mathcal{H}}_{DR}(X_U/U), \quad \overline{\mathcal{H}}_{DR}(T_U/U), \quad \text{and} \quad \overline{\mathcal{H}}_{DR}^{2n-1}(W/U),$$

and a long exact sequence relating

$$\overline{\mathcal{H}}_{DR}(Y_U/U), \quad \overline{\mathcal{H}}_{DR}(V_U/U), \quad \text{and} \quad \overline{\mathcal{H}}_{DR}^{2n-1}(W/U).$$

The shift of indices induced by Poincaré duality is $2n$, an even number, so it doesn't affect the sign of the alternating sum. Thus we obtain

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(X_U/U)) - \sum_{i=0}^{2n-2} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(T_U/U)) = \sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(W/U))$$

and

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Y_U/U)) - \sum_{i=0}^{2n-2} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(V_U/U)) = \sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(W/U)).$$

In particular, by the induction hypothesis we know the statement of the theorem for $T \rightarrow S$ and $V \rightarrow S$ because these have relative dimension $\leq n-1$. On the other hand we also know the statement of the theorem for $Y \rightarrow S$, because this is the hypothesis of our present reduction lemma. Putting these together we obtain the statement of the theorem for $X \rightarrow S$. \square

Lemma 5.4. *The statement of the theorem is true in relative dimension $n = 0$.*

Proof. Indeed, in this case by making a generically finite modification of the base, we can reduce to the case when the morphism $X_U \rightarrow U$ is a disjoint union of sections isomorphic to U . Thus the relative cohomology sheaf, nontrivial only in degree zero, is a trivial bundle. \square

Remark: It would be good to have the statement of Theorem 5.1 for families which are not necessarily projective. Our method of proof gives the statement for smooth quasiprojective families. The reduction from the general case to these cases would seem to require a more detailed utilisation of Grothendieck-Verdier duality theory than we are prepared to do here.

5.3. Lefschetz pencil and reduction to the case of the open set Z_W/U . We now assume given a projective family $f : X \rightarrow S$ of relative dimension n such that X_U is smooth over U , and assume the theorem is known for relative dimension $\leq n-1$. After modification of the base as per Lemma 3.12, arrange to have a family of Lefschetz pencils as in §4.2. Keep the same notations Z, P, W from there.

We assume that we have modified S enough so that the monodromies of the local systems which we encounter on U or W are unipotent around components of J or vertical components of $f^{-1}(J)$. In this case, the extended bundles from U to S are actual bundles rather than parabolic bundles. We may use the same notation ch for the parabolic Chern

character of parabolic bundles as that used for regular bundles, but where possible we indicate when a nontrivial parabolic structure might be involved by the notation ch^{par} .

By our inductive hypothesis, we know the statement of the theorem for the morphisms

$$\varphi : Z \rightarrow P, \quad q : P \rightarrow S.$$

Let (E^i, ∇) denote the parabolic bundles considered in §4.2. By the inductive hypothesis for the map φ we have

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(E^i) \in CH^0(P)_{\mathbb{Q}}.$$

We are assuming from the previous paragraph that the monodromies of the local system E^{∇} are unipotent around components of $f^{-1}(J)$, so the parabolic structure on E along these components of the divisor D is a trivial parabolic structure on a usual bundle. It is only in the case of a Lefschetz pencil of even fiber dimension (that is, when $n - 1$ is even) that E has nontrivial parabolic structure along the horizontal components K .

We show in the next two lemmas that the main step in the proof will be to treat the case Z_W/U .

Lemma 5.5. *Suppose we know the statement of the theorem for Z_W/U , in other words suppose we know that*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_W/U)) \in CH^0(S)_{\mathbb{Q}}.$$

Then we can conclude the statement of the theorem for Z_U/U .

Proof. Let $Z_{K,U} := Z_U - Z_W$. It is equal to the preimage $\varphi_U^{-1}(K_U)$, so it is a closed subset of Z_U .

Proceed as in the proof of Lemma 5.3. The pair $(Z_U, Z_{K,U})$ is a fibration over U . For each $u \in U$ the relative cohomology $H^*(Z_u, Z_{K,u}, \mathbb{C})$ fits into a long exact sequence with $H^*(Z_u, \mathbb{C})$ and $H^*(Z_{K,u}, \mathbb{C})$. On the other hand, $H^*(Z_u, Z_{K,u}, \mathbb{C})$ is isomorphic to the compactly supported cohomology of W_u which is Poincaré dual to the cohomology of W_u with $i \mapsto 2n - i$. This shift doesn't affect the signs in our alternating sums. As before, we obtain long exact sequences of local systems on U and then of associated parabolic bundles on S . Thus, the hypothesis of the present lemma implies that

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_U/U)) - \sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_{K,U}/U)) \in CH^0(S)_{\mathbb{Q}}.$$

On the other hand the morphism $Z_{K,U} \rightarrow U$ is projective of relative dimension n . It is not smooth, but we have taken care to include the non-smooth case in our inductive statement. Thus, by the inductive assumption we know that

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_{K,U}/U)) \in CH^0(S)_{\mathbb{Q}}.$$

We obtain the same conclusion for Z_U/U . \square

Lemma 5.6. *Suppose we know the statement of the theorem for Z_U/U , in other words suppose we know that*

$$\sum_{i=0}^{2n} (-1)^i \text{ch}(\overline{\mathcal{H}}^i(Z_U/U)) \in CH^0(S)_{\mathbb{Q}}.$$

Then we can conclude the statement of the theorem for X_U/U .

Proof. Recall that $B \subset X$ is the family of base loci for the Lefschetz pencil, and Z is obtained from X by blowing up along B , at least in the part lying over U . Let \tilde{B} denote the inverse image of B in Z . The relative dimension of B_U/U is $n - 2$ and the relative dimension of \tilde{B}_U/U is $n - 1$. Thus we can assume that we know the statement of the theorem for these families. Furthermore $Z_U - \tilde{B}_U \cong X_U - B_U$ and this is a smooth quasiprojective variety with topological fibration to U . Use the same argument as in Lemma 5.3 applying the long exact sequences for pairs and Poincaré duality on $(X_U - B_U)/U$. By this argument applied once to each side, if we know the statement of the theorem for Z_U/U then we get it for $(Z_U - \tilde{B}_U)/U = (X_U - B_U)/U$ and hence for X_U/U . \square

Corollary 5.7. *Suppose we know the statement of the theorem for Z_W/U , then we can conclude the statement of the theorem for X_U/U .*

Proof. Put together the previous two lemmas. \square

5.4. The proof for Z_W/U . Up to now we have reduced to the problem of proving the statement of the theorem for the family of open smooth varieties Z_W/U . Here is where we use the Lefschetz pencil $Z_W \xrightarrow{\varphi} W \xrightarrow{q} U$ as in §4.

Recall that we defined

$$E_W^i := \mathcal{H}_{DR}^i(Z_W/W, (\mathcal{O}_{Z_W}, d)) := \mathbb{R}^i \varphi_* DR(Z_W/W, \mathcal{O}_{Z_W}, d)$$

with its Gauss-Manin connection ∇_W on W . Then (E^i, ∇) was the associated parabolic bundle on P , with (E_U^i, ∇_U) the restriction to P_U . For any multi-index α for the divisor D we have (E_α^i, ∇) which is a vector bundle on P with logarithmic connection having singularities along D , and $(E_{\alpha,U}^i, \nabla_U)$ is the restriction to U .

We assume that $\alpha_i > 0$ for components D_i appearing in K , that is components which are subject to S , whereas we assume from now on that $\alpha_i = 0$ for vertical components D_i , that is the components which map to components of J .

By Corollary 4.9 we have

$$\sum_m (-1)^m \text{ch} \overline{\mathcal{H}}_{DR}^m(Z_W/U) = \sum_{i,j} (-1)^{i+j} \text{ch} \overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla).$$

On the left in this formula is the quantity which we are trying to show lies in $CH^0(S)_{\mathbb{Q}}$.

Recall that the quantity on the right is defined to be the parabolic bundle on S associated to the vector bundle $\mathcal{H}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$ with regular singular Gauss-Manin connection over U .

Lemma 5.8. *The extended parabolic bundle $\overline{\mathcal{H}}_{DR}^i(P_U/U, E_{\alpha,U}^j, \nabla)$ is a vector bundle with trivial parabolic structure, and it corresponds to the vector bundle $\mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla)$ obtained by taking the relative logarithmic de Rham cohomology over S .*

Proof. This is the contents of our analogue of Steenbrink's theorem, Proposition 3.10 above which is Theorem 6.1 from the next section. Note that the fact that we have chosen $\alpha_i = 0$ for vertical components of the divisor D_i means that the logarithmic connection ∇ on E_{α}^i has nilpotent residues along the vertical components D_i . Recall for this that we are assuming that we have made sufficient modification $S' \rightarrow S$ so that the monodromy transformations of (E, ∇) around vertical components of the divisor D are unipotent. \square

Corollary 5.9. *For the proof of Theorem 5.1 it suffices to showing that*

$$\sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla) \in CH^0(S)_{\mathbb{Q}}.$$

\square

We now proceed with the proof of the formula to which we have reduced in the previous corollary.

Use the inductive statement for the family $\varphi : Z \rightarrow P$. Note that this has relative dimension $n - 1$. The parabolic bundle on P associated to the higher direct image local system on the open set W is E^i . Therefore our inductive hypothesis for Z/P says that

$$\sum_i (-1)^i \text{ch}^{\text{par}}(E^i) \in CH^0(P)_{\mathbb{Q}}.$$

Next observe that by Corollary 2.9, the difference between the parabolic Chern character of the parabolic bundle, and the Chern character of any component bundle, comes from the divisor D . In fact, due to our assumption about the monodromy, E^i is just a usual bundle along the vertical components of the divisor. Thus, for multi-indices α such that $\alpha_i = 0$ along vertical components, the difference comes from the divisor K :

$$\text{ch}^{\text{par}}(E^i) - \text{ch}(E_{\alpha}^i) \in \text{Image}(CH(K)_{\mathbb{Q}} \longrightarrow CH(P)_{\mathbb{Q}}).$$

To be more precise this means that the difference is in the subspace of $CH(P)_{\mathbb{Q}}$ spanned by Chern characters of sheaves concentrated on K . Thus we get

$$\sum_i (-1)^i \text{ch}(E_{\alpha}^i) \in CH(K)_{\mathbb{Q}} + CH^0(P)_{\mathbb{Q}}.$$

Write

$$\sum_i (-1)^i \text{ch}(E_{\alpha}^i) = \mathbf{k} + \mathbf{r}$$

with $\mathbf{k} \in CH^*(K)_{\mathbb{Q}}$ and $\mathbf{r} \in CH^0(P)_{\mathbb{Q}}$. Note that \mathbf{r} is just an integer, the alternating sum of the ranks of the bundles.

Now look at the relative logarithmic de Rham complex for E_{α}^i :

$$E_{\alpha}^i \rightarrow E_{\alpha}^i \otimes_{\mathcal{O}_P} \Omega_{P/S}^1(\log D).$$

The Hodge to de Rham spectral sequence and multiplicativity for tensor products and additivity for exact sequences of the Chern character, gives the formula

$$\sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla) = \chi_{P/S} \left(\left[\sum_i (-1)^i \text{ch}(E_{\alpha}^i) \right] \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right).$$

Recall here that $\chi_{P/S}$ represents the function entering into the GRR formula for P/S calculating the relative Euler characteristic in $CH^*(S)_{\mathbb{Q}}$ for an element of $CH^*(P)_{\mathbb{Q}}$.

This formula decomposes into two pieces according to the decomposition $\mathbf{k} + \mathbf{r}$ above:

$$\begin{aligned} \sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla) &= \chi_{P/S} \left(\mathbf{k} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right) \\ &+ \chi_{P/S} \left(\mathbf{r} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right). \end{aligned}$$

Lemma 5.10. *In the above decomposition, the first piece vanishes:*

$$\chi_{P/S} \left(\mathbf{k} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right) = 0$$

Proof. We claim that the argument inside the function $\chi_{P/S}$ is already zero. Indeed, \mathbf{k} is a sum of Chern characters of sheaves supported on K , but on the other hand the divisor K is a union of components which appear in D and only intersect single vertical components of the divisor D , because $(P, K) \rightarrow S$ is a semistable family of pointed curves. Thus we have a residue isomorphism

$$\Omega_{P/S}^1(\log D)|_K \cong \mathcal{O}_K = \Omega_{P/S}^0(\log D),$$

so the difference $\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D)$ restricts to zero on K . When we multiply it by \mathbf{k} we get zero. \square

After this we are left only with the second piece:

$$\chi_{P/S} \left(\mathbf{r} \cdot \left[\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right] \right).$$

The class \mathbf{r} is just an integer considered in $CH^0(P)_{\mathbb{Q}}$. Thus, the following lemma will complete the proof.

Lemma 5.11. *In our situation,*

$$\chi_{P/S} \left(\sum_{j=0,1} (-1)^j \text{ch} \Omega_{P/S}^j(\log D) \right) \in CH^0(S)_{\mathbb{Q}}.$$

Proof. This quantity is equal to

$$\sum_i (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, (\mathcal{O}_P, d)).$$

By the same argument as above using Steenbrink's theorem for the fibration $q : P \rightarrow S$, the terms $\mathcal{H}_{DR}^i(P/S, (\mathcal{O}_P, d))$ are the vector bundles over S associated by the Deligne canonical extension to the local systems $R^i q_{U,*} \mathbb{C}_{P_U}$ defined on U . Since P/S is semistable the monodromy here is unipotent. However in this case $P_U \rightarrow U$ is a family of smooth rational curves. Thus the direct image local systems are \mathbb{C}_U for $i = 0, 2$ and 0 for $i = 1$. Their canonical extensions are trivial bundles on S so

$$\sum_i (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, (\mathcal{O}_P, d)) \in CH^0(S)_{\mathbb{Q}}.$$

□

Multiplied by the integer \mathbf{r} , the result of this lemma shows that the second term in the previous decomposition lies in $CH^0(S)_{\mathbb{Q}}$. As we saw in Lemma 5.10 that the other piece vanishes, we obtain:

Corollary 5.12.

$$\sum_{i,j} (-1)^{i+j} \text{ch} \mathcal{H}_{DR}^i(P/S, E_{\alpha}^j, \nabla) \in CH^0(S)_{\mathbb{Q}}.$$

□

In view of Corollary 5.9, this completes the proof of Theorem 5.1. □

5.5. Family of projective surfaces. Consider a family of projective surfaces

$$\pi : X \longrightarrow S$$

where S is smooth. Then, with notations as in Theorem 5.1, we have

Proposition 5.13. *The parabolic Chern character of the parabolic bundles associated to weight i de Rham bundles satisfies*

$$\text{ch}(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_{\mathbb{Q}}$$

for each $i \geq 0$.

We first consider the following case:

Lemma 5.14. *Suppose $\pi : X \longrightarrow S$ is a generically smooth morphism of relative dimension 2. Then*

$$\text{ch}(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_{\mathbb{Q}}$$

for each $i \geq 0$.

Proof. Firstly, the Hard Lefschetz theorem gives the isomorphism

$$\overline{\mathcal{H}}^1(X/S) \simeq \overline{\mathcal{H}}^3(X/S).$$

By [Es-Vi2, Theorem 4.1], we obtain

$$ch(\overline{\mathcal{H}}^1(X/S)) \in CH^0(S)_{\mathbb{Q}}.$$

Hence by Theorem 5.1, we conclude that

$$ch(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_{\mathbb{Q}}$$

for each $i \geq 0$. □

Now suppose $\pi : X \rightarrow S$ be any projective morphism. Let $\text{Spec } K$ be the generic point of S and $X_K := X \times_{\mathbb{C}} \text{Spec } K$. Since the field K is of characteristic zero, by Hironaka's theorem [Hi], there is a resolution of singularities

$$f : X'_K \rightarrow X_K$$

defined over the field K such that f is a sequence of blow-ups and X'_K is nonsingular.

This implies that there is a non-empty open subset $U \subset S$ and a commutative diagram:

$$\begin{array}{ccc} X'_U & \xrightarrow{f} & X_U \\ \downarrow \pi' & & \downarrow \pi \\ U & = & U \end{array}$$

such that π' is a smooth morphism and the variety $X'_U \rightarrow U$ is a fibrewise resolution of singularities of $X_U \rightarrow U$. By Lemma 3.12, we assume that $D := S - U$ is a normal crossing divisor.

Since a fibre X_s of π is a singular surface, the cohomology of X_s carries a mixed Hodge structure. In particular, over the open subset of S where π is a topological fibration (which again by Lemma 3.12 can be assumed to be U), we obtain a filtration of local systems with associated-graded

$$(1) \quad Gr.R^i\pi_*\mathbb{Q} = \bigoplus_{k=0}^i Gr_k R^i\pi_*\mathbb{Q}.$$

Here $Gr_k R^i\pi_*\mathbb{Q}$ is the local system corresponding to the weight k graded piece of the weight filtration $\{W_k\}$ on the cohomology $H^i(X_s, \mathbb{Q})$. More precisely, $H^i(X_s, \mathbb{Q})$ carries a weight filtration

$$(0) = W_{-1} \subset W_0 \subset \dots \subset W_i = H^i(X_s, \mathbb{Q})$$

such that $Gr_k H^i(X_s, \mathbb{Q})$ carries a polarised pure Hodge structure of weight k . In other words, $Gr_k R^i\pi_*\mathbb{Q}$ is the local system associated to a family of polarised pure Hodge structures of weight k . These statements are proved in more generality for singular schemes in [De3].

Also, notice that the morphism $f_s^* : W_i = H^i(X_s, \mathbb{Q}) \rightarrow H^i(X'_s, \mathbb{Q})$ is a morphism of mixed Hodge structures whose kernel is W_{i-1} . In particular, the i -th graded piece

$Gr_i H^i(X_s, \mathbb{Q})$ is a polarised pure sub-Hodge structure of weight i of $H^i(X'_s, \mathbb{Q})$. When $i = 2$, the complementary sub-Hodge structure is generated by the algebraic classes since $X'_s \rightarrow X_s$ is a sequence of blow-ups, for $s \in U$.

These statements put together over U corresponds to saying that

Lemma 5.15. *There is a decomposition of local systems*

$$R^2 \pi'_* \mathbb{Q} = Gr_2 R^2 \pi_* \mathbb{Q} \oplus T$$

where T is a trivial local system.

Consider the parabolic bundles $\overline{\mathcal{G}}_k(\mathcal{H}^i)$ associated to the graded pieces $Gr_k R^i \pi_* \mathbb{Q}$, for each i, k , on S .

Then we have

Lemma 5.16. *The parabolic Chern character of the above parabolic bundles satisfies*

$$ch(\overline{\mathcal{G}}_k(\mathcal{H}^i)) \in CH^0(S)_{\mathbb{Q}}$$

for $i, k \leq 2$.

Proof. By Lemma 5.14, we have $ch(\overline{\mathcal{H}}^2(X'/S)) \in CH^0(S)_{\mathbb{Q}}$. Hence by Lemma 5.15, we conclude that

$$ch(\overline{\mathcal{G}}_2(\mathcal{H}^2)) \in CH^0(S)_{\mathbb{Q}}.$$

Since the parabolic bundle $\overline{\mathcal{G}}_0(\mathcal{H}^i)$ for any i , is a trivial bundle and $\overline{\mathcal{G}}_1(\mathcal{H}^i)$ corresponds to a family of polarised pure Hodge structures of weight one, by [Es-Vi2, Theorem 4.1], we conclude that

$$ch(\overline{\mathcal{G}}_1(\mathcal{H}^2)) \in CH^0(S)_{\mathbb{Q}}.$$

□

Using (1) and Lemma 3.4, we conclude that $\overline{\mathcal{H}}^2(X/S)$ (resp. $\overline{\mathcal{H}}^1(X/S)$) has a filtration whose associated-graded is $\oplus_{k=0}^2 \overline{\mathcal{G}}_k(\mathcal{H}^2)$ (resp. $\oplus_{k=0}^1 \overline{\mathcal{G}}_k(\mathcal{H}^1)$).

By Lemma 5.16, we obtain

$$ch(\overline{\mathcal{H}}^i(X/S)) \in CH^0(S)_{\mathbb{Q}}$$

for $i = 0, 1, 2$. Now by Theorem 5.1, we obtain

$$ch(\overline{\mathcal{H}}^3(X/S)) \in CH^0(S)_{\mathbb{Q}}.$$

This completes the proof of Proposition 5.13.

6. APPENDIX: AN ANALOGUE OF STEENBRINK'S THEOREM

Suppose $f : P \rightarrow S$ is a semistable morphism of relative dimension one, with respect to a divisor J on S and its pullback $f^{-1}J$ on P , both of which have normal crossings. Assume for simplicity that the total space of P is smooth (which is possible by [dJ, Proposition 3.6]).

Semistability implies that the components of $f^{-1}J$ occur with multiplicity one, and that f is flat. Let $U := S - J$ and $V := P - f^{-1}J = f^{-1}U$ be open sets in S and P respectively. Suppose $K \subset P$ is another divisor, isomorphic to a disjoint union of copies of S on which f is the identity. Suppose that the components of K meet $f^{-1}J$ transversally at smooth points of the latter, so that $(P, K) \rightarrow S$ is a semistable family of pointed curves. Let $D := K \cup f^{-1}J$.

Suppose that (E, ∇) is a logarithmic connection on P with singularities along D with rational residues. Suppose that the residual transformations of ∇ along components of $f^{-1}J$ are nilpotent. Assume also that for every component K_i of the horizontal divisor K we have a weight α_i such that the eigenvalues of the residues of ∇ along K_i are contained in $[-\alpha_i, 1 - \alpha_i)$.

Let $E \otimes \Omega_{P/S}(\log D)$ be the relative logarithmic de Rham complex (including K in the logarithmic part). Set

$$F := \mathcal{H}_{DR}(P/S, E, \nabla) = \mathbb{R}f_*(E \otimes \Omega_{P/S}(\log D)).$$

This is a complex of \mathcal{O}_S -modules, quasi-isomorphic to a complex of vector bundles, in other words it is a *perfect complex* on S . We can now state Steenbrink's theorem in this setting:

Theorem 6.1. *Keep the above hypothesis of a semistable family and logarithmic connection (E, ∇) such that the residues of ∇ along components of $f^{-1}J$ are nilpotent and the residues along K_i have eigenvalues in $[-\alpha_i, 1 - \alpha_i)$. Then the higher direct image complex F splits locally over S into a direct sum of its cohomology sheaves $H^i(F)$. Furthermore, these cohomology sheaves are locally free on S . They have Gauss-Manin connections which are logarithmic along J , and the residues of their Gauss-Manin connections along J are nilpotent. In particular, the cohomology sheaves $H^i(F)$ are the canonical extensions of the Gauss-Manin connections over the open set U (denoted $\overline{\mathcal{H}}^i$) and*

$$\mathrm{ch}(F) = \sum_i (-1)^i \mathrm{ch}(\overline{\mathcal{H}}^i).$$

Steenbrink proved this in the context of a geometric family [St, Theorem 2.18]. Since then there have been a large number of generalizations of his theorem, see [Fa] [Fa2] [Sa] [Il] [KN] [KMN] [IKN] [Ko] [Ts] [Ts2] [El] [Og] [Cai] among others. The argument we sketch below is probably contained in some of these references in some way. For example the argument given by Illusie in [Il] uses the condition that the higher direct image is a perfect complex, although he also uses an action of Frobenius and characteristic p

reduction. Illusie doesn't treat the case of local coefficient systems, necessary for our inductive argument. He refers to Esnault for raising the question, and refers to Faltings for treating some more general cases. The result is referred to in [Fa] and [Fa2], and the techniques we use here certainly come from this theory.¹ Also the note of Cailotto [Cai] seems to use an argument similar to the one we give here. Illusie, Kato and Nakayama touch on this result in §6 and Theorem 7.1 of [IKN], and refer for the case of a system of local coefficients to Kato-Matsubara-Nakayama [KMN]. There, this result is treated for local coefficients in a variation of Hodge structure, which would be sufficient for our purposes since (E, ∇) which intervenes in our argument is a variation of Hodge structure coming from the Lefschetz pencil.

We will indicate a proof which we hope will be enlightening. The first thing to note is that it suffices to prove that the dimensions of the cohomology groups of the fibers $F_x := F \otimes_{\mathcal{O}_S} \mathbf{k}(x)$ remain constant. Therefore it suffices to look at a general curve going into each point, and since the semistability hypothesis is preserved under base change to a general curve in the base ([Ab-dJ]) we may assume that S is a curve. Note also that the Gauss-Manin connection is globally defined before we restrict to a curve, but it suffices to look over a curve in order to prove that the residues are nilpotent, so for this part also it suffices to assume that S is a curve.

The idea is to define the Gauss-Manin connection on the level of the complex F , Lemma 6.2 below. This fact and the fact that semistability of the family implies nilpotence of the residual action of the normal vector field (Lemma 6.4) are standard. We then observe in Proposition 6.5 that these standard facts imply local freeness of the cohomology sheaves by a direct calculation: any cohomology sheaves not locally free would lead to eigenvalues of the residues differing by nonzero integers which contradicts nilpotency. This part of the argument is basically the same as the main lemma in [Cai].

Let \mathcal{L} denote the sheaf of differential operators of order ≤ 1 , whose symbols in degree 1 are vector fields tangent to J . Since we are assuming that S is a curve, an action of \mathcal{L} on F will contain all of the information that we need. In order to fix the ideas, note several facts about \mathcal{L} . It has left and right structures of \mathcal{O}_S -module which are different. We have an exact sequence, compatible with both the left and right module structures:

$$(2) \quad 0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{L} \rightarrow (\Omega_S^1(\log J))^* \rightarrow 0.$$

In view of the assumption that S is a curve, so J is a collection of points, we have that

$$(\Omega_S^1(\log J))^* = T_S(-J)$$

is just the sheaf of tangent vector fields vanishing at the points of J . Thus we can write the exact sequence as

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{L} \xrightarrow{\sigma} T_S(-J) \rightarrow 0.$$

¹The second author went to Faltings' course on p -adic Hodge theory.

The fiber $T_S(-D)_x$ over a point $x \in J$ is a complex line generated by the canonical element denoted $(t \frac{\partial}{\partial t})_x$ which is independent of the choice of coordinate t at x .

In the above exact sequence, the map σ is the *symbol* of a differential operator. Using it we can describe the difference between the left and right structures of \mathcal{O}_S -module. For $v \in \mathcal{L}$ and $a \in \mathcal{O}_S$ we have

$$v \cdot a - a \cdot v = \sigma(v)(a),$$

this answer $\sigma(v)(a)$ means the derivative of a along the vector field $\sigma(v)$, and $\sigma(v)(a)$ is considered as an element of $\mathcal{O}_S \subset \mathcal{L}$.

Lemma 6.2. *There is an action of \mathcal{L} on F given by a morphism in the derived category*

$$\mathcal{L} \otimes_{\mathcal{O}_S}^{\mathbb{L}} F \rightarrow F$$

whose restriction to the scalars $\mathcal{O}_S \subset \mathcal{L}$ is equivalent to the identity.

Proof. This can be done in several ways, and it should be considered as well-known.

One way is to say that we can take the higher direct image \mathcal{D} -module, which is a complex of \mathcal{D} -modules on S with singularities along J . This is a part of the “six operations” on derived categories of \mathcal{D} -modules, for which we haven’t attempted to include references. There is a canonical piece F' of this complex of \mathcal{D} -modules which is quasi-isomorphic to F as a complex of \mathcal{O}_S -modules, and furthermore the canonical piece F' has an action of the sheaf $\mathcal{D}_{S, \log J}$ of differential operators generated by vector fields tangent to J . This sheaf contains \mathcal{L} as a subsheaf, namely the piece of the filtration corresponding to operators of degree ≤ 1 . This would give the action of \mathcal{L} we are looking for.

Another option would be to invoke the *log-crystalline site*, again with numerous references which we don’t attempt to include. The object we are looking for is basically the higher direct image from the log crystalline site of P to that of S . In one approach this comes down to saying something using formal categories [Be].

We will give a geometric description, using a truncated and simplified version of the theory of formal categories, and following the classical construction of the Gauss-Manin connection as a connecting morphism for a short exact sequence. Let $S^{(2)}$ be the first infinitesimal neighborhood of the diagonal in the morphism object of the formal category associated to the logarithmic differentials $\Omega_S^1(\log D)$. Thus $S^{(2)} \rightarrow S \times S$. For either of the structures of \mathcal{O}_S -module, $\mathcal{O}_{S^{(2)}}$ is isomorphic to the dual of \mathcal{L} .

Let

$$P^{(2)} := S^{(2)} \times_S P$$

with the second projection used as the first map in the fiber product. The first projection $S^{(2)} \rightarrow S$ gives a map $P^{(2)} \rightarrow S$ and the fiber over a general point $s \in S$ is the second infinitesimal neighborhood of the fiber P_s . The fiber over a point $x \in J$ is just a product $P_s \times \text{Spec} \mathbb{C}[t]/(t^2)$.

Given a bundle E with logarithmic connection ∇ as above, we obtain a complex of sheaves on P

$$E \rightarrow E \otimes \Omega_P^1(\log J) \rightarrow E \otimes \Omega_P^2(\log J).$$

However the differentials are not \mathcal{O}_P or even \mathcal{O}_S -linear. To remedy this last we use the second infinitesimal neighborhood $P^{(2)}$. Note that there is a morphism $P^{(2)} \rightarrow P$ so we can pull back the sheaves in the above complex to $P^{(2)}$. The differentials don't pull back however because of lack of linearity. On the other hand, a smaller modification allows one to define the differentials too. Let \mathfrak{t} denote the ideal on $P^{(2)}$ defining the subscheme P which comes from $S \subset S^{(2)}$. The ideal induces an exact sequence

$$0 \rightarrow \mathfrak{t} \cong \mathcal{O}_P \otimes_{\mathcal{O}_S} \Omega_S^1(\log J) \rightarrow \mathcal{O}_{P^{(2)}} \rightarrow \mathcal{O}_P \rightarrow 0.$$

On the other hand let $\langle d \log t \rangle$ denote the subsheaf of $\Omega_P^1(\log J)$ generated by the pullback of the element $d \log t$ where t denotes a local coordinate on S . Then look at the subsheaves

$$\mathfrak{t} \langle d \log t \rangle \subset \Omega_P^1(\log J)|_{P^{(2)}}$$

and

$$\mathfrak{t} \langle d \log t \rangle \wedge \Omega_{P/S}^1(\log J) \subset \Omega_P^2(\log J)|_{P^{(2)}}.$$

The de Rham differentials of the de Rham complex for E , pulled back to $P^{(2)}$, are well defined modulo these subsheaves. So if we let $\Omega_P^i(\log J)^{(2)}$ denote respectively the quotients by these subsheaves for $i = 1, 2$, and if we let $E^{(2)}$ denote the pullback of E to $P^{(2)}$ then the differentials defined by the logarithmic connection for E give a complex

$$DR(P^{(2)}, E, \nabla) := E^{(2)} \rightarrow E \otimes \Omega_P^1(\log J)^{(2)} \rightarrow \Omega_P^2(\log J)^{(2)}.$$

This is now \mathcal{O}_S -linear with respect to the first projection $P^{(2)} \rightarrow S$. On the other hand let $DR(P/S, E, \nabla)$ denote the relative logarithmic de Rham complex

$$DR(P/S, E, \nabla) := E \rightarrow E \otimes \Omega_{P/S}^1(\log J).$$

Similarly for the pullback morphism $P^{(2)}/S^{(2)}$ we have the complex

$$DR(P^{(2)}/S^{(2)}, E, \nabla) = \mathcal{O}_{S^{(2)}} \otimes_{\mathcal{O}_S} DR(P/S, E, \nabla),$$

where the expression in the tensor product on the right uses the second projection $S^{(2)} \rightarrow S$. On the other hand the structure of \mathcal{O}_S -module uses the first projection.

We have a surjective morphism of complexes of sheaves

$$DR(P^{(2)}, E, \nabla) \rightarrow DR(P^{(2)}/S^{(2)}, E, \nabla).$$

Its kernel is a subcomplex

$$DR(P/S, E, \nabla) \otimes \langle d \log t \rangle[-1] \subset DR(P^{(2)}, E, \nabla).$$

Recall that we have divided out by the other piece which one might expect in the kernel which might be denoted $DR(P/S, E, \nabla) \otimes \mathfrak{t} \langle d \log t \rangle[-1]$ but isn't there. The complex $\langle d \log t \rangle[-1]$ is placed in cohomological degree 1 so the cohomological degree on $DR(P/S, E, \nabla)$ is shifted by one.

All in all we obtain an exact sequence of complexes

$$0 \rightarrow DR(P/S, E, \nabla) \otimes \langle d \log t \rangle[-1] \rightarrow DR(P^{(2)}, E, \nabla) \rightarrow DR(P^{(2)}/S^{(2)}, E, \nabla) \rightarrow 0.$$

These are complexes of sheaves over $P^{(2)}$ but they are only \mathcal{O}_S -linear. Each sheaf of the complex is coherent over $P^{(2)}$ but the differentials are not linear over $\mathcal{O}_{P^{(2)}}$. Nonetheless the usual arguments as for the de Rham complex [Gr] show that one can calculate the higher direct images using an affine open covering of $P^{(2)}$. Choose one such covering and let \check{C} denote the Čech complex functor (relative to the morphism to S) for this covering. Then we obtain an exact sequence of complexes of quasicoherent \mathcal{O}_S -modules

$$\begin{aligned} 0 \rightarrow \check{C}(DR(P/S, E, \nabla)) \otimes_{\mathcal{O}_S} \langle d \log t \rangle[-1] &\rightarrow \check{C}(DR(P^{(2)}, E, \nabla)) \\ &\rightarrow \mathcal{O}_{S^{(2)}} \otimes_{\mathcal{O}_S} \check{C}(DR(P/S, E, \nabla)) \rightarrow 0. \end{aligned}$$

Here we have taken outside of the functor $\check{C}(\cdot)$ any tensor products over \mathcal{O}_S . Note that on the right of the exact sequence the structure of \mathcal{O}_S -module uses the structure on $\mathcal{O}_{S^{(2)}}$ which is different from that used in the tensor product.

Finally let Q denote the cone over the second map in the exact sequence. We have a quasiisomorphism

$$\check{C}(DR(P/S, E, \nabla)) \otimes_{\mathcal{O}_S} \langle d \log t \rangle \xrightarrow{\text{qi}} Q$$

and a morphism

$$\mathcal{O}_{S^{(2)}} \otimes_{\mathcal{O}_S} \check{C}(DR(P/S, E, \nabla)) \rightarrow Q.$$

We can represent this situation as saying that we have a morphism in the derived category,

$$\mathcal{O}_{S^{(2)}} \otimes_{\mathcal{O}_S} \check{C}(DR(P/S, E, \nabla)) \rightarrow \check{C}(DR(P/S, E, \nabla)) \otimes_{\mathcal{O}_S} \langle d \log t \rangle,$$

bearing in mind that it is represented by an explicit diagram. Finally put the differential term on the other side. We have

$$(\langle d \log t \rangle)^* \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{(2)}} \cong \mathcal{L}.$$

Thus we have constructed our morphism of complexes

$$\mathcal{L} \otimes_{\mathcal{O}_S} \check{C}(DR(P/S, E, \nabla)) \rightarrow \check{C}(DR(P/S, E, \nabla))$$

as a morphism in the derived category. The complex $\check{C}(DR(P/S, E, \nabla))$ is one possible choice for F' quasiisomorphic to F as in the discussion above. See the proof of Lemma 6.3 below for the statement about triviality of the action of the scalars. This completes the proof of Lemma 6.2. \square

The map constructed above is defined in the derived category, and more precisely it comes from a diagram

$$\mathcal{L} \otimes_{\mathcal{O}_S} F \rightarrow Q \otimes \langle d \log t \rangle^* \xleftarrow{\text{qi}} F.$$

Note is that \mathcal{L} is locally free as a right \mathcal{O}_S -module, so the tensor product is also the derived tensor product

$$\mathcal{L} \otimes_{\mathcal{O}_S} F = \mathcal{L} \otimes_{\mathcal{O}_S}^{\mathbb{L}} F.$$

Suppose F' is a complex of vector bundles of finite length over S , which is isomorphic to F in the derived category. This means that there is a chain of quasiisomorphisms going in different directions relating F' and F . However, since F' consists of projective objects locally in the Zariski topology of S , we can represent this by an actual morphism of complexes $F' \rightarrow F$. Here we allow ourselves to replace S by a smaller neighborhood of the point $x \in J$ we are interested in. Similarly, $\mathcal{L} \otimes_{\mathcal{O}_S} F'$ is a complex of vector bundles so the map

$$\mathrm{Hom}(\mathcal{L} \otimes_{\mathcal{O}_S} F', F') \rightarrow \mathrm{Hom}(\mathcal{L} \otimes_{\mathcal{O}_S} F', Q \otimes \langle d \log t \rangle^*)$$

is a quasiisomorphism of complexes over S , and again possibly after going to an open set, our map

$$\mathcal{L} \otimes_{\mathcal{O}_S} F' \rightarrow Q \otimes \langle d \log t \rangle^*$$

lifts to a map

$$\mathcal{L} \otimes_{\mathcal{O}_S} F' \rightarrow F'$$

up to addition of a map of the form $d(\kappa)$, in other words up to a homotopy of complexes. Thus, for any choice of F' which is a complex of bundles representing F up to quasiisomorphism, we get an action of \mathcal{L} . This will induce the same map as the original $\mathcal{L} \otimes F \rightarrow F$ on cohomology, and the same is true after applying a derived functor such as taking the fiber over a point $x \in J$.

Suppose $x \in J$. The fiber of \mathcal{L} over x is the same when it is calculated on the left or the right. Let \mathfrak{m}_x be the maximal ideal at x and $\mathbb{C}_x := \mathcal{O}_S / \mathfrak{m}_x$. We have

$$\mathbb{C}_x \otimes_{\mathcal{O}_S} \mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_S} \mathbb{C}_x = \mathbb{C} \oplus \langle t \frac{\partial}{\partial t} \rangle_x.$$

Denote this by \mathcal{L}_x . This is special for points $x \in J$ because we have taken differential operators generated by vector fields which are tangent to J . For general points in S the quotient of \mathcal{L} by the maximal ideal on the left and on the right are not canonically isomorphic.

For $x \in J$ we have

$$\mathbb{C}_x \otimes_{\mathcal{O}_S} (\mathcal{L} \otimes_{\mathcal{O}_S} F) = \mathcal{L}_x \otimes_{\mathbb{C}} F_x$$

where $F_x := \mathbb{C}_x \otimes_{\mathcal{O}_S}^{\mathbb{L}} F$. Our original construction of F gives it as a complex of flat quasicoherent sheaves so these can be used to calculate the derived tensor product and

$$F_x = \mathbb{C}_x \otimes_{\mathcal{O}_S}^{\mathbb{L}} F = \mathbb{C}_x \otimes_{\mathcal{O}_S} F.$$

Similarly if F' is a complex of vector bundles quasiisomorphic to F then

$$F'_x := \mathbb{C}_x \otimes_{\mathcal{O}_S}^{\mathbb{L}} F' = \mathbb{C}_x \otimes_{\mathcal{O}_S} F',$$

and again

$$\mathbb{C}_x \otimes_{\mathcal{O}_S} (\mathcal{L} \otimes_{\mathcal{O}_S} F') = \mathcal{L}_x \otimes_{\mathbb{C}} F'_x.$$

Thus our action of \mathcal{L} on F' induces an action in the fiber over x ,

$$\mathcal{L}_x \otimes_{\mathbb{C}} F'_x \rightarrow F'_x.$$

Lemma 6.3. *The action of $\mathcal{O}_S \subset \mathcal{L}$ induces the identity on the cohomology sheaves $H^i(F')$ and also on $H^i(F'_x)$ for any $x \in J$.*

Proof. This statement is invariant under composing with quasiisomorphisms so it suffices to consider the original action of \mathcal{L} on the complex F as defined in Lemma 6.2 above. To calculate the action of \mathcal{O}_S on $H^i(F)$ or $H^i(F_x)$ it suffices to view the action as being the connecting morphism in the long exact sequence of hypercohomology associated to the exact sequence of complexes

$$0 \rightarrow DR(P/S, E, \nabla) \otimes \langle d \log t \rangle[-1] \rightarrow DR(P^{(2)}, E, \nabla) \rightarrow DR(P^{(2)}/S^{(2)}, E, \nabla) \rightarrow 0.$$

recopied from above. The action of \mathcal{O}_S comes from pulling back this exact sequence to the subcomplex

$$\mathbf{t}DR(P^{(2)}/S^{(2)}, E, \nabla) \subset DR(P^{(2)}/S^{(2)}, E, \nabla).$$

The pullback of the exact sequence is equal to

$$0 \rightarrow DR(P/S, E, \nabla) \otimes \langle d \log t \rangle[-1] \rightarrow \text{Cone}(I) \rightarrow \mathbf{t}DR(P^{(2)}/S^{(2)}, E, \nabla) \rightarrow 0$$

where the middle term is the cone on the identity map

$$I : \mathbf{t}DR(P^{(2)}/S^{(2)}, E, \nabla) \xrightarrow{\cong} DR(P/S, E, \nabla) \otimes \langle d \log t \rangle.$$

Note that $\mathbf{t} \subset \mathcal{O}_{S^{(2)}}$ is isomorphic to the line bundle $\langle d \log t \rangle$ and our identity is transported by this isomorphism, which is the same as the isomorphism

$$\mathcal{L} \otimes \mathcal{O}_{S^{(2)}} \otimes \langle d \log t \rangle^*$$

which was used to define the action of \mathcal{L} , restricted to $\mathcal{O}_S \subset \mathcal{L}$.

The cone on the identity map induces the identity as connecting morphism in the long exact sequence of hypercohomology, so the action of \mathcal{O}_S on $H^i(F)$ or $H^i(F_x)$ is the identity. \square

Lemma 6.4. *For a point $x \in J$, the action of $\langle t \frac{\partial}{\partial t} \rangle_x \subset \mathcal{L}_x$ on the cohomology vector spaces $H^i(F'_x)$ is nilpotent.*

Proof. Let P_x denote the fiber of $P \rightarrow S$ over x . It is a normal crossings curve. Denote by $\Omega_{P_x}(\log)$ the restriction of $\Omega_{P/S}(\log D)$ to P_x . We have a logarithmic de Rham complex

$$DR(P_x, E, \nabla) = E|_{P_x} \rightarrow E|_{P_x} \otimes_{\mathcal{O}_{P_x}} \Omega_{P_x}^1(\log)$$

and

$$H^i(F) \cong \mathbb{H}^i(P_x, DR(P_x, E, \nabla)).$$

The statement in question concerns the action of

$$\langle t \frac{\partial}{\partial t} \rangle_x \subset \mathcal{L}_x$$

on these cohomology spaces.

Let

$$\tilde{P}_x = \coprod_i P_{x,i}$$

denote the normalization which is a union of smooth curves $P_{x,i}$. They are the vertical components of the divisor D lying over x . Let $K_{x,i}$ denote the intersection of the horizontal divisor K with $P_{x,i}$ and let $G_{x,ij} := P_{x,i} \cap P_{x,j}$. Let

$$\nu_i : P_{x,i} \rightarrow \tilde{P}_x$$

denote the normalization morphism on the component $P_{x,i}$. Then

$$\nu_i^* \Omega_{P_x}^1(\log) = \Omega_{P_{x,i}}^1(\log(K_{x,i} + G_{x,ij}))$$

so

$$\nu_i^* DR(P_x, E, \nabla) = E|_{P_x} \rightarrow E|_{P_x} \otimes_{\mathcal{O}_{P_{x,i}}} \Omega_{P_{x,i}}^1(\log(K_{x,i} + G_{x,ij})).$$

There is a short exact sequence of complexes

$$0 \rightarrow DR(P_x, E, \nabla) \rightarrow \bigoplus_i \nu_{i,*} \nu_i^* DR(P_x, E, \nabla) \rightarrow \bigoplus_{i,j} \mathcal{S}_{ij} \rightarrow 0$$

where \mathcal{S}_{ij} is a complex of skyscraper sheaves concentrated on the intersection $G_{x,ij}$.

We obtain a long exact sequence relating the hypercohomology of $DR(P_x, E, \nabla)$ with the hypercohomologies

$$\bigoplus_i \mathbb{H}^i(P_{x,i}, \nu_i^* DR(P_x, E, \nabla)) \quad \text{and} \quad \bigoplus_{i,j} \mathbb{H}^i(\mathcal{S}_{ij}).$$

The pieces in the long exact sequence have compatible actions of \mathcal{L}_x . Thus it is sufficient for the proof of the lemma, to note that the actions on the hypercohomologies of $\nu_i^* DR(P_x, E, \nabla)$ and \mathcal{S}_{ij} satisfy the required nilpotence property.

For the case of the skyscraper sheaves, note that

$$\mathcal{S}_{ij} = E|_{G_{x,ij}} \oplus E|_{G_{x,ij}}[-1]$$

because of the residue isomorphisms

$$\Omega_{P_{x,i}}^i(\log(K_{x,i} + G_{x,ij}))|_{G_{x,ij}} \cong \mathcal{O}_{G_{x,ij}}$$

for $i = 0, 1$.

In both cases, the vector bundle $E|_{P_{x,i}}$ has a logarithmic connection along $P_{x,i}$ but also an action of the transverse vector $(t \frac{\partial}{\partial t})_x$, and this action induces the action on the hypercohomologies of these pieces of the long exact sequence.² To finish the proof, we note that these actions are nilpotent. To prove that, note that by hypothesis at smooth points of P_x the action is nilpotent. On the other hand, the bundle $E|_{P_{x,i}}$ is the unique bundle on $P_{x,i}$ with logarithmic connection having singularities along $K_{x,i} + \sum_j G_{x,ij}$ such that the residues at $G_{x,ij}$ are nilpotent and the eigenvalues of the residues along the $K_{x,i}$ are contained in $[-\alpha_i, 1 - \alpha_i)$. This extension such as was considered in [De1] is canonical and compatible with extensions. Therefore nilpotence of the endomorphism along smooth points of P_x implies that there is a filtration of $E|_{P_{x,i}}$ by subbundles such that

²We are skipping something which is undoubtedly complicated to write down in full detail, namely how to show that the action when defined as in Lemma 6.2 respects the long exact sequence, and how to identify the actions on the pieces of the exact sequence with the natural actions.

the endomorphism in question respects the subbundles and is strictly upper-triangular with respect to the filtration. Thus our element $(t\frac{\partial}{\partial t})_x$ acts nilpotently on both

$$\bigoplus_i \mathbb{H}^i(P_{x,i}, \nu_i^* DR(P_x, E, \nabla)) \quad \text{and} \quad \bigoplus_{i,j} \mathbb{H}^i(\mathcal{S}_{ij}).$$

By the long exact sequence it acts nilpotently on $H^i(F) \cong \mathbb{H}^i(P_x, DR(P_x, E, \nabla))$. \square

Replace the notation F' above by the shorter notation F in what follows. See [Cai] for a similar statement with application to a Steenbrink-type theorem.

Proposition 6.5. *Suppose F is a complex of vector bundles over an affine curve or a complex disk S . Let $x \in S$ and let J be the divisor $[x]$. Define \mathcal{L} as before with respect to this divisor. Suppose that we are given an action*

$$M : \mathcal{L} \otimes_{\mathcal{O}_S} F \rightarrow F$$

where the tensor product uses the right module structure on \mathcal{L} and leaves the left module structure in the answer. Suppose that, when restricted to $\mathcal{O}_S \subset \mathcal{L}$ it gives a map

$$M_0 : F = \mathcal{O}_S \otimes_{\mathcal{O}_S} F \rightarrow F$$

which induces the identity on cohomology sheaves and on the cohomology spaces of F_x . Suppose furthermore that for our point x the morphism

$$M_x : \mathcal{L}_x \otimes_{\mathcal{O}_S} F_x \rightarrow F_x$$

induces a nilpotent action of $(t\frac{\partial}{\partial t})_x \in \mathcal{L}_x$ on F_x . Then, locally in a neighborhood of x the complex F splits as a direct sum of its cohomology sheaves, and these cohomology sheaves are locally free.

Proof. The map M gives a map

$$H^i(M) : \mathcal{L} \otimes_{\mathcal{O}_S} \mathcal{H}^i(F) \rightarrow \mathcal{H}^i(F)$$

on cohomology sheaves. Away from D , this map is a connection, and as is well-known it implies that the cohomology sheaves are locally free. Thus our only problem is to prove that they are locally free near a point $x \in J$.

Over the disc or affine curve S the complex F is quasiisomorphic to the direct sum of its cohomology sheaves. This is because there are no Ext^i terms for $i \geq 2$ over \mathcal{O}_S since S is one-dimensional.

We can then choose a minimal resolution of each cohomology sheaf. This can be done explicitly, because by the Chinese remainder theorem we have that $H^i(F)$ is a direct sum of locally free modules and modules of the form $\mathcal{O}_S/(z^n)$. For the latter pieces if they exist (our goal is to show that they don't occur) choose as resolution

$$0 \rightarrow \mathcal{O}_S \xrightarrow{z^n} \mathcal{O}_S \rightarrow \mathcal{O}_S/(z^n) \rightarrow 0.$$

In this way we obtain a quasiisomorphism $R \rightarrow F \rightarrow R$ between F and a complex R of locally free sheaves with the “minimality” property that the differentials in R vanish at our singular point $x \in D$.

Now replace F by the minimal resolution R . We still get a map

$$M^R : \mathcal{L} \otimes_{\mathcal{O}_S} R \rightarrow R.$$

Its restriction M_0^R to $\mathcal{O}_S \subset \mathcal{L}$ is a map $M_0 : R \rightarrow R$ which induces the identity on cohomology sheaves. The fiber R_x is a complex whose differentials are equal to zero, in particular

$$R_x \cong \bigoplus_i H^i(F_x).$$

Therefore, $M_{0,x}^R$ is equal to the identity by our hypothesis that M_0 induces the identity on $H^i(F_x)$. In particular, $M_{0,x}^R$ is invertible, so by Nakayama’s lemma, after possibly restricting the size of our disc or affine neighborhood of x , we may assume that $M_0^R : R \rightarrow R$ is invertible. Define

$$M' := (M_0^R)^{-1} \circ M^R : \mathcal{L} \otimes_{\mathcal{O}_S} R \rightarrow R.$$

We have now succeeded in normalizing so that the restriction of M' to $\mathcal{O}_S \subset \mathcal{L}$ is the identity. One can see, using (2), that this exactly defines a logarithmic connection denoted ∇ on the whole complex R . In particular each R^i now has a logarithmic connection and the differentials of R are compatible with these connections.

Note that the fiber \mathcal{L}_x is a vector space which is well-defined with respect to either the left or right \mathcal{O}_S -module structures of \mathcal{L} because the symbols $\sigma(v)$ of elements of \mathcal{L} vanish at $x \in D$. The exact sequence for \mathcal{L} splits over x to give just

$$\mathcal{L}_x \cong \mathbb{C} \oplus T_S(-x)_x \cong \mathbb{C} \oplus \mathbb{C}.$$

Thus M gives a well-defined map

$$M_x(t \frac{\partial}{\partial t})_x : F_x \rightarrow F_x.$$

The hypothesis of the present proposition is that this endomorphism of F_x induces nilpotent endomorphisms on cohomology. Composing with $R_x \rightarrow F_x \rightarrow R_x$ which are again quasiisomorphisms, gives the map

$$M_x^R(t \frac{\partial}{\partial t})_x : R_x \rightarrow R_x.$$

However, $R_x \cong H^i(F_x)$. Thus the action of $M_x^R(t \frac{\partial}{\partial t})_x$ on R_x is nilpotent.

We claim that the connection ∇ on the complex R has nilpotent residues at x . Indeed, the residue of ∇ at x is equal to $(M_{0,x}^R)^{-1} \circ M_x^R(t \frac{\partial}{\partial t})_x$, and we know from the hypothesis of our proposition that $M_{0,x}^R$ is the identity, so the residue of ∇ is nilpotent.

The claim implies, in particular, that each R^i is a Deligne canonical extension.

Lemma 6.6. *The differentials of the complex R are zero.*

Proof. Look at any one of the differentials

$$d_i : R^i \rightarrow R^{i+1}.$$

This map is a map of vector bundles with logarithmic connection. Thus d_i may be considered as a flat section of the vector bundle with logarithmic connection $\text{Hom}(R^i, R^{i+1})$. This latter is also a Deligne canonical extension. On the other hand, $d_i(x) = 0$ by the minimality assumption for R . We are now in the following situation: we have a vector bundle with logarithmic connection which is a Deligne canonical extension (i.e. the residue is nilpotent), and we have a flat section which vanishes at x . We claim that this implies that the section vanishes everywhere. The claim is clearly true when we are dealing with sections of the canonical extension of a trivial connection. However, as we are working over a disc, our Deligne extension is a successive extension of trivial pieces, so by an induction argument, we get that $d_i = 0$. \square

Corollary 6.7. *The complex F is locally a direct sum of its cohomology sheaves, and these cohomology sheaves are locally free.*

Proof. This statement is invariant under quasiisomorphism, and it is true for R because of the previous corollary so it is true for F . \square

This corollary completes the proof of the proposition. \square

Proof of Theorem 6.1: In Lemma 6.2 we have constructed the Gauss-Manin connection on the cohomology complex. In Lemma 6.4 we showed that it satisfied the condition that the residual actions over points $x \in J$ are nilpotent, and in Proposition 6.5 we have seen that this nilpotence property implies that the cohomology complex F is locally a direct sum of locally free cohomology sheaves. Along the way, Lemma 6.4 provides the statement of the second part of Theorem 6.1. This completes the proof when S is a curve, and as stated at the beginning of the section, it is enough to treat the case when the base is a curve. The last statement about Chern characters follows because a complex and the direct sum of its cohomology are equivalent in K -theory. \square

REFERENCES

- [Ab-dJ] Abramovich, D., de Jong, A.J. *Smoothness, semistability, and toroidal geometry*, J.Algebraic Geom. **6** (1997), 789-801.
- [AGV] Abramovich, D., Graber, T., Vistoli, A. *Algebraic orbifold quantum products*, Orbifolds in mathematics and physics (Madison, WI, 2001), 1–24, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.
- [AK] Abramovich, D., Karu, K. *Weak semistable reduction in characteristic 0*. Invent. Math. **139** (2000), 241–273.
- [Be] P. Berthelot. *Cohomologie cristalline des schémas de caractéristique $p > 0$* . LECTURE NOTES IN MATHEMATICS **407**, Springer-Verlag, Berlin-New York (1974).
- [Bi] Biswas, I. *Chern classes for parabolic bundles*, J. Math. Kyoto Univ. **37** (1997), no. **4**, 597–613.
- [Bi2] Biswas, I. *Parabolic bundles as orbifold bundles*, Duke Math. J., **88** (1997), 305-325.
- [Bi-Iy] Biswas, I., Iyer, J. *Vanishing of Chern classes of the de Rham bundles for some families of moduli spaces*, preprint 2004.

- [BE] Bloch, S., Esnault, H. *Algebraic Chern-Simons theory*, Amer. J. Math. 119 (1997), no. 4, 903–952.
- [Bo] Borne, N. *Fibrés paraboliques et champ des racines*, preprint 2005.
- [Cad] Cadman, C. *Using stacks to impose tangency conditions on curves*, Preprint math.AG/0312349.
- [Cai] Cailotto, M. *Algebraic connections on logarithmic schemes*. C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), 1089–1094.
- [Ch-Ru] Chen, W., Ruan, Y. *Orbifold Gromov-Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), 25–85, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.
- [CK] Consani, C., Kim, M. *The Picard-Lefschetz formula and a conjecture of Kato: the case of Lefschetz fibrations*. Math. Res. Lett. **9** (2002), 621–631.
- [dJ] de Jong, A.J. *Smoothness, semi-stability and alterations*, Publ. Math. I. H. E. S. **83** (1996), 51–93.
- [De1] Deligne, P. *Equations différentielles a points singuliers réguliers*. Lect. Notes in Math. **163**, 1970.
- [De2] Deligne, P. *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Publ.Math.IHES **35** (1968), 107–126.
- [De3] Deligne, P. *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. No. **44** (1974), 5–77.
- [El] Elzein, F. *Hodge-de Rham theory with degenerating coefficients*, Preprint math.AG/0311083.
- [Es1] Esnault, H. *Recent developments on characteristic classes of flat bundles on complex algebraic manifolds*, Jahresber. Deutsch. Math.-Verein. **98** (1996), no. 4, 182–191.
- [Es2] Esnault, H. *Private communication*.
- [Es-Vi1] Esnault, H., Viehweg, E. *Logarithmic De Rham complexes and vanishing theorems*, Invent.Math., **86**, 161–194, 1986.
- [Es-Vi2] Esnault, H., Viehweg, E. *Chern classes of Gauss-Manin bundles of weight 1 vanish*, K-Theory **26** (2002), no. 3, 287–305.
- [Fa] Faltings, G. *F-isocrystals on open varieties: results and conjectures*, *The Grothendieck Festschrift, Vol. II* 219–248, PROGR. MATH. **87** Birkhäuser Boston, Boston, MA, 1990.
- [Fa2] Faltings, G. *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
- [Gi] Gillet, H. *Intersection theory on algebraic stacks and Q-varieties*, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983). J. Pure Appl. Algebra **34** (1984), 193–240.
- [Gr] Grothendieck, A. *On the de Rham cohomology of algebraic varieties*, Publ. Math. I. H.E.S. **29** (1966), 95–103.
- [Ha] Hartshorne, R. *Algebraic de Rham cohomology*, Manuscripta Math., **7** (1972), 125–140.
- [Ha2] Hartshorne, R. *On the De Rham cohomology of algebraic varieties*, Publ. Math. I.H.E.S. **45** (1975), 5–99.
- [HL] Herrera, M.; Lieberman, D. *Duality and the de Rham cohomology of infinitesimal neighborhoods*, Invent. Math. **13** (1971), 97–124.
- [Hi] Hironaka, H. *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.*, Ann. of Math. (2) **79** (1964), 109–203.
- [Il] Illusie, L. *Réduction semi-stable et décomposition de complexes de de Rham à coefficients*, Duke Math. J., **60** (1990), 139–185.
- [IKN] Illusie, L., Kato, K., Nakayama, C. *Quasi-unipotent logarithmic Riemann-Hilbert correspondences*, J. Math. Sci. Univ. Tokyo, **12** (2005), 1–66.
- [In] Inaba, M. *Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence*, Preprint math.AG/0602004.
- [IIS] Inaba, M.(Kyushu), Iwasaki, K., Saito, Masa-Hiko. *Moduli of Stable Parabolic Connections, Riemann-Hilbert correspondence and Geometry of Painlevé equation of type VI, Part I*, Preprint math.AG/0309342.
- [Iy] Iyer, J. *The de Rham bundle on a compactification of moduli space of abelian varieties.*, Compositio Math. **136** (2003), no. 3, 317–321.
- [Ko] Kato, F. *The relative log Poincaré lemma and relative log de Rham theory*, Duke Math. J., **93** (1998), 179–206.

- [KMN] Kato, K., Matsubara, T., Nakayama, C. *Log C^∞ -functions and degenerations of Hodge structures*. Algebraic geometry 2000, Azumino (Hotaka), ADV. STUD. PURE MATH., **36**, Math. Soc. Japan, Tokyo (2002), 269–320
- [KN] Kato, K., Nakayama, C. *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over \mathbb{C}* . Kodai Math. J. **22** (1999), 161–186.
- [Kz] Katz, N. *Étude cohomologique des pinceaux de Lefschetz*, SGA 7, exposé 18, LNM **340**, Springer.
- [Li] Li, Jiayu. *Hermitian-Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds* Comm. Anal. Geom. 8 (2000), no. **3**, 445–475.
- [Li-Na] Li, Jiayu., Narasimhan, M. S. *Hermitian-Einstein metrics on parabolic stable bundles* Acta Math. Sin. (Engl. Ser.) **15** (1999), 93–114.
- [Man] Manin, Y. *Moduli Fuchsiani*, Annali Scuola Normale Sup. di Pisa Ser. III **19** (1965), 113–126.
- [Ma-Ol] Matsuki, K., Olsson, M. *Kawamata-Viehweg vanishing and Kodaira vanishing for stacks*, Math. Res. Lett. **12** (2005), 207–217.
- [Ma-Yo] Maruyama, M., Yokogawa, K. *Moduli of parabolic stable sheaves*, Math. Ann. 293 (1992), no. **1**, 77–99.
- [Mo] Mochizuki, T. *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Preprint math.DG/0411300.
- [Mu] Mumford, D. *Towards an Enumerative Geometry of the Moduli Space of Curves*, Arithmetic and geometry, Vol. II, 271–328, Progr. Math., **36**, Birkhäuser Boston, Boston, MA, 1983.
- [Og] A. Ogus. *F-crystals, Griffiths transversality, and the Hodge decomposition*. Astérisque **221** (1994).
- [Pa] Panov, D. *Doctoral thesis*, 2005.
- [Sa] Saito, M. *Mixed Hodge modules*, Proc. Japan Acad. Ser. A Math. Sci., **62** (1986), 360–363.
- [Se] Seshadri, C. S. *Moduli of vector bundles on curves with parabolic structures*. Bull. Amer. Math. Soc. **83** (1977), 124–126.
- [St] Steenbrink, J. *Limits of Hodge structures*, Invent. Math. 31 (1975/76), no. **3**, 229–257.
- [St-Wr] Steer, B., Wren, A. *The Donaldson-Hitchin-Kobayashi correspondence for parabolic bundles over orbifold surfaces*, Canad. J. Math. 53 (2001), no. **6**, 1309–1339.
- [To] Toen, B. *Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford K-Theory* 18 (1999), no. **1**, 33–76.
- [Ts] Tsuji, T. *On p -adic nearby cycles of log smooth families*, Bull. Soc. Math. France, **128** (2000), 529–575.
- [Ts2] Tsuji, T. *p -adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Invent. Math., **137** (1999), 233–411.
- [vdG] van der Geer, G. *Cycles on the moduli space of abelian varieties*, Moduli of curves and abelian varieties, 65–89, Aspects Math. E33, Vieweg, Braunschweig 1999.
- [Vi] Vistoli, A. *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. 97 (1989), no. **3**, 613–670.
- [Yo] Yokogawa, K. *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves*, J. Math. Kyoto Univ. 33 (1993), no. **2**, 451–504.

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