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# OPTIMAL EIGENVALUES ESTIMATE FOR THE DIRAC OPERATOR ON DOMAINS WITH BOUNDARY

SIMON RAULOT

ABSTRACT. We give a lower bound for the eigenvalues of the Dirac operator on a compact domain of a Riemannian spin manifold under the MIT bag boundary condition. The limiting case is characterized by the existence of an imaginary Killing spinor.

## 1. INTRODUCTION

Let  $\emptyset$  be a compact domain in a  $n$ -dimensional Riemannian spin manifold  $(N^n, g)$  whose boundary is denoted by  $\partial\emptyset$ . In [HMR02], the authors studied four elliptic boundary conditions for the Dirac operator  $D$  of the domain  $\emptyset$ . More precisely, they prove a Friedrich-type inequality [Fri80] which relates the spectrum of the Dirac operator and the scalar curvature of the domain  $\emptyset$ . These boundary conditions are the following: the Atiyah-Patodi-Singer (APS) condition based on the spectral resolution of the boundary Dirac operator; a modified version of the APS condition, the mAPS condition; the boundary condition CHI associated with a chirality operator; and a Riemannian version of the MIT bag boundary condition. In fact, they show that, if the boundary  $\partial\emptyset$  of  $\emptyset$  has non-negative mean curvature, then under the APS, CHI or mAPS boundary conditions, the spectrum of the classical Dirac operator of the domain  $\emptyset$  is a sequence of unbounded real numbers  $\{\lambda_k : k \in \mathbb{Z}\}$  satisfying

$$\lambda_k^2 \geq \frac{n}{4(n-1)} R_0, \quad (1)$$

where  $R_0$  is the infimum of the scalar curvature of the domain  $\emptyset$ . Moreover, equality holds only for the CHI and the mAPS conditions and in these cases,  $\emptyset$  is respectively isometric to a half-sphere or it carries a non-trivial real Killing spinor and has minimal boundary. In the case of the MIT boundary condition, they show that the spectrum of the Dirac operator on  $\emptyset$  is an unbounded discrete set of complex numbers  $\lambda^{\text{MIT}}$  with positive imaginary part satisfying

$$|\lambda^{\text{MIT}}|^2 > \frac{n}{4(n-1)} R_0, \quad (2)$$

if the mean curvature of the boundary is non-negative. This result leads to the following question: can one improve this inequality in order to obtain some boundary geometric

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invariants on the right hand side of (2)? We show in this paper that such a result can be obtained. More precisely, we prove the following theorem:

**Theorem 1.** *Let  $\mathcal{O}$  be a compact domain of an  $n$ -dimensional Riemannian spin manifold  $(N^n, g)$  whose boundary  $\partial\mathcal{O}$  satisfies  $H > 0$ . Under the MIT boundary condition  $\mathbb{B}_{\text{MIT}}^-$ , the spectrum of the classical Dirac operator  $D$  on  $\mathcal{O}$  is an unbounded discrete set of complex numbers with positive imaginary part. Any eigenvalue  $\lambda^{\text{MIT}}$  satisfies*

$$|\lambda^{\text{MIT}}|^2 \geq \frac{n}{4(n-1)} R_0 + n \operatorname{Im}(\lambda^{\text{MIT}}) H_0, \quad (3)$$

where  $H_0$  is the infimum of the mean curvature of the boundary. Moreover, equality holds if and only if the associated eigenspinor is an imaginary Killing spinor on  $\mathcal{O}$  and if the boundary  $\partial\mathcal{O}$  is a totally umbilical hypersurface with constant mean curvature.

The proof of this theorem is based on a modification of the spinorial Levi-Civita connection which leads to a spinorial Reilly-type formula. This formula can be seen as a hyperbolic version of the Reilly inequality used in [HMR02].

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## 2. GEOMETRIC PRELIMINARIES

In this section, we give some standard facts about Riemannian spin manifolds with boundary. For more details, we refer to [BBW93] or [HMR02].

On a compact domain  $\mathcal{O}$  with smooth boundary  $\partial\mathcal{O}$  in a  $n$ -dimensional Riemannian spin manifold  $(N^n, g)$ , denote by  $\Sigma\mathcal{O}$  the complex spinor bundle corresponding to the metric  $g$  and by  $\nabla$  its Levi-Civita connection acting on  $T\mathcal{O}$  as well as its lift to  $\Sigma\mathcal{O}$ . The map  $\gamma : \mathcal{Cl}(\mathcal{O}) \rightarrow \operatorname{End}(\Sigma\mathcal{O})$  is the Clifford multiplication where  $\mathcal{Cl}(\mathcal{O})$  is the Clifford bundle over  $\mathcal{O}$ . The spinor bundle is endowed with a natural Hermitian scalar product, denoted by  $\langle \cdot, \cdot \rangle$ , compatible with  $\nabla$  and  $\gamma$ . The Dirac operator is then the first order elliptic operator acting on sections of  $\Sigma\mathcal{O}$  locally given by

$$\begin{aligned} D : \Gamma(\Sigma\mathcal{O}) &\longrightarrow \Gamma(\Sigma\mathcal{O}) \\ \psi &\longmapsto \sum_{i=1}^n \gamma(e_i) \nabla_{e_i} \psi, \end{aligned}$$

where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame of  $T\mathcal{O}$ .

Consider now the boundary  $\partial\mathcal{O}$  which is an oriented hypersurface of the domain  $\mathcal{O}$  with induced orientation and Riemannian structure. Since the normal bundle of  $\partial\mathcal{O}$  is trivial, the boundary itself is a spin manifold. This spin structure on the boundary allows to construct an intrinsic spinor bundle  $\Sigma(\partial\mathcal{O})$  over  $\partial\mathcal{O}$  naturally endowed with a Hermitian metric, a Clifford multiplication  $\gamma^{\partial\mathcal{O}}$  and a spinorial Levi-Civita connection  $\nabla^{\partial\mathcal{O}}$ . Moreover the restriction  $\mathbf{S}(\partial\mathcal{O}) := \Sigma\mathcal{O}|_{\partial\mathcal{O}}$  to the boundary of the spinor bundle  $\Sigma\mathcal{O}$  is a Dirac bundle, i.e. there exist on  $\mathbf{S}(\partial\mathcal{O})$  a Hermitian metric denoted by  $\langle \cdot, \cdot \rangle$  compatible with the Levi-Civita connection  $\nabla^{\mathbf{S}}$  and the Clifford multiplication  $\gamma^{\mathbf{S}}$ . The Clifford multiplication  $\gamma^{\mathbf{S}} : \mathcal{Cl}(\partial\mathcal{O}) \rightarrow \operatorname{End}(SS)$  is given by  $\gamma^{\mathbf{S}}(X)\psi = \gamma(X)\gamma(\nu)\psi$  for all  $X \in \Gamma(T\mathcal{O})$  and  $\psi \in \Gamma(SS)$ . Similarly we can relate the Levi-Civita connection acting on  $\Sigma\mathcal{O}$  with that acting on  $\mathbf{S}(\partial\mathcal{O})$  by the spinorial Gauss formula (see [Bär98]):

$$(\nabla_X \psi)|_{\partial\mathcal{O}} = \nabla_X^{\mathbf{S}} \psi|_{\partial\mathcal{O}} + \frac{1}{2} \gamma^{\mathbf{S}}(AX) \psi|_{\partial\mathcal{O}},$$

for all  $X \in \Gamma(\mathbf{T}(\partial\mathcal{O}))$ ,  $\psi \in \Gamma(\Sigma\mathcal{O})$  and where  $AX = -\nabla_X\nu$  is the shape operator of the boundary  $\partial\mathcal{O}$  with respect to the inner normal vector field  $\nu$ . We can then define the boundary Dirac operator acting on  $\mathbf{S}(\partial\mathcal{O})$  which is an elliptic first order differential operator locally given by

$$D^{\mathbf{S}} = \sum_{j=1}^{n-1} \gamma^{\mathbf{S}}(e_j) \nabla_{e_j}^{\mathbf{S}}. \quad (4)$$

Recall that there is a standard identification

$$\mathbf{S}(\partial\mathcal{O}) \equiv \begin{cases} \Sigma(\partial\mathcal{O}) & \text{if } n \text{ is odd} \\ \Sigma(\partial\mathcal{O}) \oplus \Sigma(\partial\mathcal{O}) & \text{if } n \text{ is even} \end{cases}$$

Taking into account the relation between the Hermitian bundle  $\mathbf{S}(\partial\mathcal{O})$  and  $\Sigma(\partial\mathcal{O})$ , one can see that

$$\nabla^{\mathbf{S}} \equiv \begin{cases} \nabla^{\partial\mathcal{O}} & \text{if } n \text{ is odd} \\ \nabla^{\partial\mathcal{O}} \oplus \nabla^{\partial\mathcal{O}} & \text{if } n \text{ is even} \end{cases}$$

and

$$\gamma^{\mathbf{S}} \equiv \begin{cases} \gamma^{\partial\mathcal{O}} & \text{if } n \text{ is odd} \\ \gamma^{\partial\mathcal{O}} \oplus -\gamma^{\partial\mathcal{O}} & \text{if } n \text{ is even} \end{cases}$$

### 3. THE MIT BOUNDARY CONDITION

First, note that on a closed compact Riemannian spin manifold, the classical Dirac operator has exactly one self-adjoint  $L^2$  extension, so it has real discrete spectrum. In the setting of manifolds with boundary, a defect of self-adjointness appears. It is given by the Green formula

$$\int_{\mathcal{O}} \langle D\varphi, \psi \rangle dv(g) - \int_{\mathcal{O}} \langle \varphi, D\psi \rangle dv(g) = - \int_{\partial\mathcal{O}} \langle \gamma(\nu)\varphi, \psi \rangle ds(g), \quad (5)$$

for all  $\varphi, \psi \in \Gamma(\Sigma\mathcal{O})$ . Furthermore, in this case, the Dirac operator has a closed range of finite codimension, but an infinite-dimensional kernel, which varies depending on the choice of the Sobolev space. We refer to [BBW93], [Lop53] or [HMR02] for a careful treatment of boundary conditions for elliptic operators.

The MIT bag boundary condition has first been introduced by physicists of the Massachusetts Institute of Technology in a Lorentzian setting (see [CJJ<sup>+</sup>74], [CJJT74] or [Joh75]). The Riemannian version of this condition has been studied in [HMR02] in order to get Friedrich estimates and in [HMZ02] because of its conformal covariance to give a conformal lower bound for the first eigenvalue of the intrinsic Dirac operator of hypersurfaces bounding a compact domain in a Riemannian spin manifold. Consider the pointwise endomorphism

$$i\gamma(\nu) : \Gamma(SS) \longrightarrow \Gamma(SS)$$

acting on the restriction to the boundary  $\partial\mathcal{O}$  of the spinor bundle over  $\mathcal{O}$  and where  $i$  is the fundamental imaginary number. This map is an involution, and so the bundle  $SS$

splits into two eigensubbundles  $V^\pm$  associated with the eigenvalues  $\pm 1$ . We then have two associated orthogonal projections given by

$$\begin{aligned} \mathbb{B}_{\text{MIT}}^\pm : \mathbf{L}^2(SS) &\longrightarrow \mathbf{L}^2(V^\pm) \\ \varphi &\longmapsto \frac{1}{2}(\text{Id} \pm i\gamma(\nu))\varphi. \end{aligned}$$

which define local elliptic boundary conditions for the Dirac operator  $D$  on the domain  $\emptyset$ . So under this boundary condition, the eigenvalue problem

$$\begin{cases} D\varphi = \lambda^{\text{MIT}}\varphi & \text{on } \emptyset \\ \mathbb{B}_{\text{MIT}}^\pm\varphi = 0 & \text{along } \partial\emptyset \end{cases} \quad (6)$$

has a discrete spectrum with finite dimensional eigenspaces consisting of smooth spinor fields.

**Remark 1.** Under the MIT boundary condition  $\mathbb{B}_{\text{MIT}}^-$ , the spectrum of the Dirac operator  $D$  is contained in the upper half complex plane  $\{z \in \mathbb{C} / \text{Im}(z) > 0\}$ . Indeed, let  $\lambda^{\text{MIT}}$  be an eigenvalue of  $D$  under the MIT boundary condition and  $\varphi \in \Gamma(\Sigma\emptyset)$  the associated spinor field, then taking  $\psi = i\varphi$  in the Formula (5) leads to

$$2 \text{Im}(\lambda^{\text{MIT}}) \int_{\emptyset} |\varphi|^2 dv(g) = \int_{\partial\emptyset} |\varphi|^2 ds(g) \quad (7)$$

Two possibilities can occur: we have either  $\text{Im}(\lambda^{\text{MIT}}) > 0$  or  $\text{Im}(\lambda^{\text{MIT}}) = 0$ . If  $\text{Im}(\lambda^{\text{MIT}}) = 0$ , then the spinor field  $\varphi$  should vanish along the boundary  $\partial\emptyset$  and by the unique continuation principle (see [BBW93]), it should be identically zero on the manifold  $\emptyset$ . This is impossible because the spinor  $\varphi$  is supposed to be an eigenspinor, so a non trivial field. The first case is the only possibility, i.e.  $\text{Im}(\lambda^{\text{MIT}}) > 0$ . For the boundary condition  $\mathbb{B}_{\text{MIT}}^+$ , we can show that the imaginary part of all eigenvalues of the Dirac operator is negative.

#### 4. THE HYPERBOLIC REILLY FORMULA

In this section, we give a spinorial Reilly formula based on a modification of the spinorial Levi-Civita connection. Let  $\alpha \in \mathbb{R}$ , then we define the connection  $\nabla^\alpha$  acting on  $\Sigma\emptyset$  by

$$\nabla_X^\alpha \varphi := \nabla_X \varphi + i\alpha\gamma(X)\varphi, \quad (8)$$

for all  $\varphi \in \Gamma(\Sigma\emptyset)$  and  $X \in \Gamma(T\emptyset)$ . We can now derive an integral version of the Schrödinger-Lichnerowicz formula using the modified connection  $\nabla^\alpha$ . Indeed, we have:

**Proposition 2.** *For all spinor fields  $\varphi \in \Gamma(\Sigma\emptyset)$ , we have:*

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{\mathbf{L}^2} = \langle D^2 \varphi, \varphi \rangle_{\mathbf{L}^2} - \langle \frac{\mathbf{R}}{4} \varphi, \varphi \rangle_{\mathbf{L}^2} + n\alpha^2 \|\varphi\|_{\mathbf{L}^2}^2 - \int_{\partial\emptyset} \langle \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g), \quad (9)$$

where  $\mathbf{R}$  is the scalar curvature of the domain  $\emptyset$ .

*Proof:* First note that the  $\mathbf{L}^2$ -formal adjoint of the connection  $\nabla^\alpha$  is, by definition, given by

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{\mathbf{L}^2} = \|\nabla^\alpha \varphi\|_{\mathbf{L}^2}^2 = \sum_{j=1}^n \int_{\emptyset} \langle \nabla_{e_j}^\alpha \varphi, \nabla_{e_j}^\alpha \varphi \rangle dv(g),$$

for all  $\varphi \in \Gamma(\Sigma\mathcal{O})$  and where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame of  $T\mathcal{O}$ . An easy calculation using the compatibility properties of the Hermitian metric with the spinorial connection and the Clifford multiplication gives

$$\sum_{j=1}^n \langle \nabla_{e_j}^\alpha \varphi, \nabla_{e_j}^\alpha \varphi \rangle = \sum_{j=1}^n \left( e_j \langle \nabla_{e_j}^\alpha \varphi, \varphi \rangle - \langle \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \rangle \right),$$

and Stokes theorem leads to

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{\mathbb{L}^2} = \langle - \sum_{j=1}^n \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \rangle_{\mathbb{L}^2} - \int_{\partial\mathcal{O}} \langle \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g).$$

We can now easily compute

$$\begin{aligned} \langle - \sum_{j=1}^n \nabla_{e_j}^{-\alpha} \nabla_{e_j}^\alpha \varphi, \varphi \rangle_{\mathbb{L}^2} &= \langle - \sum_{j=1}^n \nabla_{e_j} \nabla_{e_j} \varphi, \varphi \rangle_{\mathbb{L}^2} + n\alpha^2 \|\varphi\|_{\mathbb{L}^2}^2 \\ &= \langle \nabla^* \nabla \varphi, \varphi \rangle_{\mathbb{L}^2} + n\alpha^2 \|\varphi\|_{\mathbb{L}^2}^2, \end{aligned}$$

and then the classical Schrödinger-Lichnerowicz formula (see [LM89]) leads to Identity (9).  $\square$

This formula is a first step to obtain Inequality (3). However, we have now to introduce the Dirac operator and the twistor operator associated with the connection  $\nabla^\alpha$ . The modified Dirac operator is locally defined by

$$D^\alpha \varphi = \sum_{j=1}^n \gamma(e_j) \nabla_{e_j}^\alpha \varphi, \quad (10)$$

and the associated twistor operator by

$$P_X^\alpha \varphi = \nabla_X^\alpha \varphi + \frac{1}{n} \gamma(X) D^\alpha \varphi, \quad (11)$$

for all  $X \in \Gamma(T\mathcal{O})$  and  $\varphi \in \Gamma(\Sigma\mathcal{O})$ . Note that for  $\alpha = 0$ , the operators  $D^0$  and  $P^0$  are respectively the classical Dirac operator and the classical twistor operator which satisfy the relation (see [BHMM] or [Fri00] for example)

$$|\nabla \varphi|^2 = |P\varphi|^2 + \frac{1}{n} |D\varphi|^2$$

We can then check that the modified operators satisfy the same relation, i.e.

$$|\nabla^\alpha \varphi|^2 = |P^\alpha \varphi|^2 + \frac{1}{n} |D^\alpha \varphi|^2. \quad (12)$$

Indeed, if  $\{e_1, \dots, e_n\}$  is a local orthonormal frame of  $T\mathcal{O}$ , we have

$$\begin{aligned} |\mathbb{P}^\alpha \varphi|^2 &= \sum_{j=1}^n \langle \nabla_{e_j}^\alpha \varphi + \frac{1}{n} \gamma(e_j) D^\alpha \varphi, \nabla_{e_j}^\alpha \varphi + \frac{1}{n} \gamma(e_j) D^\alpha \varphi \rangle \\ &= |\nabla^\alpha \varphi|^2 - \frac{2}{n} |D^\alpha \varphi|^2 + \frac{1}{n} |D^\alpha \varphi|^2 \\ &= |\nabla^\alpha \varphi|^2 - \frac{1}{n} |D^\alpha \varphi|^2, \end{aligned}$$

and so Identity (12) follows directly. We are now ready to establish the hyperbolic version of the spinorial Reilly formula given in [HMR02]. This formula can be seen as an analogous of the one used in [HMR03] to give a lower bound of the first eigenvalue of the intrinsic Dirac operator for hypersurfaces bounding a compact domain of a manifold with negative scalar curvature. More precisely, we prove:

**Proposition 3.** *For all  $\varphi \in \Gamma(\Sigma\mathcal{O})$ , we have:*

$$\begin{aligned} \|\mathbb{P}^\alpha \varphi\|_{L^2}^2 &= \frac{n-1}{n} \|D^\alpha \varphi\|_{L^2}^2 - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 \\ &\quad + \int_{\partial\mathcal{O}} \langle D^{\mathbb{S}} \varphi + \frac{n-1}{2} (2\alpha i\gamma(\nu)\varphi - H\varphi), \varphi \rangle ds(g), \end{aligned} \quad (13)$$

where  $H$  is the mean curvature of the boundary  $\partial\mathcal{O}$  of  $\mathcal{O}$ .

*Proof:* Observe first that the modified Dirac operator  $D^\alpha$  is not formally self-adjoint. Indeed an easy calculation using (5) gives

$$\int_{\mathcal{O}} \langle D^\alpha \varphi, \psi \rangle dv(g) = \int_{\mathcal{O}} \langle \varphi, D^{-\alpha} \psi \rangle dv(g) - \int_{\partial\mathcal{O}} \langle \gamma(\nu)\varphi, \psi \rangle ds(g), \quad (14)$$

for all  $\varphi, \psi \in \Gamma(\Sigma\mathcal{O})$ . However, we have:

$$D^2 \varphi = D^{-\alpha} D^\alpha \varphi - n^2 \alpha^2 \varphi,$$

and so substituting in Formula (9) gives

$$\langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} = \langle D^{-\alpha} D^\alpha \varphi, \varphi \rangle_{L^2} - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 - \int_{\partial\mathcal{O}} \langle \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g).$$

The integration by parts formula (14) leads to

$$\begin{aligned} \langle (\nabla^\alpha)^* \nabla^\alpha \varphi, \varphi \rangle_{L^2} &= \|D^\alpha \varphi\|_{L^2}^2 - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 \\ &\quad - \int_{\partial\mathcal{O}} \langle \gamma(\nu) D^\alpha \varphi + \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g). \end{aligned}$$

With the help of Identity (12), we have

$$\begin{aligned} \|\mathbb{P}^\alpha \varphi\|_{L^2}^2 &= \frac{n-1}{n} \|D^\alpha \varphi\|_{L^2}^2 - \langle \frac{\mathbb{R}}{4} \varphi, \varphi \rangle_{L^2} - n(n-1) \alpha^2 \|\varphi\|_{L^2}^2 \\ &\quad - \int_{\partial\mathcal{O}} \langle \gamma(\nu) D^\alpha \varphi + \nabla_\nu^\alpha \varphi, \varphi \rangle ds(g). \end{aligned}$$

However the boundary term can be written

$$-\gamma(\nu)D^\alpha\varphi - \nabla_\nu^\alpha\varphi = -\gamma(\nu)D\varphi - \nabla_\nu\varphi + (n-1)\alpha i\gamma(\nu)\varphi,$$

and using the identity

$$-\gamma(\nu)D\varphi - \nabla_\nu\varphi = D^{\mathbf{S}}\varphi - \frac{n-1}{2}H\varphi,$$

Formula (13) follows directly.  $\square$

We are now ready to prove Theorem 1.

## 5. THE ESTIMATE

*Proof of Theorem 1:* Consider now a compact domain  $\mathcal{O}$  of a Riemannian spin manifold such that the mean curvature  $H$  of the boundary satisfies  $H \geq 2\alpha$ , for  $\alpha > 0$ . By ellipticity of the MIT boundary condition  $\mathbb{B}_{\text{MIT}}^-$ , consider a smooth spinor field  $\varphi \in \Gamma(\Sigma\mathcal{O})$  solution of the eigenvalue boundary problem (6), i.e.  $\varphi$  satisfies

$$\begin{cases} D\varphi = \lambda^{\text{MIT}}\varphi & \text{on } \mathcal{O} \\ \mathbb{B}_{\text{MIT}}^-\varphi = 0 & \text{along } \partial\mathcal{O} \end{cases} \quad (15)$$

with  $\text{Im}(\lambda^{\text{MIT}}) > 0$  by Remark 1. We now apply the hyperbolic Reilly formula (13) to the spinor field  $\varphi$  to get

$$\begin{aligned} \|\mathbb{P}^\alpha\varphi\|_{\mathbf{L}^2}^2 &= \left( \frac{n-1}{n} |\lambda^{\text{MIT}} - n\alpha i|^2 - n(n-1)\alpha^2 \right) \|\varphi\|_{\mathbf{L}^2}^2 - \left\langle \frac{R}{4}\varphi, \varphi \right\rangle_{\mathbf{L}^2} \\ &\quad + \int_{\partial\mathcal{O}} \left\langle D^{\mathbf{S}}\varphi + \frac{n-1}{2}(2\alpha i\gamma(\nu)\varphi - H\varphi), \varphi \right\rangle ds(g). \end{aligned}$$

Note that since  $i\gamma(\nu)\varphi = \varphi$  along the boundary, we can compute

$$\langle D^{\mathbf{S}}\varphi, \varphi \rangle = \langle D^{\mathbf{S}}\varphi, i\gamma(\nu)\varphi \rangle = \langle i\gamma(\nu)D^{\mathbf{S}}\varphi, \varphi \rangle = -\langle D^{\mathbf{S}}(i\gamma(\nu)\varphi), \varphi \rangle = -\langle D^{\mathbf{S}}\varphi, \varphi \rangle,$$

and so the preceding formula gives

$$\begin{aligned} \|\mathbb{P}^\alpha\varphi\|_{\mathbf{L}^2}^2 + \frac{n-1}{2} \int_{\partial\mathcal{O}} (H - 2\alpha)|\varphi|^2 ds(g) &= \\ \frac{n-1}{n} (|\lambda^{\text{MIT}}|^2 - 2n\alpha \text{Im}(\lambda^{\text{MIT}})) \|\varphi\|_{\mathbf{L}^2}^2 - \left\langle \frac{R}{4}\varphi, \varphi \right\rangle_{\mathbf{L}^2} \end{aligned} \quad (16)$$

The assumption on the mean curvature gives:

$$|\lambda^{\text{MIT}}|^2 - 2n\alpha \text{Im}(\lambda^{\text{MIT}}) \geq \frac{n}{4(n-1)} R_0.$$

For  $\alpha_0 = \frac{1}{2}H_0$ , where  $H_0 = \inf_{\partial\mathcal{O}}(H)$ , we get Inequality (16). Suppose now that equality is achieved, thus

$$\|\mathbb{P}^{\alpha_0}\varphi\|_{\mathbf{L}^2}^2 = 0 \quad \text{and} \quad \frac{n-1}{2} \int_{\partial\mathcal{O}} (H - 2\alpha_0)|\varphi|^2 ds(g) = 0.$$

Moreover the spinor field  $\varphi$  is a solution of (15), so it satisfies the Killing equation

$$\nabla_X\varphi = -\frac{\lambda^{\text{MIT}}}{n}\gamma(X)\varphi, \quad \text{for all } X \in \Gamma(\text{T}\mathcal{O}).$$



Since such a spinor field has no zeroes (see [Fri00]), the mean curvature of the boundary is constant with  $H = 2\alpha_0$ . Furthermore, it is a well-known result [BFGK90] that, in this case, the eigenvalue  $\lambda^{\text{MIT}}$  has to be either real or purely imaginary. Here we have  $\text{Im}(\lambda^{\text{MIT}}) > 0$ , then  $\lambda^{\text{MIT}} \in i\mathbb{R}_*^+$ . The domain  $\mathcal{O}$  is in particular an Einstein manifold. We now show that the boundary has to be totally umbilical. Indeed, note that we have for all  $X \in \Gamma(T(\partial\mathcal{O}))$ :

$$\begin{aligned} \nabla_X(i\gamma(\nu)\varphi) &= i\gamma(\nabla_X\nu)\varphi + i\gamma(\nu)\nabla_X\varphi \\ &= i\gamma(\nabla_X\nu)\varphi + \alpha_0\gamma(\nu)\gamma(X)\varphi \\ &= i\gamma(\nabla_X\nu)\varphi - \alpha_0\gamma(X)\gamma(\nu)\varphi \\ &= i\gamma(\nabla_X\nu)\varphi + i\alpha_0\gamma(X)\varphi. \end{aligned}$$

However along the boundary we have  $i\gamma(\nu)\varphi = \varphi$ , so we obtain

$$\gamma(\nabla_X\nu)\varphi = -2\alpha_0\gamma(X)\varphi.$$

Since the spinor field  $\varphi$  has no zeros, we have  $A(X) = -\nabla_X\nu = 2\alpha X$  and the boundary is totally umbilical. We can again show that in the equality case, we have  $\text{Im}(\lambda^{\text{MIT}}) = n\alpha_0$ . In fact, just note that the boundary term can be rewritten as

$$\int_{\partial\mathcal{O}} \langle D^S\varphi - \frac{n-1}{2}H\varphi + (n-1)\alpha_0\varphi, \varphi \rangle ds(g) = - \int_{\partial\mathcal{O}} \langle \nabla_\nu\varphi + \gamma(\nu)D\varphi - (n-1)\alpha_0\varphi, \varphi \rangle ds(g).$$

This term is zero since we have equality in (16). Now using that the spinor field  $\varphi$  is an imaginary Killing spinor satisfying (6) gives

$$\nabla_\nu\varphi + \gamma(\nu)D\varphi = \frac{n-1}{n}\text{Im}(\lambda^{\text{MIT}})\varphi.$$

Substituting in the preceding identity gives

$$(n-1) \int_{\partial\mathcal{O}} \left( \alpha_0 - \frac{\text{Im}(\lambda^{\text{MIT}})}{n} \right) |\varphi|^2 ds(g) = 0,$$

and since  $\varphi$  has no zeroes,  $\text{Im}(\lambda^{\text{MIT}}) = n\alpha_0 = \frac{nH_0}{2}$ .  $\square$

## Remark 2.

- (1) The orthogonal projection  $\mathbb{B}_{\text{MIT}}^+$  defines a local elliptic boundary condition for the Dirac operator  $D$  of  $\mathcal{O}$ . We can easily check that in this case, the imaginary part of an eigenvalue  $\lambda^{\text{MIT}}$  of  $D$  satisfies  $\text{Im}(\lambda^{\text{MIT}}) < 0$ . Inequality (3) is then given by

$$|\lambda^{\text{MIT}}|^2 \geq \frac{n}{4(n-1)} R_0 - n \text{Im}(\lambda^{\text{MIT}}) H_0.$$

- (2) For  $H_0 = 0$ , we obtain Inequality (2). In fact, if we suppose that equality is achieved, Theorem 1 implies  $\text{Im}(\lambda^{\text{MIT}}) = \frac{nH_0}{2} = 0$  which is impossible by Remark 1.
- (3) Note that the Riemannian spin manifolds with an imaginary Killing spinor with Killing number  $i\alpha$  have been classified by H. Baum in [Bau89a] and [Bau89b]. Such manifolds are called pseudo-hyperbolic and they are given by

$$(\mathbb{R} \times_{\text{exp}} M_0, g) = (\mathbb{R} \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0}),$$

where  $(M_0, g_{M_0})$  is a complete Riemannian spin manifold carrying a non-trivial parallel spinor. After suitable rescaling of the metric, we can assume that the Killing number is either  $i/2$  or  $-i/2$ , i.e. we have

$$\nabla_X \phi = \pm \frac{i}{2} \gamma(X) \phi.$$

Moreover, constant mean curvature hypersurfaces in pseudo-hyperbolic manifolds are classified by the Hyperbolic Alexandrov Theorem proved in [Mon99] (see also [HMR03] for a proof using spinors). Indeed, such a hypersurface is either a round geodesic hypersphere (and, in this case,  $M_0$  is flat and  $H > 1$ ) or a slice  $\{s\} \times M_0$  (and, in this case,  $M_0$  is compact and  $H = 1$ ).

We can then prove the following corollary:

**Corollary 4.** *If the boundary of the compact domain  $\mathcal{O}$  is connected, there is no manifold satisfying the equality case in Inequality (3).*

*Proof:* If  $\mathcal{O}$  is a compact domain with connected boundary achieving equality in (3), then there exists an imaginary Killing spinor on  $\mathcal{O}$  and the boundary  $\partial\mathcal{O}$  is a totally umbilical constant mean curvature hypersurface with  $H = 2\alpha$ . However, using Remark (2).3,  $\mathcal{O}$  is a domain in a pseudo-hyperbolic space whose connected boundary is a slice  $\{s\} \times M_0$  and then  $\mathcal{O}$  is non-compact.  $\square$

**Remark 3.** With a slight modification of the boundary condition, we give a domain  $\mathcal{O}$  whose boundary has two connected components carrying an imaginary Killing spinor field  $\varphi \in \Gamma(\Sigma\mathcal{O})$  which satisfy

$$i\gamma(\nu_1)\varphi|_{\partial\mathcal{O}_1} = \varphi|_{\partial\mathcal{O}_1} \quad \text{and} \quad i\gamma(\nu_2)\varphi|_{\partial\mathcal{O}_2} = -\varphi|_{\partial\mathcal{O}_2}, \quad (17)$$

where  $\nu_1$  (resp.  $\nu_2$ ) is an inner unit vector field normal to  $\partial\mathcal{O}_1$  (resp.  $\partial\mathcal{O}_2$ ). First recall that one distinguishes two types of imaginary Killing spinors (see [Bau89a] and [Bau89b]). Indeed, if  $\varphi \in \Gamma(\Sigma\mathcal{O})$  is an imaginary Killing spinor, denote by  $f$  its length function, then the function

$$q_\varphi(x) := f(x)^2 - \frac{1}{4\alpha^2} \|\nabla f\|^2$$

satisfies  $q_\varphi$  is constant and  $q_\varphi \geq 0$ . If  $q_\varphi = 0$ ,  $\varphi$  is a Killing spinor of type I whereas if  $q_\varphi > 0$ ,  $\varphi$  is a Killing spinor of type II. If  $(N^n, g)$  is a complete connected Riemannian spin manifold with an imaginary Killing spinor of type II associated with the Killing number  $i\alpha$ , then  $(N^n, g)$  is isometric to the hyperbolic space  $\mathbb{H}_{-4\alpha^2}^n$ . If  $(N^n, g)$  admits an imaginary Killing spinor of type I, then  $(N^n, g)$  is isometric to the warped product  $(\mathbb{R} \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0})$ , where  $M_0$  is a complete Riemannian spin manifold with a non-trivial parallel spinor field. Moreover,  $q_\varphi = 0$  if and only if there exists a unit vector field  $\xi$  on  $N$  such that  $\gamma(\xi)\varphi = i\varphi$ . In fact, we can easily prove that the vector field  $\xi$  is the normal field of  $\{t\} \times M_0$  for all  $t \in \mathbb{R}$ . So consider the domain given by the warped product  $\mathcal{O} := ([a, b] \times M_0, dt^2 \oplus e^{-4\alpha t} g_{M_0})$ , where  $M_0$  is a compact spin manifold carrying a non-trivial parallel spinor field and with  $-\infty < a < b < +\infty$ . The domain  $\mathcal{O}$  carries an imaginary Killing spinor  $\varphi$  of type I, so there exists  $\xi$  normal to  $\{t\} \times M_0$  for all  $t \in [a, b]$

such that  $\gamma(\xi)\varphi = i\varphi$ . The boundary of  $\mathcal{O}$  has two connected components which are slices  $\{a\} \times M_0$  and  $\{b\} \times M_0$  of  $\mathcal{O}$  and with mean curvature  $H_a = H_b = 2\alpha$ , where  $H_t$  is the mean curvature of a slice  $\{t\} \times M_0$ . The spinor field  $\varphi$  clearly satisfies the boundary conditions (17).

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