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# A new inequality for the Hermite constants

Roland Bacher\*

*Abstract:* We prove an inequality of the form  $\gamma_n \geq C_n(\gamma_{n-1})$  between Hermite's constants  $\gamma_n$  and  $\gamma_{n-1}$ . This inequality yields also a new proof of the Minkowski-Hlawka bound  $\Delta_n \geq \zeta(n)2^{1-n}$  for the maximal density  $\Delta_n$  of  $n$ -dimensional lattice-packings. <sup>1</sup>

## 1 Introduction and main results

We denote by  $\min(\Lambda) = \min_{\lambda \in \Lambda \setminus \{0\}} \langle \lambda, \lambda \rangle$  the minimum (defined as the squared Euclidean length of a shortest non-zero element) of an  $n$ -dimensional lattice  $\Lambda \in \mathbf{E}^n$  in the Euclidean vector-space  $\mathbf{E}^n$  and define the *density* of  $\Lambda$  by

$$\Delta(\Lambda) = \sqrt{\frac{(\min \Lambda_n)^n}{4^n \det \Lambda_n}} V_n$$

where  $V_n = \frac{\pi^{n/2}}{(n/2)!}$  stands through the whole paper for the volume of the  $n$ -dimensional unit-ball in  $\mathbf{E}^n$ . The density  $\Delta(\Lambda)$  is the proportion of volume occupied by a maximal open Euclidean ball embedded in the flat torus  $\mathbf{E}^n/\Lambda$  with volume  $\sqrt{\det(\Lambda)}$  and having a shortest closed geodesic of length  $\sqrt{\min(\Lambda)}$ . The largest density  $\Delta(\Lambda)$  achieved by an  $n$ -dimensional lattice  $\Lambda$  is called the *maximal density*  $\Delta_n$  in dimension  $n$ . Related constants are the *maximal center density*  $\delta_n = \Delta_n/V_n$  and the *Hermite constant*  $\gamma_n = 4\delta_n^{2/n}$  in dimension  $n$ . The sequence  $\gamma_1, \gamma_2, \dots$  of Hermite constants satisfies for  $n \geq 3$  *Mordell's inequality*

$$\gamma_n \leq \gamma_{n-1}^{(n-1)/(n-2)}$$

which yields an upper bound for  $\gamma_n$  (if  $n \geq 3$ ) in terms of  $\gamma_{n-1}$ . Our main result is a complementary inequality bounding  $\gamma_n$  from below in terms of  $\gamma_{n-1}$ . For the convenience of the reader we state it in three equivalent ways, either in terms of densities  $\Delta_m$ , center-densities  $\delta_m$  or Hermite constants  $\gamma_m$  in dimension  $m$ . It involves the *Möbius function*  $\mu : \mathbf{N}_{>0} \rightarrow \mathbf{Z}$  defined by  $\mu(l) = (-1)^a$  for a natural integer  $l \in \mathbf{N}$  which is a product of  $a$  distinct primes and by  $\mu(l) = 0$  if  $l$  is divisible by the square of a prime number.

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**Theorem 1.1** (i) *The maximal densities  $\Delta_{n-1}$  and  $\Delta_n$  of lattice-packings in dimensions  $n-1$  and  $n \geq 2$  satisfy the inequality*

$$2^{n-1} \Delta_{n-1} \sum_{k=1}^{\lfloor 2\Delta_n V_{n-1} / (\Delta_{n-1} V_n) \rfloor} \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{1 - \left( \frac{k \Delta_{n-1} V_n}{2 \Delta_n V_{n-1}} \right)^{2^{n-1}}} \geq 1$$

where the sum  $\sum_{l|k}$  is over all positive integral divisors  $l \in \mathbf{N}$  of the natural integer  $k$ .

(ii) *The maximal center densities  $\delta_{n-1}$  and  $\delta_n$  of lattice-packings in dimensions  $n-1$  and  $n \geq 2$  satisfy the inequality*

$$2^{n-1} \delta_{n-1} \frac{\pi^{(n-1)/2}}{((n-1)/2)!} \sum_{k=1}^{\lfloor 2\delta_n / \delta_{n-1} \rfloor} \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{1 - \left( \frac{k \delta_{n-1}}{2 \delta_n} \right)^{2^{n-1}}} \geq 1.$$

(iii) *The Hermite constants  $\gamma_{n-1}$  and  $\gamma_n$  in dimensions  $n-1$  and  $n \geq 2$  satisfy the inequality*

$$\frac{\pi^{(n-1)/2}}{((n-1)/2)!} \sum_{k=1}^{\lfloor \gamma_n^{n/2} / \gamma_{n-1}^{(n-1)/2} \rfloor} \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{\gamma_{n-1} - k^2 \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{n^{n-1}}} \geq 1.$$

**Remark 1.2** (i) *The function*

$$(x, y) \mapsto F_n(x, y) = \sum_{k=1}^{\lfloor xy \rfloor} \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{x - \left( \frac{k}{y} \right)^{2^{n-1}}}$$

is  $\geq 0$ , continuous and strictly increasing in  $x > 0$  and  $y > 0$  if  $xy \geq 1$ . (It is moreover differentiable, except on the hyperbolas  $xy = n$  for  $n \in \mathbf{N}$ .) Knowledge of (a lower bound for)  $\gamma_{n-1}$  implies a lower bound  $\underline{\gamma}_n \leq \gamma_n$  defined by

$$\pi^{(n-1)/2} F_n(\gamma_{n-1}, \sqrt{\underline{\gamma}_n / \gamma_{n-1}}) = ((n-1)/2)! .$$

A completely analogous observation holds of course also for  $\Delta_m$  and  $\delta_m$ .

(ii) *The inequality of Theorem 1.1 is tight for  $n = 2$ . For  $n = 3$ , we get from  $\delta_2 = 1/2\sqrt{3}$  the lower bound  $0.1695 \leq \delta_3 = 1/4\sqrt{2} \sim 0.1768$ . For  $n = 9$ , the known value  $\delta_8 = 1/16$  gives the lower bound  $\delta_9 \geq 0.0388$  (a lattice with center-density 0.0442 is known), for  $n = 25$  the known value  $\delta_{24} = 1$  coming from the Leech lattice (see Cohn-Kumar, [3] and [4]) yields  $\delta_{25} \geq 0.657$  (a lattice with center-density 0.707 is known).*

(iii) *The above examples show that our inequality is better than the trivial inequality  $\delta_n \geq \frac{\delta_{n-1}}{2}$  obtained by considering the orthogonal sum  $\Lambda_{n-1} \oplus 4\mu_{n-1}\mathbf{Z}$  where  $\mu_{n-1}$  is the minimal length of a densest  $(n-1)$ -dimensional lattice  $\Lambda_{n-1}$ .*

(iv) The factor

$$\sum_{l|k} \frac{\mu(l)}{l^{n-1}} = \prod_{p \text{ prime}, p|k} \left(1 - \frac{1}{p^{n-1}}\right)$$

yields only a minor improvement for huge  $n$  and is the analogue of a standard trick leading to the factor  $\zeta(n)$  in the Minkowski-Hlawka bound  $\Delta_n \geq \frac{\zeta(n)}{2^{n-1}}$ .

**Theorem 1.3** For all  $\epsilon > 0$ , there exists  $N$  such that

$$\Delta_n \geq \frac{1 - \epsilon}{2^n \sum_{k=1}^{\infty} e^{-k^2 \pi}} \sim (1 - \epsilon) 23.1388 2^{-n}$$

for all  $n \geq N$ .

**Remark 1.4** Theorem 1.3 is slightly better than the Minkowski-Hlawka bound which shows the existence of lattices with density at least  $\zeta(n) 2^{1-n}$ , cf. formula (14) in [5], Chapter 1. The best known bound for densities achieved by lattice packings (together with a very nice proof) seems to be due to Keith Ball and asserts the existence of  $n$ -dimensional lattices with density at least  $2(n-1)2^{-n}\zeta(n)$ , see [2].

The paper is organized as follows:

Section 2 introduces notations and summarizes for the convenience of the reader a few well-known facts on lattices. It contains also an easy (and seemingly not very well-known) result on integral sublattices which are orthogonal to a non-zero integral vector in  $\mathbf{Z}^{n-1}$ .

In Section 3 we define of  $\mu$ -sequences which are the main tool of this paper. Theorem 3.3 of this Section gives a quantitative (and somewhat technical) statement for extending a suitable finite  $\mu$ -sequence  $(s_0, \dots, s_{n-1})$  to a  $\mu$ -sequence  $(s_0, \dots, s_{n-1}, s_n)$ . The lattice  $(s_0, \dots, s_n)^\perp \cap \mathbf{Z}^{n+1}$  associated to such an extension is obtained by an ‘‘approximate lamination’’ of its sublattice  $(s_0, \dots, s_{n-1})^\perp \cap \mathbf{Z}^n$ . Theorem 3.3 is the central result of this paper since it implies easily Theorem 1.1 as shown at the end of Section 3. The proof of Theorem 1.3 is more technical and given in Section 6.

Section 4 states and proves a weaker and easier statement than Theorem 3.3. Although not necessary for the other parts of the paper, this section describes a fairly elementary and almost effective method for constructing dense lattices. It contains moreover the main idea for proving Theorem 3.3 in a simplified form.

Section 5 describes the proof of Theorem 3.3.

Section 6 is devoted to the proof of Theorem 1.3.

Section 7 contains a few final comments and remarks.

## 2 Definitions

All facts concerning lattices needed in the sequel are collected in this Section for the convenience of the reader, see [5] and [7] for more on lattices and lattice-packings.

An  $n$ -dimensional lattice is a discret-cocompact subgroup  $\Lambda$  of the  $n$ -dimensional Euclidean vector space  $\mathbf{E}^n$ . Denoting by  $\langle \cdot, \cdot \rangle$  the scalar product and choosing a  $\mathbf{Z}$ -basis  $b_1, \dots, b_n$  of a lattice  $\Lambda = \bigoplus_{j=1}^n \mathbf{Z}b_j$ , the positive definite symmetric matrix  $G \in \mathbf{R}^{n \times n}$  with coefficients

$$G_{i,j} = \langle b_i, b_j \rangle$$

is a *Gram matrix* of  $\Lambda$ . Its determinant  $\det(G)$ , called the *determinant* of  $\Lambda$ , is independent of the chosen basis  $b_1, \dots, b_n$  and equals the squared volume of the flat torus  $\mathbf{E}^n/\Lambda$ . The *norm* of a lattice vector  $\lambda \in \Lambda$  is defined as  $\langle \lambda, \lambda \rangle$  and equals thus the square of the Euclidean norm  $\sqrt{\langle \lambda, \lambda \rangle}$ . A lattice  $\Lambda$  is *integral* if all scalar products  $\{\langle \lambda, \mu \rangle \mid \lambda, \mu \in \Lambda\}$  are integral. An integral lattice of determinant 1 is *unimodular*. An Euclidean lattice  $\Lambda$  is unimodular if and only if every group homomorphism  $\varphi : \Lambda \rightarrow \mathbf{Z}$  is of the form  $\varphi(v) = \langle v, w_\varphi \rangle$  for a suitable fixed element  $w_\varphi \in \Lambda$ . The *minimum*

$$\min \Lambda = \min_{\lambda \in \Lambda \setminus \{0\}} \langle \lambda, \lambda \rangle$$

of a lattice  $\Lambda$  is the norm of a shortest non-zero vector in  $\Lambda$ . The *density*  $\Delta(\Lambda)$  and the *center-density*  $\delta(\Lambda)$  of an  $n$ -dimensional lattice  $\Lambda$  are defined as

$$\Delta(\Lambda) = \sqrt{\frac{(\min \Lambda)^n}{4^n \det \Lambda}} V_n \quad \text{and} \quad \delta(\Lambda) = \sqrt{\frac{(\min \Lambda)^n}{4^n \det \Lambda}}$$

where  $V_n = \pi^{n/2}/(n/2)!$  denotes the volume of the  $n$ -dimensional unit-ball in  $\mathbf{E}^n$ . These two densities are proportional for a given fixed dimension  $n$  and  $\Delta(\Lambda)$  measures the (asymptotic) proportion of space occupied by the *sphere packing* of  $\Lambda$  obtained by centering  $n$ -dimensional Euclidean balls of radius  $\sqrt{\min \Lambda}/4$  at all points of  $\Lambda$ .

Given an  $n$ -dimensional lattice  $\Lambda \subset \mathbf{E}^n$  the subset

$$\Lambda^\sharp = \{x \in \mathbf{E}^n \mid \langle x, \lambda \rangle \in \mathbf{Z} \quad \forall \lambda \in \Lambda\}$$

is also a lattice called the *dual lattice* of  $\Lambda$ . The scalar product induces a natural bijection between  $\Lambda^\sharp$  and the set of homomorphisms  $\Lambda \rightarrow \mathbf{Z}$ . A lattice  $\Lambda$  is integral if and only if  $\Lambda \subset \Lambda^\sharp$ . For an integral lattice, the *determinant group*  $\Lambda^\sharp/\Lambda$  is a finite abelian group consisting of  $(\det \Lambda)$  elements.

A sublattice  $M \subset \Lambda$  is *saturated* if  $\Lambda/M$  has no torsion (or equivalently if  $M = (M \otimes_{\mathbf{Z}} \mathbf{R}) \cap \Lambda$ ).

We leave the proof of the following well-known result to the reader.

**Proposition 2.1** (cf. Chapter I, Proposition 9.8 in [7]) *Let  $M$  and  $N$  be two saturated sublattices of dimension  $m$  and  $n$  in a common unimodular lattice  $\Lambda$  of dimension  $m+n$  such that  $M$  and  $N$  are contained in orthogonal subspaces.*

*Then the two determinant groups  $M^\sharp/M$  and  $N^\sharp/N$  are isomorphic. In particular, the lattices  $M$  and  $N$  have equal determinants.*

Two lattices  $\Lambda$  and  $M$  are *similar*, if there exists a bijection  $\Lambda \rightarrow M$  which extends to an Euclidean similarity from  $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$  to  $M \otimes_{\mathbf{Z}} \mathbf{R}$ . The set of similarity classes of lattices is endowed with a natural topology: a neighbourhood of an  $n$ -dimensional lattice  $\Lambda$  is given by all lattices having a Gram matrix in  $\mathbf{R}_{>0} V(G)$  where  $V(G) \subset \mathbf{R}^{n \times n}$  is a neighbourhood of a fixed Gram matrix  $G$  of  $\Lambda$ .

Similar lattices have identical densities and the density function  $\Lambda \mapsto \Delta(\Lambda)$  is continuous with respect to the natural topology on similarity classes.

Consider the set  $\mathcal{L}_n$  of all  $n$ -dimensional integral lattices of the form

$$\Lambda = \{z \in \mathbf{Z}^{n+1} \mid \langle z, s \rangle = 0\}$$

for  $s \in \mathbf{N}^{n+1} \setminus \{0\}$ .

**Proposition 2.2** *The set  $\mathcal{L}_n$  is dense in the set of similarity classes of  $n$ -dimensional Euclidean lattices.*

There are thus lattices in  $\mathcal{L}_n$  with densities arbitrarily close to the maximal density  $\Delta_n$  of  $n$ -dimensional lattices.

**Proof of Proposition 2.2** Given a Gram matrix  $G = \langle b_i, b_j \rangle$  of an  $n$ -dimensional lattice  $\Lambda = \oplus_{j=1}^n \mathbf{Z}b_j$ , Gram-Schmidt orthogonalization of the  $\mathbf{Z}$ -basis  $b_1, \dots, b_n$  (with respect to the Euclidean scalar product) yields a matrix factorization

$$G = L L^t$$

where  $L = (l_{i,j})_{1 \leq i,j \leq n}$  is an invertible lower triangular matrix.

Choose a large real number  $\kappa > 0$  and consider the integral lower triangular matrix  $\tilde{L}(\kappa)$  whose coefficients  $\tilde{l}_{i,j} \in \mathbf{Z}$  satisfy

$$|\tilde{l}_{i,j} - \kappa l_{i,j}| \leq 1/2$$

and are obtained by rounding off each coefficient of  $\kappa L$  to a nearest integer.

Define the integral matrix

$$B(\kappa) = \begin{pmatrix} \tilde{l}_{1,1} & 1 & 0 & 0 & \dots \\ \tilde{l}_{2,1} & \tilde{l}_{2,2} & 1 & 0 & \\ \vdots & & \ddots & \ddots & \\ \tilde{l}_{n,1} & \tilde{l}_{n,2} & \dots & \tilde{l}_{n,n} & 1 \end{pmatrix}$$

of size  $n \times (n + 1)$  with coefficients

$$b_{i,j} = \begin{cases} \tilde{l}_{i,j} & \text{if } j \leq i \\ 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise .} \end{cases}$$

The rows of  $B(\kappa)$  span an integral sublattice  $\tilde{\Lambda}(\kappa)$  of dimension  $n$  in  $\mathbf{Z}^{n+1}$ . Moreover, the lattice  $\tilde{\Lambda}(\kappa)$  is saturated since deleting the first column of  $B(\kappa)$  yields an integral unimodular square matrix of size  $n \times n$ . The special form of  $B(\kappa)$  shows that there exists an integral row-vector

$$v(\kappa) = \begin{pmatrix} 1 \\ -\tilde{l}_{1,1} \\ \tilde{l}_{1,1}\tilde{l}_{2,2} - \tilde{l}_{2,1} \\ \vdots \end{pmatrix} \in \mathbf{Z}^{n+1}$$

such that  $B(\kappa)v(\kappa) = 0$ . We have thus

$$\tilde{\Lambda}(\kappa) = v(\kappa)^\perp \cap \mathbf{Z}^{n+1} \subset \mathbf{E}^{n+1} .$$

Since  $\lim_{\kappa \rightarrow \infty} \frac{1}{\kappa} B(\kappa)$  is given by the matrix  $L$  with an extra row of zeros appended, we have

$$\lim_{\kappa \rightarrow \infty} \frac{1}{\kappa^2} B(\kappa)(B(\kappa))^t = G$$

and the lattice  $\frac{1}{\kappa} \tilde{\Lambda}(\kappa)$  converges thus to the lattice  $\Lambda$  for  $\kappa \rightarrow \infty$ . Considering the integral vector  $s = (s_0, s_1, \dots) \in \mathbf{N}^{n+1}$  defined by  $s_i = |v(\kappa)_{i+1}|$  for  $i = 0, \dots, n$ , we get an integral lattice

$$\{z = (z_0, \dots, z_n) \in \mathbf{Z}^{n+1} \mid \langle z, s \rangle = 0\}$$

of  $\mathcal{L}_n$  which is isometric to  $\tilde{\Lambda}(\kappa)$ . □

### 3 $\mu$ -sequences

Let  $\mu \geq 2$  be a strictly positive integer. A  $\mu$ -sequence is a finite or infinite sequence  $s_0 = 1, s_1, s_2, \dots$  of  $(l + 1)$  strictly positive integers such that the  $n$ -dimensional lattice

$$\Lambda_n = \{(z_0, z_1, \dots, z_n) \in \mathbf{Z}^{n+1} \mid \sum_{k=0}^n s_k z_k = 0\} = (s_0, \dots, s_n)^\perp \cap \mathbf{Z}^{n+1}$$

has minimum  $\geq \mu$  for all  $n \geq 1$  which make sense (ie. for  $n \leq l$  if the sequence  $(s_0, s_1, \dots, s_l)$  has finite length  $l$ ). Since  $\Lambda_n$  is saturated in

$\mathbf{Z}^{n+1}$  by construction and orthogonal to the 1–dimensional saturated lattice  $\mathbf{Z}(s_0, \dots, s_n) \subset \mathbf{Z}^{n+1}$ , Proposition 2.1 shows that we have  $\det(\Lambda_n) = \sum_{k=0}^n s_k^2$ . We get thus a lower bound for the density

$$\Delta(\Lambda_n) = \sqrt{\frac{(\min \Lambda_n)^n}{4^n \det \Lambda_n}} V_n \geq \sqrt{\frac{\mu^n}{4^n \sum_{k=0}^n s_k^2}} V_n$$

of the  $n$ –dimensional lattice  $\Lambda_n$  associated to a  $\mu$ –sequence  $(s_0, \dots, s_n, \dots)$ . This lower bound is an equality except if the sequence  $(s_0, \dots, s_n)$  is a  $(\mu + 1)$ –sequence.

**Remark 3.1** *We hope that the double meaning of  $\mu$  will not confuse the reader:  $\mu(l) \in \{-1, 0, 1\}$  denotes always the Möbius function of a natural integer  $l$  while  $\mu$  or  $\mu_1, \mu_2, \dots$  stands for natural integers.*

**Remark 3.2** (i) *The condition  $s_0 = 1$  is of no real importance and can be omitted after minor modifications. It is of course also possible (but not very useful) to consider sequences with coefficients in  $\mathbf{Z}$ .*

(ii) *Any subsequence  $s_{i_0} = s_0, s_{i_1}, s_{i_2}, \dots$  of a  $\mu$ –sequence is again a  $\mu$ –sequence and permuting the terms of a  $\mu$ –sequence by a permutation fixing  $s_0$  yields of course again a  $\mu$ –sequence.*

(iii) *Lattices associated to  $\mu$ –sequences are generally neither perfect nor eutactic (cf. [7] for definitions). Their densities can thus generally be improved by suitable deformations.*

**Theorem 3.3** *Let  $\mu_1, \mu_2, \dots$  be a strictly increasing sequence of natural integers  $2 \leq \mu_1 < \mu_2 < \dots$ . Suppose that we have finite  $\mu_k$ –sequences  $(s(\mu_k)_0, \dots, s(\mu_k)_{n-1})$  with existing limit-density*

$$\tilde{\Delta}_{n-1} = \lim_{k \rightarrow \infty} \frac{\mu_k^{(n-1)/2}}{\sqrt{4^{n-1} \sum_{i=0}^{n-1} s(\mu_k)_i^2}} V_{n-1} > 0$$

for the sequence of orthogonal lattices  $(s(\mu_k)_0, \dots, s(\mu_k)_{n-1})^\perp \subset \mathbf{Z}^n$ .

Let  $\sigma_n$  be a positive real number such that

$$2^{n-1} \tilde{\Delta}_{n-1} \sum_{k=1}^A \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{1 - k^2 \left( 2^{n-1} \tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \sigma_n \right)^{2^{n-1}}} < 1$$

where

$$A = \left\lfloor \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \sigma_n} \right\rfloor.$$

Then there exists a natural integer  $K$  such that for all  $k \geq K$ , the  $\mu_k$ –sequence  $(s(\mu_k)_0, \dots, s(\mu_k)_{n-1})$  can be extended to a  $\mu_k$ –sequence  $(s(\mu_k)_0, \dots, s(\mu_k)_n)$  satisfying  $0 < s(\mu_k)_n < \sigma_n \mu_k^{n/2} V_n$ .

The proof of Theorem 3.3 will be given in section 5. We proceed now to prove Theorem 1.1 under the assumption that Theorem 3.3 holds.



### 3.1 Proof of Theorem 1.1

Suppose that the inequality of assertion (i) does not hold for some natural integer  $n$ . By Proposition 2.2 we can find a sequence of finite  $\mu_k$ -sequences  $(s(\mu_k)_0, \dots, s(\mu_k)_{n-1})$  (with  $\mu_k \rightarrow \infty$ ) such that

$$\lim_{k \rightarrow \infty} \frac{\mu_k^{(n-1)/2}}{\sqrt{4^{n-1} \sum_{i=0}^{n-1} s(\mu_k)_i^2}} V_{n-1} = \Delta_{n-1} .$$

Choose a positive real number  $\sigma_n$  such that

$$\begin{aligned} 2^{n-1} \Delta_{n-1} & \sum_{k=1}^{\lfloor 2\Delta_n V_{n-1}/(\Delta_{n-1} V_n) \rfloor} \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{1 - \left( \frac{k\Delta_{n-1} V_n}{2\Delta_n V_{n-1}} \right)^{2^{n-1}}} < \\ & < 2^{n-1} \Delta_{n-1} \sum_{k=1}^A \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \sqrt{1 - k^2 \left( 2^{n-1} \Delta_{n-1} \frac{V_n}{V_{n-1}} \sigma_n \right)^{2^{n-1}}} < 1 \end{aligned}$$

where  $A = \lfloor \frac{2^{1-n} V_{n-1}}{V_n \Delta_{n-1} \sigma_n} \rfloor$ .

We have thus  $2^{n-1} \sigma_n < \frac{1}{2\Delta_n}$  or, equivalently,

$$\Delta_n < \frac{1}{2^n \sigma_n} .$$

Applying Theorem 3.3 and extracting a suitable subsequence from  $\mu_1, \mu_2, \dots$ , we can suppose that all sequences  $(s(\mu_k)_0, \dots, s(\mu_k)_{n-1})$  can be extended to  $\mu_k$ -sequences  $(s(\mu_k)_0, \dots, s(\mu_k)_n)$  with

$$\lim_{k \rightarrow \infty} \frac{s(\mu_k)_n}{\mu_k^{n/2}} = \alpha \leq \sigma_n V_n .$$

Using

$$\lim_{k \rightarrow \infty} \frac{1}{\mu_k^{n-1}} \sum_{i=0}^{n-1} s(\mu_k)_i^2 = \frac{V_{n-1}^2}{4^{n-1} \Delta_{n-1}^2}$$

we have

$$\lim_{k \rightarrow \infty} \frac{1}{\mu_k^n} \sum_{i=0}^n s(\mu_k)_i^2 = \lim_{k \rightarrow \infty} \frac{1}{\sqrt{\mu_k}} \frac{V_{n-1}^2}{4^{n-1} \Delta_{n-1}^2} + \alpha^2 = \alpha^2$$

and get the existence of a sequence of  $n$ -dimensional lattices

$$(s(\mu_k)_0, \dots, s(\mu_k)_n)^\perp \subset \mathbf{Z}^{n+1}$$

with limit-density

$$\lim_{k \rightarrow \infty} \sqrt{\frac{\mu_k^n}{4^n \sum_{i=0}^n s(\mu_k)_i^2}} V_n = \frac{1}{2^n \alpha} V_n \geq \frac{1}{2^n \sigma_n V_n} V_n > \Delta_n$$

in contradiction with maximality of  $\Delta_n$ .  $\square$

## 4 A easy crude bound for the lexicographically first $\mu$ -sequence

**Theorem 4.1** *Given an integer  $\mu \geq 2$  there exists an increasing  $\mu$ -sequence  $s_0 = 1 \leq s_1 \leq \dots$  such that*

$$s_n \leq 1 + \sqrt{\mu - 2} \sqrt{\mu - 1 + n/4} \frac{\sqrt{\pi^n}}{(n/2)!} \leq \sqrt{\mu} \sqrt{\mu + n/4} \frac{\sqrt{\pi^n}}{(n/2)!}$$

for all  $n \geq 1$ .

The proof of Theorem 4.1 is elementary and consists of an analysis of the “greedy algorithm” which constructs the first  $\mu$ -sequence with respect to the lexicographic order on sequences. An easy analysis shows that the lexicographically first  $\mu$ -sequence satisfies the first inequalities of Theorem 4.1. The greedy algorithm, although very simple, is unfortunately useless for practical purposes.

A  $\mu$ -sequence satisfying the inequalities of Theorem 4.1 yields already rather dense lattices as shown by the next result.

**Corollary 4.2** *For any  $\mu \geq 2$ , there exists a  $\mu$ -sequence  $(s_0, s_1, \dots, s_n) \in \mathbf{Z}^{n+1}$  such that the density of the associated lattice  $\Lambda_n = (s_0, \dots, s_n)^\perp \cap \mathbf{Z}^{n+1}$  satisfies*

$$\Delta(\Lambda_n) \geq \frac{(1 + n/(4\mu))^{-n/2}}{2^n \sqrt{(n+1)\mu}}.$$

**Remark 4.3** *Taking  $\mu \sim n^2/4$  we get the existence of lattices in dimension  $n$  (for large  $n$ ) with density  $\Delta$  roughly at least equal to*

$$\frac{1}{2^{n-1} n \sqrt{(n+1)e}}$$

which is reasonably close to the Minkowski-Hlawka bound ensuring the existence of lattices with density at least  $\zeta(n) 2^{1-n}$ , cf. formula (14) in [5], Chapter 1.

**Lemma 4.4** *The standard Euclidean lattice  $\mathbf{Z}^n$  contains at most*

$$2 \sqrt{\mu + n/4} \frac{\pi^{n/2}}{(n/2)!}$$

vectors of (squared Euclidean) norm  $\leq \mu$ .

**Proof** We denote by

$$B_{\leq \sqrt{\rho}}(x) = \{z \in \mathbf{E}^n \mid \langle z - x, z - x \rangle \leq \rho\}$$

the closed Euclidean ball with radius  $\sqrt{\rho} \geq 0$  and center  $x \in \mathbf{E}^n$ . Given  $\sqrt{\mu}, \sqrt{\rho} \geq 0$  and  $x \in B_{\leq \sqrt{\mu}}(0)$ , the closed half-ball

$$\{z \in \mathbf{E}^n \mid \langle z, x \rangle \leq \langle x, x \rangle\} \cap B_{\leq \sqrt{\rho}}(x)$$

(obtained by intersecting the closed affine halfspace  $H_x = \{z \in \mathbf{E}^n \mid \langle z, x \rangle \leq \langle x, x \rangle\}$  with the Euclidean ball  $B_{\leq \sqrt{\rho}}(x)$  centered at  $x \in \partial H_x$ ) is contained in  $B_{\leq \sqrt{\mu+\rho}}(0)$ .

Since the regular standard cube

$$C = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbf{E}^n$$

of volume 1 is contained in a ball of radius  $\sqrt{n/4}$  centered at the origin, the intersection

$$(z + C) \cap \{x \in \mathbf{E}^n \mid \langle x, x \rangle \leq \mu + n/4\} = (z + C) \cap B_{\leq \sqrt{\mu+n/4}}(0)$$

is of volume at least  $1/2$  for any element  $z \in \mathbf{E}^n$  of norm  $\langle z, z \rangle \leq \mu$ .

Since integral translates of  $C$  tile  $\mathbf{E}^n$ , we have

$$\frac{1}{2} \#\{z \in \mathbf{Z}^n \mid \langle z, z \rangle \leq \mu\} \leq \text{Vol} \{x \in \mathbf{E}^n \mid \langle x, x \rangle \leq \mu + n/4\}.$$

Using the fact that the unit ball in Euclidean  $n$ -space has volume  $\pi^{n/2}/(n/2)!$  (cf. Chapter 1, formula 17 in [5]) we get the result.  $\square$

**Proof of Theorem 4.1** For  $n = 0$ , the first inequality boils down to  $s_0 = 1 \leq 1 + \sqrt{\mu - 2}$  and holds for  $\mu \geq 2$ . Consider now for  $n \geq 1$  a  $\mu$ -sequence  $(s_0, \dots, s_{n-1}) \in \mathbf{N}^n$ .

Introduce the set

$$\mathcal{F} = \{(a, k) \in \mathbf{N}^2 \mid \exists z = (z_0, \dots, z_{n-1}) \in \mathbf{Z}^n \setminus \{0\} \text{ such that} \\ ak = |\langle z, (s_0, \dots, s_{n-1}) \rangle| \text{ and } \langle z, z \rangle + k^2 < \mu\}.$$

Since  $\Lambda_{n-1}$  has minimum  $\geq \mu$ , the equality  $\langle z, (s_0, \dots, s_{n-1}) \rangle = 0$  implies  $\langle z, z \rangle \geq \mu$  for  $z \in \mathbf{Z}^n \setminus \{0\}$ . This shows that we have  $a, k > 0$  for  $(a, k) \in \mathcal{F}$ .

Since for a given pair of opposite non-zero vectors  $\pm z \in \mathbf{Z}^n$  with norm  $0 < \langle z, z \rangle < \mu$  there are at most  $\sqrt{\mu - 1 - \langle z, z \rangle} \leq \sqrt{\mu - 2}$  strictly positive integers  $k$  such that  $\langle z, z \rangle + k^2 < \mu$ , such a pair  $\pm z$  of vectors contributes at most  $\sqrt{\mu - 2}$  distinct elements to  $\mathcal{F}$ . The cardinality  $f = \#\mathcal{F}$  of  $\mathcal{F}$  is thus bounded by

$$f \leq \sqrt{\mu - 2} \frac{\#\{z \in \mathbf{Z}^n \mid 0 < \langle z, z \rangle \leq \mu - 1\}}{2} \leq \sqrt{\mu - 2} \sqrt{\mu - 1 + n/4} \frac{\pi^{n/2}}{(n/2)!}$$

where the last inequality follows from Lemma 3.1. There exists thus a smallest strictly positive integer

$$s_n \leq f + 1 \leq 1 + \sqrt{\mu - 2} \sqrt{\mu - 1 + n/4} \frac{\pi^{n/2}}{(n/2)!}$$

such that  $(s_n, k) \notin \mathcal{F}$  for all  $k \in \mathbf{N}$ . The strictly positive integer  $s_n$  satisfies the first inequality of Theorem 4.1 and it is straightforward to check that the  $n$ -dimensional lattice

$$\Lambda_n = \{z \in \mathbf{Z}^{n+1} \mid \sum_{i=0}^n s_i z_i = 0\}$$

has minimum  $\geq \mu$ . This shows the first inequality. Iteration of this construction yields clearly an increasing  $\mu$ -sequence.

The second inequality

$$1 + \sqrt{\mu - 2} \sqrt{\mu - 1 + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} \leq \sqrt{\mu} \sqrt{\mu + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!}$$

of Theorem 4.1 boils down to

$$1 \leq \sqrt{2} \sqrt{2 + n/4} \frac{\sqrt{\pi}^n}{(n/2)!}$$

for  $\mu = 2$ . This inequality is clearly true since the  $n$ -dimensional Euclidean ball of radius  $\sqrt{2 + n/4}$  has volume  $\sqrt{2 + n/4} \frac{\sqrt{\pi}^n}{(n/2)!}$  and contains the regular cube  $[-\frac{1}{2}, \frac{1}{2}]^n$  of volume 1.

For  $\mu \geq 3$  we have to establish the inequality  $\Phi(1) - \Phi(0) \geq 1$  where

$$\Phi(t) = \sqrt{\mu - 2 + 2t} \sqrt{\mu - 1 + t + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!}.$$

We get thus

$$\begin{aligned} \Phi(1) - \Phi(0) &\geq \inf_{\xi \in (0,1)} \Phi'(\xi) \\ &\geq \frac{1}{\sqrt{\mu}} \sqrt{\mu - 1 + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} + \frac{n}{2} \sqrt{\mu - 2} \sqrt{\mu - 1 + n/4}^{n-2} \frac{\sqrt{\pi}^n}{(n/2)!}. \end{aligned}$$

For  $n = 1$  and  $\mu \geq 2$  we have

$$\Phi(1) - \Phi(0) \geq \sqrt{1 - \frac{1}{\mu}} \frac{\sqrt{\pi}}{\sqrt{\pi}/2} \geq \frac{2}{\sqrt{2}} > 1.$$

For  $n \geq 2$  and  $\mu \geq 3$  we get

$$\Phi(1) - \Phi(0) \geq \sqrt{2 + n/4}^{n-2} \frac{\sqrt{\pi}^{n-2}}{((n-2)/2)!} \pi$$

and the right-hand side equals  $\pi > 1$  for  $n = 2$ . For  $n > 2$ , the right hand side equals  $\pi$  times the volume of the  $(n-2)$ -dimensional ball of radius  $\sqrt{2 + n/4}$  containing the regular cube  $[-\frac{1}{2}, \frac{1}{2}]^{n-2}$  of volume 1. The second inequality follows.  $\square$

**Proof of Corollary 4.2** Theorem 4.1 shows the existence of a  $\mu$ -sequence  $(s_0 = 1, \dots, s_n)$  satisfying

$$s_0, \dots, s_n \leq \sqrt{\mu} \sqrt{\mu + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!}.$$

This shows for the lattice  $\Lambda_n = (s_0, \dots, s_n)^\perp \cap \mathbf{Z}^{n+1}$  the inequality

$$\det \Lambda_n = \sum_{i=0}^n s_i^2 \leq (n+1) \mu (\mu + n/4)^n \frac{\pi^n}{((n/2)!)^2} = (n+1) \mu (\mu + n/4)^n V_n^2$$

and implies

$$\Delta(\Lambda_n) \geq \sqrt{\frac{\mu^n}{4^n (n+1) \mu (\mu + n/4)^n V_n^2}} V_n$$

which proves Corollary 4.2.  $\square$

## 5 Proof of Theorem 3.3

The main idea for proving Theorem 3.3 is to get rid of a factor  $\sqrt{\mu}$  when computing an upper bound  $f$  for the size of the finite set  $\mathcal{F}$  considered in the proof of Theorem 4.1. This is possible since the volume of the standard unit-ball of large dimension concentrates along linear hyperplanes. During the proof, we use for simplicity the slightly abusive notation  $\mu = \mu_k$  and  $(s_0, \dots, s_n) = (s(\mu_k)_0, \dots, s(\mu_k)_n)$ . Since  $\mu$  belongs to the strictly increasing integral sequence  $\mu_1 < \mu_2 < \dots$  tending to infinity, we consider sequences in the  $\mu \rightarrow \infty$  limit. This allows us to neglect boundary effects when replacing counting arguments by volume-computations.

In the sequel we write

$$g(x) \sim_{x \rightarrow \alpha} h(x), \text{ respectively } g(x) \leq_{x \rightarrow \alpha} h(x),$$

for

$$\lim_{x \rightarrow \alpha} \frac{g(x)}{h(x)} = 1, \text{ respectively } \limsup_{x \rightarrow \alpha} \frac{g(x)}{h(x)} \leq 1,$$

where  $g(x), h(x) > 0$ .

**Proof of Theorem 3.3** We prove first a weaker statement assuming the stronger inequality

$$2^{n-1} \tilde{\Delta}_{n-1} \sum_{k=1}^A \sqrt{1 - k^2 \left( 2^{n-1} \tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \sigma_n \right)^{2^{n-1}}} < 1$$

where

$$A = \lfloor \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \sigma_n} \rfloor.$$

Details for dealing with the extra factor  $\left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right)$  will be given later.

Up to replacing  $\sigma_n$  by a slightly smaller real number, we can suppose that  $\frac{2^{1-n}V_{n-1}}{V_n\tilde{\Delta}_{n-1}\tilde{\sigma}_n} \notin \mathbf{N}$ . Choose a positive real number  $\tilde{\sigma}_n < \sigma_n$  such that we have the equalities

$$A = \lfloor \frac{2^{1-n}V_{n-1}}{V_n\tilde{\Delta}_{n-1}\tilde{\sigma}_n} \rfloor = \lfloor \frac{2^{1-n}V_{n-1}}{V_n\tilde{\Delta}_{n-1}\tilde{\sigma}_n} \rfloor$$

(where  $\lfloor x \rfloor \in \mathbf{Z}$  denotes the integer part of  $x \in \mathbf{R}$ ) and the inequality

$$2^{n-1}\tilde{\Delta}_{n-1} \sum_{k=1}^A \sqrt{1 - k^2 \left(2^{n-1}\tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \tilde{\sigma}_n\right)^{2^{n-1}}} < 1.$$

We fix  $\tilde{\sigma}_n$  in the sequel and introduce  $\epsilon = \frac{\sigma_n}{\tilde{\sigma}_n} - 1 > 0$ . We prove Proposition 3.3 for all  $\mu$  huge enough by showing the existence of a  $\mu$ -sequence  $(s_0, \dots, s_{n-1}, s_n)$  with  $s_n \in I \cap \mathbf{N}$  where

$$I = [\tilde{\sigma}_n \mu^{n/2} V_n, (1 + \epsilon) \tilde{\sigma}_n \mu^{n/2} V_n] = [\tilde{\sigma}_n \mu^{n/2} V_n, \sigma_n \mu^{n/2} V_n].$$

Since our computations rely on strict inequalities involving volume-computations which are continuous in  $\tilde{\Delta}_{n-1}$ , this implies the weakened form (without the factor  $\left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right)$ ) of Theorem 3.3.

For  $k = 1, 2, \dots \in \mathbf{N}$  we define finite subsets

$$I_k = \{s \in I \cap \mathbf{N} \mid \sum_{i=0}^{n-1} s_i x_i = ks \text{ for some } (x_0, \dots, x_{n-1}) \in B_{<\sqrt{\mu-k^2}} \cap \mathbf{Z}^n\}$$

of natural integers in  $I \cap \mathbf{N}$  where  $B_{<\sqrt{\mu-k^2}} \cap \mathbf{Z}^n$  denotes the set of all integral vectors  $(x_0, \dots, x_{n-1}) \in \mathbf{Z}^n$  having (squared Euclidean) norm strictly smaller than  $\mu - k^2$ .

We have

$$\begin{aligned} \left| \sum_{i=0}^{n-1} s_i x_i \right| &\leq \sqrt{\sum_{i=0}^{n-1} s_i^2} \sqrt{\sum_{i=0}^{n-1} x_i^2} \leq_{\mu \rightarrow \infty} \sqrt{\frac{\mu^{n-1} V_{n-1}^2}{4^{n-1} \tilde{\Delta}_{n-1}^2}} \sqrt{\mu - k^2} \\ &\sim_{\mu \rightarrow \infty} \frac{2^{1-n} \mu^{n/2} V_{n-1}}{\tilde{\Delta}_{n-1}} \end{aligned}$$

for  $(x_0, \dots, x_{n-1}) \in B_{<\sqrt{\mu-k^2}}$ . This shows  $I_k = \{\emptyset\}$  if

$$k > \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \tilde{\sigma}_n} \geq A.$$

An extension  $(s_0, \dots, s_{n-1}, s_n)$  with  $s_n \in I$  of a  $\mu$ -sequence  $(s_0, \dots, s_{n-1})$  is a  $\mu$ -sequence if and only if  $s_n \notin \bigcup_{k=1}^A I_k$ .

Introducing the sets

$$X_k(a) = \{(x_0, \dots, x_{n-1}) \in \mathbf{Z}^n \mid \sum_{i=0}^{n-1} s_i x_i \in kI \cap k\mathbf{N} + a, \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\},$$

we have obviously  $\sharp(I_k) \leq \sharp(X_k(0))$ . This ensures the existence of a  $\mu$ -sequence  $(s_0, \dots, s_{n-1}, s_n)$  with  $s_n \in I \cap \mathbf{N}$  if we have

$$\sum_{k=1}^A \sharp(X_k(0)) < \sharp\{I \cap \mathbf{N}\}. \quad (1)$$

Denoting by

$$X_k(*) = \{(x_0, \dots, x_{n-1}) \in \mathbf{Z}^n \mid \frac{1}{k} \sum_{i=0}^{n-1} s_i x_i \in I, \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\}$$

the union of the disjoint sets  $X_k(0), X_k(1), \dots, X_k(k-1)$ , the following asymptotic equalities hold.

**Lemma 5.1** *We have*

$$\sharp(X_k(j)) \sim_{\mu \rightarrow \infty} \frac{1}{k} \sharp(X_k(*))$$

for  $j = 0, \dots, k-1$ .

It is thus enough to compute  $\sharp(X_k(*))$  in order to get an asymptotic estimation of  $X_k(0) \sim_{\mu \rightarrow \infty} \frac{1}{k} \sharp(X_k(*))$ . We have

$$\begin{aligned} \sharp(X_k(*)) &= \sharp\{(x_0, \dots, x_{n-1}) \in \mathbf{Z}^n \mid \frac{1}{k} \sum_{i=0}^{n-1} s_i x_i \in I, \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\} \\ &\sim_{\mu \rightarrow \infty} \text{Vol}\{(t_0, \dots, t_{n-1}) \in \mathbf{E}^n \mid \sum_{i=0}^{n-1} t_i^2 \leq \mu, \frac{1}{k} \sum_{i=0}^{n-1} s_i t_i \in I\} \end{aligned}$$

and the requirement  $\frac{1}{k} \sum_{i=0}^{n-1} s_i t_i \in I$  amounts to the inequalities

$$k\tilde{\sigma}_n \mu^{n/2} V_n \leq \sum s_i t_i \leq k\sigma_n \mu^{n/2} V_n.$$

For huge  $\mu$  (and fixed  $k$ ), the number  $k\sharp(X_k)$  is thus essentially the volume  $W_k$  of a subset of the  $n$ -dimensional ball of radius  $\sqrt{\mu}$ . More precisely, this subset is delimited by the two parallel affine hyperplanes orthogonal to  $(s_0, \dots, s_{n-1})$  which are at distance

$$k\tilde{D} = k \frac{\tilde{\sigma}_n \mu^{n/2} V_n}{\sqrt{\sum_{i=0}^n s_i^2}} \sim_{\mu \rightarrow \infty} k\sqrt{\mu} 2^{n-1} \frac{V_n}{V_{n-1}} \tilde{\Delta}_{n-1} \tilde{\sigma}_n$$

and  $(1 + \epsilon)k\tilde{D}$  of the origin.

We have thus

$$\begin{aligned} W_k &= \int_{k\tilde{D}}^{k(1+\epsilon)\tilde{D}} \sqrt{\mu - t^{2^{n-1}}} dt V_{n-1} \leq \epsilon k \tilde{D} \sqrt{\mu - k^2 \tilde{D}^{2^{n-1}}} V_{n-1} \\ &\leq_{\mu \rightarrow \infty} \epsilon k \tilde{\sigma}_n \mu^{n/2} 2^{n-1} V_n \tilde{\Delta}_{n-1} \sqrt{1 - k^2 \left( 2^{n-1} \tilde{\sigma}_n \frac{V_n}{V_{n-1}} \tilde{\Delta}_{n-1} \right)^{2^{n-1}}}. \end{aligned}$$

Using the asymptotic equalities  $\#(X_k) \sim_{\mu \rightarrow \infty} \frac{W_k}{k}$ , we get

$$\sum_{k=1}^A \#(X_k) \leq_{\mu \rightarrow \infty} \epsilon \tilde{\sigma}_n \mu^{n/2} 2^{n-1} \tilde{\Delta}_{n-1} V_n \sum_{k=1}^A \sqrt{1 - k^2 \left( 2^{n-1} \tilde{\sigma}_n \tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \right)^{2^{n-1}}}.$$

Together with the obvious estimation

$$\#\{I \cap \mathbf{N}\} \sim_{\mu \rightarrow \infty} \epsilon \tilde{\sigma}_n \mu^{n/2} V_n,$$

we have now

$$\frac{\#\{I \cap \mathbf{N}\}}{\sum_{k=1}^A \#(I_k)} \geq_{\mu \rightarrow \infty} \frac{2^{1-n}}{\tilde{\Delta}_{n-1} \sum_{k=1}^A \sqrt{1 - k^2 \left( 2^{n-1} \tilde{\sigma}_n \tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \right)^{2^{n-1}}}} > 1$$

by assumption on the choice of  $\tilde{\sigma}_n$ . This proves the weak version (without the factor  $\sum_{l|k} \frac{\mu(l)}{l^{n-1}}$ ) of Theorem 3.3 by inequation (1) since  $\#\{I \cap \mathbf{N}\} \rightarrow \infty$  if  $\mu \rightarrow \infty$ .

We consider now intersections among the sets  $I_1, I_2, \dots, I_A$  in order to deal with the factor  $\sum_{l|k} \frac{\mu(l)}{l^{n-1}}$ . This leads to a slightly better estimation of  $\#\left(\bigcup_{k=1}^A I_k\right)$  and completes the proof of Theorem 3.3.

Call an element  $x = (x_0, \dots, x_{n-1}) \in X_k(0)$  primitive if it is not of the form  $h\mathbf{Z}^n$  for an integral divisor  $h > 1$  of  $k$ . Call  $x$  imprimitive otherwise. An imprimitive element is of the form  $h\tilde{x}$  with  $\tilde{x} \in X_{k/h}(0)$  and contributes a common integer to the sets  $I_k$  and  $I_{k/h}$ . This implies the inequality

$$\#\left(\bigcup_{k=1}^A I_k\right) \leq \sum_{k=1}^A \#(X_k(0)_p)$$

where  $X_k(0)_p \subset X(k)_0$  denotes the set of all primitive elements in  $X(k)_0$ .

It is thus enough to estimate the number of primitive elements in  $X_k(0)$ . We have

$$\#(X_k(*) \cap h\mathbf{Z}^n) \sim_{\mu \rightarrow \infty} \frac{1}{h^n} \#(X_k(*)).$$

We have obviously  $X_k(a) \cap h\mathbf{Z}^n = \emptyset$  for  $a \notin h\mathbf{Z}$ . Applying Lemma 5.1, obviously modified, to the sublattice  $h\mathbf{Z}^n \subset \mathbf{Z}^n$  of index  $h^n$  shows

$$\#(X_k(\alpha h) \cap h\mathbf{Z}^n) \sim_{\mu \rightarrow \infty} \frac{1}{k/h} \#(X_k(*) \cap h\mathbf{Z}^n).$$



We get thus

$$\#(X_k(0) \cap h\mathbf{Z}^n) \sim_{\mu \rightarrow \infty} \frac{1}{kh^{n-1}} \#(X_k(*)) \sim_{\mu \rightarrow \infty} \frac{1}{h^{n-1}} \#(X_k(0)).$$

Since an element  $x \in X_k(0) \cap h\mathbf{Z}^n$  belongs also to  $X_k(0) \cap l\mathbf{Z}^n$  for any natural divisor  $l$  of  $h$  and since  $\sum_{l|h} \mu(l) = 0$  for  $h \geq 2$ , the number  $\#(X_k(0)_p)$  of primitive elements in  $X_k(0)$  is asymptotically given by

$$\left( \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \#(X_k(0)).$$

This leads to the majoration

$$\# \left( \bigcup_{k=0}^A I_k \right) \leq_{\mu \rightarrow \infty} \sum_{k=1}^A \left( \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \#(X_k(0))$$

and proves Theorem 3.3.  $\square$

**Proof of Lemma 5.1** The statement of Lemma 5.1 is equivalent to the asymptotic equalities

$$\frac{\#(X_k(j)) - \#(X_k(i))}{\#(X_k(*))} \sim_{\mu \rightarrow \infty} 0$$

for  $0 \leq i, j < k$ .

Fix  $0 \leq i < j < k$ . Associate to an element  $(x_0, x_1, \dots, x_{n-1}) \in X_k(j)$  the element  $(x_0 + i - j, x_1, \dots, x_{n-1})$  provided that it belongs to  $X_k(i)$ . This induces a bijection between subsets  $\tilde{X}_k(j)$  and  $\tilde{X}_k(i)$  of  $X_k(j), X_k(i)$ . The set of “bad” points

$$B_k(i, j) = \left( X_k(j) \setminus \tilde{X}_k(j) \right) \cup \left( X_k(i) \setminus \tilde{X}_k(i) \right)$$

consists of some integral points at bounded Euclidean distance  $< k \leq A$  from the boundary  $\partial Z_k$  of the the set

$$Z_k = \left\{ (z_0, \dots, z_{n-1}) \in \mathbf{R}^n \mid \sum_{i=0}^{n-1} s_i z_i \in kI, \sum_{i=0}^{n-1} z_i^2 \leq \mu - k^2 \right\}.$$

This shows that

$$|\#(X_k(j)) - \#(X_k(i))| \leq \#(B_k(i, j)) \leq \text{vol} \left( N_{k+\sqrt{n}/2}(\partial Z_k) \right) \sim O(\mu^{n-1})$$

where  $N_{k+\sqrt{n}/2}(\partial Z_k) \subset \mathbf{R}^n$  denotes the set of all points at distance  $\leq k + \sqrt{n}/2$  from the boundary  $\partial Z_k$  of  $Z_k$ .

Since  $\#(X_k(*)) = O(\mu^n)$  this proves the lemma.  $\square$

## 6 Proof of Theorem 1.3

### 6.1 Two auxiliary results

**Proposition 6.1** *Given a real interval  $A \subset \mathbf{R}$ , let  $f_1, f_2, \dots : A \rightarrow A$  be a sequence of uniformly converging functions with continuous and differentiable limit  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  on  $A$ . Suppose that  $f$  has a fixpoint  $\xi = f(\xi) \in A$  and suppose that we have  $\sup_{x \in A} |f'(x)| = \lambda < 1$ .*

*Then the sequence  $s_n(x)$  of functions defined recursively by  $s_0(x) = x$  and  $s_n(x) = f_n(s_{n-1}(x))$  converges pointwise to the constant function  $\xi$ .*

**Proof** Given  $\delta > 0$  there exists an integer  $N$  such that  $|f_n(x) - f(x)| < \delta(1 - \lambda)$  for all  $x \in A$  and for all  $n > N$ . We have then for  $m > N$

$$\begin{aligned} |s_m(x) - \xi| &= |f_m(s_{m-1}(x)) - \xi| \\ &< \delta(1 - \lambda) + |f(s_{m-1}(x)) - \xi| \\ &< \delta - \lambda\delta + \lambda|s_{m-1}(x) - \xi| \\ &< \delta - \lambda^2\delta + \lambda^2|s_{m-2}(x) - \xi| \\ &\quad \vdots \\ &< \delta - \lambda^{m-N}\delta + \lambda^{m-N}|s_N(x) - \xi|. \end{aligned}$$

This shows  $|s_m(x) - \xi| < 2\delta$  if

$$m > \max(N, N + \log\left(\frac{\delta}{|s_N(x) - \xi|}\right) / \log(\lambda))$$

and implies the result since we can choose  $\delta > 0$  arbitrarily small.  $\square$

**Remark 6.2** (i) *The proof of Proposition 6.1 shows in fact*

$$|s_n(x) - \xi| \leq \lambda^n |x - \xi| + \sum_{k=1}^n \lambda^{n-k} \sup_{x \in A} |f_k(x) - f(x)|.$$

*Asymptotically, we have thus  $|s_n(\xi) - \xi| = O(\sup_{x \in N(\xi)} |f_n(x) - f(x)|)$  (where  $N(\xi) \subset A$  is an arbitrarily small fixed neighbourhood of  $\xi$ ) if  $\sup_{x \in N(\xi)} |f_n(x) - f(x)|$  is decreasing at a slower rate than powers of  $\lambda$ .*

(ii) *If the sequence  $f_n(x) = F(x, 1/n)$  satisfies the hypotheses of Proposition 6.1 with  $F(x, y)$  having continuous partial derivatives of all orders up to  $k+1$  in a neighbourhood of  $(\xi, 0)$ , then there exist constants  $a_1, a_2, \dots, a_k$  such that*

$$s_n(x) = \xi + \frac{1}{1 - \partial/\partial_x F} \left( \sum_{j=1}^k \frac{a_j}{j!} n^{-j} \right) + O(n^{-(k+1)})$$

where  $\frac{\partial^{a+b}F}{\partial x^a \partial y^b}$  denotes the obvious partial derivative of  $F(x, y)$ , evaluated at  $(\xi, 0)$ . The formulae for the first three coefficients  $a_1, a_2, a_3$  are

$$\begin{aligned} a_1 &= \partial/\partial_y F \\ a_2 &= 2a_1 + (a_1 \partial/\partial_x + \partial/\partial_y)^2 F \\ a_3 &= 12a_2 - 6a_1 + 6\partial/\partial_y (a_1 \partial/\partial_x + \partial/\partial_y) F \\ &\quad + (a_1 \partial/\partial_x + \partial/\partial_y)^3 F \end{aligned}$$

In particular, for  $F(x, y)$  analytic and non-constant in  $y$ , the sequence  $s_n(x)$  is asymptotically independent from  $x$ .

(iii) Proposition 6.1 can be generalized and/or adapted to similar situations, e.g. it is enough to require  $|f'(x)| < 1$  in the interior of the interval  $A$  or by replacing  $A$  with a suitable subset of  $\mathbf{C}$  or  $B$  where  $B$  is a Banach space. In particular, assertion (ii) above associates a formal power series

$$\frac{1}{1 - f'(0)} \left( \sum_{j=1}^{\infty} \frac{\alpha_j}{j!} n^{-j} \right) \in \mathbf{C}[[\frac{1}{n}]]$$

(related to asymptotic series expansions) to a holomorphic contracting function  $f : \mathcal{O} \rightarrow \mathcal{O}$  such that  $f(0) = 0$ ,  $|f'(0)| < 1$  and  $\mathcal{O} \subset \mathbf{C}$  a neighbourhood of 0 by considering the holomorphic function  $F(x, y) = f(x) + y$  from  $\mathcal{O} \times \mathbf{C}$  into  $\mathbf{C}$ . Since the coefficients  $\alpha_1, \alpha_2, \dots$  are polynomials in  $f(0), f'(0), f''(0), \dots, f^{(n)}(0), \dots$ , the formal power series above can in fact be associated to any formal power series  $s(x) = \sum_{n=1}^{\infty} \sigma_n x^n$  such that  $\sigma_1 \neq 1$ .

For  $x \in (0, \infty)$  we consider the real analytic positive function

$$\tau(x) = \sum_{k=1}^{\infty} e^{-\pi(k/x)^2} = \frac{1}{2} \theta_3\left(\frac{i}{x^2}\right) - \frac{1}{2},$$

related to the third Jacobi-theta function  $\theta_3(z) = \sum_{k \in \mathbf{Z}} e^{i\pi k^2 z}$ , cf. for instance Equation (6), page 102 in [5]. For  $x > 0$ , we have  $\tau'(x) = \frac{2\pi}{x^3} \sum_{k=1}^{\infty} k^2 e^{-\pi(k/x)^2} > 0$  and the easy inequalities

$$\frac{x}{2} - 1 < \int_0^{\infty} e^{-\frac{\pi}{x^2} t^2} dt - \int_0^1 e^{-\frac{\pi}{x^2} t^2} dt < \sum_{k=1}^{\infty} e^{-\pi(k/x)^2} < \int_0^{\infty} e^{-\frac{\pi}{x^2} t^2} dt = \frac{x}{2}$$

for  $x > 0$  imply that  $x \mapsto \tau(x)$  is an increasing analytic diffeomorphism of  $(0, \infty)$ . The equation

$$\tau\left(\frac{\Omega(x)}{x}\right) = \frac{1}{x}$$

defines thus a real positive analytic function  $\Omega : (0, \infty) \rightarrow \mathbf{R}$ . Equivalently, the function  $\Omega$  is given by  $\Omega(x) = x\psi\left(\frac{1}{x}\right)$  where the analytic diffeomorphism  $\psi$  satisfies  $\psi(\tau(x)) = \tau(\psi(x)) = x$  for all  $x > 0$  and is the compositional inverse of  $\tau$ .

**Proposition 6.3** *The map*

$$x \mapsto \Omega(x) = x\psi\left(\frac{1}{x}\right)$$

*is an increasing diffeomorphism from  $(0, \infty)$  onto  $(2, \infty)$ . It has a unique fixpoint  $\xi = \frac{1}{\tau(1)} = \frac{1}{\sum_{k=0}^{\infty} e^{-\pi k^2}} \sim 23.13882534$  which is attracting under iteration since*

$$\Omega'(\xi) = 1 - \frac{\tau(1)}{\tau'(1)} = 1 - \frac{\sum_{k=1}^{\infty} e^{-\pi k^2}}{2\pi \sum_{k=1}^{\infty} k^2 e^{-\pi k^2}} \sim 0.9135652 < 1 .$$

**Remark 6.4** *Proposition 6.3 in its full strength is not necessary for proving Theorem 1.3 which can be deduced from the easy observation that  $\Omega$  has an attracting fixpoint  $\xi = \frac{1}{\tau(1)}$ .*

## 6.2 Proof of Theorem 1.3

Given an  $(n-1)$ -dimensional lattice of density  $\tilde{\Delta}_{n-1}$ , Theorem 3.3 implies the existence of an  $n$ -dimensional lattice with density  $\tilde{\Delta}_n$  arbitrarily close to  $\frac{1}{2^{n\bar{\sigma}}}$  for  $\bar{\sigma} > 0$  defined by

$$2^{n-1} \tilde{\Delta}_{n-1} \sum_{k=1}^A \sqrt{1 - k^2 \left(2^{n-1} \tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \bar{\sigma}\right)^2}^{n-1} = 1$$

where

$$A = \left\lfloor \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \bar{\sigma}} \right\rfloor .$$

Given a positive constant  $\epsilon > 0$  and a natural integer  $N \in \mathbf{N}$ , there exists thus a sequence of lattices  $\Lambda_1, \Lambda_2, \dots, \Lambda_N$  of dimensions  $1, 2, \dots, N$  with densities  $\tilde{\Delta}_1 = 1, \tilde{\Delta}_2, \dots, \tilde{\Delta}_N$  satisfying

$$\tilde{\Delta}_m \geq (1 - \epsilon) \frac{d_m}{2^m}, \quad m = 1, \dots, N$$

where  $d_1 = 2$  and  $d_2, d_3, \dots, d_N$  are recursively defined by the equation

$$d_{n-1} \sum_{k=1}^{A_n} \sqrt{1 - k^2 \left(\frac{d_{n-1}}{d_n} \frac{V_n}{V_{n-1}}\right)^2}^{n-1} = 1 \quad \text{with } A_n = \left\lfloor \frac{d_n V_{n-1}}{d_{n-1} V_n} \right\rfloor .$$

Equivalently, the sequence  $d_1, d_2, \dots$  is given by  $d_1 = 2, d_2 = f_1(2), d_3 = f_2(d_2), \dots, d_{n+1} = f_n(d_n), \dots$  where  $f_1, f_2, \dots : (0, \infty) \rightarrow (0, \infty)$  are the functions defined implicitly by the equations

$$x \sum_{k=0}^{\lfloor f_n(x)V_n/(xV_{n+1}) \rfloor} \sqrt{1 - k^2 \left(\frac{xV_{n+1}}{f_n(x)V_n}\right)^2}^{n-1} = 1 .$$

Stirlings formula  $n! = \sqrt{2\pi n}(n/e)^n(1 + O(1/n))$  shows

$$V_{n+1}/V_n = \sqrt{\pi} \frac{(n/2)!}{((n+1)/2)!} = \sqrt{2\pi/n} (1 + O(1/n)) .$$

We have thus asymptotically

$$\begin{aligned} & x \sum_{k=0}^{\lfloor f_n(x)V_n/(xV_{n-1}) \rfloor} \sqrt{1 - k^2 \left( \frac{xV_{n+1}}{f_n(x)V_n} \right)^2}^n \\ &= \left( x \sum_{k=1}^{\infty} e^{-k^2\pi(x/f_n(x))^2} \right) (1 + O(1/n)) \end{aligned}$$

and  $f_n(x) \rightarrow \Omega(x)$  uniformly on any compact subset of  $(0, \infty)$ . By Proposition 6.3 we can find  $\alpha < \xi = \left( \sum_{k=1}^{\infty} e^{-\pi k^2} \right)^{-1} \sim 23.14 < \beta$  such that  $\Omega'(x) \leq 19/20$  for  $x \in [\alpha, \beta]$ . We have thus uniform convergency  $f_n(x) \rightarrow \Omega(x)$  for  $x \in [\alpha, \beta]$ , and there exists an integer  $N$  such that  $f_n([\alpha, \beta]) \subset [\alpha, \beta]$  for all  $n \geq N$ . This implies Theorem 1.3.  $\square$

The following Table illustrates the convergence of the sequence  $d_1 = 2, d_2, \dots$ :

1	2.00000000	2.00000000	0
2	3.62759873	3.99997210	-0.7447467
4	8.08369319	7.92472241	0.6358831
8	18.71971890	14.38756801	34.6572071
16	30.69030131	20.71395996	159.6214617
32	29.45114255	22.98242063	206.9991014
64	25.53248635	23.13821340	153.2334688
128	24.17810739	23.13882533	133.0281029
256	23.63011883	23.13882534	125.7711333
512	23.37820694	23.13882534	122.5633803
1024	23.25703467	23.13882534	121.0463495

The first column shows the indices  $n$ , chosen as successive powers of 2. The second column shows the corresponding value of  $d_n$ . The third column shows the  $(n-1)$ -th iteration of  $\Omega$ , starting from the initial value 2. The last column is the difference between the second and third column, multiplied by  $n$  and illustrates the expected asymptotic properties.

Asymptotically, the number  $d_n$  is roughly given by

$$23.13882534 + 119.58193 \frac{1}{n} + 1473.8282 \frac{1}{n^2} + 25774.448 \frac{1}{n^3} + \dots$$

(cf assertion (ii) of Remark 6.2).

### 6.3 Proof of Proposition 6.3

Using the orientation-reversing diffeomorphism  $x = \frac{1}{\tau(Y)} \mapsto Y = \psi\left(\frac{1}{x}\right)$  of  $(0, \infty)$  we have

$$\frac{Y}{\tau(Y)} = \Omega(x) = \Omega\left(\frac{1}{\tau(Y)}\right).$$

The inequality  $\tau(Y) < \frac{Y}{2}$  shows  $\Omega(x) = \frac{Y}{\tau(Y)} > 2$  and  $\frac{2Y}{Y-2} > \frac{Y}{\tau(Y)}$  implies  $\lim_{x \rightarrow 0^+} \Omega(x) = 2$ . Since

$$\lim_{Y \rightarrow 0^+} \frac{Y}{\tau(Y)} = \lim_{Y \rightarrow 0^+} Y e^{\pi^2/Y^2} \left(1 + \sum_{k=2}^{\infty} e^{-\pi(k^2-1)/Y^2}\right)^{-1} = \infty$$

the continuous map  $\Omega$  has image  $(2, \infty)$ .

Since  $x \mapsto Y$  is orientation reversing,  $\frac{d}{dx}\Omega(x) > 0$  is equivalent to strict positivity of

$$Y^2 \frac{d}{dY} \left(\frac{\tau(Y)}{Y}\right) = Y\tau'(Y) - \tau(Y) = \sum_{k=1}^{\infty} (2\pi k^2 - Y^2) e^{-\pi(k/Y)^2} = \pi F_0\left(\frac{Y}{\sqrt{\pi}}\right)$$

where

$$F_0(Y) = \sum_{k=1}^{\infty} (2k^2 - Y^2) e^{-(k/Y)^2} = \sum_{k=1}^{\infty} k^2 \left(2 - \frac{0!}{0!} \left(\frac{Y^2}{k^2}\right)\right) e^{-(k/Y)^2}.$$

Set

$$F_n(Y) = \sum_{k=1}^{\infty} k^{2(n+1)} \left(2 - \sum_{j=0}^n \frac{n!}{j!} \left(\frac{Y^2}{k^2}\right)^{n+1-j}\right) e^{-(k/Y)^2}.$$

We have

$$\begin{aligned} \frac{Y^3}{2} F'_n(Y) &= \sum_{k=1}^{\infty} k^{2(n+1)} \left( - \sum_{j=0}^n \frac{n! k^2}{j!} (n+1-j) \left(\frac{Y^2}{k^2}\right)^{n+2-j} \right. \\ &\quad \left. + k^2 \left(2 - \sum_{j=0}^n \frac{n!}{j!} \left(\frac{Y^2}{k^2}\right)^{n+1-j}\right) \right) e^{-(k/Y)^2} \\ &= \sum_{k=1}^{\infty} k^{2(n+2)} \left( - \sum_{j=0}^n \frac{(n+1)!}{j!} \left(\frac{Y^2}{k^2}\right)^{n+2-j} + \sum_{j=1}^n \frac{n!}{(j-1)!} \left(\frac{Y^2}{k^2}\right)^{n+1-(j-1)} \right. \\ &\quad \left. + 2 - \sum_{j=0}^{n-1} \frac{n!}{j!} \left(\frac{Y^2}{k^2}\right)^{n+1-j} - \frac{n!}{n!} \left(\frac{Y^2}{k^2}\right)^{n+1-n} \right) e^{-(k/Y)^2} \\ &= \sum_{k=1}^{\infty} k^{2(n+2)} \left(2 - \sum_{j=0}^{n+1} \frac{(n+1)!}{j!} \left(\frac{Y^2}{k^2}\right)^{n+2-j}\right) e^{-(k/Y)^2} \\ &= F_{n+1}(Y). \end{aligned}$$

All functions  $F_0(Y), F_1(Y), \dots, F_n(Y), \dots$  are analytic for real  $Y > 0$  and satisfy  $\lim_{Y \rightarrow 0^+} F_n(Y) = 0$ . We have thus  $F_0(Y) > 0$  if there exists a natural integer  $n$  such that  $F_n(Y) > 0$  for all  $Y > 0$ .

Consider

$$F_4(Y) = \sum_{k=1}^{\infty} k^{10} \left( 2 - \sum_{j=0}^4 \frac{24}{j!} \left( \frac{Y^2}{k^2} \right)^{5-j} \right) e^{-(k/Y)^2}.$$

We will show that  $F_4(Y) > 0$  for  $Y > 0$ .

Denote by

$$t_{4,k}(Y) = \left( 2k^{10} - k^8 Y^2 - 4k^6 Y^4 - 12k^4 Y^6 - 24k^2 Y^8 - 24Y^{10} \right) e^{-(k/Y)^2}$$

the  $k$ -th summand involved in  $F_4(Y)$ . Let  $\rho \sim 0.60074553$  be the unique positive real root of  $t_{4,k}(kY) = k^{10} t_{4,1}(Y)$ .

For natural integers  $m, N$  with  $N \geq m$  set

$$A_{4,m} = \sum_{k=1}^m t_{4,k}((m+1)\rho) e^{(k/((m+1)\rho))^2 - (k/(m\rho))^2}$$

$$B_{4,m}(N) = \sum_{k=m+1}^N t_{4,k}((m+1)\rho).$$

For  $Y$  such that  $m\rho \leq Y \leq (m+1)\rho$  one checks easily that

$$A_{4,m} \leq A_{4,m} + B_{4,m}(N) \leq \sum_{k=1}^N t_{4,k}(Y) < F_4(Y)$$

and a computation shows that  $A_{4,m} + B_{4,m}(100) > 0$  for  $m \leq 19$  which implies  $F_4(Y) > 0$  for  $Y$  strictly positive such that  $Y \leq 12 < 20\rho \sim 12.01491$ .

Consider the function

$$g_4(x, y) = T_4(x) + T_4(y-x) e^{-y(y-2x)}$$

where

$$T_4(z) = 2z^{2(4+1)} - \sum_{j=0}^4 \frac{4!}{j!} z^{2j} = 2z^{10} - z^8 - 4z^6 - 12z^4 - 24z^2 - 24$$

is essentially the polynomial part with respect to  $k$  involved in the summands  $t_{4,k}(Y)$  of  $F_4(Y)$ . We have the following result whose proof will be given later:

**Lemma 6.5** *There exist two real numbers  $\alpha \sim 2.629623$  and  $\beta \sim 2.714446$  such that  $\beta - \alpha > \frac{1}{12}$  and  $g_4(x, y) \geq 0$  for all  $(x, y) \in \mathcal{D} = [0, \rho] \times [\alpha, \beta] \sim [0, 0.601] \times [2.63, 2.71]$ .*

Consider now a fixed positive real number  $Y \geq 12$ . Since  $Y(\beta - \alpha) \geq 12(\beta - \alpha) \sim 1.017874 > 1$  (with  $\alpha \sim 2.63, \beta \sim 2.7144$  as in Lemma 6.5) we can find a natural integer  $L$  such that  $\alpha Y < L < \beta Y$ . For  $k \leq \rho Y$  a positive integer we have now

$$t_{4,k}(Y) + t_{4,L-k}(Y) = Y^{10} e^{-(k/Y)^2} g_4\left(\frac{k}{Y}, \frac{L}{Y}\right)$$

(remark that  $k \leq \rho Y \leq \frac{\rho}{\alpha} L \sim 0.2284531 L < \frac{1}{2} L$  implies  $k < L - k$ ). Since  $0 < \frac{k}{Y} \leq \rho$  and  $\alpha < \frac{L}{Y} < \beta$ , Lemma 6.5 implies  $t_{4,k}(Y) + t_{4,L-k}(Y) > 0$ . This shows  $F_4(Y) = \sum_{k=1}^{\lfloor \rho Y \rfloor} (t_{4,k}(Y) + t_{4,L-k}(Y)) + R_Y$  where all summands involved in

$$R_Y = \sum_{k=\lfloor \rho Y \rfloor + 1}^{L-1-\lfloor \rho Y \rfloor} t_{4,k}(Y) + \sum_{k=L+1}^{\infty} t_{4,k}(Y)$$

are strictly positive. We have thus  $F_4(Y) > 0$  for all  $Y > 0$  which implies  $F_m(Y) > 0$  for all  $Y > 0$  and  $m = 0, 1, 2, 3$ . Proposition 6.3 follows.  $\square$

**Proof of Lemma 6.5** Let  $\alpha \sim 2.629623$  denote the smallest positive real root of  $y \mapsto g_4(\rho, y)$  (where, as before,  $\rho \sim 0.601$  satisfies  $t_{4,k}(k\rho) = T_4(\frac{1}{\rho}) = 0$ ). Similarly, we denote by  $\beta \sim 2.714446$  the largest real root of  $y \mapsto g_4(0, y)$ . We have  $\beta - \alpha \sim 0.084823 > \frac{1}{12} = 0.083333 \dots$

We leave it to the reader to check that  $g_4(x, y) \geq 0$  for  $(x, y) \in \partial \mathcal{D}$  where  $\partial \mathcal{D}$  denotes the boundary of the product  $[0, \rho] \times [\alpha, \beta]$ .

The computation

$$\begin{aligned} e^{y(y-2x)} \frac{\partial}{\partial y} & \left( \left( 2(y-x)^{2(n+1)} - \sum_{j=0}^n \frac{n!}{j!} (y-x)^{2j} \right) e^{-y(y-2x)} \right) \\ & = 4(n+1)(y-x)^{2n+1} - 4(y-x)^{2n+3} + 2 \frac{n!}{n!} (y-x)^{2n+1} \\ & = 2(y-x)^{2n+1} \left( 2n+3 - 2(y-x)^2 \right), \end{aligned}$$

specialized to  $n = 4$ , shows that local extrema of the function

$$g_n(x, y) = T_n(x) + T_n(y-x) e^{-y(y-2x)}$$

with  $T_n(z) = 2z^{2(n+1)} - \sum_{j=0}^n \frac{n!}{j!} z^{2j}$  are elements of the set

$$\{y = x\} \cup \{y = x + \sqrt{(2n+3)/2}\} \cup \{y = x - \sqrt{(2n+3)/2}\}.$$

The coefficients of the power-series expansions of the even analytic function  $h_n(x) = \sum_{j=0}^{\infty} \alpha_{n,j} x^{2j}$  defined by

$$h_n(x) = g_n(x, x \pm \sqrt{(2n+3)/2}) = T_n(x) + T_n\left(\sqrt{\frac{2n+3}{2}}\right) e^{x^2 - (2n+3)/2}$$



are given by

$$\alpha_{n,j} = \begin{cases} \frac{1}{j!} \left( T_n \left( \sqrt{\frac{2n+3}{2}} \right) e^{-(2n+3)/2} - n! \right) & 0 \leq j \leq n \\ 2 + \frac{1}{(n+1)!} T_n \left( \sqrt{\frac{2n+3}{2}} \right) e^{-(2n+3)/2} & j = n+1 \\ \frac{1}{j!} T_n \left( \sqrt{\frac{2n+3}{2}} \right) e^{-(2n+3)/2} & j > n+1. \end{cases}$$

We have thus  $h_n(x) > 0$  and  $h'(x) > 0$  for all  $x > 0$  if and only if

$$h_n(0) = \alpha_{n,0} = \left( T_n \left( \sqrt{\frac{2n+3}{2}} \right) e^{-(2n+3)/2} - n! \right) > 0.$$

Specializing to  $n = 4$ , we have  $h_4(x) > 0, h_4'(x) > 0$  for  $x > 0$  since

$$\alpha_{4,0} = h_4(0) = 63729/8e^{-11/2} - 24 \sim 8.5557 > 0.$$

This shows that the function  $g_4(x, y) : (0, \infty)^2 \rightarrow \mathbf{R}$  has no local extrema on the two lines  $\{y = x \pm \sqrt{(2n+3)/2}\}$ .

The line  $x = y$  does not intersect  $\mathcal{D} = [0, \rho] \times [\alpha, \beta] \sim [0, 0.601] \times [2.63, 2.71]$ . Minima and maxima of the restriction of  $g(x, y)$  to  $\mathcal{D}$  are thus achieved by points on the boundary  $\partial\mathcal{D}$  of  $\mathcal{D}$  and the restriction of  $g(x, y)$  to  $\partial\mathcal{D}$  is  $\geq 0$ . This ends the proof of Lemma 6.5.  $\square$

**Remark 6.6** *The even analytic functions*

$$h_n(x) = T_n(x) + T_n \left( \sqrt{\frac{2n+3}{2}} \right) e^{x^2 - (2n+3)/2}$$

(where  $T_n(x) = 2x^{2(n+1)} - n! \sum_{j=0}^n \frac{z^{2j}}{j!}$ ) have all coefficients non-negative for  $n \geq 3$ . For  $n \leq 9$  this can be done by direct computations. For  $n > (e^3 - 1)/2 \sim 9.54$  this follows from the easy upper bound

$$\begin{aligned} h_n(0) &= -n! + \left( 2 \left( \frac{2n+3}{2} \right)^{n+1} - n! \sum_{j=0}^n \frac{1}{j!} \left( \frac{2n+3}{2} \right)^j \right) e^{-(2n+3)/2} \\ &> -n! + \left( 2 \left( \frac{2n+3}{2} \right)^{n+1} - n! e^{(2n+3)/2} \right) e^{-(2n+3)/2} \\ &= 2 \left( \left( \frac{2n+3}{2} \right)^{n+1} e^{-(2n+3)/2} - n! \right) > 0 \end{aligned}$$

and the inequality

$$\log n! \leq \int_{1/2}^{n+1/2} \log(x) dx = \left( n + \frac{1}{2} \right) \log \left( n + \frac{1}{2} \right) - n + \frac{1}{2} \log 2$$

implied by concavity of  $x \mapsto \log x$ .

## 7 Final remarks

The inequality

$$\#(\bigcup_{k=1}^A I_k) \leq \sum_{k=1}^A \#(X_k(0)_p)$$

appearing in the proof of Theorem 3.3 is probably not sharp. A smaller upper bound for the cardinality  $\#(\bigcup_{k=1}^A I_k)$  would thus improve the results of this paper.

The inequality above can be decomposed into the two inequalities

$$\#(\bigcup_{k=1}^A I_k) \leq \sum_{k=1}^A \#(I_{k,p})$$

and

$$\#(I_{k,p}) \leq \#(X_k(0)_p)$$

where we denote by  $I_{k,p} \subset I_k$  the subset of integers corresponding to primitive elements. If the subsets  $I_{1,p}, \dots, I_{A,p}$  are asymptotically “independent” in the sense that

$$\#(\bigcap_{j=1}^l I_{k_j,p}) / \#(I \cap \mathbf{Z}) \sim_{\mu \rightarrow \infty} \prod_{j=1}^l (\#(I_{k_j,p}) / \#(I \cap \mathbf{Z})),$$

for  $\{I_{k_1,p}, \dots, I_{k_l,p}\} \subset \{I_{1,p}, \dots, I_{A,p}\}$  a subset of  $l$  distinct elements, one can neglect the contributions corresponding to  $k = 2, \dots, A$ . This would lead to a small improvement.

A probably much more important improvement would result from a better understanding of the inequality  $\#(I_{k,p}) \leq \#(X_k(0)_p)$ .

Instead of working with sublattices of  $\mathbf{Z}^{n+1}$  orthogonal to a given vector  $(s_0, \dots, s_n) \in \mathbf{Z}^{n+1}$ , it is possible to consider sublattices  $\mathbf{Z}^{n+a}$  which are orthogonal to a set of  $a \geq 2$  linearly independent vectors in  $\mathbf{Z}^{n+a}$ . One might also replace the standard lattice  $\mathbf{Z}^{n+1}$  by other lattices, e.g. sublattices of dimension  $n$  in  $\mathbf{Z}^{n+1}$  (which approximate homothetically an arbitrary lattice by Proposition 2.2) or of finite index in  $\mathbf{Z}^{n+1}$ .

Extending finite  $\mu$ -sequences in an optimal way into longer  $\mu$ -sequences amounts geometrically to the familiar process of lamination for lattices (see for instance [5] or [7]). The existence of an integer  $s \in I \setminus I_1$  implies indeed the existence of a point  $P \in \mathbf{E}^{n-1}$  which is far away from any lattice point of the affine lattice  $\{(x_0, \dots, x_{n-1}) \mid \sum x_i s_i = s\} \subset \mathbf{Z}^n$  and corresponds thus to a “hole” of the lattice.

The present version of this paper owes much to Fedor Petrov whose pertinent questions clarified and improved (and changed the title of) a preliminary version, see [1].

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