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# Some scaling limits for a brownian polymer in a Gaussian medium

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## Abstract

This paper provides information about the asymptotic behavior of a one-dimensional Brownian polymer in random medium represented by a Gaussian field  $W$  on  $\mathbb{R}_+ \times \mathbb{R}$  assumed to be white noise in time and function-valued in space. According to the behavior of the spatial covariance  $W$ , we give sharp upper and lower bounds on the partition function's exponential rate (Lyapunov exponent), and on the growth (wandering exponent) of the polymer itself when the time parameter goes to infinity.

**Key words and phrases:** Polymer model, Random medium, Lyapunov exponent, Gaussian field, Free energy, Wandering exponent.

**MSC:** 82D60, 60K37, 60G15.

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# 1 Introduction

This paper is concerned with a model of one-dimensional directed Brownian polymer in a Gaussian random environment which can be briefly described as follows: the polymer itself will be simply modeled by a Brownian motion  $b = \{b_t; t \geq 0\}$ , defined on a complete filtered probability space  $(\Omega_b, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (P^x)_{x \in \mathbb{R}})$ , where  $P^x$  stands for the Wiener measure starting from the initial condition  $x$ . The corresponding expected value will be denoted by  $E_b^x$ , or simply by  $E_b$  when  $x = 0$ .

The random environment will be represented by a Gaussian landscape  $W$  indexed by  $\mathbb{R}_+ \times \mathbb{R}$ , defined on another independent complete probability space  $(\Omega, \mathcal{G}, \mathbf{P})$ . Denoting by  $\mathbf{E}$  the expected value with respect to  $\mathbf{P}$ , the covariance structure of  $W$  is given by

$$\mathbf{E} [W(t, x)W(s, y)] = [t \wedge s] Q(x - y), \quad (1.1)$$

for a given homogeneous covariance function  $Q : \mathbb{R} \rightarrow \mathbb{R}$  satisfying some growth and regularity conditions that will be specified later on. In particular, the function  $t \mapsto [Q(0)]^{-1/2}W(t, x)$  will be a standard Brownian motion for any fixed  $x \in \mathbb{R}$ , and for every fixed  $t \in \mathbb{R}_+$ , the process  $x \mapsto t^{-1/2}W(t, x)$  is a homogeneous Gaussian field on  $\mathbb{R}$  with covariance function  $Q$ . Notice that the homogeneity assumption is made here for sake of readability, but could be weakened for almost all the results we will show. Nevertheless, we have chosen to present this paper using only homogeneous landscapes  $W$ . The interested reader can consult [10] for the types of tools needed for such generalizations.

Once  $b$  and  $W$  are defined, the polymer measure itself can be described as follows: for any  $t > 0$ , the energy of a given path (or configuration)  $b$  on  $[0, t]$  will be given by

$$-H_t(b) = \int_0^t W(ds, b_s). \quad (1.2)$$

Notice that a completely rigorous meaning for this integral will be given in the next section, but for the moment, notice that for any fixed path  $b$ ,  $H_t(b)$  is a centered Gaussian random variable with variance  $tQ(0)$ . Based on this Hamiltonian, for any  $x \in \mathbb{R}$ , and a given constant  $\beta$  (interpreted as the inverse of the temperature of the system), we will define our (random) polymer measure by

$$dG_t^x(b) = \frac{e^{-\beta H_t(b)}}{Z_t^x} d\hat{P}^x(b), \quad \text{with} \quad Z_t^x = E_b^x [e^{-\beta H_t(b)}]. \quad (1.3)$$

In the sequel, we will also consider some Gibbs averages with respect to the polymer measure: for all  $t \geq 0$ ,  $n \geq 1$ , and for any bounded measurable functional  $f : (C([0, t]; \mathbb{R}^d))^n \rightarrow \mathbb{R}$ , we set

$$\langle f \rangle_t = \frac{E_b^x [f(b^1, \dots, b^n) e^{-\beta \sum_{l \leq n} H_t(b^l)}]}{Z_t^n}, \quad (1.4)$$

where the  $b^l, 1 \leq l \leq n$ , are understood as independent Brownian configurations.

Notice that, after the early results of the Mathematical Physics literature (see [9], [11]), links between martingale theory and directed polymers in random environments have been

established in [2], [1], and over the last few years, several papers have shed some light on different type of polymer models: the case of random walks in discrete potential is treated for instance in [3], the case of Gaussian random walks in [12], [13], and the case of Brownian polymers in a Poisson potential is considered in [7]. On the other hand, we have undertaken in [15] the study of the polymer measure  $G_t$  defined by (1.3). This latter model, which is believed to behave similarly to the other directed polymers mentioned above, has at least one advantage, from our point of view: it can be tackled with a wide variety of methods. Our hope is then to exploit all the tools available in this context (scaling invariances for both  $b$  and  $W$ , stochastic analysis, Gaussian tools, relationship with Lyapunov exponents for stochastic pdes) in order to get a rather complete description of the asymptotic behavior of the measure  $G_t$ .

In the present article, we will illustrate this point of view on the topic by giving two kind of results on the asymptotic behavior of our model. The first one concerns the limiting behavior of the partition function: namely, we will see at Proposition 2.1 that  $\frac{1}{t} \log(Z_t^x)$  converges almost surely to a quantity  $p(\beta)$ , usually called the free energy of the system. In the so-called weak disorder regime of the polymer model, in which the medium  $W$  has no real influence on the polymer  $b$  (see e.g. [7] for a general result in that direction),  $p(\beta)$  is equal to  $[Q(0)\beta^2]/2$ , which coincides with the annealed free energy. On the other hand, in the strong disorder regime, one gets the strict inequality  $p(\beta) < [Q(0)\beta^2]/2$ , and the function  $\beta \mapsto p(\beta)$  is in fact believed to behave linearly in  $\beta$ . Some examples of weak and strong disorder are given in [15], and the free energy of some of the related models mentioned above are also deeply investigated in [1], [2], [3], [7], to cite just a sample of relevant references. In this paper, we will use some methods inspired by the study of Lyapunov exponents for stochastic PDEs (see [6], [18] and mostly [10]) in order to get some non-trivial bounds on  $p(\beta)$  for large  $\beta$ , that is in the low temperature regime. These results will be obtained in terms of the local regularity of  $Q$  in a neighborhood of 0. Namely, assuming some upper and lower bounds on  $Q$  of the form

$$c_0|x|^H \leq Q(0) - Q(x) \leq c_1|x|^H, \quad \text{for all } x \in \mathbb{R}, \quad (1.5)$$

for a given exponent  $H \in (0, 1]$ , we get the following conclusions:

1. If  $H \in [1/2, 1]$ , we have for some constants  $C_0$  and  $C_1$  depending only on  $Q$ , for all  $\beta \geq 1$ ,

$$C_0\beta^{4/3} \leq p(\beta) \leq C_1\beta^{2-2H/(3H+1)}.$$

2. If  $H \in (0, 1/2]$ , we have for some constants  $\beta_Q$ ,  $C'_0$ , and  $C'_1$  depending only on  $Q$ , for all  $\beta \geq \beta_Q$ ,

$$C'_0\beta^{2/(1+H)} \leq p(\beta) \leq C'_1\beta^{2-2H/(3H+1)}.$$

Condition (1.5) is equivalent to assuming that  $W$  has a specific almost-sure modulus of continuity in space, of order  $|x|^H \log^{1/2}(1/|x|)$ , i.e. barely failing to be  $H$ -Hölder continuous (see [17] for details.)

The transition at  $H = 1/2$  for the lower bounds above is continuous but not smooth, and cannot be analytic since the lower-bound power of  $\beta$  is constant for  $H \geq 1/2$ . In the

case  $H \in [1/2, 1]$ , the gap between the two estimates decreases as  $H$  increases to 1; for  $H = 1/2$ , we get bounds with the powers of  $\beta$  equal to  $4/3$  and  $8/5$ ; and for  $H = 1$ , the bounds are  $4/3$  and  $3/2$ . It should be noted that the case  $H = 1/2$  is our least sharp result, while the case  $H = 1$  yields the lowest power of  $\beta$ , and one should not expect lower powers for any potential  $W$  since even if  $W$  is so smooth that it is  $C^\infty$  in space: indeed, unless  $W$  is highly degenerate, the lower bound in (1.5) should hold with  $H = 1$ , while the upper bound will automatically be satisfied with  $H = 1$ . The case of small  $H$  is more interesting. Indeed, we can rewrite the lower and upper bounds above as

$$C'_0 \beta^{2-2H+F(H)} \leq p(\beta) \leq C'_1 \beta^{2-2H+G(H)}$$

where the functions  $F$  and  $G$  satisfy, for  $x$  near 0:

$$\begin{aligned} F(x) &= 2x^2 + O(x^3); \\ G(x) &= 6x^2 + O(x^3). \end{aligned}$$

We therefore see that the asymptotic  $\beta^{2-2H}$  is quite sharp for small  $H$ , but that the second order term in the expansion of the power of  $\beta$  for small  $H$ , while bounded, is always positive. Notice that some sharper results will be obtained in case of a logarithmic spatial regularity for  $W$ .

The second type of result that will be addressed here deals with the wandering exponent  $\xi$ , which measures the growth of the polymer when  $t$  tends to  $\infty$ , and could be defined informally by the fact that, under the measure  $G_t$ ,  $\sup_{s \leq t} |b_s|$  should behave like  $t^\xi$  for large times  $t$ . This kind of exponent has been studied in different contexts in [7], [12], [13], [14] and [20], yielding the conclusion that, for a wide number of models in dimension one, we should have  $3/5 \leq \xi \leq 3/4$ , the true exponent conjectured by the Physicists being  $\xi = 2/3$ . In this note, we will show that, for our model, we have  $\xi \geq 3/5$ , and more specifically, that for any  $\beta > 0$  and any  $\varepsilon > 0$ ,

$$\lim_{t \rightarrow \infty} \mathbf{P} \left[ \frac{1}{t^{\frac{3}{5}-\varepsilon}} \langle \sup_{s \leq t} |B_s| \rangle_t \geq 1 \right] = 1. \quad (1.6)$$

This result follows the steps of [13], where the same kind of growth bound has been established for a random walk in a Gaussian potential, and our proof will also be inspired by this latter reference. Notice however that we have been able to extend Petermann's result to a wider class of environments: indeed the relation (1.6) will be obtained as soon as  $Q$  satisfies a mild assumption on its decay at infinity, that is

$$Q(x) = O\left(\frac{1}{|x|^{3+\theta}}\right), \quad \text{as } x \rightarrow \pm\infty, \quad (1.7)$$

while [13] assumed an exponential decay for  $Q$ . On the other hand, many arguments have to be changed in order to pass from the random walk to the Brownian case, and [13] is an unpublished work. We have thus chosen to include most of the computations in our proof, for sake of readability. Notice also that the two hypothesis (1.5) and (1.7) are quite different in their nature, and one could hope that combining both conditions on the regularity at 0 and on the decay at  $\infty$  of  $Q$ , we could get some sharper results on the asymptotic behavior of  $G_t$ . We plan to report on this strategy in a subsequent communication.

## 2 Preliminaries; the partition function

In this section, we will first recall some basic facts about the partition function  $Z_t$ , and then give briefly some notions of Gaussian analysis which will be used later on. Let us recall that  $W$  is a centered Gaussian field defined on  $\mathbb{R}_+ \times \mathbb{R}$ , which can also be seen as a Gaussian family  $\{W(\varphi)\}$  indexed by tests functions  $\varphi : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ , where  $W(\varphi)$  stands for the Wiener integral of  $\varphi$  with respect to  $W$ :

$$W(\varphi) = \int_{\mathbb{R}_+ \times \mathbb{R}} \varphi(s, x) W(ds, dx),$$

whose covariance structure is given by

$$\mathbf{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}_+} \left( \int_{\mathbb{R} \times \mathbb{R}} \varphi(s, x) Q(x - y) \psi(s, y) dx dy \right) ds, \quad (2.1)$$

for two arbitrary test functions  $\varphi, \psi$ .

Let us start here by defining more rigorously the quantity  $H_t(b)$  given by (1.2), which can be done through a Fourier transform procedure: there exists (see e.g. [6] for further details) a centered Gaussian independently scattered measure  $\nu$  on  $\mathbb{R}_+ \times \mathbb{R}$  such that

$$W(t, x) = \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbf{1}_{[0, t]}(s) e^{iux} \nu(ds, du). \quad (2.2)$$

For every test function  $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$ , set now

$$\nu(f) \equiv \int_{\mathbb{R}_+ \times \mathbb{R}} f(s, u) \nu(ds, du). \quad (2.3)$$

Then the law of  $\nu$  is defined by the following covariance structure: for any test functions  $f, g : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$ , we have

$$\mathbf{E}[\nu(f)\overline{\nu(g)}] = \int_{\mathbb{R}_+ \times \mathbb{R}} f(s, u) \overline{g(s, u)} \hat{Q}(du) ds, \quad (2.4)$$

where the finite (real) measure  $\hat{Q}$  is the Fourier transform of  $Q$ . We can go back now to the definition of  $H_t(b)$ : invoking the representation (2.2), we can write

$$-H_t(b) = \int_0^t B(ds, b_s) = \int_0^t \int_{\mathbb{R}} e^{iub_s} \nu(ds, du),$$

and it can be shown (see [6]) that the right hand side of the above relation is well defined for any Hölder continuous path  $b$ , by a  $L^2$ -limit procedure.

Recall now that  $Z_t^x$  has been defined by

$$Z_t^x = E_b[e^{-\beta H_t(b)}],$$

and set

$$p_t(\beta) = \frac{1}{t} \mathbf{E}[\log(Z_t^x)], \quad (2.5)$$

usually called the free energy of the system. It is easily seen that  $p_t(\beta)$  is independent of the initial condition  $x \in \mathbb{R}$ , thanks to the spatial homogeneity of  $W$ , and it is thus usual to concentrate on this quantity for  $x = 0$ . In fact, in the remainder of the paper,  $x$  will be understood as 0 when not specified, and  $E_b, Z_t$  will stand for  $E_b^0, Z_t^0$ , etc. Let us summarize then some basic results on  $p_t(\beta)$  and  $Z_t$  shown in [15].

**Proposition 2.1.** *For all  $\beta > 0$  there exists a constant  $p(\beta) > 0$  such that*

$$p(\beta) \equiv \lim_{t \rightarrow \infty} p_t(\beta) = \sup_{t \geq 0} p_t(\beta).$$

Furthermore, the function  $p$  satisfies:

1. The map  $\beta \mapsto p(\beta)$  is a convex nondecreasing function on  $\mathbb{R}_+$ .
2. The following upper bound holds true:

$$p(\beta) \leq \frac{\beta^2}{2} Q(0). \quad (2.6)$$

3.  $\mathbf{P}$ -almost surely, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log Z_t = p(\beta). \quad (2.7)$$

### 3 Study of the partition function

This section is devoted to the analysis of the quantity  $p(\beta)$  defined at Proposition 2.1. In particular, we will show that for large  $\beta$  the function  $\beta \mapsto p(\beta)$  is always subquadratic, but grows faster than a linear function of  $\beta$ . More specifically, our results in this section will be formulated in relation to  $W$ 's regularity, or lack thereof, in the space parameter  $x$ . The hypothesis we use guarantees that there is some  $H \in (0, 1)$  such that  $W$  is no more than  $H$ -Hölder-continuous in space. Accordingly, we define the spatial *canonical metric*  $\delta$  of  $W$  by

$$\delta^2(x - y) := \mathbf{E} [(W(1, x) - W(1, y))^2] = 2 [Q(0) - Q(x - y)].$$

#### 3.1 Lower bound result

**Theorem 3.1.** *Assume there exists a number  $H \in (0, 1]$  and numbers  $c_0, r_0 > 0$  such that for all  $x, y \in \mathbb{R}$  with  $|x - y| \leq r_0$ , we have*

$$\delta(x - y) > c_0 |x - y|^H. \quad (3.1)$$

Then if  $H \leq 1/2$ , there exist constants  $C_0$  and  $\beta_0$  depending only on  $Q$  such that for all  $\beta > \beta_0$ ,

$$p(\beta) \geq C_0 \beta^{2/(1+H)}.$$

On the other hand if  $H \geq 1/2$ , there exists a constant  $\hat{C}_0$  depending only on  $Q$  such that for all  $\beta \geq 0$ ,

$$p(\beta) \geq \hat{C}_0 \beta^{4/3}.$$

**Remark 3.2.** A remarkable feature of the above theorem is the shape and values of the power function in the lower bound. Indeed, we are claiming that  $p(\beta)$  is bounded below by universal constants times  $\beta^{\theta(H)}$  where  $\theta(H)$  is a decreasing function defined on  $[0, 1]$  which is continuous, flat on  $[1/2, 1]$ , reaches its minimum value of  $4/3$  for all  $H \geq 1/2$ , and does not reach its supremum of 2 near the open endpoint 0 of its domain. In particular, the exponential rate of increase of the partition function is always significantly faster than linear in  $\beta$ , is nearly quadratic for very irregular potential, and may only be as low as the power  $\beta^{4/3}$  when the potential  $W$  has precisely a Brownian-type behavior in space, or is more regular.

*Proof of Theorem 3.1:* Let us divide this proof in several steps.

*Step 0: Strategy.* Recall that  $p_t(\beta)$  has been defined by (2.5). It was established in [15] that almost surely

$$p(\beta) = \sup_{t>0} p_t(\beta).$$

This proves that a lower bound on  $p(\beta)$  will be obtained by evaluating  $p_t(\beta)$  for any fixed value  $t$ . Additionally, by the positivity of the exponential in the definition of  $Z_t$ , one may include as a factor inside the expectation  $E_b$  the sum of the indicator functions of any disjoint family of events of  $\Omega_b$ . In fact, we will only need two events.

*Step 1: Setup.* Let  $A_+(b)$  and  $A_-(b)$  be two disjoint events defined on the probability space  $\Omega_b$  under  $P_b$ . Let  $X_b = -\beta H_{2t}(b) = \beta \int_0^{2t} W(ds, b_s)$ . Conditioning by the two events  $A_+$  and  $A_-$ , and using Jensen's inequality, we easily obtain

$$\log(Z_{2t}) \geq \log(\min\{P_b(A_+); P_b(A_-)\}) + \mathbf{E}[\max\{Z_+, Z_-\}], \quad (3.2)$$

where

$$Z_+ := E_b[X_b | \tilde{A}_+] \quad \text{and} \quad Z_- := E_b[X_b | \tilde{A}_-];$$

these two random variables form a pair of centered jointly Gaussian random variables. Indeed, they are both limits of linear combinations of values of a single centered Gaussian field. This implies that

$$\mathbf{E}[\max\{\tilde{Z}_+, \tilde{Z}_-\}] = (2\pi)^{-1/2} \mathbf{E}^{1/2} \left[ \left( \tilde{Z}_+ - \tilde{Z}_- \right)^2 \right].$$

Therefore we see that we only need to choose sets  $A_+$  and  $A_-$  that are not too small, but still big enough that condition (3.1) guarantees a certain amount of positivity in the variance of  $\tilde{Z}_+ - \tilde{Z}_-$ .

*Step 2: Choice of  $A_+$  and  $A_-$ , and their probabilities.* Let  $f$  be an arbitrary positive increasing function on  $(0, \infty)$ . We take

$$A_+ := \{2f(t) \geq b_s \geq f(t) : \forall s \in [t, 2t]\},$$

and

$$A_- := \{-2f(t) \leq b_s \leq -f(t) : \forall s \in [t, 2t]\}.$$



In other words, we force our trajectory  $b$  to be, during the entire time interval  $[t, 2t]$ , in one of two boxes of size  $f(t)$  which are at a distance of  $2f(t)$  from each other. Because these two boxes are symmetric about the starting point of  $b$ , the corresponding events have the same probability. While this probability can be calculated in an arguably explicit way, we give here a simple lower bound argument for it. Using time scaling, the Markov property of Brownian motion, the notation  $a = f(t)/\sqrt{t}$ , and some trivial lower bounds, we have

$$\begin{aligned}
P_b(A_+) &= P_b(\forall s \in [1, 2] : b_s \in [a, 2a]) \\
&= (2\pi)^{-1} \int_a^{2a} P_b(\forall s \in [0, 1] : b_s + y \in [a, 2a]) e^{-y^2/2} dy \\
&\geq (2\pi)^{-1} \int_{5a/4}^{7a/4} P_b(\forall s \in [0, 1] : b_s + y \in [y - a/4, y + a/4]) e^{-y^2/2} dy \\
&= P_b(b_1 \in [5a/4, 7a/4]) P_b(\forall s \in [0, 1] : |b_s| \leq a/4). \tag{3.3}
\end{aligned}$$

*Step 3: Recalling the estimation of  $\tilde{Z}_+ - \tilde{Z}_-$ .* It was established in [10] that

$$\mathbf{E} \left[ \left( \tilde{Z}_+ - \tilde{Z}_- \right)^2 \right] \geq \int_t^{2t} \mathbf{E} \left[ \delta(x_{s,+}^* - x_{s,-}^*)^2 \right] ds,$$

where the quantities  $x_{s,+}^*$  and  $x_{s,-}^*$  are random variables, but we have the following deterministic bounds on them for all  $s \in [t, 2t]$ :  $x_{s,+}^* \in [f(t), 2f(t)]$  and  $x_{s,-}^*$  in the interval  $[-2f(t), -f(t)]$ . As a consequence, by condition (3.1), as long as  $f(t)$  can be made smaller than  $r_0$ , we have (recall that  $c$  designates a constant that can change from line to line)

$$\mathbf{E} \left[ \left( \tilde{Z}_+ - \tilde{Z}_- \right)^2 \right] \geq ct\beta^2 (f(t))^{2H}. \tag{3.4}$$

*Step 4: The case  $H \leq 1/2$ .* It is possible, although we will spare the reader the pain of deciphering such a development, to prove that the optimal choice for  $f$  in the case  $H < 1/2$  is  $f(t) = t^{1/2}$ , which corresponds to  $a = 1$ , so that  $P_b(A_+)$  is a universal constant  $c_1$  that does not actually depend on  $t$ . Thus we have, from (3.2) and (3.4), that for any  $t > 0$ ,

$$p_{2t}(\beta) = \frac{\log(Z_{2t})}{2t} \geq -\frac{C_1}{2t} + \frac{C_2\beta}{t^{(1-H)/2}}, \tag{3.5}$$

where the universal constant  $C_1 = \log(1/c_1) > 0$ . Now we may maximize the above function over all possible values of  $t > 0$ . To make things simple, we simply choose  $t$  so that the second term in the above right-hand side equals twice the first, yielding  $t$  of the form

$$t = \frac{c}{\beta^{2/(1+H)}}$$

and therefore

$$\sup_{t>0} p_t(\beta) \geq c\beta^{2/(1+H)},$$

and the result for  $H < 1/2$  follows as announced, as long as the use of (3.4) from Step 3 can be justified, namely as long as  $f(t) \leq r_0$ . This is obviously achieved as soon as  $\beta > \beta_0$  where

$$\beta_0 = c(r_0)^{-1-H}.$$

*Step 5: The case  $H > 1/2$ .* The simple-minded choice for  $f$  that worked so well in the previous case is still applicable here, but we may try instead the following

$$f(t) = r_0 t^\alpha,$$

for a given  $\alpha \in [0, 1]$ . Clearly the case  $\alpha = 1/2$  yields the same formula as in the case  $H < 1/2$ , but we would like to do much better. It is possible to prove that  $\alpha \geq 1/2$  is suboptimal, but, again, we will spare the reader. Assuming now that  $\alpha < 1/2$ , we then have, using the notation of Step 2, that  $a = t^{\alpha-1/2}$  so that, assuming for the moment that  $t$  is small,  $a$  will be large. In this case, the result (3.3) from Step 2 yields that for some universal constant  $c_3$ ,

$$P_b(A_+) \geq \exp(-c_3 a^2) = \exp(-c_3 t^{2\alpha-1})$$

Using condition (3.1) and relation (3.4), we now get, for another constant  $c > 0$ ,

$$p_{2t}(\beta) \geq -ct^{2\alpha-2} + c\beta t^{-1/2+\alpha H}. \quad (3.6)$$

Again, choosing  $t$  so that the second term on the right-hand side equals twice the first, we obtain

$$t = \frac{c}{\beta^{1/(3/2-(2-H)\alpha)}}, \quad (3.7)$$

and therefore

$$\sup_{t>0} p_{2t}(\beta) \geq c\beta^{(2-2\alpha)/(3/2-(2-H)\alpha)}. \quad (3.8)$$

It is now clear that in order to maximize the power of  $\beta$  in the lower bound for  $\sup_{t \geq 0} p_t(\beta)$ , we should find the maximum of the function

$$k(\alpha) = \frac{2-2\alpha}{3/2-(2-H)\alpha}$$

for  $\alpha < 1/2$ . This function is monotone decreasing when  $H > 1/2$ , which implies that we should take  $\alpha$  as small as possible when  $\beta$  is large. We may now arbitrarily decide to take  $\alpha = 0$ . This yields  $k(0) = 4/3$ . In applying condition (3.1) above, we had to assume that  $t^\alpha \leq 1$ , which is now obviously satisfied. This means we can state, from (3.8),

$$\sup_{t>0} p_{2t}(\beta) \geq c\beta^{4/3},$$

which finishes the case  $H > 1/2$  for all  $\beta \geq 0$ , and the proof of the theorem.  $\square$

The reader might wonder, in view of the arbitrary choice made for  $\alpha$  in Step 5 of the above proof, why a smaller  $\alpha$ , namely  $\alpha < 0$ , does not yield a better result? Let us briefly discuss this issue, to prove that  $\alpha = 0$  is the optimal choice above, even when  $\beta$  is close to 0. Thus, assume  $\alpha < 0$ . The condition  $t^\alpha \leq 1$  then means that  $t$  must be chosen  $\geq 1$ . Hence from formula (3.7), we see that  $\beta$  must be bounded above by  $c^{3/2+(2-H)|\alpha|}$ , so a result for large  $\beta$  cannot be obtained this way. Still, one might be interested in improving the lower bound result of the theorem for small  $\beta$ . In view of (3.8), it would then be best to make  $k(\alpha)$  as small as possible; since  $k(\alpha)$  is still decreasing, we are thus forced to take  $\alpha = 0$ , since no  $\alpha < 0$  will yield a better result even for small  $\beta$ . This paragraph concludes the proof that the choice  $\alpha = 0$  in Step 5 above is optimal.

## 3.2 Upper bound result

To find an upper bound, we will use a different type of computation. We have the following.

**Theorem 3.3.** *Assume there exists a number  $H \in (0, 1]$  and numbers  $c_1, r_1 > 0$  such that for all  $x, y \in \mathbb{R}$  with  $|x - y| \leq r_1$ , we have*

$$\delta(x - y) \leq c_1 |x - y|^H. \quad (3.9)$$

*Then there exists a constant  $C_1$  depending only on  $Q$  such that for all  $\beta \geq 1$ ,*

$$p(\beta) \leq C_1 \beta^{2-2H/(3H+1)}.$$

*Proof.* Let us divide again this proof into several steps.

*Step 0: Strategy.* Exactly as in the Lower Bound theorem of the previous section, consider the quantity  $p_t(\beta)$  defined by (2.5), and recall that it was established in [15] that almost surely

$$p(\beta) \leq \limsup_{t \rightarrow \infty} p_t(\beta).$$

We will first give a discretization result, which, in spirit, was obtained originally in [6], but finds here a much simpler proof, because of our use of the function  $p_t(\beta)$ . Then it will be a matter estimating this function for the discretized path.

*Step 1: Discretization.* For each continuous function  $b$  on  $[0, t]$ , and for each fixed  $\varepsilon > 0$ , let the  $\varepsilon$ -discretization of  $b$  be the function  $\tilde{b}$  defined by letting  $T_0 = 0$ ,  $T_{i+1}$  = the first exit time after  $T_i$  of  $b - b_{T_i}$  from the interval  $[-\varepsilon; +\varepsilon]$ , and for each  $t \in [T_i, T_{i+1})$ , by letting  $x_i := \tilde{b}_t = b_{T_i}$ . Hence under  $P_b$ ,  $\tilde{b}$  is a pure jump process which visits the sites of a discrete-time simple-symmetric random walk on  $\varepsilon\mathbb{Z}$ , while the inter-jump times of  $\tilde{b}$ , which are independent of the sites visited, are independent and distributed like  $T_1$ , the first exit time of  $b$  from  $[-\varepsilon, \varepsilon]$ .

In particular, we have that for any  $t \geq 0$ ,  $|b_t - \tilde{b}_t| \leq \varepsilon$ . Now using Hölder's and Jensen's

inequalities, we obtain

$$\begin{aligned} \mathbf{E} [\log(Z_t)] &= \mathbf{E} \left[ \log E_b \left[ \exp \left( -\beta[H_t(b) - H_t(\tilde{b})] \right) \exp \left( -\beta H_t(\tilde{b}) \right) \right] \right] \\ &\leq \frac{1}{2} \mathbf{E} \left[ \log E_b \left[ \exp \left( -2\beta[H_t(b) - H_t(\tilde{b})] \right) \right] \right] + \frac{1}{2} \mathbf{E} \left[ \log E_b \left[ \exp \left( -2\beta H_t(\tilde{b}) \right) \right] \right] \\ &\leq \frac{1}{2} \log E_b \left[ \exp 2\beta^2 \int_0^t \delta^2(b_s - \tilde{b}_s) ds \right] + \frac{1}{2} \mathbf{E} \left[ \log E_b \left[ \exp \left( -2\beta H_t(\tilde{b}) \right) \right] \right]. \end{aligned}$$

Defining

$$p_t^\varepsilon(\beta) := \frac{1}{2t} \mathbf{E} \left[ \log E_b \left[ \exp \left( -2\beta H_t(\tilde{b}) \right) \right] \right],$$

and assuming  $\varepsilon \leq r_1$ , we obtain the following.

**Lemma 3.4.** *Let  $\delta^+$  be an increasing function of one positive variable such that for all  $|x - y| \leq r_1$ ,  $\delta(x - y) \leq \delta^+(|x - y|)$ . Then we have*

$$\limsup_{t \rightarrow \infty} p_t(\beta) \leq \beta^2 \delta^+(\varepsilon)^2 + \limsup_{t \rightarrow \infty} p_t^\varepsilon(\beta).$$

Of course, assuming Condition (3.9), this translates, for all  $\varepsilon \leq r_1$ , as

$$\limsup_{t \rightarrow \infty} p_t(\beta) \leq \beta^2 c_1^2 \varepsilon^{2H} + \limsup_{t \rightarrow \infty} p_t^\varepsilon(\beta). \quad (3.10)$$

*Step 2: Setup.* Let  $N_t$  be the number of jumps of  $\tilde{b}$  up to time  $t$ , and use the convention  $t_{N_t+1} = t$ . We note that  $H_t(\tilde{b}) = X \left( N_t, (T_i)_{i=1}^{N_t}, (x_i)_{i=1}^{N_t} \right)$  where for any  $\tilde{x} \in \mathcal{P}_m$  the set of nearest-neighbor path of length  $m$  in  $\varepsilon\mathbb{Z}$ , and any  $\tilde{t} \in \mathcal{S}_{m,t}$  the simplex of all sequences of length  $m$  of increasing times in the interval  $[0, t]$  with the convention  $\tilde{t}_{i+1} = t$ , we define

$$X(m, \tilde{t}, \tilde{x}) := \sum_{k=0}^{N_t-1} \{W(\tilde{t}_{i+1}, \tilde{x}_i) - W(\tilde{t}_i, \tilde{x}_i)\}.$$

As such,  $X$  is a Gaussian field indexed by the union over all positive integers  $m$  of all the sets  $J_m := \{m\} \times \mathcal{S}_{m,t} \times \mathcal{P}_m$ . Let  $\alpha$  be a fixed positive number which we will choose later. Let  $I_\alpha = \cup_{m \leq \alpha t} J_m$ , and set also

$$Y_\alpha = \sup_{I_\alpha} X.$$

We can bound  $p_t^\varepsilon(\beta)$  above as follows:

$$\begin{aligned} t p_t^\varepsilon(\beta) &\leq \mathbf{E} [\log(A + B)] \\ &\leq \mathbf{E} [(\log A)_+] + \mathbf{E} [(\log B)_+] + \log 2 \end{aligned} \quad (3.11)$$

where

$$A := P_b [N_t \leq \alpha t] \exp(\beta Y_\alpha)$$

and

$$B := \sum_{n \geq 1} P_b [n \alpha t \leq N_t \leq (n+1) \alpha t] \exp(\beta Y_{(n+1)\alpha}).$$

The following fact, which were established in [10], will be crucial: for some constant  $C_Q$  depending only on  $Q$ , for every  $\alpha > 0$ ,

$$P_b [N_t > t\alpha] \leq \exp \left( -\frac{1}{2}t\alpha^2\varepsilon^2 + t\alpha \right); \quad (3.12)$$

$$\mathbf{E} [|Y_\alpha|] \leq C_Q t\alpha^{1/2}. \quad (3.13)$$

*Step 3: The term A.* Bounding the probability  $P_b [N_t \leq \alpha t]$  by 1, we immediately have

$$\mathbf{E} [(\log A)_+] \leq \beta \mathbf{E} [|Y_\alpha|].$$

Since  $X$  is a centered Gaussian field on  $I_\alpha$ , it follows that the right-hand side above is bounded by  $2\beta \mathbf{E}[Y_\alpha]$ , which implies, via the last estimate (3.13) in Step 2,

$$\mathbf{E} [(\log A)_+] \leq C_Q \beta t \sqrt{\alpha}. \quad (3.14)$$

*Step 4: The term B.* Let  $\mu := \mathbf{E}[Y_{\alpha n}]$ . Since  $X$  is a Gaussian field on  $I_{\alpha n}$ , and because one shows easily that

$$\sigma^2 := \sup_{(m, \tilde{t}, \tilde{x}) \in I_{\alpha n}} \mathbf{Var} [X(m, \tilde{t}, \tilde{x})] \leq tQ(0),$$

the so-called *Borell-Sudakov* inequality implies that for any constant  $a > 0$ ,

$$\mathbf{E} [\exp a |Y_{\alpha n} - \mu|] \leq 2 \exp (a^2 \sigma^2 / 2) \leq 2 \exp (cta^2), \quad (3.15)$$

where here and below the constant  $c$  depending only on  $Q$  may change from line to line. Note that the last estimate above is uniform in  $\alpha$  and  $n$ . Let now  $\gamma \in (1/2, 1)$  be a fixed number, and calculate, using the estimate  $\mu \leq C_Q t(\alpha n)^{1/2}$  from (3.13) at the end of Step 2,

$$\begin{aligned} & \frac{1}{t^\gamma} \mathbf{E} [(\log B)_+] \\ & \leq \mathbf{E} \left[ \left( \frac{1}{t^\gamma} \log \sum_{n \geq 2} P_b [N_t > (n-1)\alpha t] \exp [\beta Y_{\alpha n} - \beta \mu] \exp [c't\beta(\alpha n)^{1/2}] \right)_+ \right] \\ & = \mathbf{E} \left[ \log_+ \left( \sum_{n \geq 1} P_b [N_t > n\alpha t] \exp [\beta Y_{\alpha(n+1)} - \beta \mu] \exp [c't\beta(\alpha(n+1))^{1/2}] \right)^{t^{-\gamma}} \right], \end{aligned}$$

where we have set  $\log_+(x) = [\log(x)]_+$  and where the constant  $c'$  appearing in the last line need only be chosen such that  $c' \geq C_Q$  where  $C_Q$  is given in (3.13), and thus we can assume without loss of generality that  $c' \geq 2$ . For  $t \geq 1$ , we will now use the fact that for any sequence  $(x_n)_n$  of non-negative reals,  $(\sum_n x_n)^{t^{-\gamma}} \leq \sum_n x_n^{t^{-\gamma}}$ . We will also use the estimate (3.12) on the tail of  $N_t$  from the end of Step 2. Thus, denoting

$$\hat{Y}_{\alpha n} \equiv \exp [\beta t^{-\gamma} (Y_{\alpha(n+1)} - \mu)],$$

we obtain

$$\begin{aligned} \frac{1}{t^\gamma} \mathbf{E} [(\log B)_+] &\leq \mathbf{E} \left[ \log_+ \left( \sum_{n \geq 1} (P_b [N_t > nat])^{t^{-\gamma}} |\hat{Y}_{\alpha n}| \exp (c't^{1-\gamma} \beta (\alpha (n+1))^{1/2}) \right) \right] \\ &\leq \mathbf{E} \left[ \log_+ \left( \sum_{n \geq 1} |\hat{Y}_{\alpha n}| \exp \left( -\frac{t^{1-\gamma}}{2} [\alpha^2 n^2 \varepsilon^2 - 2\alpha n - c' \beta (\alpha (n+1))^{1/2}] \right) \right) \right]. \end{aligned}$$

Now, bounding  $\log_+(x)$  above by  $\log(1+x)$  for  $x \geq 0$ , and using Jensen's inequality, we get

$$\frac{1}{t^\gamma} \mathbf{E} [(\log B)_+] \leq \log \left( 1 + \sum_{n \geq 1} \mathbf{E} [|\hat{Y}_{\alpha n}|] \exp \left( -\frac{t^{1-\gamma} u_n}{2} \right) \right),$$

where

$$u_n \equiv \alpha^2 n^2 \varepsilon^2 - 2\alpha n - c' \beta (\alpha (n+1))^{1/2}.$$

In order for the series above to converge, since the expectation in the last line above is bounded in (3.15) independently of  $n$ , it is clear that we must choose  $\alpha$  so as to compensate the negative terms in the first exponential factor. Specifically, we choose

$$\alpha^2 \varepsilon^2 = 4c' \beta \alpha^{1/2},$$

i.e.

$$\alpha = (c' \beta \varepsilon^{-2})^{2/3}. \quad (3.16)$$

Rewriting the above estimate, with use of (3.15), we now obtain

$$\frac{1}{t^\gamma} \mathbf{E} [(\log B)_+] \leq \log \left( 1 + 2 \exp (c\beta^2 t^{1-2\gamma}) \sum_{n \geq 1} \exp \left( -\frac{1}{2} t^{1-\gamma} \alpha^2 \varepsilon^2 y_n \right) \right), \quad (3.17)$$

where

$$y_n := n^2 - \frac{2n}{\alpha \varepsilon^2} - \frac{1}{4} \sqrt{(n+1)}$$

The value  $\varepsilon$  will be chosen in the next step. We will then check that we can chose  $\varepsilon$  as a function of  $\beta$  such that

$$\beta \varepsilon \geq 1. \quad (3.18)$$

This, and the value of  $\alpha$  in (3.16), imply that

$$\alpha \varepsilon^2 \geq (4c')^{2/3} \geq 4$$

Note then that, since for all  $n \geq 1$ ,  $4n^2 - n - \sqrt{n+1} \geq n/2$ , we can now estimate the quantity  $y_n$  as

$$y_n \geq n/8.$$

In this way, summing the geometric series thus obtained by this substitution for  $y$  in (3.17), we immediately get,

$$\frac{1}{t^\gamma} \mathbf{E} [(\log B)_+] \leq \log \left[ 1 + 2 \exp (c\beta^2 t^{1-2\gamma}) \left( \frac{1}{1 - \exp \left( -\frac{1}{16} t^{1-\gamma} \alpha^2 \varepsilon^2 \right)} - 1 \right) \right].$$

By choosing  $t$  large enough, the last factor above can be made less than 1. Hence we can write a final estimate on  $\mathbf{E}[(\log B)_+]$  as follows: for some constant  $c$  depending only on  $Q$ , for large  $t$ , we have

$$\begin{aligned} \frac{1}{t^\gamma} \mathbf{E}[(\log B)_+] &\leq \log [1 + 2 \exp(c\beta^2 t^{1-2\gamma})] \\ &\leq \log 2 + c\beta^2 t^{1-2\gamma}. \end{aligned} \tag{3.19}$$

*Step 5: Conclusion.* With the result (3.14) of step 3, with inequality (3.19) from step 4, inequality (3.11) from step 2, and the value of  $\alpha$  in (3.16), we obtain for some constant  $c$  depending only on  $Q$ ,

$$\begin{aligned} p_t^\varepsilon(\beta) &\leq c\beta\alpha^{1/2} + \frac{\log 2}{t^{1-\gamma}} + \frac{c\beta^2}{t^\gamma} + \frac{\log 2}{t} \\ &= c\beta^{4/3}\varepsilon^{-2/3} + o(1). \end{aligned} \tag{3.20}$$

Now we can put this result together with the discretization lemma 3.4, or more specifically the estimate (3.10), to obtain, for some constant  $c$  depending only on  $Q$ , that almost surely, for all  $\beta \geq 0$ ,

$$\limsup_{t \rightarrow \infty} p_t(\beta) \leq c(\beta^2 \varepsilon^{2H} + \beta^{4/3} \varepsilon^{-2/3}).$$

In order to make this upper bound as small as possible (ignoring any possible multiplicative factors depending only on  $H$ ), we can choose  $\varepsilon$  so that

$$\beta^2 \varepsilon^{2H} = \beta^{4/3} \varepsilon^{-2/3},$$

i.e.

$$\varepsilon = \beta^{-1/(3H+1)}$$

so that

$$\limsup_{t \rightarrow \infty} p_t(\beta) \leq c\beta^2 \beta^{-2H/(3H+1)}.$$

Also, to satisfy condition (3.18), we can see that  $\beta\varepsilon = \beta^{1-1/(3H+1)} \geq 1$  as soon as  $\beta \geq 1$ . This finishes the proof of the theorem.  $\square$

### 3.3 Sharpness of our method. The logarithmic regularity scale.

In this section we first show that when putting the upper and lower bound results together, we obtain an increasingly sharp result as the spatial regularity parameter  $H$  for the potential decreases, with a nearly optimal result as  $H$  approaches 0. This suggests that if we use a potential that is more irregular than any Hölder-continuous function, then we should get an optimal result (up to undetermined multiplicative constants.)

### 3.3.1 Comparing upper and lower bounds

Recall from the introduction (see condition (1.5) and explanations following) that if we assume that we may use both Theorems 3.1 and 3.3, i.e. if we assume that there exist positive constants  $c_0$  and  $c_1$ , and  $H \in (0, 1]$ , such that for all  $x, y$ ,

$$c_0 |x - y|^H \leq \delta(x - y) \leq c_1 |x - y|^H, \quad (3.21)$$

the case of  $H$  small can be expressed as

$$C'_0 \beta^{2-2H+F(H)} \leq p(\beta) \leq C'_1 \beta^{2-2H+G(H)}$$

where the functions  $F$  and  $G$  are, up to constants (2 and 6 respectively), of order  $x^2$ . Hence while the asymptotic  $p(\beta) \simeq \beta^{2-2H} = \beta^2/\delta^2(\beta)$  is quite sharp for small  $H$ , the lower correction  $F$  is always positive, meaning that the expression  $\beta^2/\delta^2(\beta)$  always underestimates the true value of  $p(\beta)$  for  $H > 0$ . It is therefore natural to ask ourselves if this phenomenon persists for potentials that are more irregular than those in the Hölder scale (those satisfying (3.21)). This question is the subject of the next subsection.

### 3.3.2 Logarithmic scale

We now work under the assumptions that there exist positive constants  $c_0$ ,  $c_1$ , and  $r_1$ , and  $\beta \in (0, \infty)$ , such that for all  $x, y$  with  $|x - y| \leq r_1$ ,

$$c_0 \log^{-\gamma}(1/|x - y|) \leq \delta(x - y) \leq c_1 \log^{-\gamma}(1/|x - y|), \quad (3.22)$$

and also that  $\sup_x Q(x, x)$  is finite. Assumption (3.22) implies that  $W$  is not spatially Hölder-continuous for any exponent  $H \in (0, 1]$ . Moreover, the theory of Gaussian regularity implies that, if  $\gamma > 1/2$ ,  $W$  is almost-surely continuous in space, with modulus of continuity proportional to  $\log^{-\gamma+1/2}(1/|x - y|)$ , while if  $\gamma \leq 1/2$ ,  $W$  is almost-surely not uniformly continuous on any interval in space. The case  $\gamma = 1/2$ , which is the threshold between continuous and discontinuous  $W$ , is of special interest, as the reader will find in paragraph 3.4.3. We now establish the following result, which is optimal, up to multiplicative constants.

**Theorem 3.5.** *Assume condition (3.22). We have for some constants  $C_0$  and  $C_1$  depending only on  $Q$ , for all  $\beta \geq 1$ ,*

$$C_0 \beta^2 \log^{-2\gamma}(\beta) \leq p(\beta) \leq C_1 \beta^2 \log^{-2\gamma}(\beta).$$

*Proof. Step 1: Setup.* Nearly all the calculations in the proof of Theorems 3.1 and 3.3 are valid in our situation.

*Step 2: Lower bound.* For the lower bound, reworking the argument in Step 2 in the proof of Theorem 3.1, using the function  $\log^{-\gamma}(x^{-1})$  instead of the function  $x^H$ , we obtain the following instead of (3.4):

$$\mathbf{E} \left[ \left( \tilde{Z}_+ - \tilde{Z}_- \right)^2 \right] \geq t (\beta c_0)^2 \log^{-2\gamma}(2f(t)^{-1}),$$



which implies, instead of (3.5) in Step 4 of that proof, the following

$$p_{2t}(\beta) \geq -\frac{c}{2t} + \beta \left( \frac{c_0^2 \log^{-2\gamma} (2f(t)^{-1})}{8\pi t} \right)^{1/2}.$$

In other words, now choosing  $f(t) = t^{1/2}$  as we did in the case  $H < 1/2$ , for some constant  $C_Q$  depending only on  $Q$ ,

$$p_{2t}(\beta) \geq -\frac{c}{2t} + \beta C_Q \frac{\log^{-\gamma} (t^{-1})}{t^{1/2}}.$$

Now choose  $t$  such that the second term in the right-hand side above equals twice the first, i.e.

$$t^{1/2} \log^{-\gamma} (t^{-1}) = c / (C_Q \beta).$$

For small  $t$ , the function on the left-hand side is increasing, so that the above  $t$  is uniquely defined when  $\beta$  is large. We see in particular that when  $\beta$  is large,  $t$  is small, and we have  $t^{-1} \leq \beta^2$ . This fact is then used to imply

$$\frac{1}{t} = (c\beta)^2 \log^{-2\gamma} (t^{-1}) \leq 2(c\beta)^2 \log^{-2\gamma} (\beta).$$

Therefore, for some constants  $\beta_2$  and  $c$  depending only on  $Q$ , for the  $t$  chosen above with  $\beta \geq \beta_2$ ,

$$p_{2t}(\beta) \geq c\beta^2 \log^{-2\gamma} (\beta).$$

*Step 3: Upper bound.* Here, returning to the proof of Theorem 3.3, the upper bound (3.20) in Step 5 of that proof holds regardless of  $\delta$ , and therefore, using the result of Lemma 3.4 with  $\delta^+(r) = \log^{-\gamma} (1/r)$ , we immediately get that there exists  $c$  depending only on  $Q$  such that for all  $\varepsilon < r_1$  and all  $\beta > \beta_3$ ,

$$\limsup_{t \rightarrow \infty} p_t(\beta) \leq \beta^2 \log^{-2\gamma} (1/\varepsilon) + c\beta^{4/3} \varepsilon^{-2/3},$$

as long as one is able to choose  $\varepsilon$  so that  $\beta\varepsilon \geq 1$  (condition (3.18)). By equating the two terms in the right-hand side of the last inequality above, we get

$$\varepsilon \log^{-3\gamma} (1/\varepsilon) = c^{3/2} \beta.$$

Since the function  $\varepsilon \mapsto \varepsilon \log^{-3\gamma} (1/\varepsilon)$  is increasing for small  $\varepsilon$ , the above equation defines  $\varepsilon$  uniquely when  $\beta$  is large, and in that case  $\varepsilon$  is small. We also see that for any  $\theta > 0$ , for large  $\beta$ ,  $1/\varepsilon \geq \beta^{1-\theta}$ . Therefore we can write, for  $\beta \geq \beta_3$ , almost surely,

$$\limsup_{t \rightarrow \infty} p_t(\beta) \leq 2\beta^2 \frac{1}{(1-\theta)^{2\gamma}} \log^{-2\gamma} (\beta).$$

This finishes the proof of the theorem. □

## 3.4 Conclusions

### 3.4.1 Relation between $p(\beta)$ and $\delta^2(1/\beta)$

Define the commensurability relation  $a \asymp b$  for two positive functions  $a$  and  $b$  by saying that the ratio  $a/b$  is bounded above and away from 0. The conclusions we can draw from the last theorem is that in the logarithmic regularity scale, i.e. under condition (3.22), up to multiplicative constants depending only on  $Q$ , the Lyapunov exponent  $\lim_{t \rightarrow \infty} t^{-1} \log Z_t$  of the partition function  $Z$  is of order

$$p(\beta) \asymp \beta^2 \delta^2(1/\beta), \quad (3.23)$$

since  $\log^{-\gamma}(1/x)$  is commensurate with the canonical metric  $\delta(x)$  via (3.22). Thus, our results are sharp in this logarithmic scale. But when comparing with the Hölder scale, if we write  $\delta(x) = x^H$ , then the relation (3.23) does not hold. In fact, for large  $\beta$ ,  $\liminf_{t \rightarrow \infty} \frac{1}{t} \log u(t)$  is much larger than  $\beta^2 \delta^2(1/\beta) = \beta^{2-2H}$ . Hence the Lyapunov exponent's true power of  $\beta$  remains unknown in the Hölder scale, and we cannot base a conjecture for the Hölder scale on our sharp results in the logarithmic scale.

### 3.4.2 Superlinear and subquadratic growth

In Theorem 3.1, we have found the lower bound

$$p(\beta) \geq c\beta^{4/3}$$

holds in all cases, which, as we said in Remark 3.2, means the partition function grows always significantly faster than linearly. On the other hand, in order to get the fastest possible growth in  $\beta$ , we see that we should use a potential which is as spatially irregular as possible, e.g., thanks to the lower bound in Theorem 3.5, we should work in the logarithmic scale with small  $\gamma$ . But no matter how small  $\gamma$  is, that is, even if  $W$  is not continuous anywhere in space – which is the case if and only if  $\gamma \leq 1/2$  – the upper bound in Theorem 3.5 shows that the Lyapunov exponent will always grow slightly slower than quadratically in  $\beta$ .

However, we can get arbitrarily close to quadratic growth. Indeed, one can repeat the proof of Theorem 3.5 assuming, instead of (3.22), that  $\delta$  is continuous and increasing on  $\mathbb{R}_+$ , with  $\delta(0) = 0$ , simply requiring that we are at best in the logarithmic scale, i.e. that for some  $\gamma > 0$ ,  $\delta(r) \geq \log^{-\gamma}(1/r)$ ; we then obtain that (3.23) holds in this general highly irregular case. So we see that by choosing  $\delta$  which converges to 0 at 0 extremely slowly, we can force the Lyapunov exponent to grow arbitrarily close to quadratically. Still, one can prove that for any separable homogeneous Gaussian field on  $\mathbb{R}$ , then  $\delta$  has to be continuous in a neighborhood of 0, which implies that the Lyapunov exponent can never grow quadratically in  $\beta$ .

### 3.4.3 Special case: spatially white-noise medium

A comparison with the discrete-space polymer is worthwhile. Our proof techniques for establishing the Lyapunov exponent of  $Z$  and the estimation of  $p(\beta)$  are valid if we replace

our Gibbs measure model (1.3) of Brownian paths in  $\mathbb{R}$  under the influence of the random field  $W$ , by the same model, but on  $\mathbb{Z}^d$  instead, i.e. with  $\hat{W}$  on  $\mathbb{R}_+ \times \mathbb{Z}^d$  and with the paths  $b$  as continuous-time random walks in  $\mathbb{Z}^d$  under  $P_b$ . Then, the polymer model in which  $W$  is sometimes known as *space-time white noise*, is that in which  $\{\hat{W}(1, x) : x \in \mathbb{Z}^d\}$  are IID centered Gaussian variables, with still  $t \mapsto \hat{W}(t, x)$  a Brownian motion for each  $x$ . This is the most popular model in a Gaussian environment as far as Lyapounov exponent computations for stochastic PDEs are concerned. We omit the proof of the following result. In some sense, it also follows from the calculations in [4].

**Proposition 3.6.** *For the polymer model in discrete space in the space-time white-noise environment  $\hat{W}$  described above,  $\mathbf{P}$ -almost-surely,*

$$\lim_{t \rightarrow \infty} t^{-1} \log Z_t = p(\beta) \asymp \frac{\beta^2}{\log(\beta)}.$$

In relation with the continuous space models which are the subject of this article, we see that to obtain the same behavior as with space-time white noise in discrete space, we need to use precisely the environment  $W$  in  $\mathbb{R}$  with the logarithmic regularity corresponding to  $\gamma = 1/2$ , i.e.

$$\delta(r) \asymp \log^{-\gamma}(1/r).$$

In this logarithmic case, this behavior of  $W$  happens to be precisely at the threshold in which  $W$  becomes almost-surely discontinuous at every point. Nevertheless such a  $W$  is still function-valued. Hence, for the purpose of understanding the polymer partition function, there is no need to study the space-time white noise in continuous space, for which  $W(t, \cdot)$  is not a bonafide function (only a distribution), and for which the meaning of  $Z_t$  itself is difficult to even define. Said differently, from the continuous space perspective, the appellation “*space-time white noise in  $\mathbb{R}_+ \times \mathbb{Z}^d$* ” is not well-founded, since it only has the same effect on the partition function as the first spatially discontinuous function-valued  $W$  in  $\mathbb{R}_+ \times \mathbb{R}$  that one encounters in the logarithmic regularity scale, when  $\gamma = 1/2$ .

One last way to interpret the coincidence of behaviors for “*space-time white noise in  $\mathbb{R}_+ \times \mathbb{Z}$* ” and for “ $\gamma = 1/2$ ” is to say that both models for  $W$  are function-valued and exhibit spatial discontinuity: indeed, in discrete space, one extends  $W(t, \cdot)$  to  $\mathbb{R}$  by making it piecewise constant, in order to preserve independence. This is again an argument implying state that “space-time white noise in  $\mathbb{R}_+ \times \mathbb{Z}$ ” should really *not* be called “white” in space.

#### 3.4.4 Strong disorder

We finish this section with some remarks on strong disorder, which draw a connection with the next section. We say (see [15] and references therein) that the polymer measure  $dG_t^x(b)$  exhibits *strong disorder* if its partition function  $Z$  satisfies  $\lim_{t \rightarrow \infty} t^{-1} \log Z_t < Q(0) \beta^2/2$  almost surely. The upper bounds in Theorems 3.3 and 3.5 show that our Hölder- and logarithmic regularity scales provide wide classes of polymer measures exhibiting strong disorder for  $\beta \geq 1$ . This also includes all spatially smooth  $W$ 's, for which  $H = 1$ . Paragraph 3.4.2 above shows that this strong disorder for low temperature also works with even more irregular  $W$ .

The condition  $\beta \geq 1$  is uncomfortable, however. One would prefer not having any condition on the temperature scale. This is where the polymer's growth (wandering exponent, denoted by  $\alpha$  in the next section) can be useful. Since the concept of "strong disorder" was introduced in order to determine whether the random medium has any significant influence on polymer paths  $b$ , it is generally accepted to say that a polymer with super-diffusive behavior ( $\alpha > 1/2$ ) exhibits strong disorder. Even though this second definition does not match the common one given above ( $p(\beta) = Q(0)\beta^2/2$ ), it is useful to note that the results of the next section imply the following (see Corollary 4.14): if  $W$  exhibits decorrelation that is not too slow, specifically if for large  $x$ ,  $Q(x) \leq cx^{-5/2-\vartheta}$  where  $\vartheta > 0$ , then the polymer is superdiffusive with exponent any  $\alpha < \min\{\frac{1}{2} + \frac{\vartheta}{6-2\vartheta}; 3/5\}$ , and this form of strong disorder holds for all  $\beta > 0$ . The specific order of decorrelation  $x^{-5/2-\vartheta} \ll x^{-5/2}$  can be quantified by saying that  $W$ 's decorrelation is certainly faster than the well-known order  $x^{-2+2H}$  for the increments of fractional Brownian motion, but the class of such  $W$ 's still qualifies as containing long-range correlations (polynomial with moderate power).

## 4 Polymer growth

In this section, we will specialize our environment in the following way:

**Hypothesis 4.1.** *We assume that  $Q : \mathbb{R} \rightarrow \mathbb{R}$  defined by (1.1) is a symmetric positive function, decreasing on  $\mathbb{R}_+$  and satisfying*

1. *The function  $Q$  is an element of  $L^1(\mathbb{R})$  and  $\int_{\mathbb{R}} Q(x)dx = 1$ .*
2. *There exists a strictly positive constant  $\theta$  such that*

$$Q(x) = O\left(\frac{1}{|x|^{3+\theta}}\right), \quad \text{as } x \rightarrow \pm\infty.$$

**Remark 4.2.** *The normalization  $\int_{\mathbb{R}} Q(x)dx = 1$  is not physically relevant, and is just here in order to simplify some of our formulae. On the other hand, it does represent a kind of non-degeneracy condition, which says that the decorrelation of  $W$  at distinct sites is not immediate. A typical situation is that of a  $W$  which is separable; one can then prove that  $Q$  is continuous at the origin. Since  $Q(0) = \mathbf{E}[W(1,0)^2]$  is finite and strictly positive,  $\int_{\mathbb{R}} Q(x)dx$  is then strictly positive. Part 2 of Hypothesis 4.1 implying that  $\int_{\mathbb{R}} Q(x)dx$  is finite, Part 1 of Hypothesis 4.1 is thus indeed really only a normalization in this situation.*

Recall also that the polymer measure  $G_t = G_t^0$  has been defined by (1.3). Then, under the conditions of Hypothesis 4.1, we will be able to prove the

**Theorem 4.3.** *Let  $\beta$  be any strictly positive real number. Then, for any  $\varepsilon > 0$ , we have*

$$\lim_{t \rightarrow \infty} \mathbf{P} \left[ \frac{1}{t^{\frac{3}{5}-\varepsilon}} \langle \sup_{s \leq t} |b_s| \rangle_t \geq 1 \right] = 1.$$

**Remark 4.4.** *This theorem gives an indication of the asymptotic speed of our polymer. Indeed, if we could write that  $\sup_{s \leq t} |b_s| \sim t^\alpha$  under  $G_t$  as  $t \rightarrow \infty$ , then Theorem 4.3 would state that  $\alpha \geq \frac{3}{5}$ .*

*Strategy of the proof for Theorem 4.3.* For  $t, \epsilon > 0$ , set

$$A_{t,\epsilon} = \left\{ \text{there exists } s_0 \in [t/2, t] \text{ such that } |b_{s_0}| \geq t^{\frac{3}{5}-\frac{\epsilon}{2}} \right\}.$$

Then we can write

$$\begin{aligned} \frac{\langle \sup_{s \leq t} |b_s| \rangle_t}{t^{\frac{3}{5}-\epsilon}} &\geq \frac{t^{\frac{\epsilon}{2}}}{t^{\frac{3}{5}-\frac{\epsilon}{2}}} \left\langle \sup_{s \leq t} |b_s| \mathbf{1}_{A_{t,\epsilon}} \right\rangle_t \\ &\geq t^{\frac{\epsilon}{2}} G_t(A_{t,\epsilon}), \end{aligned}$$

since  $\sup_{s \leq t} |b_s| \geq t^{\frac{3}{5}-\frac{\epsilon}{2}}$  on  $A_{t,\epsilon}$ . Thus

$$\frac{\langle \sup_{s \leq t} |b_s| \rangle_t}{t^{\frac{3}{5}-\epsilon}} \geq t^{\frac{\epsilon}{2}} \left( 1 - G_t(\tilde{A}_{t,\epsilon}) \right), \quad (4.1)$$

where  $\tilde{A}_{t,\epsilon}$  is defined by

$$\tilde{A}_{t,\epsilon} = \left\{ b; \sup_{s \in [t/2, t]} |b_s| \leq t^{\frac{3}{5}-\frac{\epsilon}{2}} \right\} = A_{t,\epsilon}^c.$$

We will start now a discretization procedure in space: for an arbitrary integer  $k$ , and  $\alpha > 0$ , set

$$I_k^\alpha = t^\alpha [2k - 1, 2k + 1), \quad \text{and} \quad L_k^\alpha = \{b; b_s \in I_k^\alpha \text{ for all } s \in [t/2, t]\}.$$

Then  $\tilde{A}_{t,\epsilon} = L_0^{3/5-\epsilon/2}$ , and equation (4.1) can be rewritten as

$$\frac{\langle \sup_{s \leq t} |b_s| \rangle_t}{t^{\frac{3}{5}-\epsilon}} \geq t^{\frac{\epsilon}{2}} \left( 1 - G_t \left( L_0^{\frac{3}{5}-\frac{\epsilon}{2}} \right) \right).$$

Set now

$$Z_t^\alpha(k) := E_b \left[ \mathbf{1}_{L_k^\alpha} \exp(-\beta H_t(b)) \right]. \quad (4.2)$$

We have

$$\frac{\langle \sup_{s \leq t} |b_s| \rangle_t}{t^{\frac{3}{5}-\epsilon}} \geq t^{\frac{\epsilon}{2}} \left( 1 - \frac{Z_t^{\frac{3}{5}-\frac{\epsilon}{2}}(0)}{E_b [\exp(-\beta H_t(b))]} \right),$$

by definition of  $G_t$ . On the other hand, since the events  $L_k$  are disjoint sets we have

$$E_b [\exp(-\beta H_t(b))] \geq \sum_{k \in \mathbb{Z}} Z_t^{\frac{3}{5}-\frac{\epsilon}{2}}(k).$$

Therefore, we have established that

$$\frac{\langle \sup_{s \leq t} |b_s| \rangle_t}{t^{\frac{3}{5}-\epsilon}} \geq t^{\frac{\epsilon}{2}} \left( 1 - \frac{Z_t^{\frac{3}{5}-\frac{\epsilon}{2}}(0)}{Z_t^{\frac{3}{5}-\frac{\epsilon}{2}}(0) + Z_t^{\frac{3}{5}-\frac{\epsilon}{2}}(k)} \right), \quad (4.3)$$

for any integer  $k \neq 0$ . Suppose now that  $W \in \mathcal{A}_t$ , where  $\mathcal{A}_t$  is defined as

$$\mathcal{A}_t := \{W ; \text{ There exists } k^* \neq 0 \text{ such that } Z_t^\alpha(k^*) > Z_t^\alpha(0)\}.$$

Then, choosing  $k = k^*$  in (4.3), it is easily seen that

$$\frac{\langle \sup_{s \leq t} |b_s| \rangle_t}{t^{\frac{3}{5}-\epsilon}} \geq t^{\frac{\epsilon}{2}} \left( 1 - \frac{1}{2} \right) \geq 1,$$

whenever  $t$  is large enough. The proof is now easily finished if we can prove the following lemma:

**Lemma 4.5.** *Given a positive real number  $\alpha \in (1/2; 3/5)$  and an environment  $W$  satisfying Hypothesis 4.1, then*

$$\liminf_{t \rightarrow \infty} \mathbf{P}(\mathcal{A}_t) = 1. \quad (4.4)$$

The remainder of this section will now be devoted to the proof Lemma 4.5. □

## 4.1 Preliminary results

In order to prove Lemma 4.5, we shall first go into a series of preliminary results, and we will start by a lemma on the covariance structure of  $W$ .

### 4.1.1 Covariance computations

For a given  $k \in \mathbb{Z}$  and  $\alpha > 0$ , recall that  $I_k^\alpha = t^\alpha[2k - 1, 2k + 1)$ , and set

$$\tilde{\eta}_k^\alpha \equiv \frac{1}{t^{(\alpha+1)/2}} \int_{\frac{t}{2}}^t \int_{I_k^\alpha} W(ds, x) dx. \quad (4.5)$$

Then  $\{\tilde{\eta}_k^\alpha; k \in \mathbb{Z}\}$  is a centered Gaussian vector, whose covariance matrix will be called  $C(t) = (C_{\ell, k}(t))_{\ell, k \in \mathbb{Z}}$ , where

$$C_{\ell, k}(t) = \mathbf{E}[\tilde{\eta}_\ell^\alpha \tilde{\eta}_k^\alpha] = \mathbf{Cov}(\tilde{\eta}_\ell^\alpha; \tilde{\eta}_k^\alpha). \quad (4.6)$$

We will now proceed to estimate this matrix, and show in particular that  $\lim_{t \rightarrow \infty} C(t) = \text{Id}$ . This can be interpreted as saying that the amount of decorrelation of the potential at distant locations implied by Hypothesis 4.1, is enough to guarantee independence of the  $\tilde{\eta}_k^\alpha$  asymptotically.

**Proposition 4.6.** *Let  $\theta$  be the strictly positive constant defined in Hypothesis 4.1, and consider  $k \in \mathbb{Z}$ ,  $\alpha > 0$  and  $\tau < \theta$ . Set also*

$$\lambda \equiv \frac{1}{C_{0,0}(t)} = \frac{1}{C_{k,k}(t)},$$

where  $C(t)$  has been defined at (4.6). Then, the elements of  $C(t)$  satisfy the following properties:

(i)  $\lambda = 1 + O\left(\frac{1}{t^\alpha}\right)$ .

(ii)  $\lambda \sum_{\ell \neq k} |\ell - k|^\tau |C_{\ell,k}(t)| = O\left(\frac{1}{t^\alpha}\right)$ .

*Proof.* We will only consider the case  $k = 0$ , the other ones being easily deduced by homogeneity of  $W$ . Let us first evaluate  $C_{\ell,0}(t)$  for  $\ell \geq 0$  (here again, the case  $\ell < 0$  is similar, since  $Q$  is a symmetric function). Then, a direct application of (2.1) gives

$$C_{\ell,0}(t) = \frac{1}{2t^\alpha} \int_{t^\alpha(2\ell-1)}^{t^\alpha(2\ell+1)} \int_{-t^\alpha}^{t^\alpha} Q(x-y) dx dy.$$

Moreover,  $\int_{\mathbb{R}} Q(x-y) dx = 1$  for any  $y \in \mathbb{R}$ , and thus it is easily checked that

$$C_{\ell,0}(t) = 1 - \frac{1}{2t^\alpha} \int_{t^\alpha(2\ell-1)}^{t^\alpha(2\ell+1)} \left[ \int_{-\infty}^{-t^\alpha} Q(x-y) dx + \int_{t^\alpha}^{\infty} Q(x-y) dx \right] dy. \quad (4.7)$$

Set now

$$(I) = \frac{1}{2t^\alpha} \left[ \int_{t^\alpha(2\ell-1)}^{t^\alpha(2\ell+1)} \int_{-\infty}^{-t^\alpha} Q(x-y) dx dy + \int_{t^\alpha(2\ell-1)}^{t^\alpha(2\ell+1)} \int_{t^\alpha}^{\infty} Q(x-y) dx dy \right].$$

Then, a series of changes of variable yields

$$\begin{aligned} (I) &= \frac{1}{2t^\alpha} \left[ \int_{t^\alpha(2\ell-1)}^{t^\alpha(2\ell+1)} \int_{-\infty}^{-t^\alpha-y} Q(u) du dy + \int_{t^\alpha(2\ell-1)}^{t^\alpha(2\ell+1)} \int_{t^\alpha-y}^{\infty} Q(u) du dy \right] \\ &= \frac{1}{2t^\alpha} \left[ \int_{-t^\alpha(2\ell+2)}^{-t^\alpha(2\ell)} \int_{-\infty}^{\hat{z}} Q(u) du d\hat{z} + \int_{-t^\alpha(2\ell)}^{-t^\alpha(2\ell-2)} \int_z^{\infty} Q(u) du dz \right], \end{aligned}$$

where we have set  $\hat{z} = -t^\alpha - y$  and  $z = t^\alpha - y$ . Thus, denoting by  $F(z)$  the quantity  $\int_{-\infty}^z Q(u) du$ , we get

$$\begin{aligned} (I) &= \frac{1}{2t^\alpha} \left[ \int_{-t^\alpha(2\ell+2)}^{-t^\alpha(2\ell)} F(\hat{z}) d\hat{z} + \int_{-t^\alpha(2\ell)}^{-t^\alpha(2\ell-2)} (1 - F(z)) dz \right] \\ &= 1 + \frac{1}{2t^\alpha} \int_{-t^\alpha(2\ell+2)}^{-t^\alpha(2\ell)} F(z) dz - \frac{1}{2t^\alpha} \int_{-t^\alpha(2\ell)}^{-t^\alpha(2\ell-2)} F(z) dz. \end{aligned} \quad (4.8)$$

Hence, setting  $\bar{F}(z) = \int_z^\infty Q(u)du = 1 - F(z)$  and putting together (4.7) and (4.8) one obtains, for any  $\ell \geq 0$ ,

$$\begin{aligned} C_{\ell,0}(t) &= \frac{1}{2t^\alpha} \left[ \int_{-t^\alpha(2\ell)}^{-t^\alpha(2\ell-2)} F(z)dz - \int_{-t^\alpha(2\ell+2)}^{-t^\alpha(2\ell)} F(z)dz \right] \\ &= \frac{1}{2t^\alpha} \left[ \int_{-t^\alpha(2\ell+2)}^{-t^\alpha(2\ell)} \bar{F}(z)dz - \int_{-t^\alpha(2\ell)}^{-t^\alpha(2\ell-2)} \bar{F}(z)dz \right]. \end{aligned} \quad (4.9)$$

We are now ready to prove item (i): for  $\ell = 0$ , equation (4.9) becomes

$$\begin{aligned} C_{0,0}(t) &= \frac{1}{2t^\alpha} \left[ \int_0^{2t^\alpha} F(z)dz - \int_{-2t^\alpha}^0 F(z)dz \right] \\ &= \frac{1}{2t^\alpha} \left[ \int_0^{2t^\alpha} \left( 1 - \int_z^\infty Q(u)du \right) dz - \int_{-2t^\alpha}^0 F(z)dz \right], \end{aligned} \quad (4.10)$$

by definition of  $F$ . Thus

$$C_{0,0}(t) = 1 - \frac{1}{2t^\alpha} \left[ \int_0^{2t^\alpha} \left( \int_z^\infty Q(u)du \right) dz + \int_{-2t^\alpha}^0 F(z)dz \right] \quad (4.11)$$

$$= 1 - \frac{1}{t^\alpha} \int_0^{2t^\alpha} \bar{F}(z)dz, \quad (4.12)$$

where we have used the symmetry of  $Q$ . Now, using the fact that

$$\bar{F}(z) \leq c(1 \wedge |z|^{-(2+\theta)}), \quad (4.13)$$

which follows directly from Hypothesis 4.1, it is easily seen that  $C_{0,0}(t) = 1 + O(t^{-\alpha})$ , which ends the proof of item (i).

In order to show item (ii), notice that equation (4.9) yields

$$\sum_{l=1}^k \ell^\tau |C_{\ell,0}(t)| \leq \frac{1}{2t^\alpha} \sum_{\ell=1}^k \ell^\tau \int_{t^\alpha(2\ell-2)}^{t^\alpha(2\ell+2)} \bar{F}(z)dz.$$

However, invoking Hypothesis 4.1, there exists a constant  $\kappa$  (that may change from line to line) such that

$$\begin{aligned} \sum_{l=1}^k \ell^\tau |C_{\ell,0}(t)| &\leq \frac{1}{2t^\alpha} \sum_{\ell=1}^k \ell^\tau \left[ \frac{\kappa}{[t^\alpha(2\ell+2)]^{\theta+1}} + \frac{\kappa}{[t^\alpha(2\ell-2)]^{\theta+1}} \right] \\ &\leq \sum_{\ell=1}^k \frac{\kappa}{\ell^{\theta+1-\tau} t^{\alpha(\theta+1)}} = O\left(\frac{1}{t^\alpha}\right), \end{aligned} \quad (4.14)$$

thanks to the fact that  $\theta + 1 - \tau > 1$ . The same kind of estimate being true for  $\sum_{l=-k}^{-1} \ell^\tau |C_{\ell,0}(t)|$  by symmetry of  $Q$ , the result now follows using item (i).  $\square$



### 4.1.2 Interaction between $b$ and $W$

We will now try to get some quantitative information about the way  $b$  interacts with with the random medium  $W$  when the Brownian motion is localized by the event  $L_k^\alpha$ . This will be done by introducing two quantities: first a family of random variables  $\{\eta_\ell, \ell \in \mathbb{Z}\}$  which will be obtained by a normalization of  $\tilde{\eta}$ , that is

$$\eta_\ell = \frac{t^{\frac{1-\alpha}{2}}}{2} \tilde{\eta}_\ell. \quad (4.15)$$

We will also need a vector  $v = v(b_s; t/2 \leq s \leq t)$  of  $\mathbb{R}^{\mathbb{Z}}$ , defined for each  $\ell \in \mathbb{Z}$  by

$$v_\ell = 4t^{\alpha-1} \mathbf{Cov} \left( \int_{\frac{t}{2}}^t W(ds, b_s); \eta_\ell \right) = 4t^{\alpha-1} \mathbf{E} \left[ \int_{\frac{t}{2}}^t W(ds, b_s) \eta_\ell \right]. \quad (4.16)$$

Then we will prove, in a sense, that  $v_\ell$  looks like  $\mathbf{1}_{k=\ell}$  on  $L_k^\alpha$ . To this purpose, for a fixed  $k \in \mathbb{Z}$ , and  $\tau < \theta$  (remember that  $\theta$  is defined at Hypothesis 4.1), let us consider the norm  $\|\cdot\|_{\tau,k}$  defined on  $\mathbb{R}^{\mathbb{Z}}$  by

$$\|x\|_{\tau,k} = |x_k| + \sum_{i \neq k} |x_i| |i - k|^\tau. \quad (4.17)$$

**Remark 4.7.** *It will be essential in the sequel to control the decay of  $v_\ell$ , and also of  $\delta_\ell$  (defined at Proposition 4.9) when  $|\ell| \rightarrow \infty$ . It will be used for instance in relations (4.33) and (4.38). This is why we have introduced here this norm  $\|\cdot\|_{\tau,k}$ .*

We are now ready to state a first result about the interaction between  $b$  and  $W$ .

**Proposition 4.8.** *Suppose  $b \in L_k^\alpha$ . Then the vector  $v$  given by (4.16) satisfies the following properties:*

(i) *Let  $\|\cdot\|_{\tau,k}$  be the norm defined at (4.17). Then*

$$\|v\|_{\tau,k} - v_k = O\left(\frac{1}{t^\alpha}\right).$$

(ii) *For  $t$  large enough, there exist two strictly positive real numbers  $\underline{c}$  and  $\bar{c}$  such that*

$$\underline{c} \leq v_k \leq \bar{c}.$$

*Proof.* Let us start with item (i), and observe that our computations will be easier here if we use the environment representation (2.2). Recall also that  $\tilde{\eta}_k$  is defined by (4.5) and  $\eta_k$  by (4.15). Then

$$\begin{aligned} v_\ell &= \frac{2}{t} \mathbf{E} \left[ \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \exp(iub_s) \nu(ds, du) \int_{I_\ell} \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \exp(iux) \nu(ds, du) dx \right] \\ &= \frac{2}{t} \int_{I_\ell} \mathbf{E} \left[ \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \exp(iub_s) \nu(ds, du) \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \exp(iux) \nu(ds, du) \right] dx. \end{aligned}$$

Thanks to (2.4), and according to the fact that  $\hat{Q}$  is the Fourier transform of  $Q$ , we thus have

$$\begin{aligned} v_\ell &= \frac{2}{t} \int_{I_\ell} \int_{\frac{t}{2}}^t \int_{\mathbb{R}} \exp(iu(b_s - x)) \hat{Q}(du) ds dx \\ &= \frac{2}{t} \int_{\frac{t}{2}}^t \int_{I_\ell} Q(b_s - x) dx ds \end{aligned} \quad (4.18)$$

$$\leq \sup_{s \in [t/2, t]} \int_{I_\ell} Q(b_s - x) dx. \quad (4.19)$$

However, on the event  $L_k^\alpha$ , it is easily checked that, for  $s \in [t/2, t]$ , we have

$$(2|\ell - k| - 2)t^\alpha \leq b_s - x \leq (2|\ell - k| + 2)t^\alpha.$$

Assume now that  $\ell \neq k$ . According to the fact that  $Q$  is a positive decreasing function on  $\mathbb{R}_+$ , for each  $s \in [t/2, t]$ , we can conclude that

$$\int_{I_\ell} Q(b_s - x) dx \leq \int_{t^\alpha(2|\ell - k| - 2)}^{t^\alpha(2|\ell - k| + 2)} Q(x) dx \leq 4t^\alpha Q(t^\alpha(2|\ell - k| - 2)). \quad (4.20)$$

Consequently, putting together equations (4.19) and (4.20), we get

$$\begin{aligned} \|v\|_{\tau, k} &= v_k + \sum_{\ell \neq k} |\ell - k|^\tau v_\ell \leq v_k + 4t^\alpha \sum_{\ell \neq k} |\ell - k|^\tau Q(t^\alpha(2|\ell - k| - 2)) \\ &\leq v_k + \frac{\kappa}{t^{\alpha(2+\theta)}} \sum_{\ell \neq k} |\ell - k|^{-(3+\theta-\tau)} \leq v_k + \frac{\kappa}{t^{\alpha(2+\theta)}}, \end{aligned} \quad (4.21)$$

where  $\kappa$  is a positive constant that can change from line to line, and where we have used again Hypothesis 4.1. It is now readily checked that  $\|v\|_{\tau, k} \leq v_k + 0(t^{-\alpha})$ , which ends the proof of item (i).

Let us prove now the item (ii): go back to equation (4.18) and set  $\ell = k$ . Then we get

$$\inf_{s \in [\frac{t}{2}, t]} \int_{I_k} Q(b_s - x) dx \leq v_k \leq \sup_{s \in [\frac{t}{2}, t]} \int_{I_k} Q(b_s - x) dx \leq \int_{\mathbb{R}} Q(u) du = 1$$

To find a lower bound on the left-hand side, we now make use of Part 1 of Hypothesis 4.1, which will serve as a non-degeneracy assumption, as announced in Remark 4.2. Since  $Q$  is an even function, we get  $\int_0^\infty Q(x) dx = 1/2$ . But if  $b \in L_k^\alpha$ , then for any  $s \in [t/2, t]$ , we have that the interval  $b_s - I_k$  contains either  $[0, t^\alpha]$  or  $[-t^\alpha, 0]$ , so that, again by the evenness of  $Q$ ,

$$\int_{I_k} Q(b_s - x) dx \geq \int_0^{t^\alpha} Q(x) dx.$$

The latter quantity, which tends to  $1/2$  when  $t \rightarrow \infty$ , can be made to exceed  $1/4$  for  $t$  large enough. This finishes the proof of item (ii) with  $\underline{c} = 1/4$  and  $\bar{c} = 1$ , and of the proposition.  $\square$

### 4.1.3 Inversion of $C(t)$

In this section, we will be concerned with the operator  $C^{-1}(t)$ , where  $C(t)$  has been defined by relation (4.6), and more specifically, we will try to get some information about the solution  $\delta$  to the system  $C(t)x = v$ . The importance of  $\delta$  stems from the fact that the variables  $\eta_k$  will be independent of  $-H_t(b) - \sum_{j \in \mathbb{Z}} \delta_j \eta_j$ , which will be useful for further computations (see Proposition 4.13). However, we have already seen that  $C(t)$  behaves asymptotically like the identity matrix, and thus the vector  $\delta$  should be of the same kind as  $v$ . This is indeed the case, and will be proved in the following Proposition:

**Proposition 4.9.** *Assume Hypothesis 4.1 holds true, and that  $b \in L_k^\alpha$ . Set  $l_{\tau,k} = \{x \in \mathbb{R}^{\mathbb{Z}}; \|x\|_{\tau,k} < \infty\}$ . Then*

(i) *The operator  $C(t)$  is invertible in  $l_{\tau,k}$ . We will set then  $\delta = C^{-1}(t)v$ .*

(ii) *There exist some strictly positive real numbers  $\underline{d}$  and  $\bar{d}$  such that*

$$\underline{d} \leq \delta_k \leq \bar{d}.$$

(iii) *The following relation holds true:*

$$\|\delta\|_{\tau,k} - \delta_k = O\left(\frac{1}{t^\alpha}\right). \quad (4.22)$$

(iv) *On the probability space  $(\Omega, \mathcal{G}, \mathbf{P})$ , the family  $\{\eta_l; l \in \mathbb{Z}\}$  is independent of  $-H_t(b) - \sum_{j \in \mathbb{Z}} \delta_j \eta_j$ .*

**Remark 4.10.** *Notice that Proposition 4.9 contains a considerable amount of the information which will be used for the proof of Lemma 4.5. Indeed, inequality (4.41) will be obtained thanks to item (iv), item (iii) will be invoked for inequality (4.38), and item (ii) will be essential in order to define the random variables  $\check{\eta}_0$  and  $\check{\eta}_k$  at (4.36).*

*Proof of Proposition 4.9.* (i) We will first choose an appropriate operator norm on  $l_{\tau,k}$ . In fact this norm is given by

$$\|A\|_{\tau,k} := \sup_{r \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} \frac{f_{\tau,k}(i)}{f_{\tau,k}(r)} |A_{ir}|, \quad \text{where} \quad f_{\tau,k}(i) = \mathbf{1}_{i=k} + |i-k|^\tau \mathbf{1}_{i \neq k}.$$

We will also call  $\mathcal{L}_{\tau,k}$  the corresponding operator space. Then, on one hand, the following relations are easily checked, for  $D_1, D_2 \in \mathcal{L}_{\tau,k}$  and  $x \in l_{\tau,k}$ :

$$\|D_1 x\|_{\tau,k} \leq \|D_1\|_{\tau,k} \|x\|_{\tau,k}, \quad \text{and} \quad \|D_1 + D_2\|_{\tau,k} \leq \|D_1\|_{\tau,k} + \|D_2\|_{\tau,k}. \quad (4.23)$$

On the other hand, setting  $A(t) = \text{Id} - \lambda C(t)$ , Proposition 4.6 directly yields  $\|A(t)\|_{\tau,k} = O(t^{-\alpha})$ , and thus  $\|A(t)\|_{\tau,k} < 1$  if  $t$  is large enough. This allow us to define  $C^{-1}(t)$  in  $\mathcal{L}_{\tau,k}$  by a Von Neumann type series of the form

$$C^{-1}(t) = \lambda \sum_{j \geq 0} A^j. \quad (4.24)$$

(ii) For  $t$  large enough, set  $\delta = C^{-1}(t)v$ , which makes sense since  $v \in l_{\tau,k}$ . Then, thanks to the fact that  $C^{-1}(t)$  can be defined by relation (4.24), we have

$$\begin{aligned}\delta_k &= \lambda \left( v_k + \sum_{j \geq 1} (A^j v)_k \right) \geq \lambda \left( v_k - \sum_{j \geq 1} \|A^j v\|_{\tau,k} \right) \\ &\geq \lambda \left( v_k - \sum_{j \geq 1} \|A\|_{\tau,k}^j \|v\|_{\tau,k} \right),\end{aligned}$$

where we have used the relations  $x_k \geq -\|x\|_{\tau,k}$  and (4.23). Hence, since  $\|A(t)\|_{\tau,k} = O(t^{-\alpha})$ , we obtain

$$\begin{aligned}\delta_k &\geq \lambda \left( v_k - \frac{\|A\|_{\tau,k}}{1 - \|A\|_{\tau,k}} \|v\|_{\tau,k} \right) \geq \lambda \left( v_k + O\left(\frac{1}{t^\alpha}\right) \right) \\ &\geq \underline{d} + O\left(\frac{1}{t^\alpha}\right),\end{aligned}\tag{4.25}$$

according to the properties of  $v$  shown at Proposition 4.8. The upper bound on  $\delta_k$  can now be shown by the same type of argument, which ends the proof of our claim.

(iii) Let us evaluate now the quantity  $\|\delta\|_{\tau,k} - \delta_k$ : thanks to relations (4.23) and (4.25), we get

$$\begin{aligned}\|\delta\|_{\tau,k} - \delta_k &\leq \|C(t)^{-1}\|_{\tau,k} \|v\|_{\tau,k} - \delta_k \\ &\leq \left( \|C(t)^{-1}\|_{\tau,k} \|v\|_{\tau,k} - \lambda v_k + \frac{\lambda \|A\|_{\tau,k}}{1 - \|A\|_{\tau,k}} \|v\|_{\tau,k} \right).\end{aligned}$$

Thus, using again that fact that  $C^{-1}(t)$  is defined by equation (4.24) and relation (4.23), we obtain

$$\begin{aligned}\|\delta\|_{\tau,k} - \delta_k &\leq \lambda \left( \frac{1 + \|A\|_{\tau,k}}{1 - \|A\|_{\tau,k}} \|v\|_{\tau,k} - v_k \right) \\ &= \lambda (\|v\|_{\tau,k} - v_k) + O\left(\frac{1}{t^\alpha}\right) = O\left(\frac{1}{t^\alpha}\right),\end{aligned}$$

where in the last two steps, we have invoked, respectively, item (i) and Proposition 4.8. This concludes our proof.

(iv) Recall that  $C(t) = \mathbf{Cov}(\eta)$ . Hence, by definition of  $v$  and  $\eta$ ,

$$\begin{aligned}\delta_j &= (C^{-1}(t)v)_j = \frac{1}{4} t^{1-\alpha} \sum_{k \in \mathbb{Z}} [\mathbf{Cov}(\eta)]_{jk}^{-1} v_k \\ &= \sum_{k \in \mathbb{Z}} [\mathbf{Cov}(\eta)]_{jk}^{-1} \mathbf{E} \left[ \int_{\frac{t}{2}}^t W(ds, b_s) \eta_k \right] \\ &= \sum_{k \in \mathbb{Z}} [\mathbf{Cov}(\eta)]_{jk}^{-1} \mathbf{E} [(-H_t(b)) \eta_k],\end{aligned}$$

we have the following standard calculation for any  $k \in \mathbb{Z}$

$$\begin{aligned}
& \mathbf{E} \left[ \left( -H_t(b) - \sum_{j \in \mathbb{Z}} \delta_j \eta_j \right) \eta_\ell \right] \\
&= -\mathbf{E} [H_t(b) \eta_\ell] + \mathbf{E} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} [\mathbf{Cov}(\eta)]_{jk}^{-1} \mathbf{E} [H_t(b) \eta_k] \eta_j \eta_\ell \\
&= -\mathbf{E} [H_t(b) \eta_\ell] + \mathbf{E} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} [\mathbf{Cov}(\eta)]_{jk}^{-1} [\mathbf{Cov}(\eta)]_{j\ell} \mathbf{E} [H_t(b) \eta_k] \\
&= -\mathbf{E} [H_t(b) \eta_\ell] + \sum_{k \in \mathbb{Z}} \delta_{k\ell} \mathbf{E} [H_t(b) \eta_k] = 0.
\end{aligned}$$

Now since for fixed  $b$ ,  $H_t(b)$  and  $\eta$  are both linear functionals of a same Gaussian field, they form a jointly Gaussian vector, and are thus independent.  $\square$

#### 4.1.4 Application of Girsanov's theorem

In our context, the cost of having  $b$  living in the interval  $I_k = [t^\alpha(2k-1), t^\alpha(2k+1)]$  instead of  $I_0 = [-t^\alpha, t^\alpha]$  can be calculated explicitly thanks to Girsanov's theorem: given an integer  $k$ , a real number  $t$  and a realization of the environment  $W$ , we define a new environment by

$$W^{k,t}(s, x) \equiv W(s, x + \min(2s/t, 1)2kt^\alpha). \quad (4.26)$$

We will also need to introduce a modified partition function  $\tilde{Z}$  defined by

$$\tilde{Z}_t^\alpha(k) = E_b \left[ \mathbf{1}_{I_k^\alpha} \exp \left( \beta \left( \int_0^t W(ds, b_s) - \sum_{j \in \mathbb{Z}} \delta_j \eta_j \right) \right) \right].$$

In the sequel, we will have to stress the dependence of these partition functions on the environment under consideration. We will thus set  $\tilde{Z}_t^\alpha(k) = \tilde{Z}_t^\alpha(k, W)$ . With these notations in mind, we can prove the following proposition:

**Proposition 4.11.** *Given two positive real numbers  $\alpha$  and  $t$ , there exists a positive integer  $\zeta = \zeta(k)$  such that*

$$\tilde{Z}_t^\alpha(k, W) \geq \exp(-\zeta t^{2\alpha-1}) \tilde{Z}_t^\alpha(0, W^{k,t}). \quad (4.27)$$

*Proof.* Given  $k$  and  $t$ , we associate to a path  $b$  a shifted path  $b'$  by the relation

$$b'_s \equiv b_s - \min(2s/t, 1)2kt^\alpha, \quad \text{for } s \in \mathbb{R}.$$

Notice that this shift transforms a path which lives in the interval  $I_k$  for all  $s \in [t/2, t]$  into a path which belongs to  $I_0$  in the same time interval. Now, a standard application of Girsanov's theorem yields

$$\begin{aligned}
\tilde{Z}_t^\alpha(k, W) &= E_b \left[ \mathbf{1}_{L_0} \exp \left( -b_{t/2} 4kt^{\alpha-1} - 4k^2 t^{2\alpha-1} \right) \right. \\
&\quad \left. \exp \left( \beta \left( \int_0^t W^{k,t}(ds, b_s) - \sum_{j \in \mathbb{Z}} \delta_j \eta_j(W^{k,t}) \right) \right) \right] \\
&\geq \exp \left( -(1+k)4kt^{2\alpha-1} \right) \tilde{Z}_t^\alpha(0, W^{k,t}),
\end{aligned}$$

where in the last step, we have used the fact that  $L_0$  is realized. This finishes our proof.  $\square$

## 4.2 Proof of Lemma 4.5

Recall that we have reduced our problem to the evaluation of  $\mathbf{P}(\mathcal{B}_t)$ , where

$$\mathcal{B}_t = \mathcal{A}_t^c = \{\text{For all } k \in \mathbb{Z}, Z_t^\alpha(k) \leq Z_t^\alpha(0)\},$$

and one wishes to show that  $\lim_{t \rightarrow \infty} \mathbf{P}(\mathcal{B}_t) = 0$ . Then a first step in order to prove this claim is to truncate  $\mathcal{B}_t$ : for a positive integer  $M$  let  $\mathbb{Z}_M$  and  $\bar{\mathbb{Z}}_M$  be the sets defined respectively by

$$\bar{\mathbb{Z}}_M = \{-M, -M+1, \dots, M-1, M\} \quad \text{and} \quad \mathbb{Z}_M = \bar{\mathbb{Z}}_M \setminus \{0\}, \quad (4.28)$$

and  $\mathcal{B}_{M,t}$  the event defined by

$$\mathcal{B}_{M,t} = \{\text{For all } k \in \mathbb{Z}_M, Z_t^\alpha(k) \leq Z_t^\alpha(0)\}.$$

Then obviously,  $\mathbf{P}(\mathcal{B}_t) \leq \mathbf{P}(\mathcal{B}_{M,t})$ , and we only need to prove that  $\mathbf{P}(\mathcal{B}_{M,t})$  tends to 0 as  $t \rightarrow \infty$ .

Here is a brief account on the strategy we will follow in order to complete our proof.

(1) Recall that we are trying to bound

$$\mathbf{P}(\mathcal{B}_{M,t}) = \mathbf{P}\left(E_b[\mathbf{1}_{L_k} e^{-H_t(b)}] < E_b[\mathbf{1}_{L_0} e^{-H_t(b)}] \text{ for all } k \in \mathbb{Z}_M\right). \quad (4.29)$$

A natural idea is then split the conditions  $E_b[\mathbf{1}_{L_k} e^{-H_t(b)}] < E_b[\mathbf{1}_{L_0} e^{-H_t(b)}]$  in terms of a condition involving the random variables  $\eta_l$  introduced at (4.15), on which we have a reasonable control, and another set of conditions involving some random variables independent of the family  $\{\eta_l; l \in \mathbb{Z}\}$ . However, we have already seen at Proposition 4.9 that  $-H_t(b) - \sum_{j \in \mathbb{Z}} \delta_j \eta_j$  is independent of  $\{\eta_l; l \in \mathbb{Z}\}$ . Thus, a natural choice will be to replace  $e^{-H_t(b)}$  by  $e_t(b)$  in the expression (4.29), where  $e_t(b)$  is defined by

$$e_t(b) = \exp\left(-\beta \left(H_t(b) + \sum_{j \in \mathbb{Z}} \delta_j \eta_j\right)\right).$$

Of course, this induces a correction term  $\exp(\beta \sum_{j \in \mathbb{Z}} \delta_j \eta_j)$ , but this term can be controlled, since the covariance structure of the family  $\{\eta_l; l \in \mathbb{Z}\}$  is given by Proposition 4.6, and the vector  $\delta$  is controlled by means of Proposition 4.9. Thus, up to a negligible term, we will be allowed to bound  $\mathbf{P}(\mathcal{B}_{M,t})$  by

$$\mathbf{P}\left(\text{For any } k \in \mathbb{Z}_M; \frac{\tilde{Z}_t^\alpha(k)}{\tilde{Z}_t^\alpha(0)} < \exp(2\gamma t^{2\alpha-1} + \check{\eta}_0 - \hat{\eta}_k)\right), \quad (4.30)$$

where  $\tilde{Z}_t^\alpha(k) = E_b[\mathbf{1}_{L_k} e_t(b)]$ , and the random variables  $\check{\eta}_k, \hat{\eta}_k$  are defined in (4.36) in terms of the random variables  $\eta$ .

(2) Notice that up to now, we have chosen our parameters carefully in order to get a penalization of order  $\exp(2\gamma t^{2\alpha-1})$  in (4.30). Indeed, Proposition 4.11 gives a sharp estimate on the correction we have to impose on  $b$  if we wish that it lives the second half of his life in  $I_k^\alpha$ , and this penalization is exactly of order  $\exp(\zeta t^{2\alpha-1})$ . In fact, we will be able to bound  $\mathbf{P}(\mathcal{B}_{M,t})$  by  $\mathbf{P}(F_M)$ , where the event  $F_M$  is defined by

$$F_M = \left\{ \text{For any } k \in \mathbb{Z}_M; \frac{\tilde{Z}_t^\alpha(0, W^{k,t})}{\tilde{Z}_t^\alpha(0, W)} < \exp(\hat{\gamma} t^{2\alpha-1} + \tilde{\eta}_0 - \hat{\eta}_k) \right\},$$

for a constant  $\hat{\gamma} = \hat{\gamma}(M)$ , where the shifted environments  $W^{k,t}$  are defined at (4.26), and where the random variables  $\tilde{\eta}_0 - \hat{\eta}_k$ , defined via (4.36), are going to be of the order  $t^{(1-\alpha)/2}(\tilde{\eta}_0 - \tilde{\eta}_k)$ . See also Part 2 of point (3) that follows.

(3) We are now considering a set  $F_M$  involving the random variables  $\tilde{Z}_t$  and  $\tilde{\eta}, \hat{\eta}$ , and this will allow us to take advantage of the following facts:

1. The ratio  $\tilde{Z}_t^\alpha(0, W^{k,t})/\tilde{Z}_t^\alpha(0, W)$  cannot be too small at many different sites  $k \in \mathbb{Z}_M$ , by translation invariance in space of  $W$ .
2. The difference  $\tilde{\eta}_0 - \hat{\eta}_k$  can be highly negative at many different sites, since it is of the order  $t^{(1-\alpha)/2}(\tilde{\eta}_0 - \tilde{\eta}_k)$ , and Proposition 4.6 asserts that  $\{\tilde{\eta}_k; k \in \mathbb{Z}_M\}$  is asymptotically a standard Gaussian vector, so that we are allowed to expect that  $\exp(\hat{\gamma} t^{2\alpha-1} + \tilde{\eta}_0 - \hat{\eta}_k)$  is much smaller than 1 at many different sites of  $\mathbb{Z}_M$ .
3. The random variables  $\tilde{Z}_t^\alpha$  are independent of  $\tilde{\eta}_0, \hat{\eta}_k$ , and hence the two effects alluded to above can be taken into account separately.

These heuristic considerations will be formalized at Step 3 through the introduction of an intricate family of subsets of  $\mathbb{Z}_M$ , but let us mention that the exponent  $3/5$  comes out already at this stage: indeed, the above considerations only make sense if the magnitude  $t^{(1-\alpha)/2}$  of the  $\eta_k$  is greater than the magnitude  $t^{2\alpha-1}$  of the penalization, which occurs obviously whenever  $\alpha < 3/5$ . In this sense, our estimates are quite sharp: they mainly rely on the covariance structure of  $\tilde{\eta}$  and on Girsanov's theorem applied to  $b$ .

Before going into the details of our calculations, let us introduce a new set  $\hat{\mathcal{B}}_{M,t}$ : as mentioned above, our computations will bring out some expressions of the form  $u_t \equiv t^{(1-\alpha)/2} \sum_{j \in \mathbb{Z}} \delta_j \tilde{\eta}_j$ , and it will be convenient to keep this kind of term of order  $O(t^{2\alpha-1})$ , which is also the order of the exponential correction term appearing in (4.27). However, since  $\delta$  satisfies Proposition 4.9, it is easily checked that  $u_t$  is of the desired order if  $\tilde{\eta}_j \leq |j - k|^\tau t^{(7\alpha-3)/2}$  on  $L_k^\alpha$ . These considerations motivate the introduction of the event

$$\hat{\mathcal{B}}_{M,t} \equiv \{ \text{There exists } \ell \in \bar{\mathbb{Z}}_M \text{ and } j \in \mathbb{Z} \setminus \{\ell\}; |\tilde{\eta}_j| \geq |j - \ell|^\tau t^{(7\alpha-3)/2} \},$$

and we will trivially bound  $\mathbf{P}(\mathcal{B}_{M,t})$  by

$$\mathbf{P}(\mathcal{B}_{M,t}) \leq \mathbf{P}(\hat{\mathcal{B}}_{M,t}) + \mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}). \quad (4.31)$$

We will now prove that the two terms in the right hand side of (4.31) are vanishing as  $t \rightarrow \infty$ , whenever  $M$  is large enough.

*Step 1: Estimation of  $\mathbf{P}(\hat{\mathcal{B}}_{M,t})$*

Let  $\Phi$  be the distribution function of a standard Gaussian random variable, i.e. if  $Z \sim \mathcal{N}(0, 1)$ , then

$$\Phi(x) = \mathbf{P}(Z \leq x), \quad (4.32)$$

and set  $\bar{\Phi} = 1 - \Phi$ . Then let us bound simply  $\mathbf{P}(\hat{\mathcal{B}}_{M,t})$  by

$$\begin{aligned} \mathbf{P}(\hat{\mathcal{B}}_{M,t}) &\leq \sum_{\ell \in \bar{\mathbb{Z}}_M} \sum_{j \neq \ell} \mathbf{P}(|\tilde{\eta}_j| \geq |j - \ell| \tau t^{\frac{7\alpha-3}{2}}) \\ &\leq 2 \sum_{\ell \in \bar{\mathbb{Z}}_M} \sum_{j \neq \ell} \bar{\Phi} \left( \frac{|j - \ell| \tau t^{\frac{7\alpha-3}{2}}}{C_{0,0}^{1/2}(t)} \right), \end{aligned}$$

where  $C_{0,0}(t)$  is defined at (4.6). Recall now that  $\bar{\Phi}(x) \leq e^{-x^2/2}$  for  $x$  large enough, and that  $C(t)$  satisfies Proposition 4.6. Thus, for two constants  $c_1, c_2 > 0$ , we get

$$\mathbf{P}(\hat{\mathcal{B}}_{M,t}) \leq c_1 M \sum_{j \geq 1} \exp(-c_2 j^{2\tau} t^{7\alpha-3}). \quad (4.33)$$

The following facts are now easily seen:

- The series in the right hand side of (4.33) is convergent, since  $\tau > 0$ , which explains the choice of the norm  $\|x\|_{\tau, \ell}$  in order to bound  $\tilde{\eta}_j$ .
- Since we have assumed  $\alpha > 1/2 > 3/7$ , we have  $7\alpha - 3 > 0$ , and thus, an elementary application of the dominated convergence theorem yields

$$\lim_{t \rightarrow \infty} \mathbf{P}(\hat{\mathcal{B}}_{M,t}) = 0, \quad (4.34)$$

which proves our first claim.

*Step 2: Estimation of  $\mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t})$*

Recall that the vector  $\delta$  has been introduced because  $-H_t(b) - t^{(1-\alpha)/2} \sum_{j \in \mathbb{Z}} \delta_j \tilde{\eta}_j$  is independent of the family  $\tilde{\eta}$ , and for sake of compactness of notations, set

$$\eta_j = t^{(1-\alpha)/2} \tilde{\eta}_j, \quad \text{and} \quad e_t(b) = \exp \left( -\beta \left( H_t(b) + \sum_{j \in \mathbb{Z}} \delta_j \eta_j \right) \right). \quad (4.35)$$

Now we have

$$\begin{aligned} \mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}) &= \mathbf{P} \left( \hat{\mathcal{B}}_{M,t}^c \text{ and } E_b [\mathbf{1}_{L_k} e^{-H_t(b)}] < E_b [\mathbf{1}_{L_0} e^{-H_t(b)}] \text{ for all } k \in \mathbb{Z}_M \right) \\ &= \mathbf{P} \left( \hat{\mathcal{B}}_{M,t}^c \text{ and } E_b \left[ \mathbf{1}_{L_k} e_t(b) \exp \left( \sum_{j \in \mathbb{Z}} \beta \delta_j \eta_j \right) \right] \right. \\ &\quad \left. < E_b \left[ \mathbf{1}_{L_0} e_t(b) \exp \left( \sum_{j \in \mathbb{Z}} \beta \delta_j \eta_j \right) \right] \text{ for all } k \in \mathbb{Z}_M \right). \end{aligned}$$



It is worth noticing at this point that  $v$ , and thus  $\delta$ , depend on the path  $b$ , as is easily seen from definition (4.16). In order to get rid of the term  $\sum_{j \in \mathbb{Z}} \delta_j \eta_j$ , we will then set

$$\check{\eta}_0 = \max(\beta \underline{d} \eta_0, \beta \bar{d} \eta_0), \quad \text{and} \quad \hat{\eta}_k = \min(\beta \underline{d} \eta_k, \beta \bar{d} \eta_k), \quad (4.36)$$

where the constants  $\underline{d}, \bar{d}$  have been introduced in Proposition 4.9. Then, according to the definition of  $\hat{\mathcal{B}}_{M,t}^c$ , we get

$$\begin{aligned} & \mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}) \\ & \leq \mathbf{P} \left( \text{For any } k \in \mathbb{Z}_M, E_b \left[ \mathbf{1}_{L_k} e_t(b) \exp \left( \sum_{j \in \mathbb{Z}} \beta |\delta_j| |j - k|^\tau t^{3\alpha-1} + \hat{\eta}_k \right) \right] \right. \\ & \quad \left. < E_b \left[ \mathbf{1}_{L_0} e_t(b) \exp \left( \sum_{j \in \mathbb{Z}} \beta |\delta_j| |j|^\tau t^{3\alpha-1} + \check{\eta}_0 \right) \right] \right). \end{aligned}$$

Now, invoking Proposition 4.9 item (iii), we obtain that for any integer  $k$ , there exists a constant  $\gamma$  (possibly depending on  $\beta$ ) such that  $\sum_{j \in \mathbb{Z}} \beta |\delta_j| |j - \ell|^\tau \leq \gamma/t^\alpha$  on  $L_k$ . Thus, thanks to the fact that the random variables  $\eta$  only depend on  $W$ , and observing that  $\tilde{Z}_t^\alpha(k) = E_b[\mathbf{1}_{L_k} e_t(b)]$ , we get

$$\begin{aligned} & \mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}) \quad (4.37) \\ & \leq \mathbf{P} \left( \text{For any } k \in \mathbb{Z}_M; \tilde{Z}_t^\alpha(k) \exp(-\gamma t^{2\alpha-1} + \hat{\eta}_k) < \exp(\gamma t^{2\alpha-1} + \check{\eta}_0) \tilde{Z}_t^\alpha(0) \right) \end{aligned}$$

$$= \mathbf{P} \left( \text{For any } k \in \mathbb{Z}_M; \frac{\tilde{Z}_t^\alpha(k)}{\tilde{Z}_t^\alpha(0)} < \exp(2\gamma t^{2\alpha-1} + \check{\eta}_0 - \hat{\eta}_k) \right). \quad (4.38)$$

Let us apply now Proposition 4.11 in order to conclude that

$$\mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}) \leq \mathbf{P} \left( \text{For any } k \in \mathbb{Z}_M; \frac{\tilde{Z}_t^\alpha(0, W^{k,t})}{\tilde{Z}_t^\alpha(0, W)} < \exp(\hat{\gamma} t^{2\alpha-1} + \check{\eta}_0 - \hat{\eta}_k) \right),$$

where  $\hat{\gamma} = \hat{\gamma}(M) = \sup\{2\gamma + \zeta(k); k \in \mathbb{Z}_M\}$  and  $\zeta(k) = 4k(k+1)$ . We have thus proved that

$$\mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}) \leq \mathbf{P}(F_M),$$

where

$$F_M = \left\{ \text{For any } k \in \mathbb{Z}_M; \frac{\tilde{Z}_t^\alpha(0, W^{k,t})}{\tilde{Z}_t^\alpha(0, W)} < \exp(\hat{\gamma} t^{2\alpha-1} + \check{\eta}_0 - \hat{\eta}_k) \right\}.$$

*Step 3: Evaluation of  $\mathbf{P}(F_M)$*

We can see now that the probability of  $F_M$  will be expressed in terms of a balance between the values of  $\check{\eta}_0 - \hat{\eta}_k$  (which will be assumed to be highly negative) and the ratio  $\tilde{Z}_t^\alpha(0, W^{k,t})/\tilde{Z}_t^\alpha(0, W)$ , which cannot be too small at many different sites  $k$ . In order to quantify this heuristic statement, we will introduce a family  $\bar{\mathcal{S}}_{M,m}$  of subsets of  $\bar{\mathbb{Z}}_M$  which

will be used to construct a large symmetric set  $L$  around 0 such that  $\check{\eta}_0 - \hat{\eta}_\ell < -t^{2\alpha-1+\rho}$  for all  $\ell \in L$ : for a given  $\rho > 0$  and integer numbers  $m$  and  $M$ , define the sets

$$\begin{aligned}\mathcal{S}_{M,m} &= \bigcup_{k, \hat{k} \in D_{M,m}} k\mathbb{Z}_{\hat{k}}, \quad \text{with} \quad D_{M,m} = \left\{ k \geq 1, \hat{k} \geq m; k\mathbb{Z}_{\hat{k}} \subset \mathbb{Z}_M \right\} \\ \bar{\mathcal{S}}_{M,m} &= \{L \subset \mathbb{Z}_M; \text{ There exists } S \in \mathcal{S}_{M,m} \text{ such that } S \subset L\}.\end{aligned}\tag{4.39}$$

In relation with these families of subsets of  $\mathbb{Z}_M$ , set also

$$\hat{F}_{M,m,\rho} = \bigcup_{L \in \bar{\mathcal{S}}_{M,m}} \hat{F}_{\rho,L},\tag{4.40}$$

with

$$\hat{F}_{\rho,L} = \{\check{\eta}_0 - \hat{\eta}_\ell < -t^{2\alpha-1+\rho}, \text{ for all } \ell \in L, \check{\eta}_0 - \hat{\eta}_{\hat{\ell}} > -t^{2\alpha-1+\rho}, \text{ for all } \hat{\ell} \in \mathbb{Z}_M \setminus L\}.$$

Then one can bound trivially  $\mathbf{P}(F_M)$  by

$$\mathbf{P}(F_M) \leq 1 - \mathbf{P}(\hat{F}_{M,m,\rho}) + \mathbf{P}(F_M \cap \hat{F}_{M,m,\rho}).$$

Furthermore, for  $t$  large enough,  $\hat{\gamma}t^{2\alpha-1} - t^{2\alpha-1+\rho} < 0$  – which explains the need for the constant  $\rho > 0$  – and thus

$$\begin{aligned}F_M \cap \hat{F}_{M,m,\rho} &\subseteq \bigcup_{L \in \bar{\mathcal{S}}_{M,m}} \bigcap_{\ell \in L} \left\{ \frac{\tilde{Z}_t^\alpha(0, W^{\ell,t})}{\tilde{Z}_t^\alpha(0, W)} < \exp(\hat{\gamma}t^{2\alpha-1} - 2t^{2\alpha-1+\rho}) \right\} \cap \hat{F}_{\rho,L} \\ &\subseteq \bigcup_{L \in \bar{\mathcal{S}}_{M,m}} \left\{ \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in L \right\} \cap \hat{F}_{\rho,L}.\end{aligned}$$

Hence, since the events  $\hat{F}_{\rho,L}$  are disjoint, we get

$$\begin{aligned}\mathbf{P}(F_M) &\leq 1 - \mathbf{P}(\hat{F}_{M,m,\rho}) + \sum_{L \in \bar{\mathcal{S}}_{M,m}} \mathbf{P} \left( \left\{ \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in L \right\} \cap \hat{F}_{\rho,L} \right) \\ &\leq 1 - \mathbf{P}(\hat{F}_{M,m,\rho}) + \sum_{L \in \bar{\mathcal{S}}_{M,m}} \mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in L \right) \mathbf{P} \left( \hat{F}_{\rho,L} \right),\end{aligned}\tag{4.41}$$

where in the last step, we have just used the independence between the random variables  $\tilde{Z}_t^\alpha$  and the sequence  $\{\eta_\ell; \ell \in \bar{\mathbb{Z}}_M\}$ .

The end of our proof will then rely on the following propositions, whose proofs will be postponed until the next subsections:

**Proposition 4.12.** *Let  $m$  be a fixed positive even integer, and  $M > m$ . Then, for any  $L \in \bar{\mathcal{S}}_{M,m}$ , we have*

$$\mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in L \right) \leq \frac{1}{m}.$$

**Proposition 4.13.** *Let  $m$  be a fixed positive integer. Let  $\rho$  be a strictly positive number such that  $\frac{5}{2}(\alpha - \frac{3}{5}) + \rho < 0$ . Then, for  $t$  large enough, there exists a  $M$  large enough such that*

$$\mathbf{P}(\hat{F}_{M,m,\rho}) \geq 1 - \frac{1}{m}. \quad (4.42)$$

With these results in mind, let us finish now the proof of our Theorem: take  $t, M$  large enough so that (4.42) is satisfied. Then (4.41) yields directly, invoking Proposition 4.12 and the fact that the events  $\hat{F}_{\rho,L}$  are disjoint,

$$\mathbf{P}(F_M) \leq \frac{1}{m} + \frac{1}{m} \sum_{L \in \mathcal{S}_{M,m}} \mathbf{P}(\hat{F}_{\rho,L}) \leq \frac{1}{m} + \frac{1}{m} = \frac{2}{m}.$$

which tends to 0 as  $m \rightarrow \infty$ , and ends the proof of the Theorem.  $\square$

Before proceeding with the proofs of Propositions 4.12 and 4.13, we discuss the consequences of weakening Part 2 of Hypothesis 4.1. If we assume only that

$$Q(x) \leq x^{-2-\theta}, \quad (4.43)$$

can we find values of  $\theta \leq 1$  such that we still get superdiffusive behavior for the polymer, i.e.  $\alpha > 1/2$ ? Since the result of the Girsanov theorem, Proposition 4.11, is not effected by the value of  $\theta$  above, this means that the penalization from Girsanov's theorem, of order  $t^{2\alpha-1}$ , cannot be made smaller by a different choice of decorrelation speed in  $Q$ . Therefore we should expect not to be able to preserve the threshold  $\alpha < 3/5$ . To see exactly what happens to this threshold under condition (4.43), we first state, and leave it to the reader to check, that we can rework the proof of Proposition 4.9 item (iii) to obtain instead

$$|\delta|_{\tau,k} - \delta_k = o(t^{\alpha\theta}).$$

It is then simple to check that (4.38) becomes

$$\mathbf{P}(\hat{\mathcal{B}}_{M,t}^c \cap \mathcal{B}_{M,t}) \leq \mathbf{P}\left(\text{For any } k \in \mathbb{Z}_M; \frac{\tilde{Z}_t^\alpha(k)}{\tilde{Z}_t^\alpha(0)} < \exp(2\gamma t^{3\alpha-1-\theta} + \check{\eta}_0 - \hat{\eta}_k)\right).$$

Hence the application of Proposition 4.11 still works, but we can no longer make the corresponding Girsanov penalization of the same order, since for  $\theta < 1$ ,  $3\alpha - 1 - \alpha\theta > 2\alpha - 1$ . Having thus convinced ourselves that Part 2 of Hypothesis 4.1 is the only way to get the entire proof to be efficient in terms of using comparable penalizations throughout, we can now ignore this inefficiency, and answer the question at the beginning of this paragraph. The reader will check that any other occurrences of the use of Hypothesis 4.1 are not further effected by switching (4.43): the entire proof can still be used if we only require that the magnitude of the  $\eta_k$ 's, namely  $t^{(1-\alpha)/2}$ , is larger than the new penalization  $t^{3\alpha-1-\alpha\theta}$ . This yields

$$\alpha < \frac{3}{7-2\theta}.$$

We have already argued that  $\theta$  should be no greater than 1 (or else we revert to the original Hypothesis 4.1), but now we see that to get a super-diffusive behavior, we need

$3/(7-2\theta) > 1/2$ , i.e.  $\theta > 1/2$ . We also see that the weakest hypothesis required for such behavior is  $Q(x) \leq x^{-5/2-\vartheta}$  for  $\vartheta > 0$ . We state these findings formally, using the reparametrization  $\theta = \vartheta + 1/2$ .

**Corollary 4.14.** *Assume instead of Part 2 of Hypothesis 4.1 that there exists  $\vartheta \in (0, 1/2]$  such that as  $|x| \rightarrow \infty$ ,*

$$Q(x) = O\left(|x|^{-5/2-\vartheta}\right).$$

*Then for any  $\varepsilon > 0$  we obtain the following specific super-diffusive behavior for the polymer measure:*

$$\lim_{t \rightarrow \infty} \mathbf{P} \left[ \left\langle \sup_{s \leq t} |b_s| \right\rangle_t \geq t^{\frac{1}{2} + \frac{\vartheta}{6-2\vartheta} - \varepsilon} \right] = 1.$$

### 4.3 Proof of Proposition 4.12

Let  $L \in \bar{\mathcal{S}}_{M,m}$ . Then, by definition (4.39) of  $\bar{\mathcal{S}}_{M,m}$ , there exists  $k \geq 1$  such that  $k\mathbb{Z}_m \subset L$ . Then

$$\begin{aligned} & \mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in L \right) \\ & \leq \mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in k\mathbb{Z}_m \right). \end{aligned}$$

It is thus sufficient to estimate the right hand side in the above inequality.

Given an even integer  $m \leq M$ , recall that  $\bar{\mathbb{Z}}_m$  has been defined at (4.28). Set also  $\hat{m} = m/2$ , and for each  $i \in k\bar{\mathbb{Z}}_{\hat{m}}$ , we associate the following event:

$$\Omega^{(i)} \equiv \left\{ \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W^{i,t}) \text{ for all } \ell \in k\bar{\mathbb{Z}}_{\hat{m}} \setminus \{i\} \right\}.$$

Then these events are disjoint, and since  $|k\bar{\mathbb{Z}}_{\hat{m}}| = 2\hat{m} + 1$ , we get trivially the existence of  $i_0 \in k\bar{\mathbb{Z}}_{\hat{m}}$  such that

$$\mathbf{P}(\Omega^{(i_0)}) \leq \frac{1}{2\hat{m} + 1} \leq \frac{1}{m}. \quad (4.44)$$

However, the translation invariance of the environment  $W$  yields

$$\begin{aligned} & \mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W) \text{ for all } \ell \in k\mathbb{Z}_m \right) \\ & = \mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell+i_0,t}) < \tilde{Z}_t^\alpha(0, W^{i_0,t}) \text{ for all } \ell \in k\mathbb{Z}_m \right) \\ & \leq \mathbf{P} \left( \tilde{Z}_t^\alpha(0, W^{\ell,t}) < \tilde{Z}_t^\alpha(0, W^{i_0,t}) \text{ for all } \ell \in k\bar{\mathbb{Z}}_{\hat{m}} \setminus \{i_0\} \right) \\ & = \mathbf{P}(\Omega^{(i_0)}). \end{aligned} \quad (4.45)$$

Observe that the last inequality is just due to the fact that  $k\bar{\mathbb{Z}}_{\hat{m}} \setminus \{i_0\} \subset i_0 + k\mathbb{Z}_m$  whenever  $i_0 \in k\bar{\mathbb{Z}}_{\hat{m}}$ , a fact that can simply be proved as follows: if  $i' \in k\bar{\mathbb{Z}}_{\hat{m}} \setminus \{i_0\}$  then for some  $j$  such that  $|j| \leq m/2$ ,  $i' = jk$ ; similarly,  $i_0 = kj_0$ , and  $j_0 \neq j$  and  $|j_0| \leq m/2$  as well; therefore  $i' = i_0 + (j - j_0)k$  where  $j - j_0 \neq 0$  and  $|j - j_0| \leq |j| + |j_0| \leq m$ ; this does then indeed mean that  $i' \in i_0 + k\mathbb{Z}_m$ . Hence, putting together (4.44) and (4.45), we get the announced result.  $\square$

#### 4.4 Proof of Proposition 4.13

Recall that  $\hat{F}_{M,m,\rho}$  is defined by (4.40), and define the quantity

$$\tau(t) = \beta^{-1} t^{\frac{5}{2}(\alpha - \frac{3}{5}) + \rho},$$

which tends to 0 as  $t \rightarrow \infty$  if  $\alpha < \frac{3}{5}$  and  $\rho$  is small enough. The following inequality

$$\mathbf{P}(\hat{F}_{M,m,\rho}) \geq \mathbf{P}\left(\bigcup_{L \in \bar{\mathcal{S}}_{M,m}} \{t^{(\alpha-1)/2}(\check{\eta}_0 - \hat{\eta}_\ell) \leq -\beta\tau(t) \text{ for all } \ell \in L\}\right) \quad (4.46)$$

is then easily established by an elementary inclusion argument, which we detail here. Indeed, assume that for some  $L \in \bar{\mathcal{S}}_{M,m}$ , for all  $\ell \in L$ ,  $\eta$  satisfies

$$t^{(\alpha-1)/2}(\check{\eta}_0 - \hat{\eta}_\ell) \leq -\beta\tau(t)$$

which is equivalent to

$$\check{\eta}_0 - \hat{\eta}_\ell \leq -t^{2\alpha-1+\rho}$$

To justify the above inequality, we only need to prove that for some other  $L' \in \bar{\mathcal{S}}_{M,m}$ , the same  $\eta$  also satisfies the above inequality for all  $\ell \in L'$ , while for all  $\ell \in \mathbb{Z}_M \setminus L'$ , the contrary holds, namely

$$\check{\eta}_0 - \hat{\eta}_\ell > -t^{2\alpha-1+\rho}.$$

Let then  $\Lambda$  be the subset of  $\mathbb{Z}_M$  defined by

$$\Lambda = \{\ell \in \mathbb{Z}_M; \check{\eta}_0 - \hat{\eta}_\ell > -t^{2\alpha-1+\rho}\},$$

and set  $L' = \mathbb{Z}_M \setminus \Lambda$ . Then, by construction  $L'$  has the required properties defined above, and since  $L' \supset L$ , by definition of  $\bar{\mathcal{S}}_{M,m}$ , we have  $L' \in \bar{\mathcal{S}}_{M,m}$ .

In order to get a lower bound on the right hand side of (4.46) above, we will construct now a large enough collection of symmetric and disjoint sets in  $\mathbb{Z}_M$ : with  $m < M$ , consider the collection  $\{P_q(m)\mathbb{Z}_m; q < q^*\}$ , where the integers  $P_q(m)$  are defined by

$$P_1(m) = 1, \quad P_{q+1}(m) = mP_q(m) + 1, \quad q^* = \inf\{q; P_q(m) > M\}.$$

This collection is the sequence

$$\mathbb{Z}_m, (m+1)\mathbb{Z}_m, [m(m+1)+1]\mathbb{Z}_m, \dots, P_q(m)\mathbb{Z}_m, \dots, P_{q^*-1}(m)\mathbb{Z}_m,$$

which are non-overlapping annuli in  $\mathbb{Z}_M$ , and therefore are indeed symmetric and disjoint subsets of  $\mathbb{Z}_M$ . Since  $P_q(m)\mathbb{Z}_m$  is certainly of the form  $k\mathbb{Z}_{\hat{k}}$  with  $k \geq 1$  and  $\hat{k} \geq m$ , and is a subset of  $\mathbb{Z}_M$  as soon as  $q < q^*$ , by definition  $P_q(m)\mathbb{Z}_m \in \bar{\mathcal{S}}_{M,m}$ . Thus, using the notation  $\check{\eta}_0, \hat{\eta}_\ell$  and  $\eta_\ell$  defined in (4.35) and (4.36), we get

$$\mathbf{P}(\hat{F}_{M,m,\rho}) \geq \mathbf{P}\left(\bigcup_{q < q^*} \left\{ \max(\underline{d}\check{\eta}_0, \bar{d}\check{\eta}_0) - \min(\underline{d}\hat{\eta}_\ell, \bar{d}\hat{\eta}_\ell) \leq -\tau(t) \text{ for all } \ell \in P_q(m)\mathbb{Z}_m \right\}\right).$$

Let us call now  $A_\ell$  the event

$$A_\ell = \{ \max(\underline{d}\tilde{\eta}_0, \bar{d}\tilde{\eta}_0) - \min(\underline{d}\tilde{\eta}_\ell, \bar{d}\tilde{\eta}_\ell) \leq -\tau(t) \},$$

and we distinguish two cases according to the values of  $\tilde{\eta}_0$ :

(a) If  $\tilde{\eta}_0 \geq 0$ , then  $\max(\underline{d}\tilde{\eta}_0, \bar{d}\tilde{\eta}_0) = \bar{d}\tilde{\eta}_0$ , and hence  $A_\ell$  is the event defined by the relation

$$\min(\underline{d}\tilde{\eta}_\ell, \bar{d}\tilde{\eta}_\ell) \geq \tau(t) + \bar{d}\tilde{\eta}_0.$$

In particular,  $\tilde{\eta}_\ell$  has to be positive, and thus  $A_\ell$  can be written as

$$\{ \bar{d}\tilde{\eta}_0 - \underline{d}\tilde{\eta}_\ell < -\tau(t) \}.$$

(b) If  $\tilde{\eta}_0 \leq -\tau(t)/\underline{d} \leq 0$ , then  $\max(\underline{d}\tilde{\eta}_0, \bar{d}\tilde{\eta}_0) = \underline{d}\tilde{\eta}_0$ . Thus  $A_\ell$  can be written as the event defined by the relation

$$\min(\underline{d}\tilde{\eta}_\ell, \bar{d}\tilde{\eta}_\ell) \geq \tau(t) + \underline{d}\tilde{\eta}_0, \quad (4.47)$$

and if  $\tilde{\eta}_0 \leq -\tau(t)/\underline{d}$ , the quantity  $\tau(t) + \underline{d}\tilde{\eta}_0$  is negative. Hence, (4.47) is implied by  $\tilde{\eta}_\ell \geq 0$ .

Summarizing the considerations above, we get

$$\mathbf{P}(\hat{F}_{M,m,\rho}) \geq \mathbf{P}(D^+) + \mathbf{P}(D^-),$$

with

$$D^+ = \bigcup_{q < q^*} \{ \bar{d}\tilde{\eta}_0 - \underline{d}\tilde{\eta}_\ell \leq -\tau(t) \text{ for all } \ell \in P_q(m)\mathbb{Z}_m \} \cap \{ \tilde{\eta}_0 > 0 \}$$

$$D^- = \bigcup_{q < q^*} \{ \tilde{\eta}_\ell \geq 0 \text{ for all } \ell \in P_q(m)\mathbb{Z}_m \} \cap \{ \tilde{\eta}_0 \leq -\tau(t)/\underline{d} \}.$$

We will now prove that  $\mathbf{P}(D^+)$  is close to  $1/2$ . Entirely similar arguments, left to the reader, lead to showing that  $\mathbf{P}(D^-)$  can also be made arbitrarily close to  $1/2$ , concluding the proof of the proposition.

Observe that, according to Proposition 4.6 the random variables  $\{\tilde{\eta}_\ell; l \in \bar{\mathbb{Z}}_M\}$  converge in distribution to a family of independent standard Gaussian random variables  $\{\Upsilon_\ell; l \in \bar{\mathbb{Z}}_M\}$ . Consequently, and using the fact that  $-\tau(t) \rightarrow 0$  as  $t \rightarrow \infty$ ,

$$\mathbf{P}(D^+) = \mathbf{P} \left( \bigcup_{q < q^*} \{ \bar{d}\Upsilon_0 - \underline{d}\Upsilon_\ell \leq 0 \text{ for all } \ell \in P_q(m)\mathbb{Z}_m \} \cap \{ \Upsilon_0 > 0 \} \right) + \varepsilon_M(t),$$

where, for a fixed  $M \in \mathbb{N}$ , we have  $\lim_{t \rightarrow \infty} \varepsilon_M(t) = 0$ . Furthermore, since the  $\Upsilon_\ell$  are independent random variables, we get

$$\begin{aligned} \mathbf{P}(D^+) &= \int_0^\infty \mathbf{P} \left( \bigcup_{q < q^*} \{ \bar{d}x - \underline{d}\Upsilon_\ell \leq 0 \text{ for all } \ell \in P_q(m)\mathbb{Z}_m \} \right) \frac{e^{-\frac{x^2}{2}}}{(2\pi)^{1/2}} dx + \varepsilon_M(t) \\ &= \frac{1}{2} - \int_0^\infty \mathbf{P} \left( \bigcap_{q < q^*} \hat{D}_q \right) \frac{e^{-\frac{x^2}{2}}}{(2\pi)^{1/2}} dx + \varepsilon_M(t), \end{aligned} \quad (4.48)$$

where

$$\hat{D}_q = \{ \text{There exists } \ell \in P_q(m)\mathbb{Z}_m; \bar{d}x - \underline{d}\Upsilon_\ell \geq 0 \}.$$

In order to take advantage of the independence of the  $\Upsilon_\ell$ , it is convenient to pick some disjoint sets out of  $\mathbb{Z}_M$ , which explains the choice of disjoint subsets  $P_q(m)\mathbb{Z}_m$ . Now, it is easily seen that, for a fixed value  $q_0$ , if one desires to have  $q^* > q_0$ , it is sufficient to take  $M$  of order  $m^{q_0}$ . Let us assume that we are in this situation; this means that, setting  $\kappa = \bar{d}/\underline{d}$ , we have

$$\begin{aligned} \mathbf{P}\left(\bigcap_{q < q^*} \hat{D}_q\right) &\leq \mathbf{P}\left(\bigcap_{q \leq q_0} \{ \text{There exists } \ell \in P_q(m)\mathbb{Z}_m; \Upsilon_\ell \leq \kappa x \} \right) \\ &= \mathbf{P}^{q_0}(\text{There exists } \ell \in \mathbb{Z}_m; \Upsilon_\ell \leq \kappa x) \\ &= [1 - \mathbf{P}^{2m}(\Upsilon_1 \geq \kappa x)]^{q_0}. \end{aligned}$$

Plugging these inequalities into (4.48), we obtain

$$\mathbf{P}(D^+) \geq \frac{1}{2} - \int_0^\infty [1 - \mathbf{P}^{2m}(\Upsilon_1 \geq \kappa x)]^{q_0} \frac{e^{-\frac{x^2}{2}}}{(2\pi)^{1/2}} dx + \varepsilon_M(t).$$

Recall that the functions  $\Phi$  has been defined by relation (4.32). Then the last inequality yields,

$$\mathbf{P}(D^+) \geq \frac{1}{2} - \int_0^\infty [1 - \Phi(\kappa x)^{2m}]^{q_0} \frac{e^{-\frac{x^2}{2}}}{(2\pi)^{1/2}} dx + \varepsilon_M(t).$$

It is now easily seen that this probability can be made as close as we wish to  $\frac{1}{2}$  by taking  $q_0 \rightarrow \infty$ , because  $0 < \Phi(x) < 1$  for all  $x \in \mathbb{R}$ , this asymptotic being equivalent to  $M \rightarrow \infty$ .  $\square$

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