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► **To cite this version:**

Valeriu Vrabie, Pierre Granjon, Christine Serviere. Spectral kurtosis: from definition to application. 6th IEEE International Workshop on Nonlinear Signal and Image Processing (NSIP 2003), 2003, Grado-Trieste, Italy. hal-00021302

**HAL Id: hal-00021302**

**<https://hal.science/hal-00021302>**

Submitted on 20 Mar 2006

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# SPECTRAL KURTOSIS: FROM DEFINITION TO APPLICATION

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## ABSTRACT

The objective of this paper is to present the Spectral Kurtosis (SK) as a spectral analysis tool complementary to the classical power spectrum density (PSD). Theoretical definition of the SK is firstly justified, followed by a study of its behavior with respect to the spectral components of signals. An unbiased estimator is thus proposed, with its exact variance, and its signal to noise ratio sensitivity. Finally, results are illustrated by applying SK and PSD on synthetic and real signals.

## 1 Introduction

In the past few years, higher order statistics have been an extensive field of research [1, 2]. These works lead to several analysis tools, complementary to classical second order methods. One useful tool is the fourth-order cumulant based kurtosis, providing a measure of distance to gaussianity. In frequency domain, the Spectral Kurtosis (SK) of a signal is defined as the kurtosis of its frequency components. It was initially defined and used to detect “randomly occurring signals” in [3, 4]. In this work, the kurtosis of the real and imaginary parts of the frequential components of signals are estimated separately. This allows dealing with real variables, but leads to an incomplete definition and estimation. In [5, 6], the SK approach is generalized by using the frequency components modulus. Unfortunately, the authors don't give a theoretical definition and use only a moment based estimator instead of the cumulant based one. Moreover, they present the SK only in ambient gaussian noise context. The correct theoretical definition of SK, using cumulants of complex random variables, was finally given in a source separation context [7, 8]. In this work, the SK is used to measure the distance to gaussianity of different spectral components, but a biased estimator is provided.

In this paper, an unbiased estimator of the SK with

its theoretical variance is proposed. The influence of the signal to noise ratio is studied for this new estimator. This tool provides complementary informations with respect to the classical power spectral density (PSD), which are very useful in signal analysis.

## 2 Definition

Let  $x(n)$  be a real discrete time random process, and  $X(m)$  its  $N$ -point Discrete Fourier Transform (DFT). The spectral kurtosis (SK) of  $x(n)$  is defined as the kurtosis of the complex random variable  $X(m)$  at each frequency bin  $m$ :

$$\mathcal{K}_x(m) = \frac{\kappa_4\{X^\dagger(m), X^\dagger(m), X^\dagger(m), X^\dagger(m)\}}{[\kappa_2\{X^\dagger(m), X^\dagger(m)\}]^2} \quad (1)$$

where  $X^\dagger(m) \in \{X(m), X^*(m)\}$ ,  $X^*(m)$  is the complex conjugate of  $X(m)$  and  $\kappa_r$  is the  $r^{th}$  order cumulant.

Unfortunately, Eq. (1) leads to several different definitions. In order to solve this problem,  $x(n)$  is supposed to be a stationary random process. Therefore, the frequency component  $X(m)$  is a *circular* complex random variable at each frequency bin  $m$  [9, 10]. This particularity implies that the only non-null cumulants of  $X(m)$  must have as many complex conjugate terms as complex non-conjugate terms. Hence, the only definition of the SK that does not vanish for stationary signals is:

$$\mathcal{K}_x(m) = \frac{\kappa_4\{X(m), X^*(m), X(m), X^*(m)\}}{[\kappa_2\{X(m), X^*(m)\}]^2} \quad (2)$$

In the stationary case, using the simplifications due to the circularity of complex random variables [9, 10], the definition of SK in terms of expectation is:

$$\mathcal{K}_x(m) = \frac{E\{|X(m)|^4\} - 2[E\{|X(m)|^2\}]^2}{[E\{|X(m)|^2\}]^2} \quad (3)$$

### 3 Properties

Let  $b(n)$  be a stationary random process with absolutely summable  $k^{th}$  order multicorrelations, whatever  $k$ . This means that  $b(n)$  is sufficiently random such that samples which are well separated in time can be regarded as approximately independent. This property is verified by the most commonly encountered stochastic processes.

It was shown in [1, pp. 94-98] that for this kind of process, often called "mixing process", its  $N$ -points DFT  $B(m)$  tends toward a centered gaussian circular complex random variable at each frequency bin  $m$ , with  $N \rightarrow +\infty$ . Its PSD is denoted  $\gamma_b(m) = \kappa_2\{B(m), B^*(m)\} = E\{|B(m)|^2\}$ .

Knowing that all  $r^{th}$  order cumulant, with  $r \geq 3$ , are zero for a gaussian (complex) random variable, its SK becomes:

$$\mathcal{K}_b(m) = 0 \quad (4)$$

Let  $s(n) = Ae^{j(2\pi\frac{m_0}{N}n + \varphi)}$  be a stationary harmonic process, with  $\varphi$  a random variable uniformly distributed over  $[-\pi, \pi]$ . Since its  $N$ -points DFT is  $S(m) = A\delta(m_0)e^{j\varphi}$ , where  $\delta(m)$  denotes the Dirac function, it is a simple exercise to prove, using Eq. (3), that the SK of this process at the frequency bin  $m_0$  is:

$$\mathcal{K}_s(m_0) = -1 \quad (5)$$

Let  $x(n)$  be a stationary process given by the sum of the two previous processes:

$$x(n) = s(n) + b(n) \quad (6)$$

In frequency domain, at the frequency bin  $m_0$ ,  $S(m_0) = Ae^{j\varphi}$  is a non-gaussian circular complex random variable and  $B(m_0)$  is a gaussian circular complex random variable, independent of  $S(m_0)$ . Defining the local Signal to Noise Ratio (SNR) as

$$SNR(m_0) = \frac{A^2}{\gamma_b(m_0)} \quad (7)$$

the SK of  $x(n)$  at the frequency bin  $m_0$  is (see appendix A.1):

$$\mathcal{K}_x(m_0) = -\left(\frac{SNR(m_0)}{1 + SNR(m_0)}\right)^2 \quad (8)$$

This equation reduces to Eq. (4), respectively (5), when  $SNR(m_0)$  tends toward 0, respectively  $\infty$ .

### 4 Estimation

In practice, we usually have only one realization of length  $N_t$  of the stationary process  $x(n)$  defined in

Eq. (6). In this case, the signal is divided into  $M$  unoverlapped blocks, each of length  $N$ . A  $N$ -point DFT is performed on each segment giving the random variables  $X_i(m)$ , ( $i = 1, \dots, M$ ).

Using the  $k$ -statistics as in [11], an unbiased estimator of the SK of  $x(n)$  is (see also appendix A.2):

$$\hat{\mathcal{K}}_x(m) = \frac{M}{M-1} \left[ \frac{(M+1) \sum_{i=1}^M |X_i(m)|^4}{\left(\sum_{i=1}^M |X_i(m)|^2\right)^2} - 2 \right] \quad (9)$$

This equation is illustrated in Fig. 1, which shows the theoretical and estimated values of the SK versus  $SNR$  at the frequency bin  $m_0$  for the process  $x(n)$  defined in eq. (6). Hence, this estimator is unbiased for the sum of two processes  $b(n)$  and  $s(n)$  and also for each process separately.

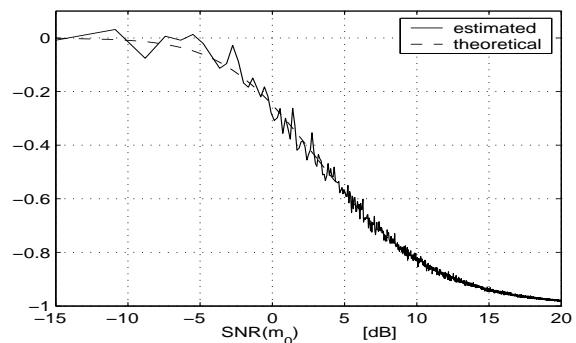


Figure 1: Spectral kurtosis versus  $SNR(m_0)$

For the random process  $b(n)$  defined in section 3, the variance of the previous estimator at each frequency bin  $m$  is (see appendix A.3):

$$var\{\hat{\mathcal{K}}_b(m)\} = \frac{4M^2}{(M-1)(M+2)(M+3)} < \frac{4}{M} \quad (10)$$

This allows to obtain a theoretical threshold of gaussianity in frequency domain, and to detect processes as  $b(n)$ . In Fig. 2 are plotted the theoretical variance and its estimate versus the number of unoverlapped blocks  $M$ , as well as its asymptotic limit  $4/M$ .

### 5 Behavior for nonstationary signals

Let  $c(n)$  be a nonstationary signal, defined as a component that may occasionally appear over a small time interval with a constant energy.

Assuming the signal is divided into  $M$  unoverlapped blocks, the contribution of  $c(n)$  in frequency domain can be described in each block  $i$  ( $i = 1, \dots, M$ ) as:

$$\alpha_i |C(m)| e^{j\theta_i}, \quad \forall m \in Supp\{C(m)\} \quad (11)$$

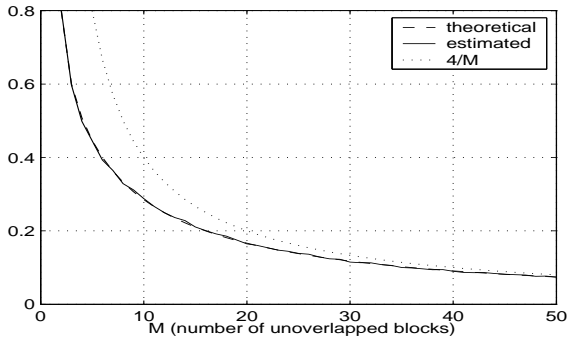


Figure 2: Estimator variance versus  $M$

where  $\text{Supp}\{C(m)\}$  is the spectral support of  $c(n)$ .  $\alpha_i$  is the  $i^{\text{th}}$  realization of a random variable  $\alpha = \{0, 1\}$ , representing the random occurrence of the unsteady component (see [5, 6]).

Assuming that the unsteady component involves only, let say,  $K$  blocks, the SK of  $c(n)$  is:

$$\hat{\mathcal{K}}_c(m) = \frac{M}{M-1} \left[ \frac{M+1}{K} - 2 \right], \quad \forall m \in \text{Supp}\{C(m)\} \quad (12)$$

which becomes positive if only a few blocks (*i.e.*  $K < \frac{M+1}{2}$ ) are involved.

Hence, using Eqs. (8), (10) and (12), the SK indicates that, at each frequency bin  $m$ , the analyzed signal contains a sine wave, a mixing process or a nonstationary component.

## 6 Applications

This section illustrates previous results by applying the SK on two different signals.

The first one is a synthetic signal, composed of four different components:

- a first nonstationary component which is a linear frequency chirp between frequencies 0.03 and 0.07, present over all time support of the signal,
- a second nonstationary component consisting of a sine wave with a frequency 0.1, and a slowly time-varying amplitude,
- a pure sine wave with a frequency 0.16,
- a white gaussian noise filtered by a resonant system, with an important resonance at frequency 0.24.

These different components can be visualized on the PSD shown in Fig. 3, and the corresponding SK is shown in Fig. 4. These two spectral quantities

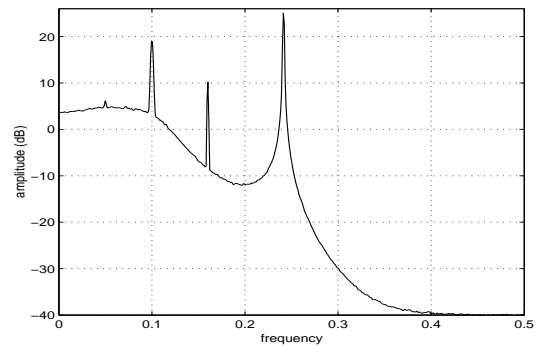


Figure 3: PSD of the synthetic signal

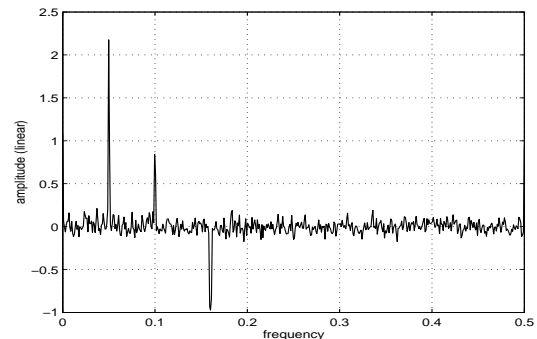


Figure 4: SK of the synthetic signal

were calculated by using  $N = 1024$ -points FFT and  $M = 900$  unoverlapped blocs. As expected, the SK estimator is a function of the nature of the spectral components. Its value is positive for the nonstationary spectral components (frequencies 0.05 and 0.1). The negative value obtained for the pure sine wave (frequency 0.16) verifies Eq. (8). At the other frequency bins, the SK tends to zero with a variance verifying Eq. (10) (even for the important resonance at frequency 0.24).

The second signal is a supply current of an induction machine running at a constant rotation speed. This real signal was sampled during 60 seconds, with a sampling frequency of 10 kHz. Its PSD, SK and spectrogram are represented in Fig. 5, 6 and 7 respectively. All these spectral quantities were calculated by using  $N = 3000$ -points DFT and  $M = 200$  unoverlapped blocs. For the sake of simplicity, the studied frequency range is limited between 2950 Hz and 3200 Hz. The PSD in Fig. 5 shows that in this frequency range, this signal is composed of two important spectral lines at frequencies 3010 Hz and 3110 Hz (dashed arrows), four small spectral lines (solid arrows), and a wide band noise. The SK in Fig. 6 equals -1 for the two important spectral lines (dashed arrows), which indicates that these compo-

nents are stationary and harmonic. Moreover, the positive values of the SK for the four small spectral lines (solid arrows) shows that these components are nonstationary. Finally, the wide band noise can be viewed as a mixing process since its SK is null.

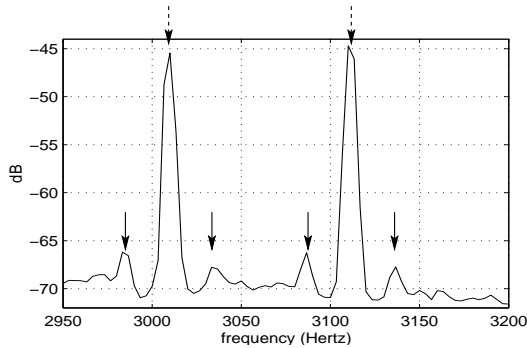


Figure 5: PSD of the real signal

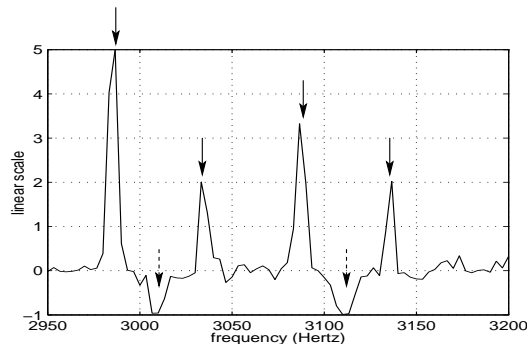


Figure 6: SK of the real signal

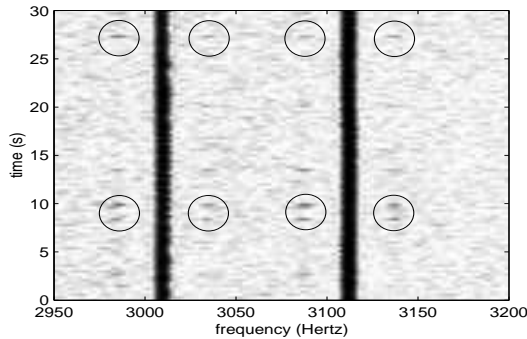


Figure 7: Spectrogram of the real signal

All these results are confirmed by the spectrogram in Fig. 7. Indeed, the stationarity of the two important spectral lines is evident. Moreover, the nonstationarity of the small spectral lines is shown by the presence of several components that appear occasionally at these frequencies (see solid circles).

These two applications show that the SK provides

complementary useful informations on the nature of the signal components with respect to the classical PSD.

## 7 Conclusion

Finally, the SK can be viewed as a complementary spectral analysis tool with respect to the PSD. Indeed, it provides further informations about the spectral nature of the analyzed signal. More specifically, it indicates at each frequency bin, if the signal contains nonstationary, stationary harmonic or mixing process.

## 8 Acknowledgement

The authors wish to thank Schneider Electric for providing the real signals presented in this paper.

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## A Appendices

For simplification, we denote in the following:

$$\begin{aligned}\kappa_{X(4)}(m) &= \kappa_4\{X(m), X^*(m), X(m), X^*(m)\} \\ \kappa_{X(2)}(m) &= \kappa_2\{X(m), X^*(m)\}\end{aligned}$$

### A.1

At the frequency bin  $m_0$ , the components  $S(m_0)$  and  $B(m_0)$  are two independent circular complex random variables. Knowing that the cumulant of a sum of two independent (complex) random variables equals the sum of cumulants of each variable [1, pp. 19], the SK of  $x$  is:

$$\mathcal{K}_x(m_0) = \frac{\kappa_{S(4)}(m_0) + \kappa_{B(4)}(m_0)}{[\kappa_{S(2)}(m_0) + \kappa_{B(2)}(m_0)]^2}$$

By using the property that all  $r^{\text{th}}$  order cumulant, with  $r \geq 3$ , are zero for a gaussian (complex) random variable:

$$\mathcal{K}_x(m_0) = \frac{\kappa_{S(4)}(m_0)}{[\kappa_{S(2)}(m_0) + \kappa_{B(2)}(m_0)]^2}$$

These complex random variables being circular, the relation above in terms of expectation is (see [9]):

$$\mathcal{K}_x(m_0) = \frac{E\{|S(m_0)|^4\} - 2[E\{|S(m_0)|^2\}]^2}{[E\{|S(m_0)|^2\} + E\{|B(m_0)|^2\}]^2}$$

With  $E\{|S(m_0)|^4\} = A^4$ ,  $E\{|S(m_0)|^2\} = A^2$ ,  $E\{|B(m_0)|^2\} = \gamma_b(m_0)$  and using Eq. (7), the relation (8) is proved.

### A.2

As in the real case, an estimator of the SK is given by:

$$\hat{\mathcal{K}}_x(m) = \frac{\hat{\kappa}_{X(4)}(m)}{[\hat{\kappa}_{X(2)}(m)]^2} \quad (13)$$

with  $\hat{\kappa}_{X(4)}(m)$  and  $\hat{\kappa}_{X(2)}(m)$  unbiased estimators of the fourth, respectively, second order cumulant.

The numerator  $\kappa_{X(4)}(m)$  in Eq. (2) in terms of expectation is:

$$\kappa_{X(4)}(m) = E\{|X(m)|^4\} - 2[E\{|X(m)|^2\}]^2$$

hence, an estimator of  $\kappa_{X(4)}(m)$  will be given by:

$$\begin{aligned}\hat{\kappa}_{X(4)}(m) &= f(M) \sum_{i=1}^M |X_i(m)|^4 + \\ &+ g(M) \sum_{i,j=1}^M |X_i(m)|^2 |X_j(m)|^2\end{aligned}$$

Developing the second sum, we obtain:

$$\begin{aligned}\hat{\kappa}_{X(4)}(m) &= (f(M) + g(M)) \sum_{i=1}^M |X_i(m)|^4 + \\ &+ g(M) \sum_{i,j=1; i \neq j}^M |X_i(m)|^2 |X_j(m)|^2\end{aligned}$$

Supposing the realizations  $X_i(m)$  and  $X_j(m)$  are independent and taking the expectation in the relation above, we get:

$$\begin{aligned}E\{\hat{\kappa}_{X(4)}(m)\} &= M(f(M) + g(M))E\{|X(m)|^4\} + \\ &+ M(M-1)g(M) [E\{|X(m)|^2\}]^2\end{aligned}$$

Thus, an unbiased estimator of the numerator is provided when  $M(M-1)g(M) = -2$  and  $M(f(M) + g(M)) = 1$ , giving:

$$\begin{aligned}\hat{\kappa}_{X(4)}(m) &= \frac{M+1}{M(M-1)} \sum_{i=1}^M |X_i(m)|^4 + \\ &+ \frac{-2}{M(M-1)} \sum_{i,j=1}^M |X_i(m)|^2 |X_j(m)|^2\end{aligned}$$

In the same manner, an unbiased estimator of the second order cumulant is:

$$\hat{\kappa}_{X(2)}(m) = \frac{1}{M} \sum_{i=1}^M |X_i(m)|^2$$

Replacing the last two equations in Eq. (13), we obtain the estimator of SK given in Eq. (9).

If  $b(n)$  is a mixing process (see section 3), then:

$$\frac{\hat{\kappa}_{B(4)}(m)}{[\hat{\kappa}_{B(2)}(m)]^2} \text{ and } \hat{\kappa}_{B(2)}(m) \text{ are independents} \quad (14)$$

at each frequency bin  $m$ . This property is not proved here; for more details see [11, Chap. 12]. Thus:

$$E\left\{\frac{\hat{\kappa}_{B(4)}(m)}{[\hat{\kappa}_{B(2)}(m)]^2}\right\} = \frac{E\{\hat{\kappa}_{B(4)}(m)\}}{E\{[\hat{\kappa}_{B(2)}(m)]^2\}} = \frac{\kappa_{B(4)}(m)}{E\{[\hat{\kappa}_{B(2)}(m)]^2\}} = 0 \quad (15)$$

If  $s(n)$  is a harmonic process (see section 3), then:

$$\hat{\mathcal{K}}_s(m_0) = -1 \text{ leading to } E\{\hat{\mathcal{K}}_s(m_0)\} = \mathcal{K}_s(m_0)$$

Hence, from these two last equations and using appendix A.1, the estimator of the SK of  $x(n)$  given in Eq. (9) is unbiased.

### A.3

For a mixing process  $b(n)$ , the variance of the estimator given in Eq. (9) is:

$$\text{var}\{\hat{\mathcal{K}}_b(m)\} = E\left\{\frac{[\hat{\kappa}_{B(4)}(m)]^2}{[\hat{\kappa}_{B(2)}(m)]^4}\right\} - \left[E\left\{\frac{\hat{\kappa}_{B(4)}(m)}{[\hat{\kappa}_{B(2)}(m)]^2}\right\}\right]^2$$

Using Eqs. (14) and (15), it follows that:

$$\text{var}\{\hat{\mathcal{K}}_b(m)\} = E\left\{\frac{[\hat{\kappa}_{B(4)}(m)]^2}{[\hat{\kappa}_{B(2)}(m)]^4}\right\} = \frac{E\{[\hat{\kappa}_{B(4)}(m)]^2\}}{E\{[\hat{\kappa}_{B(2)}(m)]^4\}}$$

After some mathematical manipulation and knowing that:  $E\{|B(m)|^4\} = 2\gamma_b^2(m)$ ,  $E\{|B(m)|^6\} = 6\gamma_b^3(m)$  and  $E\{|B(m)|^8\} = 24\gamma_b^4(m)$  for a gaussian complex circular random variable  $B(m)$ , we obtain:

$$E\{[\hat{\kappa}_{B(4)}(m)]^2\} = \frac{4(M+1)}{M(M-1)}\gamma_b^4(m)$$

$$E\{[\hat{\kappa}_{B(2)}(m)]^4\} = \frac{(M+1)(M+2)(M+3)}{M^3}\gamma_b^4(m)$$

which prove the relation (10).