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# The quaternion group as a subgroup of the sphere braid groups

DACIBERG LIMA GONÇALVES

Departamento de Matemática - IME-USP,  
Caixa Postal 66281 - Ag. Cidade de São Paulo,  
CEP: 05311-970 - São Paulo - SP - Brazil.  
e-mail: [dlgoncal@ime.usp.br](mailto:dlgoncal@ime.usp.br)

JOHN GUASCHI

Laboratoire de Mathématiques Emile Picard, UMR CNRS 5580,  
UFR-MIG, Université Toulouse III, 31062 Toulouse Cedex 9, France.  
e-mail: [guaschi@picard.ups-tlse.fr](mailto:guaschi@picard.ups-tlse.fr)

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## Abstract

*Let  $n \geq 3$ . We prove that the quaternion group of order 8 is realised as a subgroup of the sphere braid group  $B_n(\mathbb{S}^2)$  if and only if  $n$  is even. If  $n$  is divisible by 4 then the commutator subgroup of  $B_n(\mathbb{S}^2)$  contains such a subgroup. Further, for all  $n \geq 3$ ,  $B_n(\mathbb{S}^2)$  contains a subgroup isomorphic to the dicyclic group of order  $4n$ .*

The braid groups  $B_n$  of the plane were introduced by E. Artin in 1925 [A1, A2], and were generalised by Fox to braid groups of arbitrary topological spaces using the notion of configuration space [FoN]. Van Buskirk showed that the braid groups of a compact connected surface  $M$  possess torsion elements if and only if  $M$  is the sphere  $\mathbb{S}^2$  or the real projective plane  $\mathbb{R}P^2$  [VB]. Let us recall briefly some of the properties of the braid groups of the sphere [FVB, GVB, VB].

If  $\mathbb{D}^2 \subseteq \mathbb{S}^2$  is a topological disc, there is a group homomorphism  $\iota: B_n \rightarrow B_n(\mathbb{S}^2)$  induced by the inclusion. If  $\beta \in B_n$  then we shall denote its image  $\iota(\beta)$  simply by  $\beta$ . Then  $B_n(\mathbb{S}^2)$  is generated by  $\sigma_1, \dots, \sigma_{n-1}$  which are subject to the following relations:

$$\begin{aligned}\sigma_i \sigma_j &= \sigma_j \sigma_i \text{ if } |i - j| \geq 2 \text{ and } 1 \leq i, j \leq n - 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } 1 \leq i \leq n - 2, \text{ and} \\ \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2 \sigma_{n-2} \cdots \sigma_1 &= 1.\end{aligned}$$

Consequently,  $B_n(\mathbb{S}^2)$  is a quotient of  $B_n$ . The first three sphere braid groups are finite:  $B_1(\mathbb{S}^2)$  is trivial,  $B_2(\mathbb{S}^2)$  is cyclic of order 2, and  $B_3(\mathbb{S}^2)$  is a ZS-metacyclic group (a group

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whose Sylow subgroups, commutator subgroup and commutator quotient group are all cyclic) of order 12. The Abelianisation of  $B_n(\mathbb{S}^2)$  is isomorphic to the cyclic group  $\mathbb{Z}_{2(n-1)}$ . The kernel of the associated projection  $\xi: B_n(\mathbb{S}^2) \rightarrow \mathbb{Z}_{2(n-1)}$  (which is defined by  $\xi(\sigma_i) = \bar{1}$  for all  $1 \leq i \leq n-1$ ) is the commutator subgroup  $\Gamma_2(B_n(\mathbb{S}^2))$ . If  $w \in B_n(\mathbb{S}^2)$  then  $\xi(w)$  is the exponent sum (relative to the  $\sigma_i$ ) of  $w$  modulo  $2(n-1)$ .

The torsion elements of the braid groups of  $\mathbb{S}^2$  and  $\mathbb{R}P^2$  were classified by Murasugi [M]: if  $M = \mathbb{S}^2$  and  $n \geq 3$ , they are all conjugates of powers of the three elements  $\alpha_0 = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}$  (which is of order  $2n$ ),  $\alpha_1 = \sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}^2$  (of order  $2(n-1)$ ) and  $\alpha_2 = \sigma_1 \cdots \sigma_{n-3} \sigma_{n-2}^2$  (of order  $2(n-2)$ ) which are respectively  $n^{\text{th}}$ ,  $(n-1)^{\text{th}}$  and  $(n-2)^{\text{th}}$  roots of  $T_n$ , where  $T_n$  is the so-called ‘full twist’ of  $B_n(\mathbb{S}^2)$ , defined by  $T_n = (\sigma_1 \cdots \sigma_{n-1})^n$ . If  $n \geq 3$ ,  $T_n$  is the unique element of  $B_n(\mathbb{S}^2)$  of order 2 and generates the centre of  $B_n(\mathbb{S}^2)$ . In [GG2], we showed that  $B_n(\mathbb{S}^2)$  is generated by  $\alpha_0$  and  $\alpha_1$ .

For  $n \geq 4$ ,  $B_n(\mathbb{S}^2)$  is infinite. It is an interesting question as to which finite groups are realised as subgroups of  $B_n(\mathbb{S}^2)$  (apart of course from the cyclic groups  $\langle \alpha_i \rangle$ ). In [GG2], we proved that  $B_n(\mathbb{S}^2)$  contains an isomorphic copy of the finite group  $B_3(\mathbb{S}^2)$  of order 12 if and only if  $n \not\equiv 1 \pmod{3}$ . The quaternion group  $\mathcal{Q}_8$  of order 8 appears in the study of braid groups of non-orientable surfaces, being isomorphic to the 2-string pure braid group  $P_2(\mathbb{R}P^2)$ . Further, since the projection  $F_3(\mathbb{R}P^2) \rightarrow F_2(\mathbb{R}P^2)$  of configuration spaces of  $\mathbb{R}P^2$  onto the first two coordinates admits a section [VB], it follows using the Fadell-Neuwirth short exact sequence that  $P_3(\mathbb{R}P^2)$  is a semi-direct product of a free group of rank 2 by  $\mathcal{Q}_8$  [GG1].

While studying the lower central and derived series of the sphere braid groups, we showed that  $\Gamma_2(B_4(\mathbb{S}^2))$  is isomorphic to a semi-direct product of  $\mathcal{Q}_8$  by a free group of rank 2 [GG3]. After having proved this result, we noticed that the question of the realisation of  $\mathcal{Q}_8$  as a subgroup of  $B_n(\mathbb{S}^2)$  was explicitly posed by R. Brown in connection with the fact that the fundamental group of  $\text{SO}(3)$  is isomorphic to  $\mathbb{Z}_2$  [ATD]. In this paper, we give a complete answer to this question:

**THEOREM.** *Let  $n \in \mathbb{N}$ ,  $n \geq 3$ .*

- (a) *If  $n$  is a multiple of 4 then  $\Gamma_2(B_n(\mathbb{S}^2))$  contains a subgroup isomorphic to  $\mathcal{Q}_8$ .*
- (b) *If  $n$  is an odd multiple of 2 then  $B_n(\mathbb{S}^2)$  contains a subgroup isomorphic to  $\mathcal{Q}_8$ .*
- (c) *If  $n$  is odd then  $B_n(\mathbb{S}^2)$  contains no subgroup isomorphic to  $\mathcal{Q}_8$ .*

**PROOF.** We first suppose that  $n$  is even, so that  $n = 2m$  with  $m \in \mathbb{N}$ . Let  $H$  be the subgroup of  $B_{2m}(\mathbb{S}^2)$  generated by  $x$  and  $y$ , where

$$\begin{aligned} x &= (\sigma_1 \cdots \sigma_{2m-1})(\sigma_1 \cdots \sigma_{2m-2}) \cdots (\sigma_1 \sigma_2) \sigma_1, \\ y &= (\sigma_1 \cdots \sigma_{m-1})(\sigma_1 \cdots \sigma_{m-2}) \cdots (\sigma_1 \sigma_2) \sigma_1 \cdot \sigma_{2m-1}^{-1} (\sigma_{2m-2}^{-1} \sigma_{2m-1}^{-1}) \cdots \\ &\quad \cdots (\sigma_{m+2}^{-1} \cdots \sigma_{2m-1}^{-1}) (\sigma_{m+1}^{-1} \cdots \sigma_{2m-1}^{-1}). \end{aligned}$$

Geometrically,  $x$  may be interpreted as the ‘half twist’ or Garside element of  $B_{2m}$  [Bi]. Further,  $y$  may be considered as the commuting product of the positive half twist of the first  $m$  strings with the negative half twist of the last  $m$  strings. Then  $x^2 = T_{2m}$  and  $y^2 = (\sigma_1 \cdots \sigma_{m-1})^m (\sigma_{m+1}^{-1} \cdots \sigma_{2m-1}^{-1})^m = T_{2m}$  in  $B_{2m}(\mathbb{S}^2)$  (cf. [FVB, GVB]). It is well known that  $x \sigma_i x^{-1} = \sigma_{2m-i}$  in  $B_{2m}$  [Bi], and thus in  $B_{2m}(\mathbb{S}^2)$ , from which we obtain  $xyx^{-1} = y^{-1}$ . Hence  $H$  is isomorphic to a quotient of  $\mathcal{Q}_8$ . But  $x$  is of order 4, and the induced permutation of  $y$  on the symmetric group  $S_{2m}$  is different from that of the elements of  $\langle x \rangle$ . It follows that  $H$  contains the five distinct elements of  $\langle x \rangle \cup \{y\}$ , and

so  $H \cong \mathcal{Q}_8$ . If  $m$  is even then  $x, y \in \text{Ker}(\xi)$ , and thus  $H \subseteq \Gamma_2(B_n(\mathbb{S}^2))$ . This proves parts (a) and (b).

To prove part (c), suppose that  $n$  is odd, and suppose that  $x, y \in B_n(\mathbb{S}^2)$  generate a subgroup  $H$  isomorphic to  $\mathcal{Q}_8$ , so  $x^2 = y^2$  and  $xyx^{-1} = y^{-1}$ . In particular,  $x$  and  $y$  are of order 4, and thus are conjugates of  $\alpha_1^{\pm(n-1)/2}$  by Murasugi's classification. By considering a conjugate of  $H$  if necessary, we may suppose that  $x = \alpha_1^{\varepsilon_1(n-1)/2}$  and  $y = w\alpha_1^{\varepsilon_2(n-1)/2}w^{-1}$ , where  $w \in B_n(\mathbb{S}^2)$  and  $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ . Replacing  $x$  by  $x^{-1}$  if necessary, we may suppose further that  $\varepsilon_1 = -\varepsilon_2$ . Thus  $xy = [\alpha_1^{\varepsilon_1(n-1)/2}, w]$ , and is of exponent sum zero. On the other hand,  $xy$  is an element of  $H$  of order 4, and so is conjugate to  $\alpha_1^{\pm(n-1)/2}$  by Murasugi's classification. But  $\xi(\alpha_1^{\pm(n-1)/2}) = \pm \frac{n(n-1)}{2}$ , which is non zero modulo  $2(n-1)$ . This yields a contradiction, and proves part (c).  $\square$

REMARK. Let  $n \geq 3$ . Using techniques similar to those of the proof of the Theorem, one may show that the subgroup of  $B_n(\mathbb{S}^2)$  generated by  $\sigma_1 \cdots \sigma_{n-1}$  and the half twist  $x$  is isomorphic to the dicyclic group of order  $4n$ . In particular, if  $n$  is a power of two then  $B_n(\mathbb{S}^2)$  contains a subgroup isomorphic to the generalised quaternion group of order  $4n$ . Further investigation into the finite subgroups of  $B_n(\mathbb{S}^2)$  and  $B_n(\mathbb{R}P^2)$  will appear elsewhere.

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