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The Projective Line Over the Finite Quotient Ring $\mathbf{GF(2)}[x]/\langle x^3 - x\rangle$ and Quantum Entanglement II. The Mermin "Magic" Square/Pentagram

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Abstract
In 1993, Mermin (Rev. Mod. Phys. 65, 803–815) gave lucid and strikingly simple proofs of the Bell-Kochen-Specker (BKS) theorem in Hilbert spaces of dimensions four and eight by making use of what has since been referred to as the Mermin(-Peres) "magic square" and the Mermin pentagram, respectively. The former is a 3×3 array of nine observables commuting pairwise in each row and column and arranged so that their product properties contradict those of the assigned eigenvalues. The latter is a set of ten observables arranged in five groups of four lying along five edges of the pentagram and characterized by similar contradiction. An interesting one-to-one correspondence between the operators of the Mermin-Peres square and the points of the projective line over the product ring $\mathrm{GF}(2) \otimes \mathrm{GF}(2)$ is established. Under this mapping, the concept "mutually commuting" translates into "mutually distant" and the distinguishing character of the third column's observables has its counterpart in the distinguished properties of the coordinates of the corresponding points, whose entries are both either zero-divisors, or units. The ten operators of the Mermin pentagram answer to a specific subset of points of the line over $\mathrm{GF}(2)[x]/\langle x^3 - x\rangle$. The situation here is, however, more intricate as there are two different configurations that seem to serve equally well our purpose. The first one comprises the three distinguished points of the (sub)line over GF(2), their three "Jacobson" counterparts and the four points whose both coordinates are zero-divisors; the other features the neighbourhood of the point $(1, 0)$ (or, equivalently, that of $(0, 1)$). Some other ring lines that might be relevant for BKS proofs in higher dimensions are also mentioned.

Keywords: Projective Ring Lines – Neighbour/Distant Relation – Mermin's Square/Pentagram Quantum Entanglement

1 Introduction

In Part I of our paper [1], after highlighting the fundamental properties of commutative rings with unity, we introduced an important algebraic geometrical concept, namely that of a projective line defined over a (finite) *ring*. This concept was then illustrated in detail on the structure of two rather elementary kinds of projective ring line; the line defined over the factor ring $\mathrm{GF}(2)[x]/\langle x^3 - x\rangle$ and that over the elementary product ring $\mathrm{GF}(2) \otimes \mathrm{GF}(2)$. Both the lines feature finite number of points, eighteen and nine respectively, shown to form three distinguished groups in term of the so-called neighbour and/or distant relation and endowed with a number of interesting properties. In this part we aim at demonstrating that these two remarkable ring geometries can be employed in mimicking the structure of the Mermin(-Peres) "magic square" and the Mermin pentagram—the two essential ingredients in one of the simplest proofs of the Bell-Kochen-Specker (BKS) theorem furnished up to date [2],[3].

Some five years ago, Aravind [4] pointed out that the 24 quantum states (or rays) employed by Peres in [5] to prove the BKS theorem are intimately linked with Reye's configuration of twelve points and sixteen lines in the classical projective space, i.e. space defined over a *field*. The dodecahedron of Zimba and Penrose [6] is another configuration with many interesting classical projective properties which turned out to be of relevance for BKS proofs. The present paper may thus be regarded as a natural, qualitative extension and/or generalization of the spirit of Aravind's and Zimba-Penrose's geometrical reasoning into the domain of more abstract projective geometries, where fields are replaced by rings.

2 The Bell-Kochen-Specker Theorem and the Two Mermin's Configurations

Quantum mechanics imposes, in general, only statistical restrictions on the results of measurements and so one is naturally tempted to assume that it is an incomplete theory, being the result of a more complete description in terms of so-called hidden variables, or "elements of reality" [7]. The Bell-Kochen-Specker theorem [8],[9] renders such a description impossible. It shows that any hidden-variable theory in the Hilbert spaces of dimensions three and higher must be contextual, i.e. relying not only on hidden states in the quantum system under study, but also on those in the measuring devices. There have been a large number of proofs of this theorem, differing from each other in philosophy and a degree of technicalities involved, but those given by Mermin in [2],[3] for dimensions four and eight stand out as most straightforward and succinct ever furnished. Here, we shall not be interested in these proofs themselves, but focus merely on (the properties of) two remarkable configurations of observables which play a key/decisive role in them.

Let us start in four dimensions and, so, with the configuration usually referred to as the Mermin(-Peres) "magic square" (see [10] for statements of credits). This configuration, depicted in Fig. 1, represents a 3×3 array of nine observables for a system of two-qubits, with the superscripts on the operators referring to the qubits and with σ_x, σ_y, σ_z denoting the Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

One can easily verify that each row/column features three pairwise commuting observables and that the product of the three operators in each row and each of the first two columns is $+I$ (the identity matrix), but the product of those in the *third* column is $-I$. Now, each of the operators in the square can be assigned an eigenvalue ± 1, and since these eigenvalues must obey the same identities as the operators themselves, their product in each row and the first two columns must be $+1$, whilst in the last column -1. This is, however, impossible, since according to the rows the parity is even, yet according to the columns, odd; hence, the name "magic square." Going to eight dimensions, Mermin [2] considers three qubits instead of two, and employs the ten relevant observables located at the vertices of a pentagram, as shown in Fig. 2. The four observables in each of the five edges of the pentagram are mutually commuting, and their product on every edge is $+I$, except for the *horizontal* one, where it is $-I$. As the same identities must also be satisfied by the eigenvalues assigned to the observables, such a configuration cannot exist because each observable, and so its eigenvalue, is shared by two edges.

$$\begin{array}{ccc} \sigma_x^1 & \sigma_x^2 & \sigma_x^1\sigma_x^2 \\ \sigma_y^2 & \sigma_y^1 & \sigma_y^1\sigma_y^2 \\ \sigma_x^1\sigma_y^2 & \sigma_x^2\sigma_y^1 & \sigma_z^1\sigma_z^2 \end{array}$$

Figure 1: The Mermin(-Peres) "magic square" [2]. This configuration is unique up to a transposition of its rows and/or columns.

$$\sigma_y^1$$

$$\sigma_x^1\sigma_x^2\sigma_x^3 \qquad \sigma_y^1\sigma_y^2\sigma_x^3 \qquad \sigma_y^1\sigma_x^2\sigma_y^3 \qquad \sigma_x^1\sigma_y^2\sigma_y^3$$

$$\sigma_x^3 \qquad \sigma_y^3$$

$$\sigma_x^1$$

$$\sigma_y^2 \qquad \sigma_x^2$$

Figure 2: The Mermin pentagram [2]. The symbols are the same as in the previous figure, with the superscripts now referring to three distinct qubits.

The two configurations just introduced are very similar to each other as they both consist of a finite number of sets of pairwise commuting observables of the same cardinality, where one of the sets stands on a qualitatively different footing than the rest of them. A natural question emerges: Can one find any other algebraic geometrical configurations behaving in a similar way? The answer is yes; and to justify this answer, we only need to return to [1; Sec. 4].

3 The Mermin(-Peres) Square and the Projective Line Over GF(2)⊗GF(2)

We shall deal first with a ring geometrical analogue of Mermin's square, which turns out to be nothing but $P\widetilde{R}_{\clubsuit}(1)$ — the projective line defined over $\mathrm{GF}(2)\otimes\mathrm{GF}(2)$ [1]. The nine points of this line [1; Eqs. (20)–(22)] can be arranged into a 3×3 array as shown in Fig. 3. This array has an important property that all the points in the same row and/or column are pairwise *distant*. Moreover, a closer look at Fig. 3 reveals that one triple of mutually distant points, that located in the third column, differs from all the others in having both the entries in the coordinates of all the three points of the same character, namely either zero-divisors (the points $(x, x+1)$ and $(x+1, x)$), or units (the point $(1, 1)$). After comparing Fig. 1 with Fig. 3, and identifying, in an obvious way, the observables of the Mermin(-Peres) square with the points of $P\widetilde{R}_{\clubsuit}(1)$, one immediately sees that the concept *mutually commuting* translates ring geometrically into *mutually distant* and that the "peculiar" character of the third column's observables has its geometrical counterpart in the above-mentioned distinguishing properties of the coordinates of the corresponding points. It is, however, worth pointing out that it is also the third row's observables that get a piece of recognition in our picture, for they correspond to the points which represent nothing but the embedding in $P\widetilde{R}_{\clubsuit}(1)$ of the *ordinary* projective line over GF(2), $PG(1, 2)$.

$$\begin{array}{ccc} (x+1,1) & (1,x) & (x,x+1) \\ (x,1) & (1,x+1) & (x+1,x) \\ (1,0) & (0,1) & (1,1) \end{array}$$

Figure 3: An arrangement of the points of $P\widetilde{R}_{\clubsuit}(1)$ into a square array in such a way that any two points in each row/column are distant.

4 The Mermin Pentagram and the Projective Line Over $\mathbf{GF(2)}[x]/\langle x^3 - x\rangle$

The structure of the Mermin pentagram is obviously more intricate and complex than that of the square, and these properties must obviously carry over onto its ring geometrical sibling(s). The relevant ring geometry is now that of $PR_{\clubsuit}(1)$, the projective line defined over the finite factor ring $R_{\clubsuit} \equiv \mathrm{GF}(2)[x]/\langle x^3 - x\rangle$ [1]. In fact, we have here two different ten-point configurations that seem to serve equally well our purpose. The first one comprises the point $(1,0)$ (or, equivalently, $(0,1)$) and the nine points of its neighbourhood [1; Eq. (15)], arranged as shown in Fig. 4. The other one features the three distinguished points of the (sub)line over GF(2), viz. $(1,0)$, $(0,1)$ and $(1,1)$, their three "Jacobson" counterparts, $(1, x^2+x)$, $(x^2+x, 1)$ and $(1, x^2+x+1)$, respectively, and the four points whose both coordinates are zero-divisors, and arranged as depicted in Fig. 5.

$$(1, x^2+x)$$

$$(x, x+1) \qquad (x^2+1, x) \qquad (x, x^2+1) \qquad (x+1, x)$$

$$(1, x^2) \qquad (1, x^2+1)$$

$$(1, 0)$$

$$(1, x) \qquad (1, x+1)$$

Figure 4: A pentagram-forming subset of ten points of $PR_{\clubsuit}(1)$, consisting of the point $(1,0)$ and its neighbourhood. The uppermost vertex of the pentagram is the "Jacobson" point of the neighbourhood [1].

$$(1, 1)$$

$$(x, x+1) \qquad (x^2+x, 1) \qquad (1, x^2+x) \qquad (x+1, x)$$

$$(x, x^2+1) \qquad (x^2+1, x)$$

$$(1, x^2+x+1)$$

$$(1, 0) \qquad (0, 1)$$

Figure 5: A ten-point subset of $PR_{\clubsuit}(1)$ made of three points of $PG(1,2)$, their three "Jacobson" counterparts, and the four points whose both coordinates are zero-divisors.

In the former case, even a passing look at Fig. 4 reveals that the distinct character of the horizonal edge is due to the sole zero-divisor entries in the coordinates of all the four points. In the latter case, the "prominent" character of the horizontal edge is not so readily discernible and one has to invoke the neighbour/distant relation [1] between the relevant points to spot it. We find, in particular, that the quadruples of points at each of the remaining edges have the property that one of the points is distant to each of the remaining three; as per the top-to-bottom-right/left edges it is the point $(1,1)$, whereas for the right/left-to-bottom-left/right ones this role is played by the point $(1, x^2+x+1)$; there exists, however, *no* such point for the horizontal edge.

Is there any means of discriminating between the two configurations? An affirmative answer is provided by the ring-induced homomorphism from $PR_{\clubsuit}(1)$ into $P\widetilde{R}_{\clubsuit}(1)$ [1; Eq. (23)]. Under this homomorphism, the four "horizontal" points of the first configuration (Fig. 4) map into only two distinct points, namely the $(x, x+1)$ and $(x+1, x)$ ones, whilst for the second configuration (Fig. 5) we get four distinct points — the above-given two points and the points $(1, 0)$ and $(0, 1)$. Hence, the "neighbourhood-generated" analogue of the Mermin pentagram seems to be more appealing, for its horizontal edge is homomorphic to (a portion of) the distinguished, third column of the ring geometric analogue of the Mermin(-Peres) magic square (Fig. 3) and, as a whole, it "condenses" into the set of all the four points distant to $(1, 1)$.

5 Zero-Divisors and Quantum (Entanglement)?

In order to fully appreciate the meaning of our *ring* geometrical analogues of the Mermin's "magic" configurations, we shall show that they cannot be reproduced by any classical, i.e. field projective lines. We shall examine the square case only, as the procedure can readily be extended to the pentagram case.

To this end in view, we simply notice that only four distinct marks, viz. 0, 1, x and $x+1$, appear in Fig. 3; these are, of course, the elements of the ring $\widetilde{R}_{\clubsuit}$ [1; Eq. (7)]. If we consider the projective line over the field featuring the same number of elements, $\mathrm{GF}(4) \cong \mathrm{GF}(2)[x]/\langle x^2+x+1\rangle$ [11] then, repeating the strategy and reasoning of [1; Sect 4], we find that such a line features *only five* points, namely $(1, 0)$, $(1, 1)$, $(1, x)$, $(1, x+1)$ and $(0, 1)$, because the elements/marks x and $x+1$ represent now *units*.[1] In order to get the required number of points, we have to take the line defined over $\mathrm{GF}(8) \cong \mathrm{GF}(2)[x]/\langle x^3+x+1\rangle$; the price to be paid for this move is, however, introducing additional, superfluous marks — those featuring second powers of x — into our scheme.

This illustration makes thus explicit what all the preceding discussions seem to evince, namely that it is thanks to the presence of *zero-divisors* in our approach that we are able to shed some light on the *quantum* intricacies embodied in the two Mermin's configurations. This claim can be made substantially stronger by the following observations. We can rephrase the "magic" of the Mermin(-Peres) square (Fig. 1) by noticing that the three rows have joint (mutually unbiased to each other) orthogonal bases of *un*entangled eigenstates and the same holds for the first two columns; on the contrary, the operators in the *third column* share a base of *maximally entangled* states. And looking at Fig. 3, we see that it is the *third column* of its geometrical counterpart where the presence of zero-divisors is *most pronounced*. The same applies to the Mermin pentagram (Fig. 3), with the horizontal edge's operators sharing a base of *maximally entangled* states, and its "neighbourhood-induced" analogue (Fig. 4), with the zero-divisor-dominated horizontal edge.

6 Conclusion

We have drawn a remarkable, though at this stage rather subtle, analogy between the structure of the two "magic" operator-valued configurations employed by Mermin in [2] to prove the BKS theorem in dimensions four and eight and the point sets represented by the projective ring line defined over $\mathrm{GF}(2) \otimes \mathrm{GF}(2)$ and sub-configurations of the line defined over $\mathrm{GF}(2)[x]/\langle x^3 - x\rangle$. A cornerstone algebraic geometrical concept of this correspondence is that of *zero-divisors* and its closest ally, the *neighbour/distant* relation [1], which may well turn out to lead to a deeper understanding of quantum entanglement. In order to test this hypothesis, it will necessitate examining more general kinds of projective ring line, in particular those defined over $\mathrm{GF(q)} \otimes \ldots \otimes \mathrm{GF(q)}$ and/or $\mathrm{GF(q)}[x]/\langle x^s - x\rangle$, with $q > 2$ and $s > 3$, and see whether their properties are indeed relevant for proofs of the BKS theorem in higher-dimensional Hilbert spaces.

As a final note, we would like to stress that our simple ring geometries may turn out to be just the right starting point to a more systematic, *geometrically*-oriented modelling of entangled quantum states/systems. In this sense our current findings, supplemented by the role of other

[1] Let us recall (see, e.g., [11]) that a field does not contain any zero-divisor — except for the trivial one (zero).

finite ring geometries, namely those of Hjelmslev, were found to play in addressing the properties of mutually unbiased bases [12],[13], lend some support to the "Relational Block World" view of quantum mechanics, recently proposed and advocated by Stuckey *et al.* [14], which is a novel paradigm resting uniquely on non-dynamical, *geometric* explanation of quantum phenomena.

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References

[1] Saniga M, Planat M. The projective line over the finite quotient ring $\mathrm{GF}(2)[x]/\langle x^3 - x\rangle$ and quantum entanglement I. Theoretical background. Available from ⟨quant-phys/0603051⟩.

[2] Mermin ND. Hidden variables and two theorems of John Bell. Rev Mod Phys 1993;65(3):803–15.

[3] Mermin ND. Simple unified form for no-hidden-variables theorems. Phys Rev Lett 1990;65:3373–6.

[4] Aravind PK. How Reye's configuration helps in proving the Bell-Kochen-Specker theorem: a curious geometrical tale. Found Phys Lett 2000;13(6):499–519.

[5] Peres A. Two simple proofs of the Kochen-Specker theorem. J Phys A 1991;24:174–8.

[6] Zimba J, Penrose R. On Bell non-locality without probabilities: more curious geometry. Stud Hist Phil Sci 1993;24:697–720.

[7] Einstein A, Podolsky B and Rosen N. Can quantum mechanical description of physical reality be considered complete? Phys Rev 1935;47:777–80.

[8] Bell JS. On the problem of hidden variables in quantum mechanics. Rev Mod Phys 1966;38:447–52.

[9] Kochen S, Specker EP. The problem of hidden variables in quantum mechanics. J Math Mech 1976;17:59–88.

[10] Aravind PK. Quantum mysteries revisited again. Am J Phys 2004;72:1303–7.

[11] Hirschfeld JWP. Projective geometries over finite fields. Oxford: Oxford University Press; 1998.

[12] Saniga M, Planat M. Hjelmslev geometry of mutually unbiased bases. J Phys A: Math Gen 2006;39:435–40. Available from ⟨math-ph/0506057⟩.

[13] Saniga M, Planat M. Finite geometries in quantum theory: from Galois (fields) to Hjelmslev (rings), Int J Modern Phys B 2006; in press.

[14] Stuckey WM, Silberstein M and Cifone M. Deflating quantum mysteries via the relational blockworld. Phys Essays 2006; in press. Available from ⟨quant-ph/0503065⟩.