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The Schrödinger-Virasoro Lie group and algebra : from geometry to representation theory.

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Abstract

This article is concerned with an extensive study of a infinite-dimensional Lie algebra \mathfrak{sv} , introduced in [14] in the context of non-equilibrium statistical physics, containing as subalgebras both the Lie algebra of invariance of the free Schrödinger equation and the central charge-free Virasoro algebra $\text{Vect}(S^1)$. We call \mathfrak{sv} the *Schrödinger-Virasoro Lie algebra*. We choose to present \mathfrak{sv} from a Newtonian geometry point of view first, and then in connection with conformal and Poisson geometry. We turn afterwards to its representation theory: realizations as Lie symmetries of field equations, coadjoint representation, coinduced representations in connection with Cartan's prolongation method (yielding analogues of the tensor density modules for $\text{Vect}(S^1)$), and finally Verma modules with a Kac determinant formula. We also present a detailed cohomological study, providing in particular a classification of deformations and central extensions; there appears a non-local cocycle.

in memory of Daniel Arnaudon

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0 Introduction

There is, in the physical literature of the past decades - without mentioning the pioneering works of Wigner for instance -, a deeply rooted belief that physical systems - macroscopic systems for statistical physicists, quantum particles and fields for high energy physicists - could and should be classified according to which group of symmetries act on them and how this group acts on them.

Let us just point at two very well-known examples : elementary particles on the $(3 + 1)$ -dimensional Minkowski space-time, and two-dimensional conformal field theory.

From the point of view of 'covariant quantization', introduced at the time of Wigner, elementary particles of relativistic quantum mechanics (of positive mass, say) may be described as irreducible unitary representations of the Poincaré group $\mathfrak{p}_4 \simeq \mathfrak{so}(3, 1) \ltimes \mathbb{R}^4$, which is the semi-direct product of the Lorentz group of rotations and relativistic boosts by space-time translations : that is to say, the physical states of a particle of mass $m > 0$ and spin $s \in \frac{1}{2}\mathbb{N}$ are in bijection with the states of the Hilbert space corresponding to the associated irreducible representation of \mathfrak{p}_4 ; the indices (m, s) characterizing positive square mass representations come from the two Casimir of the enveloping algebra $\mathcal{U}(\mathfrak{p}_4)$.

This 'covariant quantization' has been revisited by the school of Souriau in the 60'es and 70'es as a particular case of geometric quantization; most importantly for us, the physicists J.-J. Lévy-Leblond and C. Duval introduced the so-called *Newton-Cartan* manifolds (which provide the right geometric frame for Newtonian mechanics, just as Lorentz manifolds do for relativistic mechanics) and applied the tools of geometric quantization to construct wave equations in a geometric context.

Two-dimensional conformal field theory is an attempt at understanding the universal behaviour of two-dimensional statistical systems at equilibrium and at the critical temperature, where they undergo a second-order phase transition. Starting from the basic assumption of translational and rotational invariance, together with the fundamental hypothesis (confirmed by the observation of the fractal structure of the systems and the existence of long-range correlations, and made into a cornerstone of renormalization-group theory) that scale invariance holds at criticality, one is³ naturally led to the idea that invariance under the whole conformal group $\text{Conf}(d)$ should also hold. This group is known to be finite-dimensional as soon as the space dimension d is larger than or equal to three, so physicists became very interested in dimension $d = 2$, where local conformal transformations are given by holomorphic or anti-holomorphic functions. A systematic investigation of the theory of representations of the Virasoro algebra (considered as a central extension of the algebra of infinitesimal holomorphic transformations) in the 80'es led to introduce a class of physical models (called *unitary minimal models*), corresponding to the unitary highest weight representations of the Virasoro algebra with central charge less than one. Miraculously, covariance alone is enough to allow the computation of the statistic correlators - or so-called ' n -point functions' - for these highly constrained models.

We shall give here a tentative mathematical foundation (though very sketchy at present time, and not pretending to have physical applications!) to closely related ideas, developed since the mid-nineties (see short survey [17]), and applied to two related fields : strongly anisotropic critical systems and out-of-equilibrium statistical physics (notably ageing phenomena). Theoretical studies and numerical models coming from both fields have been developed, in which invariance under space rotations and *anisotropic dilations* $(t, r) \rightarrow (e^{\lambda z}t, e^{\lambda}r)$ ($\lambda \in \mathbb{R}$) plays a central rôle. Here $r \in \mathbb{R}^d$ is considered as a space coordinate and $t \in \mathbb{R}$ is (depending on the context) either the time coordinate or an extra (longitudinal, say) space coordinate; the parameter $z \neq 1$ is called the *anisotropy* or *dynamical exponent*.

³for systems with sufficiently short-ranged interactions

We shall here restrict (at least most of the time) to the value $z = 2$. Then the simplest wave equation invariant under translations, rotations *and* anisotropic dilations is the free Schrödinger equation $2\mathcal{M}\partial_t\psi = \Delta_d\psi$, where $\Delta_d := \sum_{i=1}^d \partial_{r_i}^2$ is the Laplacian in spatial coordinates. So it is natural to believe that this equation should play the same rôle as the Klein-Gordon equation in the study of relativistic quantum particles, or the Laplace equation in conformal field theory, whose maximal group of Lie symmetries is the conformal group; in other words, one may also say that we are looking for symmetry groups arising naturally in a non-relativistic setting, while hoping that their representations might be applied to a classification of non-relativistic systems, or, more or less equivalently, to ($z = 2$) anisotropic systems.

This program, as we mentioned earlier in this introduction, was partially carried out by Duval, Künzle and others through the 70'es and 80'es (see for instance [Duv1,Duv2,Duv3,Duv4]). We shall discuss it briefly in the first part and see how the maximal group of Lie symmetries of the Schrödinger group, Sch_d , called the *Schrödinger group*, appears in some sense as the natural substitute for the conformal group in Newtonian mechanics. Unfortunately, it is finite-dimensional for every value of d , and its unitary irreducible representations are well-understood and classified (see [27]), giving very interesting though partial informations on two- and three-point functions in anisotropic and out-of-equilibrium statistical physics at criticality that have been systematically pursued in the past ten years of so (see [31, 32, 33, 15, 16, 2]) but relying on rather elementary mathematics, so this story could well have stopped here short of further arguments.

Contrary to the conformal group though, which corresponds to a rather 'rigid' riemannian or Lorentzian geometry, the Schrödinger group is only one of the groups of symmetries that come out of the much more 'flexible' Newtonian geometry, with its loosely related time and space directions. In particular (restricting here to one space dimension for simplicity, although there are straightforward generalizations in higher dimension), there arises a new group SV, which will be our main object of study, and that we shall call the *Schrödinger-Virasoro* group for reasons that will become clear shortly. Its Lie algebra \mathfrak{sv} was originally introduced by M. Henkel in 1994 (see [14]) as a by-product of the computation of n -point functions that are covariant under the action of the Schrödinger group. It is given abstractly as

$$\mathfrak{sv} = \langle L_n \rangle_{n \in \mathbb{Z}} \oplus \langle Y_m \rangle_{m \in \frac{1}{2} + \mathbb{Z}} \oplus \langle M_p \rangle_{p \in \mathbb{Z}} \quad (0.1)$$

with relations

$$[L_n, L_p] = (n - p)L_{n+p} \quad (0.2)$$

$$[L_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m}, \quad [L_n, M_p] = -pM_p \quad (0.3)$$

$$[Y_m, Y_{m'}] = (m - m')M_{m+m'}, \quad [Y_m, M_p] = 0, \quad [M_n, M_p] = 0 \quad (0.4)$$

($n, p \in \mathbb{Z}, m, m' \in \frac{1}{2} + \mathbb{Z}$). Denoting by $\text{Vect}(S^1) = \langle L_n \rangle_{n \in \mathbb{Z}}$ the Lie algebra of vector fields on the circle (with brackets $[L_n, L_p] = (n - p)L_{n+p}$), \mathfrak{sv} may be viewed as a semi-direct product $\mathfrak{sv} \simeq \text{Vect}(S^1) \ltimes \mathfrak{h}$, where $\mathfrak{h} = \langle Y_m \rangle_{m \in \frac{1}{2} + \mathbb{Z}} \oplus \langle M_p \rangle_{p \in \mathbb{Z}}$ is a two-step nilpotent Lie algebra, isomorphic to $\mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_0$ as a $\text{Vect}(S^1)$ -module (see Definition 1.3 for notations).

This articles aims at motivating the introduction of the \mathfrak{sv} -algebra and related objects, and at studying them from a mathematical point of view. It is not an easy task to choose the best order of exposition since, as usual in mathematics, the best motivation for introducing a new object is often provided by what can be done with it, and also because the \mathfrak{sv} algebra actually appears in many contexts (not only in the geometric approach chosen in this introduction), and it is actually a matter of taste to decide which definition is most fundamental. Generally speaking, though, we shall be more concerned

with giving various definitions of \mathfrak{sv} in the two or three first parts, and with the study of \mathfrak{sv} proper and its representations in the rest of the article.

Here is the plan of the article.

Chapter 1 will be devoted to a geometric introduction to \mathfrak{sv} and SV in the frame of Newton-Cartan manifolds, as promised earlier in this introduction.

In Chapter 2, we shall prove that $\mathfrak{sch}^d = Lie(\text{Sch}^d)$ appears as a real subalgebra of $\mathfrak{conf}(d+2)_{\mathbb{C}}$, extending results contained in a previous article of M. Henkel and one of the authors (see [31]). We shall also give a 'no-go' theorem proving that this embedding cannot be extended to \mathfrak{sv} .

In Chapter 3, we shall decompose \mathfrak{sv} as a sum of tensor density modules for $\text{Vect}(S^1)$. Introducing its central extension $\widehat{\mathfrak{sv}} \simeq \text{Vir} \ltimes \mathfrak{h}$ which contains both the Virasoro algebra and the Schrödinger algebra (hence its name!), we shall study its coadjoint action on its regular dual $(\widehat{\mathfrak{sv}})^*$. We shall also study the action of \mathfrak{sv} on a certain space of Schrödinger-type operators and on some other spaces of operators related to field equations.

In Chapter 4, we shall see that 'half of \mathfrak{sv} ' can be interpreted as a Cartan prolongation $\bigoplus_{k=-1}^{\infty} \mathfrak{g}_k$ with $\mathfrak{g}_{-1} \simeq \mathbb{R}^3$ and \mathfrak{g}_0 three-dimensional solvable, and study the related co-induced representations by analogy with the case of the algebra of formal vector fields on \mathbb{R} , where this method leads to the tensor density modules of the Virasoro representation theory.

Chapter 5 is devoted to a systematic study of deformations and central extensions of \mathfrak{sv} .

Finally, we shall study in Chapter 6 the Verma modules of \mathfrak{sv} and of some related algebras, and the associated Kac determinant formulae.

Let us remark, for the sake of completeness, that we voluntarily skipped a promising construction of \mathfrak{sv} and a family of supersymmetric extensions of \mathfrak{sv} in terms of quotients of the Poisson algebra on the torus or of the algebra of pseudo-differential operators on the line that appeared elsewhere (see [33]). Also, representations of \mathfrak{sv} into vertex algebras have been investigated (see [34]).

1 Geometric definitions of \mathfrak{sv}

1.1 From Newtonian mechanics to the Schrödinger-Virasoro algebra

Everybody knows, since Einstein's and Poincaré's discoveries in the early twentieth century, that Lorentzian geometry lies at the heart of relativistic mechanics; the geometric formalism makes it possible to define the equations of general relativity in a coordinate-free, covariant way, in all generality.

Owing to the success of the theory of general relativity, it was natural that one should also think of geometrizing Newtonian mechanics. This gap was filled in about half a century later, by several authors, including J.-M. Lévy-Leblond, C. Duval, H.P. Künzle and others (see for instance [3, 21, 6, 7, 8, 9]), leading to a geometric reformulation of Newtonian mechanics on the so-called Newton-Cartan manifolds, and also to the discovery of new fundamental field equations for Newtonian particles.

Most Lie algebras and groups that will constitute our object of study in this article appear to be closely related to the Newton-Cartan geometry. That is why - although we shall not produce any new result in that particular field - we chose to give a very short introduction to Newton-Cartan geometry, whose main objective is to lead as quickly as possible to a definition of the Schrödinger group and its infinite-dimensional generalization, the *Schrödinger-Virasoro group*.

Definition 1.1

A Newton-Cartan manifold of dimension $(d + 1)$ is a C^∞ manifold M of dimension $(d + 1)$ provided with a closed one-form τ , a degenerate symmetric contravariant non-negative two-tensor $Q \in \text{Sym}(T^2M)$ with one-dimensional kernel generated by τ and a connection ∇ preserving τ and Q .

A right choice of local coordinates makes it clear why these data provide the right framework for Newtonian mechanics. Locally, one may put $\tau = dt$ for a certain time coordinate t , and $Q = \sum_{i=1}^d \partial_{r_i}^2$ for a choice of d space coordinate vector fields (r_i) on the hyperplane orthogonal to dt . The standard Newton-Cartan manifold is the flat manifold $\mathbb{R} \times \mathbb{R}^d$ with coordinates (t, r_1, \dots, r_d) and τ, Q given globally by the above formulas.

The analogue of the Lie algebra of infinitesimal isometries in Riemannian or Lorentzian geometry is here the Lie algebra of infinitesimal vector fields X preserving τ, Q and ∇ . In the case of the flat manifold $\mathbb{R} \times \mathbb{R}^d$, it is equal to the well-known Lie algebra of Galilei transformations \mathfrak{gal}_d , namely, can be written

$$\mathfrak{gal}_d = \langle X_{-1} \rangle \oplus \langle Y_{-\frac{1}{2}}^i, Y_{\frac{1}{2}}^i \rangle_{1 \leq i \leq d} \oplus \langle \mathcal{R}_{ij} \rangle_{1 \leq i < j \leq d}, \quad (1.1)$$

including the time and space translations $X_{-1} = -\partial_t$, $Y_{-\frac{1}{2}}^i = -\partial_{r_i}$, the generators of motion with constant speed $Y_{\frac{1}{2}}^i = -t\partial_{r_i}$ and rotations $\mathcal{R}_{ij} = r_i\partial_{r_j} - r_j\partial_{r_i}$.

It was recognized, mainly by Souriau, that (from the point of view of symplectic geometry) many original features of classical mechanics stem from the existence of the two-cocycle $c \in H^2(\mathfrak{gal}_d, \mathbb{R})$ of the Galilei Lie algebra defined by

$$c(Y_{-\frac{1}{2}}^i, Y_{\frac{1}{2}}^j) = \delta_{i,j}, \quad i, j = 1, \dots, d \quad (1.2)$$

leading to the definition of the mass as a central charge. We shall denote by $\widetilde{\mathfrak{gal}}_d$ the centrally extended Lie algebra. Its has a one-dimensional center, generated by M_0 , and a modified Lie bracket such that $[Y_{\frac{1}{2}}^i, Y_{-\frac{1}{2}}^j] = \delta_{i,j}M_0$, while all the other relations remain unmodified. Schur's Lemma implies that the generator M_0 is scalar on irreducible representations of $\widetilde{\mathfrak{gal}}_d$; the next Proposition shows that it is natural to call *mass* the value of M_0 .

Definition 1.2 *The Schrödinger Lie algebra in d dimensions, denoted by \mathfrak{sch}_d , is the Lie algebra with generators*

$$X_{-1}, X_0, X_1, Y_{-\frac{1}{2}}^i, Y_{\frac{1}{2}}^i \quad (i = 1, \dots, d), \quad \mathcal{R}_{ij} \quad (1 \leq i < j \leq d),$$

isomorphic to the semi-direct product $\mathfrak{sl}(2, \mathbb{R}) \ltimes \widetilde{\mathfrak{gal}}_d \simeq \langle X_{-1}, X_0, X_1 \rangle \ltimes \widetilde{\mathfrak{gal}}_d$, with the following choice of relations for $\mathfrak{sl}(2, \mathbb{R})$

$$[X_0, X_{-1}] = X_1, [X_0, X_1] = -X_1, [X_1, X_{-1}] = 2X_0 \quad (1.3)$$

and an action of $\mathfrak{sl}(2, \mathbb{R})$ on \mathfrak{gal}_d defined by

$$[X_n, Y_{-\frac{1}{2}}^i] = \left(\frac{n}{2} + \frac{1}{2}\right)Y_{n-\frac{1}{2}}^i, [X_n, Y_{\frac{1}{2}}^i] = \left(\frac{n}{2} - \frac{1}{2}\right)Y_{n+\frac{1}{2}}^i \quad (n = -1, 0, 1); \quad (1.4)$$

$$[X_n, \mathcal{R}^{ij}] = 0, \quad [X_n, M_0] = 0. \quad (1.5)$$

Note that the generator M_0 remains central in the semi-direct product: hence, it still makes sense to speak about the mass of an irreducible representation of \mathfrak{sch}_d .

The motivation for this definition (and also the reason for this name) lies in the following classical Proposition.

We denote by $\Delta_d = \sum_{i=1}^d \partial_{r_i}^2$ the usual Laplace operator on \mathbb{R}^d .

Proposition 1.1

- (see [7]) *The Lie algebra of projective (i.e. conserving geodesics) vector fields X on $\mathbb{R} \times \mathbb{R}^d$ such that there exists a function $\lambda \in C^\infty(\mathbb{R} \times \mathbb{R}^d)$ with*

$$\mathcal{L}_X(dt) = \lambda dt, \quad \mathcal{L}_X\left(\sum_{i=1}^d \partial_{r_i}^2\right) = -\lambda \sum_{i=1}^d \partial_{r_i}^2 \quad (1.6)$$

is generated by the vector fields

$$L_{-1} = -\partial_t, L_0 = -t\partial_t - \frac{1}{2} \sum_i r_i \partial_{r_i}, L_1 = -t^2 \partial_t - t \sum_i r_i \partial_{r_i}$$

$$Y_{-\frac{1}{2}}^i = -\partial_{r_i}, Y_{\frac{1}{2}}^i = -t\partial_{r_i}$$

$$\mathcal{R}^{ij} = r_i \partial_{r_j} - r_j \partial_{r_i}, \quad 1 \leq i < j \leq d \quad (1.7)$$

that define a massless representation of the Schrödinger Lie algebra \mathfrak{sch}_d .

- (see [35]) *Let $\mathcal{M} \in \mathbb{C}$. The Lie algebra of differential operators \mathcal{X} on $\mathbb{R} \times \mathbb{R}^d$, of order at most one, preserving the space of solutions of the free Schrödinger equation*

$$(2\mathcal{M}\partial_t - \Delta_d)\psi = 0, \quad (1.8)$$

that is, verifying

$$(2\mathcal{M}\partial_t - \Delta_d)\mathcal{X}\psi = 0$$

for every function ψ such that $(2\mathcal{M}\partial_t - \Delta_d)\psi = 0$, gives a representation $d\pi_{d/4}^d$ of mass $-\mathcal{M}$ of the Schrödinger algebra \mathfrak{sch}_d , with the following realization:

$$d\pi_{d/4}^d(L_{-1}) = -\partial_t, \quad d\pi_{d/4}^d(L_0) = -t\partial_t - \frac{1}{2} \sum_i r_i \partial_{r_i} - \frac{d}{4}, \quad d\pi_{d/4}^d(L_1) = -t^2 \partial_t - t \sum_i r_i \partial_{r_i} - \frac{1}{2} \mathcal{M}r^2 - \frac{d}{2}t$$

$$d\pi_{d/4}^d(Y_{-\frac{1}{2}}^i) = -\partial_{r_i}, \quad d\pi_{d/4}^d(Y_{\frac{1}{2}}^i) = -t\partial_{r_i} - \mathcal{M}r_i, \quad d\pi_{d/4}^d(M_0) = -\mathcal{M}$$

$$d\pi_{d/4}^d(\mathcal{R}^{ij}) = r_i \partial_{r_j} - r_j \partial_{r_i}, \quad 1 \leq i < j \leq d. \quad (1.9)$$

Let us mention, anticipating on Section 1.2, that realizing instead L_0 by the operator $-t\partial_t - \frac{1}{2} \sum_i r_i \partial_{r_i} - \lambda$ and L_1 by $-t^2 \partial_t - t \sum_i r_i \partial_{r_i} - \frac{1}{2} \mathcal{M}r^2 - 2\lambda t$ ($\lambda \in \mathbb{R}$), leads to a family of representations $d\pi_\lambda^d$ of \mathfrak{sch}_d . This accounts for the parameter $\frac{d}{4}$ in our definition of $d\pi_{d/4}^d$. The parameter λ may be interpreted physically as the *scaling dimension* of the field on which \mathfrak{sch}_d acts, $\lambda = \frac{d}{4}$ for solutions of the free Schrödinger equation in d space dimensions.

We shall also frequently use the realization $d\tilde{\pi}_\lambda^d$ of the Schrödinger algebra given by the Laplace transform of the above generators with respect to the mass, which is formally equivalent to replacing the parameter \mathcal{M} in the above formulas by ∂_ζ . This simple transformation leads to a representation $d\tilde{\pi}_\lambda^d$ of \mathfrak{sch}^d ; let us consider the particular case $\lambda = 0$ for simplicity. Then, $d\tilde{\pi}^d := d\tilde{\pi}_0^d$ gives a realization of

\mathfrak{sch}^d by vector fields on \mathbb{R}^{d+2} with coordinates t, r_i ($i = 1, \dots, d$) and ζ . Let us write the action of the generators for further reference (see [31]):

$$\begin{aligned} d\tilde{\pi}^d(L_{-1}) &= -\partial_t, d\tilde{\pi}^d(L_0) = -t\partial_t - \frac{1}{2} \sum_i r_i \partial_{r_i}, d\tilde{\pi}^d(L_1) = -t^2\partial_t - t \sum_i r_i \partial_{r_i} - \frac{1}{2} r^2 \partial_\zeta \\ d\tilde{\pi}^d(Y_{-\frac{1}{2}}^i) &= -\partial_{r_i}, d\tilde{\pi}^d(Y_{\frac{1}{2}}^i) = -t\partial_{r_i} - r_i \partial_\zeta, d\tilde{\pi}^d(M_0) = -\partial_\zeta \\ d\tilde{\pi}^d(\mathcal{R}^{ij}) &= r_i \partial_{r_j} - r_j \partial_{r_i}, \quad 1 \leq i < j \leq d. \end{aligned} \tag{1.10}$$

So, according to the context, one may use either the representation by differential operators on \mathbb{R}^{d+1} of order one, or the representation by vector fields on \mathbb{R}^{d+2} . Both points of view prove to be convenient. We shall see later (see Section 1.2) that these representations extend to representations of the Schrödinger-Virasoro algebra, and give a more intrinsic interpretation of this formal Laplace transform.

Proposition 1.2.

The Lie algebra of vector fields X on $\mathbb{R} \times \mathbb{R}^d$ - not necessarily preserving the connection - such that equations (1.6) are verified is generated by the following set of transformations :

(i)

$$L_f = -f(t)\partial_t - \frac{1}{2}f'(t) \sum_{i=1}^d r_i \partial_{r_i} \quad (\text{Virasoro - like transformations})$$

(ii)

$$Y_{g_i}^i = -g_i(t)\partial_{r_i} \quad (\text{time - dependent space translations})$$

(iii)

$$\mathcal{R}_{h_{ij}}^{ij} = -h_{ij}(t)(r_i \partial_{r_j} - r_j \partial_{r_i}), \quad 1 \leq i < j \leq d \quad (\text{time - dependent space rotations})$$

where f, g_i, h_{ij} are arbitrary functions of t .

Proof. Put $r = (r_1, \dots, r_d)$. Let $X = f(t, r)\partial_t + \sum_{i=1}^d g_i(t, r)\partial_{r_i}$ verifying the conditions of Proposition 1.1. Then $\mathcal{L}_X dt = df$ is compatible with the first condition if f depends on time only, so $\lambda = f'$ is also a function of time only. Hence

$$\begin{aligned} \mathcal{L}_X \left(\sum_{j=1}^d \partial_{r_j}^2 \right) &= -2 \sum_{i,j} \partial_{r_j} \otimes \partial_{r_j} g_i \partial_{r_i} \\ &= -2 \sum_j \partial_{r_j} \otimes \partial_{r_j} g_j \partial_{r_j} - 2 \sum_{i \neq j} \partial_{r_j} \otimes \partial_{r_j} g_i \partial_{r_i} \end{aligned}$$

so $\partial_{r_j} g_i = -\partial_{r_i} g_j$ if $i \neq j$, which gives the time-dependent rotations, and $2\partial_{r_j} g_j = f'$ for $j = 1, \dots, d$, which gives the Virasoro-like transformations and the time-dependent translations. \square

Note that the Lie algebra of Proposition 1.1 (1), corresponds to $f(t) = 1, t, t^2$, $g_i(t) = 1, t$ and $h_{ij}(t) = 1$.

One easily sees that the $\mathcal{R}_{h_{ij}}^{ij}$ generate the algebra of currents on $\mathfrak{so}(d)$, while the $\mathcal{R}_{h_{ij}}^{ij}$ and the $Y_{g_i}^i$ generate together the algebra of currents on the Euclidean Lie algebra $\mathfrak{eucl}(d) = \mathfrak{so}(d) \times \mathbb{R}^d$. The

transformations L_f generate a copy of the Lie algebra of tangent vector fields on \mathbb{R} , denoted by $\text{Vect}(\mathbb{R})$, so one has

$$[L_f, L_g] = L_{\{f,g\}} \quad (1.11)$$

where $\{f, g\} = f'g - fg'$. So this Lie algebra can be described algebraically as a semi-direct product $\text{Vect}(\mathbb{R}) \ltimes \mathfrak{eucl}(d)_{\mathbb{R}}$, where $\mathfrak{eucl}(d)_{\mathbb{R}}$ stands for the Lie algebra of currents with values in $\mathfrak{eucl}(d)$. In our realization, it is embedded as a subalgebra of $\text{Vect}(\mathbb{R} \times \mathbb{R}^d)$.

For both topological and algebraic reasons, we shall from now on compactify the t coordinate. So we work now on $S^1 \times \mathbb{R}^d$, and $\text{Vect}(\mathbb{R}) \ltimes \mathfrak{eucl}(d)_{\mathbb{R}}$ is replaced by $\text{Vect}(S^1) \ltimes \mathfrak{eucl}(d)_{S^1}$, where $\text{Vect}(S^1)$ stands for the famous (centerless) Virasoro algebra.

It may be the right place to recall some well-known facts about the Virasoro algebra, that we shall use throughout the article.

We represent an element of $\text{Vect}(S^1)$ by the vector field $f(z)\partial_z$, where $f \in \mathbb{C}[z, z^{-1}]$ is a Laurent polynomial. Vector field brackets $[f(z)\partial_z, g(z)\partial_z] = (fg' - f'g)\partial_z$, may equivalently be rewritten in the basis $(l_n)_{n \in \mathbb{Z}}$, $l_n = -z^{n+1}\partial_z$ (also called Fourier components), which yields $[l_n, l_m] = (n - m)l_{n+m}$. Notice the unusual choice of signs, justified (among other arguments) by the precedence of [14] on our subject.

The Lie algebra $\text{Vect}(S^1)$ has only one non-trivial central extension (see [12] or [18] for instance), given by the so-called Virasoro cocycle $c \in Z^2(\text{Vect}(S^1), \mathbb{R})$ defined by

$$c(f\partial_z, g\partial_z) = \int_{S^1} f'''(z)g(z) dz, \quad (1.12)$$

or, in Fourier components,

$$c(l_n, l_m) = \delta_{n+m,0}(n+1)n(n-1). \quad (1.13)$$

The resulting centrally extended Lie algebra, called *Virasoro algebra*, will be denoted by \mathfrak{vir} .

The Lie algebra $\text{Vect}(S^1)$ has a one-parameter family of representations \mathcal{F}_λ , $\lambda \in \mathbb{R}$.

Definition 1.3.

We denote by \mathcal{F}_λ the representation of $\text{Vect}(S^1)$ on $\mathbb{C}[z, z^{-1}]$ given by

$$\ell_n \cdot z^m = (\lambda n - m)z^{n+m}, \quad n, m \in \mathbb{Z}. \quad (1.14)$$

An element of \mathcal{F}_λ is naturally understood as a $(-\lambda)$ -density $\phi(z)dz^{-\lambda}$, acted by $\text{Vect}(S^1)$ as

$$f(z)\partial_z \cdot \phi(z)dz^{-\lambda} = (f\phi' - \lambda f'\phi)(z)dz^{-\lambda}. \quad (1.15)$$

In the bases $\ell_n = -z^n\partial_z$ and $a_m = z^m dz^{-\lambda}$, one gets $\ell_n \cdot a_m = (\lambda n - m)a_{n+m}$.

Replacing formally t by the compactified variable z in the formulas of Proposition 1.2, and putting $f(z) = -z^{n+1}$, $g_i(z) = -z^{n+\frac{1}{2}}$, $h_{ij}(z) = -z^n$, one gets a realization of $\text{Vect}(S^1) \ltimes \mathfrak{eucl}(d)_{S^1}$ as a Lie subalgebra of $\text{Vect}(S^1 \times \mathbb{R}^d)$ generated by $L_n, Y_{m_i}^i, \mathcal{R}_{p_{ij}}^{ij}$ (with integer indices n and p_{ij} and half-integer indices m_i), with the following set of relations:

$$\begin{aligned} [L_n, L_p] &= (n - p)L_{n+p} \\ [L_n, Y_m^i] &= \left(\frac{n}{2} - m\right)Y_{n+m}^i, \quad [L_n, \mathcal{R}_p^{ij}] = -p\mathcal{R}_{n+p}^{ij} \end{aligned}$$

$$\begin{aligned}
[Y_m^i, Y_{m'}^j] &= 0, [\mathcal{R}_p^{ij}, Y_m^k] = \delta_{j,k} Y_{m+p}^i - \delta_{i,k} Y_{m+p}^j \\
[\mathcal{R}_n^{ij}, \mathcal{R}_p^{kl}] &= \delta_{j,k} \mathcal{R}_{n+p}^{il} + \delta_{i,l} \mathcal{R}_{n+p}^{jk} - \delta_{j,l} \mathcal{R}_{n+p}^{ik} - \delta_{i,k} \mathcal{R}_{n+p}^{jl}
\end{aligned} \tag{1.16}$$

With the above definitions, one sees immediately that, under the action of $\langle L_f \rangle_{f \in C^\infty(S^1)} \simeq \text{Vect}(S^1)$, the $(Y_m^i)_{m \in \frac{1}{2} + \mathbb{Z}}$ behave as elements of the module $\mathcal{F}_{\frac{1}{2}}$, while the $(\mathcal{R}_m^{ij})_{m \in \mathbb{Z}}$ and the $(M_m)_{m \in \mathbb{Z}}$ define several copies of \mathcal{F}_0 .

The commutative Lie algebra generated by the Y_n^i has an infinite family of central extensions. If we want to leave unchanged the action of $\text{Vect}(S^1)$ on the Y_n^i and to extend the action of $\text{Vect}(S^1)$ to the central charges, though, the most natural possibility (originally discovered by M. Henkel, see [14], by extrapolating the relations (1.4) and (1.5) to integer or half-integer indices), containing \mathfrak{sch}_d as a Lie subalgebra, is the Lie algebra \mathfrak{sv}^d defined as follows.

Definition 1.4. *We denote by \mathfrak{sv}^d the Lie algebra with generators $X_n, Y_m^i, M_n, \mathcal{R}_n^{ij}$ ($n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}$) and following relations (where $n, p \in \mathbb{Z}, m, m' \in \frac{1}{2} + \mathbb{Z}$):*

$$\begin{aligned}
[L_n, L_p] &= (n - p)L_{n+p} \\
[L_n, Y_m^i] &= \left(\frac{n}{2} - m\right)Y_{n+m}^i, [L_n, \mathcal{R}_p^{ij}] = -p\mathcal{R}_{n+p}^{ij} \\
[Y_m^i, Y_{m'}^j] &= (m - m')M_{m+m'}, [\mathcal{R}_p^{ij}, Y_m^k] = \delta_{j,k} Y_{m+p}^i - \delta_{i,k} Y_{m+p}^j \\
[Y_m^i, M_p] &= 0, [\mathcal{R}_n^{ij}, M_p] = 0, [M_n, M_p] = 0 \\
[\mathcal{R}_n^{ij}, \mathcal{R}_p^{kl}] &= \delta_{j,k} \mathcal{R}_{n+p}^{il} + \delta_{i,l} \mathcal{R}_{n+p}^{jk} - \delta_{j,l} \mathcal{R}_{n+p}^{ik} - \delta_{i,k} \mathcal{R}_{n+p}^{jl}
\end{aligned} \tag{1.17}$$

One sees immediately that \mathfrak{sv}^d has a semi-direct product structure $\mathfrak{sv}^d \simeq \text{Vect}(S^1) \ltimes \mathfrak{h}^d$, with $\text{Vect}(S^1) \simeq \langle L_n \rangle_{n \in \mathbb{Z}}$ and $\mathfrak{h}^d = \langle Y_m^i \rangle_{m \in \mathbb{Z}, i \leq d} \oplus \langle M_p \rangle_{p \in \mathbb{Z}} \oplus \langle \mathcal{R}_m^{ij} \rangle_{m \in \mathbb{Z}, 1 \leq i < j \leq d}$.

Note that the Lie subalgebra $\langle X_1, X_0, X_{-1}, Y_{-\frac{1}{2}}^i, Y_{\frac{1}{2}}^i, \mathcal{R}_0^{ij}, M_0 \rangle \subset \mathfrak{sv}^d$ is isomorphic to \mathfrak{sch}_d . So the following Proposition gives a positive answer to a most natural question.

Proposition 1.3 (see [14]) *The realization $d\tilde{\pi}^d$ of \mathfrak{sch}_d (see (1.10)) extends to the following realization $d\tilde{\pi}^d$ of \mathfrak{sv}^d as vector fields on $S^1 \times \mathbb{R}^{d+1}$:*

$$\begin{aligned}
d\tilde{\pi}^d(L_f) &= -f(z)\partial_z - \frac{1}{2}f'(z)\left(\sum_{i=1}^d r_i \partial_{r_i}\right) - \frac{1}{4}f''(z)r^2 \partial_\zeta \\
d\tilde{\pi}^d(Y_{g_i}^i) &= -g_i(z)\partial_{r_i} - g_i'(z)r_i \partial_\zeta \\
d\tilde{\pi}^d(\mathcal{R}_{k_{ij}}^{ij}) &= -k_{ij}(z)(r_i \partial_{r_j} - r_j \partial_{r_i}) \\
d\tilde{\pi}^d(M_h) &= -h(z)\partial_\zeta
\end{aligned} \tag{1.18}$$

Let us rewrite this action in Fourier components for completeness. In the following formulas, $n \in \mathbb{Z}$ while $m \in \frac{1}{2} + \mathbb{Z}$:

$$\begin{aligned}
d\tilde{\pi}^d(L_n) &= -z^{n+1}\partial_z - \frac{1}{2}(n+1)z^n\left(\sum_{i=1}^d r_i \partial_{r_i}\right) - \frac{1}{4}(n+1)nz^{n-1}r^2 \partial_\zeta \\
d\tilde{\pi}^d(Y_m) &= -z^{m+\frac{1}{2}}\partial_{r_i} - \left(m + \frac{1}{2}\right)z^{m-\frac{1}{2}}r_i \partial_\zeta
\end{aligned}$$

$$\begin{aligned}
d\tilde{\pi}^d(\mathcal{R}_n^{ij}) &= -z^n(r_i\partial_{r_j} - r_j\partial_{r_i}) \\
d\tilde{\pi}^d(M_n) &= -z^n\partial_\zeta
\end{aligned} \tag{1.19}$$

We shall restrict to the case $d = 1$ in the rest of the article and write \mathfrak{sv} for \mathfrak{sv}^1 , $d\pi$ for $d\pi^1$, \mathfrak{h} for \mathfrak{h}^1 , Y_f for Y_f^1 , Y_m for Y_m^1 to simplify notations.

Then (as one sees immediately) $\mathfrak{sv} \simeq \langle L_n \rangle_{n \in \mathbb{Z}} \times \langle Y_m, M_p \rangle_{m \in \frac{1}{2} + \mathbb{Z}, p \in \mathbb{Z}}$ is generated by three fields, L, Y and M , with commutators

$$\begin{aligned}
[L_n, L_p] &= (n - p)L_{n+p}, \quad [L_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m}, \quad [L_n, M_p] = -pM_{n+m} \\
[Y_m, Y_{m'}] &= (m - m')M_{m+m'}, \quad [Y_m, M_p] = 0, \quad [M_n, M_p] = 0
\end{aligned} \tag{1.20}$$

where $n, p \in \mathbb{Z}, m, m' \in \frac{1}{2} + \mathbb{Z}$, and $\mathfrak{h} = \langle Y_m, M_p \rangle_{m \in \frac{1}{2} + \mathbb{Z}, p \in \mathbb{Z}}$ is a two-step nilpotent infinite dimensional Lie algebra.

1.2 Integration of the Schrödinger-Virasoro algebra to a group

We let $\text{Diff}(S^1)$ be the group of orientation-preserving C^∞ -diffeomorphisms of the circle. Orientation is important since we shall need to consider the square-root of the jacobian of the diffeomorphism (see Proposition 1.5).

Theorem 1.4.

1. Let $H = C^\infty(S^1) \times C^\infty(S^1)$ be the product of two copies of the space of infinitely differentiable functions on the circle, with its group structure modified as follows:

$$(\alpha_2, \beta_2) \cdot (\alpha_1, \beta_1) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2 + \frac{1}{2}(\alpha'_1\alpha_2 - \alpha_1\alpha'_2)). \tag{1.21}$$

Then H is a Fréchet-Lie group which integrates \mathfrak{h} .

2. Let $SV = \text{Diff}(S^1) \times H$ be the group with semi-direct product given by

$$(1; (\alpha, \beta)) \cdot (\phi; 0) = (\phi; (\alpha, \beta)) \tag{1.22}$$

and

$$(\phi; 0) \cdot (1; (\alpha, \beta)) = (\phi; ((\phi')^{\frac{1}{2}}(\alpha \circ \phi), \beta \circ \phi)). \tag{1.23}$$

Then SV is a Fréchet-Lie group which integrates \mathfrak{sv} .

Proof.

1. From Hamilton (see [13]), one easily sees that H is a Fréchet-Lie group, its underlying manifold being the Fréchet space $C^\infty(S^1) \times C^\infty(S^1)$ itself.

One sees moreover that its group structure is unipotent.

By computing commutators

$$(\alpha_2, \beta_2)(\alpha_1, \beta_1)(\alpha_2, \beta_2)^{-1}(\alpha_1, \beta_1)^{-1} = (0, \alpha'_1\alpha_2 - \alpha_1\alpha'_2) \tag{1.24}$$

one recovers the formulas for the nilpotent Lie algebra \mathfrak{h} .

2. It is a well-known folk result that the Fréchet-Lie group $\text{Diff}(S^1)$ integrates the Lie algebra $\text{Vect}(S^1)$ (see Hamilton [13], or [12], chapter 4, for details). Here the group H is realized (as $\text{Diff}(S^1)$ -module) as a product of modules of densities $\mathcal{F}_{\frac{1}{2}} \times \mathcal{F}_0$, hence the semi-direct product $\text{Diff}(S^1) \ltimes H$ integrates the semi-direct product $\text{Vect}(S^1) \ltimes \mathfrak{h}$.

□

The representation $d\tilde{\pi}$, defined in Proposition 1.3, can be exponentiated into a representation of SV , given in the following Proposition :

Proposition 1.5. (see [31])

1. Define $\tilde{\pi} : SV \rightarrow \text{Diff}(S^1 \times \mathbb{R}^2)$ by

$$\tilde{\pi}(\phi; (\alpha, \beta)) = \tilde{\pi}(1; (\alpha, \beta)) \cdot \tilde{\pi}(\phi; 0)$$

and

$$\tilde{\pi}(\phi; 0)(z, r, \zeta) = (\phi(z), r\sqrt{\phi'(z)}, \zeta - \frac{1}{4} \frac{\phi''(z)}{\phi'(z)} r^2).$$

Then $\tilde{\pi}$ is a representation of SV .

2. The infinitesimal representation of $\tilde{\pi}$ is equal to $d\tilde{\pi}$.

Proof.

Point (a) may be checked by direct verification (note that the formulas were originally derived by exponentiating the vector fields in the realization $d\tilde{\pi}$).

For (b), it is plainly enough to show that, for any $f \in C^\infty(S^1)$ and $g, h \in C^\infty(\mathbb{R})$,

$$\frac{d}{du} \Big|_{u=0} \tilde{\pi}(\exp uL_f) = d\tilde{\pi}(L_f), \quad \frac{d}{du} \Big|_{u=0} \tilde{\pi}(\exp uY_g) = d\tilde{\pi}(Y_g), \quad \frac{d}{du} \Big|_{u=0} \tilde{\pi}(\exp uM_h) = d\tilde{\pi}(M_h).$$

Put $\phi_u = \exp uL_f$, so that $\frac{d}{du} \Big|_{u=0} \phi_u(z) = f(z)$. Then

$$\begin{aligned} \frac{d}{du} r(\phi'_u)^{\frac{1}{2}} &= \frac{1}{2} r(\phi'_u)^{-\frac{1}{2}} \frac{d}{du} \phi'_u \rightarrow_{u \rightarrow 0} \frac{1}{2} r f'(z), \\ \frac{d}{du} \left(r^2 \frac{\phi''_u}{\phi'_u} \right) &= r^2 \left(\frac{\frac{d}{du} \phi''_u}{\phi'_u} - \frac{\phi''_u}{(\phi'_u)^2} \frac{d}{du} \phi'_u \right) \rightarrow_{u \rightarrow 0} r^2 f''(z) \end{aligned}$$

so the equality $\frac{d}{du} \Big|_{u=0} \tilde{\pi}(\exp uL_f) = d\tilde{\pi}(L_f)$ holds. The two other equalities can be proved in a similar way. □

Let us introduce another related representation, using the 'triangular' structure of the representation $\tilde{\pi}$. The action $\tilde{\pi} : SV \rightarrow \text{Diff}(S^1 \times \mathbb{R}^2)$ can be projected onto an action $\bar{\pi} : SV \rightarrow \text{Diff}(S^1 \times \mathbb{R})$ by 'forgetting' the coordinate ζ , since the way coordinates (t, r) are transformed does not depend on ζ . Note also that $\tilde{\pi}$ acts by (time- and space- dependent) translations on the coordinate ζ , so one may define a function $\Phi : SV \rightarrow C^\infty(\mathbb{R}^2)$ with coordinates (t, r) by

$$\tilde{\pi}(g)(t, r, \zeta) = (\bar{\pi}(g)(t, r), \zeta + \Phi_g(t, r))$$

(independently of $\zeta \in \mathbb{R}$). This action may be further projected onto $\bar{\pi}_{S^1} : SV \rightarrow \text{Diff}(S^1)$ by 'forgetting' the second coordinate r this time, so

$$\bar{\pi}_{S^1}(\phi; (\alpha, \beta)) = \phi.$$

Proposition 1.6.

1. One has the relation

$$\Phi_{g_2 \circ g_1}(t, r) = \Phi_{g_1}(t, r) + \Phi_{g_2}(\bar{\pi}(g_1)(t, r)).$$

In other words, Φ is a trivial π -cocycle: $\Phi \in Z^1(G, C^\infty(\mathbb{R}^2))$.

2. The application $\pi_\lambda : SV \rightarrow \text{Hom}(C^\infty(S^1 \times \mathbb{R}), C^\infty(S^1 \times \mathbb{R}))$ defined by

$$\pi_\lambda(g)(\phi)(t, r) = (\bar{\pi}'_{S^1} \circ \bar{\pi}_{S^1}^{-1}(t))^\lambda e^{\mathcal{M}\Phi_g(\bar{\pi}(g)^{-1} \cdot (t, r))} \phi(\bar{\pi}(g)^{-1} \cdot (t, r))$$

defines a representation of SV in $C^\infty(S^1 \times \mathbb{R})$.

Proof.

Straightforward. \square

Note that the function Φ comes up naturally when considering projective representations of the Schrödinger group in one space dimension $Sch^1 \simeq SL(2, \mathbb{R}) \times \text{Gal}_d$, where Gal_d is the Lie group naturally associated to \mathfrak{gal}_d (see [27]).

Let us look at the associated infinitesimal representation. Introduce the function Φ' defined by $\Phi'(X) = \frac{d}{du}|_{u=0} \Phi(\exp uX)$, $X \in \mathfrak{sv}$. If now $g = \exp X$, $X \in \mathfrak{sv}$, then

$$\frac{d}{du}|_{u=0} \pi_\lambda(\exp uX)(\phi)(t, r) = (\mathcal{M}\Phi'(X) + \lambda(d\bar{\pi}_{S^1}(X))'(t) + d\bar{\pi}(X)) \phi(t, r) \quad (1.25)$$

so $d\pi_\lambda(X)$ may be represented as the differential operator of order one

$$d\bar{\pi}(X) + \mathcal{M}\Phi'(X) + \lambda(d\bar{\pi}_{S^1}(X))'(t).$$

So all this amounts to replacing formally ∂_ζ by \mathcal{M} in the formulas of Proposition 1.3 in the case $\lambda = 0$. Then, for any λ ,

$$d\pi_\lambda(Lf) = d\pi_0(Lf) - \lambda f', \quad (1.26)$$

while $d\pi_\lambda(Y_g) = d\pi_0(Y_g)$ and $d\pi_\lambda(M_h) = d\pi_0(M_h)$.

Let us write explicitly the action of all generators, both for completeness and for future reference :

$$d\pi_\lambda(L_f) = -f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}f''(t)\mathcal{M}r^2 - \lambda f'(t)$$

$$d\pi_\lambda(Y_g) = -g(t)\partial_r - \mathcal{M}g'(t)r$$

$$d\pi_\lambda(M_h) = -\mathcal{M}h(t) \quad (1.27)$$

Remarks:

1. One may check easily that one gets by restriction a representation π_λ of the Schrödinger group in one dimension Sch^1 whose infinitesimal representation coincides with (1.9). In particular, $\pi_{1/4}|_{\text{Sch}^1}$ acts on the space of solutions of the free Schrödinger equation in one space dimension (see Proposition 1.1).
2. Taking the Laplace transform of (1.27) with respect to \mathcal{M} , one gets a realization $d\tilde{\pi}_\lambda$ of \mathfrak{sv} as differential operators of order one acting on functions of t, r and ζ , extending the formulas of Proposition 1.3 in the case $d = 1$.

1.3 About graduations and deformations of the Lie algebra \mathfrak{sv}

We shall say in this paragraph a little more on the algebraic structure of \mathfrak{sv} and introduce another related Lie algebra \mathfrak{tsv} ('*twisted Schrödinger-Virasoro algebra*').

The reader may wonder why we chose half-integer indices for the field Y . The shift in the indices is due to the fact that Y behaves as a $(-\frac{1}{2})$ -density, or, in other words, Y has conformal weight $\frac{3}{2}$ under the action of the Virasoro field L (see e.g. [18] or [4] for a mathematical introduction to conformal field theory and its terminology).

Note in particular that, although its weight is a half-integer, Y is a bosonic field, which would contradict spin-statistics theorem, were it not for the fact that Y is not meant to represent a relativistic field (and also that we are in a one-dimensional context).

Nevertheless, as in the case of the double Ramond/Neveu-Schwarz superalgebra (see [19]), one may define a 'twisted' Schrödinger-Virasoro algebra \mathfrak{tsv} which is a priori equally interesting, and exhibits to some respects quite different properties (see Chapter 5).

Definition 1.5.

Let \mathfrak{tsv} be the Lie algebra generated by $(L_n, Y_m, M_p)_{n,m,p \in \mathbb{Z}}$ with relations

$$[L_n, L_m] = (n - m)L_{n+m}, \quad [L_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m}, \quad [L_n, M_m] = -mM_{n+m} \quad (1.28)$$

$$[Y_n, Y_m] = (n - m)M_{n+m}, \quad [Y_n, M_m] = 0, \quad [L_n, Y_m] = 0, \quad (1.29)$$

where n, m are integers.

Notice that the relations are exactly the same as for \mathfrak{sv} (see equations (1.1)-(1.4)), except for the values of the indices.

The simultaneous existence of two linearly independent graduations on \mathfrak{sv} or \mathfrak{tsv} sheds some light on this ambiguity in the definition.

Definition 1.6. Let δ_1 , resp. δ_2 , be the graduations on \mathfrak{sv} or \mathfrak{tsv} defined by

$$\delta_1(L_n) = n, \quad \delta_1(Y_m) = m, \quad \delta_1(M_p) = p \quad (1.30)$$

$$\delta_2(L_n) = n, \quad \delta_2(Y_m) = m - \frac{1}{2}, \quad \delta_2(M_p) = p - 1 \quad (1.31)$$

with $n, p \in \mathbb{Z}$ and $m \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.

One immediately checks that both δ_1 and δ_2 define graduations and that they are linearly independent.

Proposition 1.7. *The graduation δ_1 , defined either on \mathfrak{sv} or on \mathfrak{tsv} , is given by the inner derivation $\delta_1 = \text{ad}(-X_0)$, while δ_2 is an outer derivation, $\delta_2 \in Z^1(\mathfrak{sv}, \mathfrak{sv}) \setminus B^1(\mathfrak{sv}, \mathfrak{sv})$ and $\delta_2 \in Z^1(\mathfrak{tsv}, \mathfrak{tsv}) \setminus B^1(\mathfrak{tsv}, \mathfrak{tsv})$.*

Remark. As we shall see in Chapter 5, the space $H^1(\mathfrak{sv}, \mathfrak{sv})$ or $H^1(\mathfrak{tsv}, \mathfrak{tsv})$ of outer derivations modulo inner derivations is three-dimensional, but only δ_2 defines a graduation on the basis (L_n, Y_m, M_p) .

Proof.

The only non-trivial point is to prove that δ_2 is not an inner derivation. Suppose (by absurd) that $\delta_2 = \text{ad}X$, $X \in \mathfrak{sv}$ or $X \in \mathfrak{tsv}$ (we treat both cases simultaneously). Then $\delta_2(M_0) = 0$ since M_0 is central in \mathfrak{sv} and in \mathfrak{tsv} . Hence the contradiction. \square

Note that the graduation δ_2 is given by the Lie action of the Euler vector field $t\partial_t + r\partial_r + \zeta\partial_\zeta$ in the representation $d\tilde{\pi}$ (see Proposition 1.3).

Let us introduce a natural deformation of \mathfrak{sv} , anticipating on Chapter 5 (we shall need the following definition in paragraph 3.5, see Theorem 3.10, and chapter 5):

Definition 1.7

Let $\mathfrak{sv}_\varepsilon$, $\varepsilon \in \mathbb{R}$ (resp. $\mathfrak{tsv}_\varepsilon$) be the Lie algebra generated by L_n, Y_m, M_p , $n, p \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}$ (resp. $m \in \mathbb{Z}$), with relations

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m}, \quad [L_n, Y_m] = \left(\frac{(1 + \varepsilon)n}{2} - m\right)Y_{n+m}, \quad [L_n, M_m] = (\varepsilon n - m)M_{n+m} \\ [Y_n, Y_m] &= (n - m)M_{n+m}, \quad [Y_n, M_m] = 0, \quad [L_n, Y_m] = 0, \end{aligned} \tag{1.32}$$

One checks immediately that this defines a Lie algebra, and that $\mathfrak{sv} = \mathfrak{sv}_0$.

All these Lie algebras may be extended by using the trivial extension of the Virasoro cocycle of Section 1.1, yielding Lie algebras denoted by $\tilde{\mathfrak{sv}}, \tilde{\mathfrak{tsv}}, \tilde{\mathfrak{sv}}_\varepsilon, \tilde{\mathfrak{tsv}}_\varepsilon$.

2 About the conformal embedding of the Schrödinger algebra

2.1 The conformal embedding

The idea of embedding \mathfrak{sch}^d into $\mathfrak{conf}(d + 2)_{\mathbb{C}}$ comes naturally when considering the wave equation

$$(2i\mathcal{M}\partial_t - \partial_r^2)\psi(\mathcal{M}; t, r) = 0 \tag{2.1}$$

where \mathcal{M} is viewed no longer as a parameter, but as a coordinate. Then the Fourier transform of the wave function with respect to the mass

$$\tilde{\psi}(\zeta; t, r) = \int_{\mathbb{R}} \psi(\mathcal{M}; t, r) e^{-i\mathcal{M}\zeta} d\mathcal{M} \tag{2.2}$$

satisfies the equation

$$(2\partial_\zeta\partial_t - \partial_r^2)\psi(\zeta; t, r) = 0 \tag{2.3}$$

which is none but a zero mass Klein-Gordon equation on $(d + 2)$ -dimensional space-time, put into light-cone coordinates $(\zeta, t) = (x + y, x - y)$.

This simple idea has been developed in a previous article (see [31]) for $d = 1$, in which case \mathfrak{sch}_1 appears as a (non-trivial) maximal parabolic subalgebra of $\mathfrak{conf}(d+2)$ - which is no longer true when $d > 1$. Let us here give an explicit embedding for any dimension d .

We need first to fix some notations. Consider the conformal algebra in its standard representation as infinitesimal conformal transformations on \mathbb{R}^{d+2} with coordinates $(\xi_1, \dots, \xi_{d+2})$. Then there is a natural basis of $\mathfrak{conf}(d+2)$ made of $(d+2)$ translations P_μ , $\frac{1}{2}(d+1)(d+2)$ rotations $\mathcal{M}_{\mu,\nu}$, $(d+2)$ inversions K_μ and the Euler operator D : in coordinates, one has

$$P_\mu = \partial_{\xi_\mu} \quad (2.4)$$

$$\mathcal{M}_{\mu,\nu} = \xi_\mu \partial_\nu - \xi_\nu \partial_\mu \quad (2.5)$$

$$K_\mu = 2\xi_\mu \left(\sum_{\nu=1}^{d+2} \xi_\nu \partial_\nu \right) - \left(\sum_{\nu=1}^{d+2} \xi_\nu^2 \right) \partial_\mu \quad (2.6)$$

$$D = \sum_{\nu=1}^{d+2} \xi_\nu \partial_\nu. \quad (2.7)$$

Proposition 2.1.

The formulas

$$Y_{-\frac{1}{2}}^j = -2^{\frac{1}{2}} e^{-i\pi/4} P_j \quad (2.8)$$

$$Y_{\frac{1}{2}}^j = -2^{-\frac{1}{2}} e^{i\pi/4} (\mathcal{M}_{d+2,j} + i\mathcal{M}_{d+1,j}) \quad (2.9)$$

$$\mathcal{R}_{j,k} = \mathcal{M}_{j,k} \quad (2.10)$$

$$X_{-1} = i(P_{d+2} - iP_{d+1}) \quad (2.11)$$

$$X_0 = -\frac{D}{2} + \frac{i}{2} \mathcal{M}_{d+2,d+1} \quad (2.12)$$

$$X_1 = -\frac{i}{4} (K_{d+2} + iK_{d+1}) \quad (2.13)$$

give an embedding of \mathfrak{sch}^d into $\mathfrak{conf}(d+2)_\mathbb{C}$.

Proof. Put

$$t = \frac{1}{2}(-\xi_{d+1} + i\xi_{d+2}), \zeta = \frac{1}{2}(\xi_{d+1} + i\xi_{d+2}), r_j = 2^{-\frac{1}{2}} e^{i\pi/4} \xi_j \quad (j = 1, \dots, d).$$

Then the previous definitions yield the representation $d\tilde{\pi}_{d/4}^d$ of \mathfrak{sch}^d (see (1.10)). \square

2.2 Relations between \mathfrak{sp} and the Poisson algebra on T^*S^1 and 'no-go' theorem

The relation between the Virasoro algebra and the Poisson algebra on T^*S^1 has been investigated in [26]. We shall consider more precisely the Lie algebra $\mathcal{A}(S^1)$ of smooth functions on $\dot{T}^*S^1 = T^*S^1 \setminus S^1$, the total space of the cotangent bundle with zero section removed, which are Laurent series on the fibers. So $\mathcal{A}(S^1) = C^\infty(S^1) \otimes \mathbb{R}[\partial, \partial^{-1}]$ and $F \in \mathcal{A}(S^1)$ is of the following form:

$$F(t, \partial, \partial^{-1}) = \sum_{k \in \mathbb{Z}} f_k(t) \partial^k,$$

with $f_k = 0$ for large enough k . The Poisson bracket is defined as usual, following:

$$\{F, G\} = \frac{\partial F}{\partial \partial} \frac{\partial G}{\partial t} - \frac{\partial G}{\partial \partial} \frac{\partial F}{\partial t}.$$

(The reader should not be afraid by notation $\partial\partial$!). In terms of densities on the circle, one has the natural decomposition: $\mathcal{A}(S^1) = \bigoplus_{k \geq 0} \mathcal{F}_k \oplus \left(\prod_{k \leq 0} \mathcal{F}_k \right)$. The Poisson bracket turns out to be homogeneous with respect to that decomposition: $\{\mathcal{F}_k, \mathcal{F}_l\} \subset \mathcal{F}_{k+l-1}$, and more explicitly $\{f(x)\partial^k, g(x)\partial^l\} = (kf'g - lf'g)\partial^{k+l-1}$. One recovers the usual formulae for the Lie bracket on $\mathcal{F}_1 = \text{Vect}(S^1)$ and its representations on modules of densities; one has as well the embedding of the semi-direct product $\text{Vect}(S^1) \ltimes C^\infty(S^1) = \mathcal{F}_1 \ltimes \mathcal{F}_0$ as a Lie subalgebra of $\mathcal{A}(S^1)$, representing differential operators of order ≤ 1 .

Remark: One can also consider the subalgebra of $\mathcal{A}(S^1)$, defined as $\mathbb{C}[z, z^{-1}] \otimes \mathbb{C}[\partial, \partial^{-1}]$; it gives the usual description of the Poisson algebra on the torus \mathbb{T}^2 , sometimes denoted $SU(\infty)$. We also have to consider half densities and developments into Laurent series in z and \sqrt{z} . Geometrically speaking, the half densities can be described as spinors: let E be a vector bundle over S^1 , square root of T^*S^1 ; in other words one has $E \otimes E = T^*S^1$. Then the space of Laurent polynomials on the fibers of E (minus the zero-section) is exactly the Poisson algebra $\tilde{\mathcal{A}}(S^1) = C^\infty(S^1) \otimes \mathbb{C}[\partial^{1/2}, \partial^{-1/2}]$. Moreover, one also needs half-integer power series or polynomials in z as coefficients of the Laurent series in ∂ ; one can obtain the corresponding algebra globally, using the pull-back through the application $S^1 \rightarrow S^1$ defined as $z \rightarrow z^2$.

Finally one has obtained the subalgebra $\hat{\mathcal{A}}(S^1) \subset \tilde{\mathcal{A}}(S^1)$ generated by terms $z^m \partial^n$ where m and n are either integers or half-integers. One can represent such generators as the points with coordinates (m, n) in the plane \mathbb{R}^2 .

So our algebra $SV = \text{Vect}(S^1) \ltimes \mathfrak{h}$, with $\mathfrak{h} \simeq \mathcal{F}_{1/2} \ltimes \mathcal{F}_0$ as a $\text{Vect}(S^1)$ -module, can be naturally embedded into $\hat{\mathcal{A}}(S^1)$,

$$\begin{array}{cccccccc}
 \cdots & & L_{-2} & & L_{-1} & & L_0 & & L_1 & & L_2 & & \cdots \\
 & \cdots & & Y_{-\frac{3}{2}} & & Y_{-\frac{1}{2}} & & Y_{\frac{1}{2}} & & Y_{\frac{3}{2}} & & \cdots & \\
 \cdots & & M_{-2} & & M_{-1} & & M_0 & & M_1 & & M_2 & & \cdots
 \end{array} \tag{2.14}$$

The above scheme represents pictorially the embedding. For the twisted Schrödinger-Virasoro algebra, one considers the Y_m field with integer powers, or as described in the following scheme:

$$\begin{array}{cccccccc}
 \cdots & & L_{-2} & & L_{-1} & & L_0 & & L_1 & & L_2 & & \cdots \\
 & \cdots & & Y_{-2} & & Y_{-1} & & Y_0 & & Y_1 & & Y_2 & & \cdots \\
 \cdots & & M_{-2} & & M_{-1} & & M_0 & & M_1 & & M_2 & & \cdots
 \end{array}$$

One can naturally ask whether this defines a Lie algebra embedding, just as in the case of $\text{Vect}(S^1) \ltimes \mathcal{F}_0$. The answer is no:

Proposition 2.2.

The natural vector space embedding $\mathfrak{sv} \hookrightarrow \widehat{\mathcal{A}}(S^1)$ is not a Lie algebra homomorphism.

Proof: One sees immediately that on the one hand $[Y_n, M_m] = 0$ and $[M_n, M_m] = 0$, while on the other hand $\{\mathcal{F}_{1/2}, \mathcal{F}_0\} \subset \mathcal{F}_{-1/2}$ is in general non trivial. The vanishing of $\{\mathcal{F}_0, \mathcal{F}_0\}$ which makes the embedding of $\text{Vect}(S^1) \ltimes \mathcal{F}_0$ as a Lie subalgebra possible was in some sense an accident. In fact, one can show that from the image of the generators of \mathfrak{sv} and computing successive Poisson brackets, one can generate all the \mathcal{F}_λ with $\lambda \leq 0$.

Let $\widehat{\mathcal{A}}(S^1)_{(1)} = \mathcal{F}_1 \oplus \mathcal{F}_{1/2} \oplus \mathcal{F}_0 \bigoplus_{\lambda \in \frac{\mathbb{Z}}{2}, \lambda < 0} \mathcal{F}_\lambda$, it defines a Poisson subalgebra of $\widehat{\mathcal{A}}(S^1)$, and it is in fact the

smallest possible Poisson algebra which contains the image of \mathfrak{sv} . Now, let $\widehat{\mathcal{A}}(S^1)_{(0)} = \prod_{\lambda \in \frac{\mathbb{Z}}{2}, \lambda < 0} \mathcal{F}_\lambda$ the

Poisson subalgebra of $\widehat{\mathcal{A}}(S^1)$ which contains only negative powers of ∂ (its quantum analogue is known as Volterra algebra of integral operators, see [12], chap. X). One easily sees that it is an ideal of $\widehat{\mathcal{A}}(S^1)_{(1)}$, as a Lie algebra, but of course not an associative ideal; if one considers the quotient $\widehat{\mathcal{A}}(S^1)_{(1)}/\widehat{\mathcal{A}}(S^1)_{(0)}$, then all the obstruction for the embedding to be a homomorphism disappears. So one has:

Proposition 2.3.

There exists a natural Lie algebra embedding of \mathfrak{sv} into the quotient $\widehat{\mathcal{A}}(S^1)_{(1)}/\widehat{\mathcal{A}}(S^1)_{(0)}$. One can say briefly that \mathfrak{sv} is a subquotient of the Poisson algebra $\widehat{\mathcal{A}}(S^1)$.

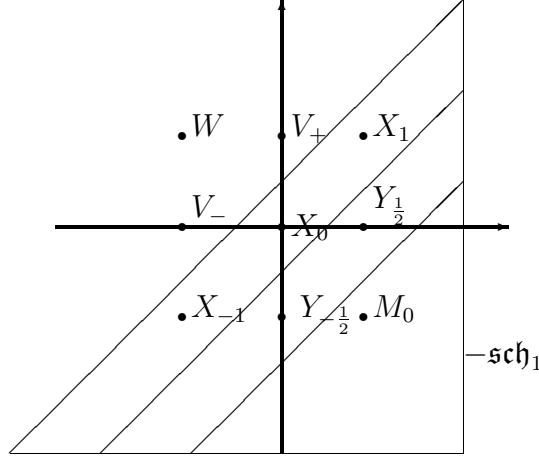
Now a natural question arises: the conformal embedding of Schrödinger algebra described in paragraph 3.1 yields $\mathfrak{sch}_1 \subset \mathfrak{conf}(3)_{\mathbb{C}}$, so one would like to extend the construction of \mathfrak{sv} as generalization of \mathfrak{sch}_1 , in order that it contain $\mathfrak{conf}(3)$: we are looking for an hypothetic Lie algebra \mathcal{G} making the following diagram of embeddings complete:

$$\begin{array}{ccc} \mathfrak{sch}_1 & \hookrightarrow & \mathfrak{conf}(3) \\ \downarrow & & \downarrow \\ \mathfrak{sv} & \hookrightarrow & \mathcal{G} \end{array} \tag{2.15}$$

In the category of abstract Lie algebras, one has an obvious solution to this problem: simply take the amalgamated sum of \mathfrak{sv} and $\mathfrak{conf}(3)$ over \mathfrak{sch}_1 . Such a Lie algebra is defined through generators and relations, and is generally untractable. We are looking here for a natural, geometrically defined construction of such a \mathcal{G} ; we shall give some evidence of its non-existence, a kind of "no-go theorem", analogous to those well-known in gauge theory, for example⁴ (see [20]).

⁴Simply recall that this theorem states that there doesn't exist a common non-trivial extension containing both the Poincaré group and the external gauge group.

Let us consider the root diagram of $\mathfrak{conf}(3)$ as drawn in [31]:



Comparing with (2.14), one sees that the successive diagonal strips are contained in $\mathcal{F}_1, \mathcal{F}_{1/2}, \mathcal{F}_0$ respectively. So the first idea might be to try to add $\mathcal{F}_{3/2}$ and \mathcal{F}_2 , as an infinite prolongation of the supplementary part to \mathfrak{sch}_1 in $\mathfrak{conf}(3)$, so that $V_- \longrightarrow t^{-1/2}\partial^{3/2}$, $V_+ \longrightarrow t^{1/2}\partial^{3/2}$, $W \longrightarrow \partial^2$.

Unfortunately, this construction fails at once for two reasons: first, one doesn't get the right brackets for $\mathfrak{conf}(3)$ with such a choice, and secondly the elements of \mathcal{F}_λ , $\lambda \in \{0, \frac{1}{2}, 1, 3/2, 2\}$, taken together with their successive brackets generate the whole Poisson algebra $\widehat{\mathcal{A}}(S^1)$.

Another approach could be the following: take two copies of \mathfrak{h} , say \mathfrak{h}^+ and \mathfrak{h}^- and consider the semi direct product $\mathcal{G} = \text{Vect}(S^1) \ltimes (\mathfrak{h}^+ \oplus \mathfrak{h}^-)$, so that \mathfrak{h}^+ extends the $\{Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\}$ as in \mathfrak{sv} before, and \mathfrak{h}^- extends $\{V_-, V_+, W\}$. Then \mathcal{G} is obtained from density modules, but doesn't extend $\mathfrak{conf}(3)$, but only a contraction of it: all the brackets between $\{Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\}$ on one hand and $\{V_-, V_+, W\}$ on the other are vanishing. Now, we can try to deform \mathcal{G} in order to obtain the right brackets for $\mathfrak{conf}(3)$. Let Y_m^+, M_m^+ and Y_m^-, M_m^- be the generators of \mathfrak{h}^+ and \mathfrak{h}^- ; we want to find coefficients $a_{p,m}$ such that $[Y_m^+, Y_p^-] = a_{p,m}L_{m+p}$ defines a Lie bracket. So let us check Jacobi identity for (L_n, Y_m^+, Y_p^-) . One obtains $(m - \frac{n}{2})a_{p,n+m} + (n - m - p)a_{p,m} + (p - \frac{n}{2})a_{p+n,m} = 0$. If one tries $a_{pm} = \lambda p + \mu m$, one deduces from this relation: $n\lambda(p - \frac{n}{2}) + n\mu(m - \frac{n}{2}) = 0$ for every $n \in \mathbb{Z}, p, m \in \frac{1}{2}\mathbb{Z}$, so obviously $\lambda = \mu = 0$.

So our computations show there doesn't exist a geometrically defined construction of \mathcal{G} satisfying the conditions of diagram (2.16). The two possible extensions of \mathfrak{sch}_1 , \mathfrak{sv} and $\mathfrak{conf}(3)$ are shown to be incompatible, and this is our "no-go" theorem.

3 On some natural representations of \mathfrak{sv}

We introduce in this chapter several natural representations of \mathfrak{sv} that split into two classes : the (centrally extended) coadjoint action on the one hand; some apparently unrelated representations on spaces of functions or differential operators that can actually all be obtained as particular cases of the general coinduction method for \mathfrak{sv} (see chapter 4).

It is interesting by itself that the coadjoint action should not belong to the same family of representations as the others. We shall come back to this later on in this chapter.

3.1 Coadjoint action of \mathfrak{sv}

Let us recall some facts about coadjoint actions of centrally extended Lie groups and algebras, referring to [12], chapter 6, for details. So let G be a Lie group with Lie algebra \mathfrak{g} , and let us consider central extensions of them, in the categories of groups and algebras respectively:

$$(1) \longrightarrow \mathbb{R} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow (1) \quad (3.1)$$

$$(0) \longrightarrow \mathbb{R} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow (0) \quad (3.2)$$

with $\tilde{\mathfrak{g}} = \text{Lie}(\tilde{G})$, the extension (3.2) representing the tangent spaces at the identity of the extension (3.1) (see [12], II 6.1.1. for explicit formulas). Let $C \in Z_{\text{diff}}^2(G, \mathbb{R})$ and $c \in Z^2(\mathfrak{g}, \mathbb{R})$ the respective cocycles. We want to study the coadjoint action on the dual $\tilde{\mathfrak{g}}^* = \mathfrak{g}^* \times \mathbb{R}$. We shall denote by Ad^* and $\widetilde{\text{Ad}}^*$ the coadjoint actions of G and \tilde{G} respectively, and ad^* and $\widetilde{\text{ad}}^*$ the coadjoint actions of \mathfrak{g} and $\tilde{\mathfrak{g}}$. One then has the following formulas

$$\widetilde{\text{Ad}}^*(g, \alpha)(u, \lambda) = (\text{Ad}^*(g)u + \lambda\Theta(g), \lambda) \quad (3.3)$$

and

$$\widetilde{\text{ad}}^*(\xi, \alpha)(u, \lambda) = (\text{Ad}^*(\xi)u + \lambda\theta(\xi), 0) \quad (3.4)$$

where $\Theta : G \rightarrow \hat{\mathfrak{g}}^*$ and $\theta : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}^*$ are the Souriau cocycles for differentiable and Lie algebra cohomologies respectively; for θ one has the following formula : $\langle \theta(\xi), \eta \rangle = c(\xi, \eta)$. For details of the proof, as well as 'dictionaries' between the various cocycles, the reader is referred to [12], chapter 6.

Note that formulas (3.3) and (3.4) define affine actions of G and \mathfrak{g} respectively, different from their coadjoint actions when $\lambda \neq 0$. The actions on hyperplanes $\mathfrak{g}_\lambda^* = \{(u, \lambda) \mid u \in \mathfrak{g}^*\} \subset \tilde{\mathfrak{g}}^*$ with fixed second coordinate will be denoted by ad_λ^* and Ad_λ^* respectively.

Here we shall consider the central extension $\tilde{\mathfrak{sv}}$ of \mathfrak{sv} inherited from Virasoro algebra, defined by the cocycle c such that

$$\begin{aligned} c(L_n, L_p) &= \delta_{n+p,0} n(n+1)(n-1) \\ c(L_n, Y_m) &= c(L_n, M_p) = c(Y_m, Y_{m'}) = 0 \end{aligned} \quad (3.5)$$

(with $n, p \in \mathbb{Z}$ and $m, m' \in \frac{1}{2} + \mathbb{Z}$). Note that we shall prove in chapter 5 that this central extension is universal (a more 'pedestrian' proof was given in [14]).

As usual in infinite dimension, the algebraic dual of $\tilde{\mathfrak{sv}}$ is untractable, so let us consider the regular dual, consisting of sums of modules of densities of $\text{Vect}(S^1)$ (see Definition 1.3): the dual module \mathcal{F}_μ^* is identified with $\mathcal{F}_{-1-\mu}$ through

$$\langle u(dx)^{1+\mu}, f dx^{-\mu} \rangle = \int_{S^1} u(x)f(x) dx. \quad (3.6)$$

So, in particular, $\text{Vect}(S^1)^* \simeq \mathcal{F}_{-2}$, and (as a $\text{Vect}(S^1)$ -module)

$$\mathfrak{sv}^* = \mathcal{F}_{-2} \oplus \mathcal{F}_{-\frac{3}{2}} \oplus \mathcal{F}_{-1}; \quad (3.7)$$

we shall identify the element $\Gamma = \gamma_0 dx^2 + \gamma_1 dx^{\frac{3}{2}} + \gamma_2 dx \in \mathfrak{sv}^*$ with the triple $\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} \in (C^\infty(S^1))^3$. In other words,

$$\left\langle \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, L_{f_0} + Y_{f_1} + M_{f_2} \right\rangle = \sum_{i=0}^2 \int_{S^1} (\gamma_i f_i)(z) dz. \quad (3.8)$$

The following Lemma describes the coadjoint representation of a Lie algebra that can be written as a semi-direct product.

Lemma 3.1. *Let $\mathfrak{s} = \mathfrak{s}_0 \ltimes \mathfrak{s}_1$ be a semi-direct product of two Lie algebra \mathfrak{s}_0 and \mathfrak{s}_1 . Then the coadjoint action of \mathfrak{s} on \mathfrak{s}^* is given by*

$$\text{ad}_{\mathfrak{s}}^*(f_0, f_1) \cdot (\gamma_0, \gamma_1) = \langle \text{ad}_{\mathfrak{s}_0}^*(f_0)\gamma_0 - \tilde{g}_1 \cdot \gamma_1, \tilde{f}_0^*(\gamma_1) + \text{ad}_{\mathfrak{s}_1}^*(f_1)\gamma_1 \rangle$$

where by definition

$$\langle \tilde{f}_1 \cdot \gamma_1, X_0 \rangle_{\mathfrak{s}_0^* \times \mathfrak{s}_0} = \langle \gamma_1, [X_0, f_1] \rangle_{\mathfrak{s}_1^* \times \mathfrak{s}_1}$$

and

$$\langle \tilde{f}_0^*(\gamma_1), X_1 \rangle_{\mathfrak{s}_1^* \times \mathfrak{s}_1} = \langle \gamma_1, [f_0, X_1] \rangle_{\mathfrak{s}_1^* \times \mathfrak{s}_1}.$$

Proof. Straightforward.

Theorem 3.2.

The coadjoint action of \mathfrak{sv} on the affine hyperplane \mathfrak{sv}_λ^ is given by the following formulas:*

$$\text{ad}^*(L_{f_0}) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} cf_0''' + 2f_0'\gamma_0 + f_0\gamma_0' \\ f_0\gamma_1' + \frac{3}{2}f_0'\gamma_1 \\ f_0\gamma_2' + f_0'\gamma_2 \end{pmatrix} \quad (3.9)$$

$$\text{ad}^*(Y_{f_1}) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\gamma_1 f_1' + \frac{1}{2}\gamma_1' f_1 \\ 2\gamma_2 f_1' + \gamma_2' f_1 \\ 0 \end{pmatrix} \quad (3.10)$$

$$\text{ad}^*(M_{f_2}) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -\gamma_2 f_2' \\ 0 \\ 0 \end{pmatrix}. \quad (3.11)$$

Proof.

The action of $\text{Vect}(S^1) \subset \mathfrak{sv}$ follows from the identification of \mathfrak{sv}_λ^* with $\mathfrak{vir}_\lambda^* \oplus \mathcal{F}_{-\frac{3}{2}} \oplus \mathcal{F}_{-1}$.

Applying the preceding Lemma, one gets now

$$\begin{aligned} \langle \text{ad}^*(Y_{f_1}) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, L_{h_0} \rangle &= -\langle \tilde{Y}_{f_1} \cdot \begin{pmatrix} 0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, L_{h_0} \rangle \\ &= \left\langle \begin{pmatrix} 0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, Y_{\frac{1}{2}h_0'f_1 - h_0f_1'} \right\rangle \\ &= \int_{S^1} \gamma_1 \left(\frac{1}{2}h_0'f_1 - h_0f_1' \right) dz \\ &= \int_{S^1} h_0 \left(-\frac{3}{2}\gamma_1 f_1' - \frac{1}{2}\gamma_1' f_1 \right) dz; \end{aligned}$$

$$\begin{aligned}
\langle \text{ad}^*(Y_{f_1}) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, Y_{h_1} \rangle &= \langle \text{ad}_{\mathfrak{g}}^*(Y_{f_1}) \cdot \begin{pmatrix} 0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, Y_{h_1} \rangle \\
&= -\langle \begin{pmatrix} 0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, M_{f_1' h_1 - f_1 h_1'} \rangle \\
&= -\int_{S^1} \gamma_2 (f_1' h_1 - f_1 h_1') dz \\
&= \int_{S^1} h_1 (-2\gamma_2 f_1' - \gamma_2' f_1) dz
\end{aligned}$$

and

$$\langle \text{ad}^*(Y_{f_1}) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, M_{h_2} \rangle = 0.$$

Hence the result for $\text{ad}^*(Y_{f_1})$.

For the action of $\text{ad}^*(M_{f_2})$, one gets similarly

$$\begin{aligned}
\langle \text{ad}^*(M_{f_2}) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, L_{h_0} \rangle &= -\langle \begin{pmatrix} 0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, M_{f_2' h_0} \rangle \\
&= -\int_{S^1} \gamma_2 f_2' h_0 dz
\end{aligned}$$

and

$$\langle \text{ad}^*(M_{f_2}) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, Y_{h_1} \rangle = \langle \text{ad}^*(M_{f_2}) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, M_{h_2} \rangle = 0.$$

Hence the result for $\text{ad}^*(M_{f_2})$. \square

One can now easily construct the coadjoint action of the group SV , which "integrates" the above defined coadjoint action of \mathfrak{sv} ; as usual in infinite dimension, such an action should not be taken for granted and one has to construct it explicitly case by case. The result is given by the following.

Theorem 3.3.

The coadjoint action of SV on the affine hyperplane $\tilde{\mathfrak{sv}}_\lambda^$ is given by the following formulas:*

Let $(\varphi, \alpha, \beta) \in SV$, then:

$$Ad^*(\varphi) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \lambda\Theta(\varphi) + (\gamma_0 \circ \varphi)(\varphi')^2 \\ (\gamma_1 \circ \varphi)(\varphi')^{\frac{3}{2}} \\ (\gamma_2 \circ \varphi)\varphi' \end{pmatrix} \quad (3.12)$$

$$Ad^*(\alpha, \beta) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_0 + \frac{3}{2}\gamma_1\alpha' + \frac{\gamma_1'}{2}\alpha + \gamma_2\beta' - \frac{\gamma_2}{2}(3\alpha'^2 + \alpha\alpha'') - \frac{3}{2}\gamma_2'\alpha\alpha' - \frac{\gamma_2''}{4}\alpha^2 \\ \gamma_1 + 2\gamma_2\alpha' + \gamma_2'\alpha \\ \gamma_2 \end{pmatrix} \quad (3.13)$$

Proof:

The first part (3.12) is easily deduced from the natural action of $\text{Diff}(S^1)$ on $\tilde{\mathfrak{sv}}_\lambda^* = \mathfrak{vir}_\lambda^* \oplus \mathcal{F}_{-3/2} \oplus \mathcal{F}_{-1}$. Here $\Theta(\varphi)$ denotes the Schwarzian derivative of φ . Let's only recall that it is the Souriau cocycle in $H^1(\text{Vect}(S^1), \mathfrak{vir}^*)$ associated to Bott-Virasoro cocycle in $H^2(\text{Diff}(S^1), \mathbb{R})$, referring to [12], Chap. IV, VI for details.

The problem of computing the coadjoint action of $(\alpha, \beta) \in H$ can be split into two pieces; the coadjoint action of H on \mathfrak{h}^* is readily computed and one finds:

$$Ad^*(\alpha, \beta) \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 + 2\gamma_2\alpha' + \gamma_2'\alpha \\ \gamma_2 \end{pmatrix}$$

The most delicate part is to compute the part of coadjoint action of $(\alpha, \beta) \in H$ coming from the adjoint action on $\text{Vect}(S^1)$, by using:

$$\langle Ad^*(\alpha, \beta) \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, f\partial \rangle = \langle \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}, Ad(\alpha, \beta)^{-1}(f\partial, 0, 0) \rangle.$$

One can now use conjugation in the group SV and one finds

$$Ad(\alpha, \beta)^{-1}(f\partial, 0, 0) = \left(f\partial, f\alpha' - \frac{1}{2}\alpha f', f\beta' + \frac{1}{2}(f\alpha''\alpha + \frac{f'}{2}\alpha\alpha' - \frac{\alpha^2}{2}f'' - f\alpha'^2 + \frac{f'}{2}\alpha\alpha'') \right).$$

Now, using integration by part, one finds easily the formula (3.13) above. \square

3.2 Action of \mathfrak{sv} on the affine space of Schrödinger operators

The next three sections aim at generalizing an idea that appeared at a crossroads between projective geometry, integrable systems and the theory of representations of $\text{Diff}(S^1)$. We shall, to our own regret, give some new insights on SV from the latter point of view exclusively, leaving aside other aspects of a figure that will hopefully soon emerge.

Let $\partial = \frac{\partial}{\partial x}$ be the derivation operator on the torus $\mathbb{T} = [0, 2\pi]$. A *Hill operator* is by definition a second order operator on \mathbb{T} of the form $\mathcal{L}_u := \partial^2 + u$, $u \in C^\infty(\mathbb{T})$. Let π_λ be the representation of $\text{Diff}(S^1)$ on the space of $(-\lambda)$ -densities \mathcal{F}_λ (see Definition 1.3). One identifies the vector spaces $C^\infty(\mathbb{T})$ and \mathcal{F}_λ in the natural way, by associating to $f \in C^\infty(\mathbb{T})$ the density $f dx^{-\lambda}$. Then, for any couple $(\lambda, \mu) \in \mathbb{R}^2$, one has an action $\Pi_{\lambda, \mu}$ of $\text{Diff}(S^1)$ on the space of differential operators on \mathbb{T} through the left-and-right action

$$\Pi_{\lambda, \mu}(\phi) : D \rightarrow \pi_\lambda(\phi) \circ D \circ \pi_\mu(\phi)^{-1},$$

with corresponding infinitesimal action

$$d\Pi_{\lambda, \mu}(\phi) : D \rightarrow d\pi_\lambda(\phi) \circ D - D \circ d\pi_\mu(\phi).$$

For a particular choice of λ, μ , namely, $\lambda = -\frac{3}{2}, \mu = \frac{1}{2}$, this representation preserves the affine space of Hill operators; more precisely,

$$\pi_{-3/2}(\phi) \circ (\partial^2 + u) \circ \pi_{1/2}(\phi)^{-1} = \partial^2 + (\phi')^2(u \circ \phi') + \frac{1}{2}\Theta(\phi) \quad (3.14)$$

where Θ stands for the Schwarzian derivative. In other words, u transforms as an element of $\mathfrak{vir}_{\frac{1}{2}}^*$ (see section 3.1). One may also - taking an opposite point of view - say that Hill operators define a $\text{Diff}(S^1)$ -equivariant morphism from $\mathcal{F}_{\frac{1}{2}}$ into $\mathcal{F}_{-\frac{3}{2}}$.

This program may be completed for actions of SV on several affine spaces of differential operators. This will lead us to introduce several representations of SV that may all be obtained by the general method of coinduction (see Chapter 4). Quite remarkably, when one thinks of the analogy with the case of the action of $\text{Diff}(S^1)$ on Hill operators, the coadjoint action of SV on \mathfrak{sv}^* does not appear in this context, and moreover cannot be obtained by the coinduction method, as one concludes easily from the formulas of Chapter 4 (see Theorem 4.2).

Definition 3.1. Let \mathcal{S}^{lin} be the vector space of second order operators on \mathbb{R}^2 defined by

$$D \in \mathcal{S}^{lin} \Leftrightarrow D = h(2\mathcal{M}\partial_t - \partial_r^2) + V(r, t), \quad h, V \in C^\infty(\mathbb{R}^2)$$

and $\mathcal{S}^{aff} \subset \mathcal{S}^{lin}$ the affine subspace of 'Schrödinger operators' given by the hyperplane $h = 1$.

In other words, an element of \mathcal{S}^{aff} is the sum of the free Schrödinger operator $\Delta_0 = 2\mathcal{M}\partial_t - \partial_r^2$ and of a potential V .

The following theorem proves that there is a natural family of actions of the group SV on the space \mathcal{S}^{lin} : more precisely, for every $\lambda \in \mathbb{R}$, and $g \in SV$, there is a 'scaling function' $F_{g,\lambda} \in C^\infty(S^1)$ such that

$$\pi_\lambda(g)(\Delta_0 + V)\pi_\lambda(g)^{-1} = F_{g,\lambda}(t)(\Delta_0 + V_{g,\lambda}) \quad (3.15)$$

where $V_{g,\lambda} \in C^\infty(\mathbb{R}^2)$ is a 'transformed potential' depending on g and on λ (see Section 1.2, Proposition 1.6 and commentaries thereafter for the definition of π_λ). Taking the infinitesimal representation of \mathfrak{sv} instead, this is equivalent to demanding that the 'adjoint' action of $d\pi_\lambda(\mathfrak{sv})$ preserve \mathcal{S}^{lin} , namely

$$[d\pi_\lambda(X), \Delta_0 + V](t, r) = f_{X,\lambda}(t)(\Delta_0 + V_{X,\lambda}), \quad X \in \mathfrak{sv} \quad (3.16)$$

for a certain infinitesimal 'scaling' function $f_{X,\lambda}$ and with a transformed potential $V_{X,\lambda}$.

We shall actually prove that this last property even characterizes in some sense the differential operators of order one that belong to $d\pi_\lambda(\mathfrak{sv})$.

Theorem 3.4.

1. *The Lie algebra of differential operators of order one \mathcal{X} on \mathbb{R}^2 preserving the space \mathcal{S}^{lin} , i.e., such that*

$$[\mathcal{X}, \mathcal{S}^{lin}] \subset \mathcal{S}^{lin}$$

is equal to the image of \mathfrak{sv} by the representation $d\pi_\lambda$ (modulo the addition to \mathcal{X} of operators of multiplication by an arbitrary function of t).

2. *The action of $d\pi_{\lambda+1/4}(\mathfrak{sv})$ on the free Schrödinger operator Δ_0 is given by*

$$[d\pi_{\lambda+1/4}(L_f), \Delta_0] = f'\Delta_0 + \frac{\mathcal{M}^2}{2}f'''r^2 + 2\mathcal{M}\lambda f'' \quad (3.17)$$

$$[d\pi_{\lambda+1/4}(Y_g), \Delta_0] = 2\mathcal{M}^2rg'' \quad (3.18)$$

$$[d\pi_{\lambda+1/4}(M_h), \Delta_0] = 2\mathcal{M}^2h' \quad (3.19)$$

Proof.

Let $\mathcal{X} = f\partial_t + g\partial_r + h$ preserving the space \mathcal{S}^{lin} : this is equivalent to the existence of two functions $\phi(t, r), V(t, r)$ such that $[\mathcal{X}, \Delta_0] = \phi(\Delta_0 + V)$. It is clear that $[h, \mathcal{S}_{lin}] \subset \mathcal{S}_{lin}$ if h is a function of t only.

By considerations of degree, one must then have $[\mathcal{X}, \partial_r] = a(t, r)\partial_r + b(t, r)$, hence f is a function of t only. Then

$$[f\partial_t, 2\mathcal{M}\partial_t - \partial_r^2] = -2\mathcal{M}f'\partial_t \quad (3.20)$$

$$[g\partial_r, 2\mathcal{M}\partial_t - \partial_r^2] = -2\mathcal{M}\partial_t g\partial_r + 2\partial_r g\partial_r^2 + \partial_r^2 g\partial_r \quad (3.21)$$

$$[h, -\partial_r^2] = 2\partial_r h\partial_r + \partial_r^2 h \quad (3.22)$$

so, necessarily,

$$f' = 2\partial_r g = -\phi$$

and

$$(2\mathcal{M}\partial_t - \partial_r^2)g = -2\partial_r h.$$

By putting together these relations, one gets points 1 and 2 simultaneously. \square

Using a left-and-right action of \mathfrak{sv} that combines $d\pi_\lambda$ and $d\pi_{1+\lambda}$, one gets a new family of representations $d\sigma_\lambda$ of \mathfrak{sv} which map the affine space \mathcal{S}^{aff} into differential operators of order zero (that is to say, into functions) :

Proposition 3.5.

Let $d\sigma_\lambda : \mathfrak{sv} \rightarrow Hom(\mathcal{S}^{lin}, \mathcal{S}^{lin})$ defined by the left-and-right infinitesimal action

$$d\sigma_\lambda(X) : D \rightarrow d\pi_{1+\lambda}(X) \circ D - D \circ d\pi_\lambda(X).$$

Then $d\sigma_\lambda$ is a representation of \mathfrak{sv} and $d\sigma_\lambda(\mathfrak{sv})(\mathcal{S}^{aff}) \subset C^\infty(\mathbb{R}^2)$.

Proof.

Let $X_1, X_2 \in \mathfrak{sv}$, and put $d\pi_{\mathcal{S}^1}(X_i) = f_i(t)$, $i = 1, 2$: then, with a slight abuse of notations, $d\sigma_\lambda(X_i) = \text{ad } d\pi_\lambda(X_i) + f'_i$, so

$$[d\sigma_\lambda(X_1), d\sigma_\lambda(X_2)] = [\text{ad}(d\pi_\lambda(X_1)) + f'_1, \text{ad}(d\pi_\lambda(X_2)) + f'_2] \quad (3.23)$$

$$= \text{ad } d\pi_\lambda([X_1, X_2]) + ([d\pi_\lambda(X_1), f'_2(t)] - [d\pi_\lambda(X_2), f'_1(t)]). \quad (3.24)$$

Now $\text{ad } d\pi_\lambda$ commutes with operators of multiplication by any function of time $g(t)$ if $X \in \mathfrak{h}$, and

$$[d\pi_\lambda(L_f), g(t)] = [f(t)\partial_t, g(t)] = f(t)g'(t)$$

so as a general rule

$$[d\pi_\lambda(X_i), g(t)] = f_i(t)g'(t).$$

Hence

$$[d\sigma_\lambda(X_1), d\sigma_\lambda(X_2)] = \text{ad } d\pi_\lambda([X_1, X_2]) + (f_1(t)f_2''(t) - f_2(t)f_1''(t)) \quad (3.25)$$

$$= \text{ad } d\pi_\lambda([X_1, X_2]) + (f_1f_2' - f_2f_1')'(t) \quad (3.26)$$

$$= d\sigma_\lambda([X_1, X_2]). \quad (3.27)$$

By the preceding Theorem, it is now clear that $d\sigma_\lambda(\mathfrak{sv})$ sends \mathcal{S}^{aff} into differential operators of order zero. \square

Remark: choosing $\lambda = \frac{1}{4}$ leads to a representation of Sch^1 preserving the kernel of Δ_0 (as already known from Proposition 1.1). So, in some sense, $\lambda = \frac{1}{4}$ is the 'best' choice.

Clearly, the (affine) subspace $\mathcal{S}_{\leq 2}^{aff} \subset \mathcal{S}^{aff}$ of Schrödinger operators with potentials that are at most quadratic in r , that is,

$$D \in \mathcal{S}_{\leq 2}^{aff} \Leftrightarrow D = 2\mathcal{M}\partial_t - \partial_r^2 + g_0(t)r^2 + g_1(t)r + g_2(t)$$

is mapped into potentials of the same form under $d\sigma_\lambda(SV)$.

Let us use the same vector notation for elements of $\mathcal{S}_{\leq 2}^{aff}$ and for potentials that are at most quadratic in r (what is precisely meant will be clearly seen from the context): set $D = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$, respectively

$V = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$ for $D = \Delta_0 + g_0(t)r^2 + g_1(t)r + g_2(t) \in \mathcal{S}_{\leq 2}^{aff}$, respectively $V(t, r) = g_0(t)r^2 + g_1(t)r + g_2(t) \in C^\infty(\mathbb{R}^2)$. Then one can give an explicit formula for the action of $d\sigma_\lambda$ on \mathcal{S}_2^{aff} .

Proposition 3.6.

1. Let $D = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{S}_{\leq 2}^{aff}$ and $f_0, f_1, f_2 \in C^\infty(\mathbb{R})$. Then the following formulas hold:

$$d\sigma_{\lambda+1/4}(L_{f_0})(D) = - \begin{pmatrix} \frac{-\mathcal{M}^2}{2}f_0''' + 2f_0'g_0 + f_0g_0' \\ f_0g_1' + \frac{3}{2}f_0'g_1 \\ f_0g_2' + f_0'g_2 - 2\mathcal{M}\lambda f_0'' \end{pmatrix} \quad (3.28)$$

$$d\sigma_{\lambda+1/4}(Y_{f_1})(D) = - \begin{pmatrix} 0 \\ 2f_1g_0 - 2\mathcal{M}^2f_1'' \\ f_1g_1 \end{pmatrix} \quad (3.29)$$

$$d\sigma_{\lambda+1/4}(M_{f_2})(D) = \begin{pmatrix} 0 \\ 0 \\ 2\mathcal{M}^2f_2' \end{pmatrix} \quad (3.30)$$

2. Consider the restriction of $d\sigma_{1/4}$ to $\text{Vect}(S^1) \subset \mathfrak{sv}$. Then $d\sigma_{1/4}|_{\text{Vect}(S^1)}$ acts diagonally on the 3-vectors $\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$ and its restriction to the subspaces $\mathcal{S}_i^{aff} := \{\Delta_0 + g(t)r^i \mid g \in C^\infty(\mathbb{R})\}$, $i = 0, 1, 2$, is equal to the coadjoint action of $\text{Vect}(S^1)$ on the affine hyperplane $\mathfrak{vir}_{1/4}^*$ ($i = 2$), and to the usual action of $\text{Vect}(S^1)$ on $\mathcal{F}_{-3/2} \simeq \mathcal{F}_{1/2}^*$ (when $i = 1$), respectively on $\mathcal{F}_{-1} \simeq \mathcal{F}_0^*$ (when $i = 0$). Taking $\lambda \neq 0$ leads to an affine term proportional to f_0'' on the third coordinate, corresponding to the non-trivial affine cocycle in $H^1(\text{Vect}(S^1), \mathcal{F}_{-1})$.

In other words, if one identifies $\mathcal{S}_{\leq 2}^{aff}$ with $\mathfrak{sv}_{\frac{1}{4}}^*$ by

$$\left\langle \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}, (f_0 \ f_1 \ f_2) \right\rangle_{\mathcal{S}_{\leq 2}^{aff} \times \mathfrak{sv}} = \sum_{i=0}^2 \int_{S^1} (g_i f_i)(z) dz,$$

then the restriction of $d\sigma_{1/4}$ to $\text{Vect}(S^1)$ is equal to the restriction of the coadjoint action of \mathfrak{sv} on $\mathfrak{sv}_{\frac{1}{4}}^*$.

But mind that $d\sigma_{1/4}$ is *not* equal to the coadjoint action of \mathfrak{sv} .

Proof. Point 2 is more or less obvious, and we shall only give some of the computations for the first one. One has $[d\pi_{\lambda+1/4}(L_{f_0}), \Delta_0] = f_0' \Delta_0 - \frac{\mathcal{M}}{2} f_0''' r^2 + 2\mathcal{M}\lambda f_0''$, $[d\pi_{\lambda+1/4}(L_{f_0}), g_2(t)] = -f_0(t)g_2'(t)$, $[d\pi_{\lambda+1/4}(L_{f_0}), g_1(t)r] = -(f_0(t)g_1'(t) + \frac{1}{2}f_0'(t)g_1(t))r$, $[d\pi_{\lambda+1/4}(L_{f_0}), g_0(t)r^2] = -(f_0(t)g_0'(t) + f_0'(t)g_0(t))r^2$, so

$$\begin{aligned} d\sigma_{\lambda+1/4}(L_{f_0})(D) &= -f_0'.D + [d\pi_{\lambda+1/4}(L_{f_0}), D] \\ &= -(f_0g_2' + f_0'g_2 - 2\mathcal{M}\lambda f_0'') - (f_0g_1' + \frac{3}{2}f_0'g_1)r - (\frac{\mathcal{M}^2}{2}f_0''' + 2f_0'g_0 + f_0g_0')r^2. \end{aligned}$$

Hence the result for $d\sigma_{\lambda+1/4}(L_{f_0})$. The other computations are similar though somewhat simpler. \square

This representation is easily integrated to a representation σ of the group SV . We let $\Theta(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2} \left(\frac{\phi''}{\phi'} \right)^2$ ($\phi \in \text{Diff}(S^1)$) be the Schwarzian of the function ϕ .

Proposition 3.7.

Let $D = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{S}_{\leq 2}^{aff}$, then

$$\sigma_{\lambda+1/4}(\phi; (a, b))D = \sigma_{\lambda+1/4}(1; (a, b))\sigma_{\lambda+1/4}(\phi; 0)D \quad (3.31)$$

$$\sigma_{\lambda+1/4}(\phi; 1).D = \begin{pmatrix} (\phi')^2.(g_0 \circ \phi) - \frac{\mathcal{M}^2}{2}\Theta(\phi) \\ (\phi')^{\frac{3}{2}}(g_1 \circ \phi) \\ \phi'(g_2 \circ \phi) - 2\mathcal{M}\lambda \frac{\phi''}{\phi} \end{pmatrix} \quad (3.32)$$

$$\sigma_{\lambda+1/4}(1; (a, b)).D = \begin{pmatrix} g_0 \\ g_1 - 2ag_0 + 2\mathcal{M}^2a'' \\ g_2 - ag_1 + a^2g_0 + \mathcal{M}^2(2b' - aa'') \end{pmatrix} \quad (3.33)$$

defines a representation of SV that integrates $d\sigma$, and maps the affine space $\mathcal{S}_{\leq 2}^{aff}$ into itself.

In other words, elements of $\mathcal{S}_{\leq 2}^{aff}$ define an SV -equivariant morphism from \mathcal{H}_λ into $\mathcal{H}_{\lambda+1}$, where \mathcal{H}_λ , respectively $\mathcal{H}_{\lambda+1}$, is the space $C^\infty(\mathbb{R}^2)$ of functions of t, r that are at most quadratic in r , equipped with the action π_λ , respectively $\pi_{\lambda+1}$ (see 1.27).

Proof.

Put $SV = G \ltimes H$. Then the restrictions $\sigma|_G$ and $\sigma|_H$ define representations (this is a classical result for the first action, and may be checked by direct computation for the second one). The associated infinitesimal representation of \mathfrak{sv} is easily seen to be equal to $d\sigma$. \square

In particular, the orbit of the free Schrödinger operator Δ_0 is given by the remarkable formula

$$\sigma_{\lambda+1/4}(\phi; (a, b))\Delta_0 = \begin{pmatrix} -\frac{\mathcal{M}^2}{2}\Theta(\phi) \\ 2\mathcal{M}^2a'' \\ \mathcal{M}^2(2b' - aa'') \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ -2\mathcal{M}\frac{\phi''}{\phi} \end{pmatrix}, \quad (3.34)$$

mixing a third-order cocycle with coefficient \mathcal{M}^2 which extends the Schwarzian cocycle $\phi \rightarrow \Theta(\phi)$ with a second-order cocycle with coefficient $-2\mathcal{M}\lambda$ which extends the well-known cocycle $\phi \rightarrow \frac{\phi''}{\phi}$ in

$H^1(\text{Vect}(S^1), \mathcal{F}_{-1})$. The following paragraph shows that all affine cocycles of \mathfrak{sv} with coefficients in the (linear) representation space $\mathcal{S}_{\leq 2}^{aff}$ are of this form.

3.3 Affine cocycles of \mathfrak{sv} and SV on the space of Schrödinger operators

The representations of groups and Lie algebras described above are of affine type; if one has linear representations of the group G and Lie algebra \mathcal{G} on a module M , one can deform these representations into affine ones, using the following construction.

Let $C : G \rightarrow M$ (resp $c : \mathcal{G} \rightarrow M$) be a 1-cocycle in $Z_{diff}^1(G, M)$ (resp $Z^1(\mathcal{G}, M)$); it defines an affine action of G (resp \mathcal{G}) by deforming the linear action as follows:

$$g * m = g.m + C(g)$$

$$\xi * m = \xi.m + c(\xi)$$

respectively. Here the dot indicates the original linear action and $*$ the affine action. One deduces from the formulas given in propositions 3.5 and 3.6 that the above representations are of this type; the first cohomology of SV (resp \mathfrak{sv}) with coefficients in the module $\mathcal{S}_{\leq 2}^{aff}$ (equipped with the linear action) classifies all the affine deformations of the action, up to isomorphism. They are given by the following theorem:

Theorem 3.8.

The degree-one cohomology of the group SV (resp Lie algebra \mathfrak{sv}) with coefficients in the module $\mathcal{S}_{\leq 2}^{aff}$ (equipped with the linear action) is two-dimensional and can be represented by the following cocycles:

- for SV :

$$C_1(\phi, (a, b)) = \begin{pmatrix} -\frac{1}{2}\Theta(\phi) \\ 2a'' \\ 2b' - aa'' \end{pmatrix}$$

$$C_2(\phi, (a, b)) = \begin{pmatrix} 0 \\ 0 \\ \phi'' \end{pmatrix}$$

- for \mathfrak{sv} :

$$c_1(L_{f_0} + Y_{f_1} + M_{f_2}) = \begin{pmatrix} \frac{1}{2}f_0''' \\ -2f_1'' \\ -2f_2' \end{pmatrix}$$

$$c_2(L_{f_0} + Y_{f_1} + M_{f_2}) = \begin{pmatrix} 0 \\ 0 \\ f_0'' \end{pmatrix}$$

(one easily recognizes that C_1 and c_1 correspond to the representations given in prop.3.7 and prop.3.6 respectively.)

Proof: One shall first make the computations for the Lie algebra and then try to integrate explicitly; here, the "heuristic" version of Van-Est theorem, generalized to the infinite-dimensional case, guarantees the isomorphism between the H^1 groups for SV and \mathfrak{sv} (see [12], chapter IV).

So let us compute $H^1(\mathfrak{sv}, M) \simeq H^1(\mathcal{G} \ltimes \mathfrak{h}, M)$ for $M = \mathcal{S}_{\leq 2}^{aff}$ (equipped with the linear action). Let $c : \mathcal{G} \times \mathfrak{h} \rightarrow M$ be a cocycle and set $c = c' + c''$ where $c' = c|_{\mathcal{G}}$ and $c'' = c|_{\mathfrak{h}}$. One has $c' \in Z^1(\mathcal{G}, M)$ and $c'' \in Z^1(\mathfrak{h}, M)$, and these two cocycles are linked together by the compatibility relation

$$c''([X, \alpha]) - X.(c''(\alpha)) + \alpha.(c'(X)) = 0 \quad (3.35)$$

As a \mathcal{G} -module, $M = \mathcal{F}_{-2} \oplus \mathcal{F}_{-3/2} \oplus \mathcal{F}_{-1}$, so one determines easily that $H^1(\mathcal{G}, \mathcal{F}_{-2})$ and $H^1(\mathcal{G}, \mathcal{F}_{-1})$ are one-dimensional, generated by $L_{f_0} \rightarrow f_0''' dx^2$ and $L_{f_0} \rightarrow f_0'' dx$ respectively, and $H^1(\mathcal{G}, \mathcal{F}_{-3/2}) = 0$ (see [12], chapter IV). One can now readily compute the 1-cohomology of the nilpotent part \mathfrak{h} ; one easily remarks that the linear action on $\mathcal{S}_{\leq 2}^{aff}$ is defined as follows:

$$(Y_{f_1} + M_{f_2}) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2f_1\gamma_0 \\ f_1\gamma_1 \end{pmatrix}$$

Direct computation shows that there are several cocycles, but compatibility condition (3.35) destroys all of them but one:

$$c''(Y_{f_1} + M_{f_2}) = \begin{pmatrix} 0 \\ f_1'' \\ f_2'' \end{pmatrix}$$

The compatibility condition gives

$$c''([L_{f_0}, Y_{f_1} + M_{f_2}]) - L_{f_0} (c''(Y_{f_1} + M_{f_2})) = -\frac{1}{2}f_1f_0''''$$

On the other hand one finds:

$$(Y_{f_1} + M_{f_2}).(c'(L_{f_0})) = (Y_{f_1} + M_{f_2}) \cdot \begin{pmatrix} f_0'''' \\ 0 \\ f_0'' \end{pmatrix} = \begin{pmatrix} 0 \\ 2f_1f_0'''' \\ 0 \end{pmatrix}$$

Hence the result, with the right proportionality coefficients, and one obtains the formula for c_1 .

One also remarks that the term with $f_0'' dx$ disappears through the action of \mathfrak{h} , so it will induce an independent generator in $H^1(\mathfrak{sv}, \mathcal{S}^{aff})$, precisely c_2 .

Finally the cocycles C_1 and C_2 in $H^1(SV, \mathcal{S}_{\leq 2}^{aff})$ are not so hard to compute, once we have determined the action of H on $\mathcal{S}_{\leq 2}^{aff}$, which is unipotent as follows:

$$(a, b) \cdot \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 + 2a\gamma_0 \\ \gamma_2 + a\gamma_1 - a^2\gamma_0 \end{pmatrix}$$

□

3.4 Action on Dirac-Lévy-Leblond operators

Lévy-Leblond introduced in [21] a matrix differential operator \mathcal{D}_0 on \mathbb{R}^{d+1} (with coordinates t, r_1, \dots, r_d) of order one, similar to the Dirac operator, whose square is equal to $-\Delta_0 \otimes \text{Id} = - \begin{pmatrix} \Delta_0 & & \\ & \ddots & \\ & & \Delta_0 \end{pmatrix}$

for $\Delta_0 = 2\mathcal{M}\partial_t - \sum_{i=1}^d \partial_{r_i}^2$. So, in some sense, \mathcal{D}_0 is a square-root of the free Schrödinger operator, just as the Dirac operator is a square-root of the D'Alembertian. The group of Lie invariance of \mathcal{D}_0 has been studied in [35], it is isomorphic to Sch^d with a realization different but close to that of Proposition 1.1.

Let us restrict to the case $d = 1$ (see [32] for details). Then \mathcal{D}_0 acts on spinors, or couples of functions $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of two variables t, r , and may be written as

$$\mathcal{D}_0 = \begin{pmatrix} \partial_r & -2\mathcal{M} \\ \partial_t & -\partial_r \end{pmatrix}. \quad (3.36)$$

One checks immediately that $\mathcal{D}_0^2 = -\Delta_0 \otimes \text{Id}$.

From the explicit realization of \mathfrak{sch} on spinors (see [32]), one may easily guess a realization of \mathfrak{sv} that extends the action of \mathfrak{sch} , and, more interestingly perhaps, acts on an affine space \mathcal{D}^{aff} of Dirac-Lévy-Leblond operators with potential, in the same spirit as in the previous section. More precisely, one has the following theorem (we need to introduce some notations first).

Definition 3.2.

Let \mathcal{D}^{lin} be the vector space of first order matrix operators on \mathbb{R}^2 defined by

$$D \in \mathcal{D}^{lin} \Leftrightarrow D = h(r, t)\mathcal{D}_0 + \begin{pmatrix} 0 & 0 \\ V(r, t) & 0 \end{pmatrix}, \quad h, V \in C^\infty(\mathbb{R}^2)$$

and \mathcal{D}^{aff} , $\mathcal{D}_{\leq 2}^{aff}$ be the affine subspaces of \mathcal{D}^{lin} such that

$$D \in \mathcal{D}^{aff} \Leftrightarrow D = \mathcal{D}_0 + \begin{pmatrix} 0 & 0 \\ V(r, t) & 0 \end{pmatrix},$$

$$D \in \mathcal{D}_{\leq 2}^{aff} \Leftrightarrow D = \mathcal{D}_0 + \begin{pmatrix} 0 & 0 \\ g_0(t)r^2 + g_1(t)r + g_2(t) & 0 \end{pmatrix}.$$

We shall call *Dirac potential* a matrix of the form $\begin{pmatrix} 0 & 0 \\ V(r, t) & 0 \end{pmatrix}$, with $V \in C^\infty(\mathbb{R}^2)$.

Definition 3.3.

Let $d\pi_\lambda^\sigma$ ($\lambda \in \mathbb{C}$) be the infinitesimal representation of \mathfrak{sv} on the space $\tilde{\mathcal{H}}_\lambda^\sigma \simeq (C^\infty(\mathbb{R}^2))^2$ with coordinates t, r , defined by

$$\begin{aligned} d\pi_\lambda^\sigma(L_f) &= (-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}\mathcal{M}f''(t)r^2) \otimes \text{Id} \\ &\quad - f'(t) \otimes \begin{pmatrix} \lambda - \frac{1}{4} & \\ & \lambda + \frac{1}{4} \end{pmatrix} - \frac{1}{2}f''(t)r \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \end{aligned} \quad (3.37)$$

$$d\pi_\lambda^\sigma(Y_g) = (-g(t)\partial_r - \mathcal{M}f'(t)r) \otimes \text{Id} - f'(t) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad (3.38)$$

$$d\pi_\lambda^\sigma(M_h) = -\mathcal{M}f(t) \otimes \text{Id}. \quad (3.39)$$

Theorem 3.9.

1. Let $d\sigma : \mathfrak{sv} \rightarrow \text{Hom}(\mathcal{D}^{lin}, \mathcal{D}^{lin})$ defined by the left-and-right infinitesimal action

$$d\sigma(X) : D \rightarrow d\pi_1^\sigma(X) \circ D - D \circ d\pi_{\frac{1}{2}}^\sigma(X).$$

Then $d\sigma$ maps $\mathcal{D}_{\leq 2}^{aff}$ into the vector space of Dirac potentials.

2. If one represents the Dirac potential $V = \begin{pmatrix} 0 & 0 \\ g_0(t)r^2 + g_1(t)r + g_2(t) & 0 \end{pmatrix}$ or, indifferently, the Dirac operator $\mathcal{D}_0 + V$, by the vector $\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$, then the action of $d\sigma$ on $\mathcal{D}_{\leq 2}^{aff}$ is given by the same formula as in Proposition 3.6, except for the affine terms (with coefficient proportional to \mathcal{M} or \mathcal{M}^2) that should all be divided by $2\mathcal{M}$.

We shall skip the proof (in the same spirit as Theorem 3.4, Proposition 3.5 and Proposition 3.6) which presents no difficulty, partly for lack of space, partly because the action on Dirac operators doesn't give anything new by comparison with the case of Schrödinger operators.

Note that, as in the previous section, one may define a 'shifted' action

$$d\sigma_\lambda(X) : D \rightarrow d\pi_{\lambda+1}^\sigma(X) \circ D - D \circ d\pi_{\lambda+\frac{1}{2}}^\sigma(X) \quad (3.40)$$

which will only modify the coefficients of the affine cocycles.

As a concluding remark of these two sections, let us emphasize two points:

–contrary to the case of the Hill operators, there is a free parameter λ in the left-and-right actions on the affine space of Schrödinger or Dirac operators;

–looking at the differences of indices between the left action and the right action, one may consider somehow that Schrödinger operators are of order one, while Dirac operators are of order $\frac{1}{2}$! (recall the difference of indices was $2 = \frac{3}{2} - (-\frac{1}{2})$ in the case of the Hill operators, which was the signature of operators of order 2 – see [5]).

So Schrödinger operators are somehow reminiscent of the operators $\partial + u$ of order one on the line, which intertwine \mathcal{F}_λ with $\mathcal{F}_{1+\lambda}$ for any value of λ . The case of the Dirac operators, on the other hand, has no counterpart whatsoever for differential operators on the line.

3.5 About multi-diagonal differential operators and some Virasoro-solvable Lie algebras

The original remark that prompted the introduction of multi-diagonal differential operators in our context (see below for a definition) was the following. Consider the space \mathbb{R}^3 with coordinates r, t, ζ as in Chapter 1. We introduce the two-dimensional Dirac operator

$$\tilde{\mathcal{D}}_0 = \begin{pmatrix} \partial_r & -2\partial_\zeta \\ \partial_t & -\partial_r \end{pmatrix}, \quad (3.41)$$

acting on spinors $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in (C^\infty(\mathbb{R}^3))^2$ - the reader will have noticed that $\tilde{\mathcal{D}}_0$ can be obtained from the Dirac-Lévy-Leblond operator \mathcal{D}_0 of section 3.3 by taking a formal Laplace transform with respect to the mass. The kernel of $\tilde{\mathcal{D}}_0$ is given by the equations of motion obtained from the Lagrangian density

$$(\bar{\phi}_2(\partial_r\phi_1 - 2\partial_\zeta\phi_2) - \bar{\phi}_1(\partial_t\phi_1 - \partial_r\phi_2)) dt dr d\zeta.$$

Let $d\tilde{\pi}_{\frac{1}{2}}^\sigma$ be the Laplace transform with respect to \mathcal{M} of the infinitesimal representation of \mathfrak{sv} given in Definition 3.3. Then $d\tilde{\pi}_{\frac{1}{2}}^\sigma(\mathfrak{sv})$ preserves the space of solutions of the equation $\tilde{\mathcal{D}}_0 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$, $\phi_1, \phi_2 \in C^\infty(\mathbb{R}^3)$. Now, by computing $\tilde{\mathcal{D}}_0 \left(d\tilde{\pi}_{\frac{1}{2}}^\sigma(X) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right)$ for $X \in \mathfrak{sv}$ and $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ in the kernel of $\tilde{\mathcal{D}}_0$, it clearly appears (do it!) that if one adds the constraint $\partial_\zeta \phi_1 = 0$, then $\tilde{\mathcal{D}}_0 \left(d\tilde{\pi}_{\frac{1}{2}}^\sigma(X) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right) = 0$ for every $X \in \mathfrak{sv}$, and, what is more, the transformed spinor $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = d\tilde{\pi}_{\frac{1}{2}}^\sigma(X) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ also satisfies the same constraint $\partial_\zeta \psi_1 = 0$. One may realize this constraint by adding to the Lagrangian density the Lagrange multiplier term $(h\partial_\zeta \bar{\phi}_1 - \bar{h}\partial_\zeta \phi_1) dt dr d\zeta$. The new equations of motion read then $\nabla \begin{pmatrix} -h/2 \\ \phi_2 \\ -\phi_1 \end{pmatrix} = 0$, with

$$\nabla = \begin{pmatrix} 2\partial_\zeta & \partial_r & \partial_t \\ & 2\partial_\zeta & \partial_r \\ & & 2\partial_\zeta \end{pmatrix}. \quad (3.42)$$

This is our main example of a multi-diagonal differential operator. Quite generally, we shall call *multi-diagonal* a function- or operator-valued matrix $M = (M_{i,j})_{0 \leq i,j \leq d-1}$ such that $M_{i,j} = M_{i+k,j+k}$ for every admissible triple of indices i, j, k . So M is defined for instance by the d independent coefficients $M_{0,0}, \dots, M_{0,j}, \dots, M_{0,d-1}$, with $M_{0,j}$ located on the j -shifted diagonal.

An obvious generalization in d dimensions leads to the following definition.

Definition 3.4.

Let ∇^d be the $d \times d$ matrix differential operator of order one, acting on d -uples of functions $H = \begin{pmatrix} h_0 \\ \vdots \\ h_{d-1} \end{pmatrix}$ on \mathbb{R}^d with coordinates $t = (t_0, \dots, t_{d-1})$, given by

$$\nabla^d = \begin{pmatrix} \partial_{t_{d-1}} & \partial_{t_{d-2}} & \cdots & \partial_{t_1} & \partial_{t_0} \\ 0 & \ddots & & & \partial_{t_1} \\ \vdots & \ddots & & & \vdots \\ 0 & \cdots & & 0 & \partial_{t_{d-2}} \\ & & & & \partial_{t_{d-1}} \end{pmatrix} \quad (3.43)$$

So ∇^d is upper-triangular, with coefficients $\nabla_{i,j}^d = \partial_{t_{i-j+d-1}}$, $i \leq j$. The kernel of ∇^d is defined by a system of equations linking h_0, \dots, h_{d-1} . The set of differential operators of order one of the form

$$X = X_1 + \Lambda = \left(\sum_{i=0}^{d-1} f_i(t) \partial_{t_i} \right) \otimes \text{Id} + \Lambda, \quad \Lambda = (\lambda_{i,j}) \in \text{Mat}_{d \times d}(C^\infty(\mathbb{R}^d)) \quad (3.44)$$

preserving the equation $\nabla^d H = 0$ forms a Lie algebra, much too large for our purpose.

Suppose now (this is a very restrictive condition) that $\Lambda = \text{diag}(\lambda_0, \dots, \lambda_{d-1})$ is diagonal. Since ∇^d is an operator with constant coefficients, $[X_1, \nabla^d]$ has no term of zero order, whereas $[\Lambda, \nabla^d]_{i,j} = \lambda_i \partial_{t_{i-j+d-1}} - \partial_{t_{i-j+d-1}} \lambda_j$ ($i \leq j$) does have terms of zero order in general. One possibility to solve this

constraint, motivated by the preceding examples (see for instance the representation $d\pi_\lambda$ of (1.27), with \mathcal{M} replaced by ∂_ζ), but also by the theory of scaling in statistical physics (see commentary following Proposition 1.1.), is to impose $\lambda_i = \lambda_i(t_0)$, $i = 0, \dots, d-2$, and $\lambda_{d-1} = 0$. Since $[X, \nabla^d]$ is of first order, preserving $\text{Ker}\nabla^d$ is equivalent to a relation of the type $[X, \nabla^d] = A\nabla^d$, with $A = A(X) \in \text{Mat}_{d \times d}(C^\infty(\mathbb{R}^d))$. Then the matrix operator $[X_1, \nabla^d]$ is upper-triangular, and multi-diagonal, so this must also hold for $A\nabla^d - [\Lambda, \nabla^d]$. By looking successively at the coefficients of $\partial_{t_{d-1-l}}$ on the l -shifted diagonals, $l = 0, \dots, d-1$, one sees easily that A must also be upper-triangular and multi-diagonal, and that one must have $\lambda_i(t_0) - \lambda_{i+1}(t_0) = \lambda(t_0)$ for a certain function λ independent of i ,

so $\Lambda = \begin{pmatrix} (d-1)\lambda & & & \\ & \ddots & & \\ & & \lambda & \\ & & & 0 \end{pmatrix}$. Also, denoting by $a_0 = A_{0,0}, \dots, a_{d-1} = A_{0,d-1}$ the coefficients of the first line of the matrix A , one obtains :

$$\partial_{t_i} f_j = 0 \quad (i > j); \quad (3.45)$$

$$a_0 = \partial_{t_0} f_0 + (d-1)\lambda = \partial_{t_1} f_1 + (d-2)\lambda = \dots = \partial_{t_{d-1}} f_{d-1}; \quad (3.46)$$

$$a_i = \partial_{t_0} f_i = \partial_{t_1} f_{i+1} = \dots = \partial_{t_{d-i-1}} f_{d-1} \quad (i = 1, \dots, d-1). \quad (3.47)$$

In particular, f_0 depends only on t_0 .

From all these considerations follows quite naturally the following definition. We let $\Lambda_0 \in \text{Mat}_{d \times d}(\mathbb{R})$

be the diagonal matrix $\Lambda_0 = \begin{pmatrix} d-1 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$.

Lemma 3.10.

Let $\mathfrak{m}\mathfrak{d}_\varepsilon^d$ ($\varepsilon \in \mathbb{R}$) be the set of differential operators of order one of the type

$$X = \left(f_0(t_0)\partial_{t_0} + \sum_{i=1}^{d-1} f_i(t)\partial_{t_i} \right) \otimes \text{Id} - \varepsilon f'_0(t_0) \otimes \Lambda_0 \quad (3.48)$$

preserving $\text{Ker}\nabla^d$.

Then $\mathfrak{m}\mathfrak{d}_\varepsilon$ forms a Lie algebra.

Proof.

Let \mathcal{X} be the Lie algebra of vector fields X of the form

$$X = X_1 + X_0 = \left(f_0(t_0)\partial_{t_0} + \sum_{i=1}^{d-1} f_i(t)\partial_{t_i} \right) \otimes \text{Id} + \Lambda, \quad \Lambda = \text{diag}(\lambda_i)_{i=0, \dots, d-1} \in \text{Mat}_{d \times d}(C^\infty(\mathbb{R}^d))$$

preserving $\text{Ker}\nabla^d$. Then the set $\{Y = \sum_{i=0}^{d-1} f_i(t)\partial_{t_i} \mid \exists \lambda \in \text{Mat}_{d \times d}(C^\infty(\mathbb{R}^d)), Y + \Lambda \in \mathcal{X}\}$ of the differential parts of order one of the elements of \mathcal{X} forms a Lie algebra, say \mathcal{X}_1 . Define

$$\mathcal{X}_1^\varepsilon := \left\{ \sum_{i=0}^{d-1} f_i(t)\partial_{t_i} - \varepsilon f'_0(t) \otimes \Lambda_0 \mid \sum_{i=0}^{d-1} f_i(t)\partial_{t_i} \in \mathcal{X}_1 \right\}.$$

Let $Y = (\sum f_i(t)\partial_{t_i}) \otimes \text{Id} - \varepsilon f'_0(t) \otimes \Lambda_0$, $Z = (\sum g_i(t)\partial_{t_i}) \otimes \text{Id} - \varepsilon g'_0(t) \otimes \Lambda_0$ be two elements of $\mathcal{X}_1^\varepsilon$: then

$$[Y, Z] = ((f_0g'_0 - f'_0g_0)(t_0)\partial_{t_0} + \dots) \otimes \text{Id} - \varepsilon(f_0g'_0 - f'_0g_0)'(t_0) \otimes \Lambda_0$$

belongs to $\mathcal{X}_1^\varepsilon$, so $\mathcal{X}_1^\varepsilon$ forms a Lie algebra. Finally, $\mathfrak{md}_\varepsilon^d$ is the Lie subalgebra of $\mathcal{X}_1^\varepsilon$ consisting of all differential operators preserving $\text{Ker}\nabla^d$.

□

It is quite possible to give a family of generators and relations for $\mathfrak{md}_\varepsilon^d$. The surprising fact, though, is the following : for $d \geq 4$, one finds by solving the equations that f''_0 is necessarily zero if $\varepsilon \neq 0$ (see proof of Theorem 3.11). So in any case, the only Lie algebra that deserves to be considered for $d \geq 4$ is \mathfrak{md}_0^d .

The algebras $\mathfrak{md}_\varepsilon^d$ ($d = 2, 3$), \mathfrak{md}_0^d ($d \geq 4$) are semi-direct products of a Lie subalgebra isomorphic to $\text{Vect}(S^1)$, with generators

$$L_{f_0}^{(0)} = (-f_0(t_0)\partial_{t_0} + \dots) \otimes \text{Id} - \varepsilon f'_0(t_0) \otimes \Lambda_0$$

and commutators $[L_{f_0}^{(0)}, L_{g_0}^{(0)}] = L_{f'_0g_0 - f_0g'_0}^{(0)}$, with a nilpotent Lie algebra consisting of all generators with coefficient of ∂_{t_0} vanishing. When $d = 2, 3$, one retrieves realizations of the familiar Lie algebras $\text{Vect}(S^1) \ltimes \mathcal{F}_{1+\varepsilon}$ and $\mathfrak{sv}_\varepsilon$.

Theorem 3.11. (structure of $\mathfrak{md}_\varepsilon^d$).

1. (case $d = 2$). Put $t = t_0, r = t_1$: then $\mathfrak{md}_\varepsilon^2 = \langle L_f^{(0)}, L_g^{(1)} \rangle_{f,g \in C^\infty(S^1)}$ with

$$L_f^{(0)} = (-f(t)\partial_t - (1 + \varepsilon)f'(t)r\partial_r) \otimes \text{Id} + \varepsilon f'(t) \otimes \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad (3.49)$$

$$L_g^{(1)} = -g(t)\partial_r. \quad (3.50)$$

It is isomorphic to $\text{Vect}(S^1) \ltimes \mathcal{F}_{1+\varepsilon}$.

2. (case $d = 3$). Put $t = t_0, r = t_1, \zeta = t_2$: then $\mathfrak{md}_\varepsilon^3 = \langle L_f^{(0)}, L_g^{(1)}, L_h^{(2)} \rangle_{f,g,h \in C^\infty(S^1)}$ with

$$L_f^{(0)} = \left(-f(t)\partial_t - (1 + \varepsilon)f'(t)r\partial_r - \left[(1 + 2\varepsilon)f'(t)\zeta + \frac{1 + \varepsilon}{2}f''(t)r^2 \right] \partial_\zeta \right) \otimes \text{Id} + \varepsilon f'(t) \otimes \begin{pmatrix} 2 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad (3.51)$$

$$L_g^{(1)} = -g(t)\partial_r - g'(t)r\partial_\zeta, \quad (3.52)$$

$$L_h^{(2)} = -h(t)\partial_\zeta. \quad (3.53)$$

The Lie algebra obtained by taking the modes

$$L_n = L_{t^{n+1}}^{(0)}, \quad Y_m = L_{t^{m+1+\varepsilon}}^{(1)}, \quad M_p = L_{t^{p+1+2\varepsilon}}^{(2)} \quad (3.54)$$

is isomorphic to $\mathfrak{sv}_{1+2\varepsilon}$ (see Definition 1.7). In particular, the differential parts give three independent copies of the representation $d\tilde{\pi}$ of \mathfrak{sv} when $\varepsilon = -\frac{1}{2}$.

3. (case $\varepsilon = 0, d \geq 2$) Then $\mathfrak{md}_\varepsilon^d \simeq \text{Vect}(S^1) \otimes \mathbb{R}[\eta]/\eta^d$ is generated by the

$$L_g^{(k)} = -g(t_0)\partial_{t_k} - \sum_{i=1}^{d-1-k} g^{(i)}(t_0)t_1^{i-1} \left(\frac{1}{i!}t_1\partial_{t_{i+k}} + \frac{1}{(i-1)!} \sum_{j=2}^{d-i-k} t_j\partial_{t_{i+j+k-1}} \right), \quad g \in C^\infty(S^1) \quad (3.55)$$

$k = 0, \dots, d-1$, with commutators $[L_g^{(i)}, L_h^{(j)}] = L_{g'h - gh'}^{(i+j)}$ if $i+j \leq d-1$, 0 else.

Proof.

Let $X = -\left(f_0(t_0)\partial_{t_0} + \sum_{i=1}^{d-1} f_i(t)\partial_{t_i}\right) \otimes \text{Id} + \varepsilon f'_0(t) \otimes \Lambda_0$: a set of necessary and sufficient conditions for X to be in $\mathfrak{m}\mathfrak{d}_\varepsilon^d$ has been given before Lemma 3.10, namely

$$\partial_{t_i} f_j = 0 \text{ if } i > j,$$

$$(1 + \varepsilon(d-1))f'_0(t_0) = \partial_{t_1} f_1(t_0, t_1) + \varepsilon(d-2)f'_0(t_0) = \dots = \partial_{t_{d-1}} f_{d-1}(t_0, \dots, t_{d-1})$$

and

$$\partial_{t_0} f_i = \partial_{t_1} f_{i+1} = \dots = \partial_{t_{d-i-1}} f_{d-1} \quad (i = 1, \dots, d-1).$$

Solving successively these equations yields

$$f_i(t_0, \dots, t_i) = (1 + \varepsilon i)f'_0(t_0) \cdot t_i + f_i^{[1]}(t_0, \dots, t_{i-1}), \quad i \geq 1; \quad (3.56)$$

$$f_i^{[1]}(t_0, \dots, t_{i-1}) = \partial_{t_0} f_1^{[1]}(t_0) \cdot t_{i-1} + (1 + \varepsilon)f''_0(t_0) \int_0^{t_{i-1}} t_1 dt_{i-1} + f_i^{[2]}(t_0, \dots, t_{i-2}); \quad (3.57)$$

At the next step, the relation $\partial_{t_0} f_2 = \partial_{t_1} f_3$ yields the equation

$$(1 + 2\varepsilon)f''_0(t_0) \cdot t_2 + (f_1^{[1]})''(t_0) \cdot t_1 + 2(1 + \varepsilon)f'''_0(t_0) \cdot t_1 + (f_2^{[2]})'(t_0) = (1 + \varepsilon)f''_0(t_0) \cdot t_2 + \partial_{t_1} f_3^{[2]}(t_0, t_1)$$

which has no solution as soon as $\varepsilon \neq 0$ and $f''_0 \neq 0$. So, as we mentioned without proof before the theorem, the most interesting case is $\varepsilon = 0$ when $d \geq 4$.

The previous computations completely solve the cases $d = 2$ and $d = 3$. So let us suppose that $d \geq 4$ and $\varepsilon = 0$.

Then, by solving the next equations, one sees by induction that f_0, \dots, f_{d-1} may be expressed in terms of d arbitrary functions of t_0 , namely, $f_0 = f_0^{[0]}, f_1 = f_1^{[1]}, f_2 = f_2^{[2]}, \dots, f_{d-1} = f_{d-1}^{[d-1]}$, and that generators satisfying $f_i^{[i]} = 0$ for every $i \neq k$, k fixed, are necessarily of the form

$$f_k^{[k]}(t_0)\partial_{t_k} + \sum_{j=1}^{d-1-k} g_{k+j}(t_0, \dots, t_j)\partial_{t_{k+j}}$$

for functions g_{k+j} that may be expressed in terms of $f_k^{[k]}$ and its derivatives.

One may then easily check that $L_{-f_k^{[k]}}^{(k)}$ is of this form and satisfies the conditions for being in $\mathfrak{m}\mathfrak{d}_\varepsilon^d$, so we have proved that the $L_f^{(k)}$, $k = 0, \dots, d-1$, $f \in C^\infty(S^1)$, generate $\mathfrak{m}\mathfrak{d}_\varepsilon^d$.

All there remains to be done is to check for commutators. Since $L_f^{(i)}$ is homogeneous of degree $-i$ for the Euler-type operator $\sum_{k=0}^{d-1} k t_k \partial_{t_k}$, one necessarily has $[L_f^{(i)}, L_g^{(j)}] = L_{C(f,g)}^{(i+j)}$ for a certain function C (depending on f and g) of the time-coordinate t_0 . One gets immediately $[L_f^{(0)}, L_g^{(0)}] = L_{f'g - fg'}$. Next (supposing $l > 0$), since

$$L_g^{(0)} = -\sum_{i=0}^{l-1} E_i^0(g)\partial_{t_i} - (g'(t_0)t_l + F_l^0(t_0, \dots, t_{l-1}))\partial_{t_l} + \dots$$

where $E_i^0(g)$, $i = 0, \dots, l-1$ do not depend on t_l , and

$$L_h^{(l)} = -h(t_0)\partial_{t_l} + \dots,$$

one gets $[L_g^{(0)}, L_h^{(l)}] = (gh' - g'h)(t_0)\partial_{t_l} + \dots$, so $[L_g^{(0)}, L_h^{(l)}] = L_{g'h - gh'}^{(l)}$. Considering now $k, l > 0$, then one has

$$L_g^{(k)} = - \sum_{i=0}^{l-1} E_i^k(g)\partial_{t_{i+k}} - (h'(t_0)t_l + F_l^k(t_0, \dots, t_{l-1}))\partial_{t_{l+k}}$$

where $E_i^k(g)$, $i = 0, \dots, l-1$, do not depend on t_l , and a similar formula for $L_h^{(l)}$, which give together the right formula for $[L_g^{(k)}, L_h^{(l)}]$.

□

Let us come back to the original motivation, that is, finding new representations of \mathfrak{sv} arising in a geometric context. Denote by $d\pi^{(3,0)}$ the realization of \mathfrak{sv} given in Theorem 3.11.

Definition 3.5

Let $d\pi^\nabla$ be the infinitesimal representation of \mathfrak{sv} on the space $\tilde{\mathcal{H}}^\nabla \simeq (C^\infty(\mathbb{R}^2))^3$ with coordinates t, r , defined by

$$d\tilde{\rho}(L_f) = \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r \right) \otimes \text{Id} + f'(t) \otimes \begin{pmatrix} -1 & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix} \quad (3.58)$$

$$+ \frac{1}{2}f''(t)r \otimes \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \frac{1}{4}f'''(t)r^2 \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}; \quad (3.59)$$

$$d\tilde{\rho}(Y_f) = -f(t)\partial_r \otimes \text{Id} + f'(t) \otimes \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + f''(t)r \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}; \quad (3.60)$$

$$d\tilde{\rho}(M_f) = f'(t) \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \quad (3.61)$$

Proposition 3.12.

For every $X \in \mathfrak{sv}$, $d\pi^\nabla(X) \circ \nabla - \nabla \circ d\pi^{(3,0)}(X) = 0$.

Proof.

Let $X \in \mathfrak{sv}$; put $d\pi^{(3,0)}(X) = -(f_0(t)\partial_t + f_1(t, r)\partial_r + f_2(t, r, \zeta)\partial_\zeta) \otimes \text{Id} - f'_0(t) \otimes \begin{pmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 0 \end{pmatrix}$.

The computations preceding Lemma 3.10 prove that $[d\pi^{(3,0)}(X), \nabla^d] = A(X)\nabla^d$, $A(X)$ being the upper-triangular, multi-diagonal matrix defined by

$$A(X)_{0,0} = \partial_\zeta f_2, \quad A(X)_{0,1} = \partial_r f_2, \quad A(X)_{0,2} = \partial_t f_2.$$

Hence one has

$$A(L_f) = \frac{r}{2}f''(t) \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \frac{r^2}{4}f'''(t) \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix},$$

$$A(Y_g) = g'(t) \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + rg''(t) \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix},$$

and

$$A(M_h) = f'(t) \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Hence the result.

□

Remark.

Consider the affine space

$$\mathcal{H}_{\nabla}^{aff} = \left\{ \nabla + \begin{pmatrix} g_0 & g_1 & g_2 \\ & g_0 & g_1 \\ & & g_0 \end{pmatrix} \mid g_0, g_1, g_2 \in C^\infty(S^1 \times \mathbb{R}^2) \right\}.$$

Then one may define an infinitesimal left-and-right action $d\sigma$ of \mathfrak{sv} on $\mathcal{H}_{\nabla}^{aff}$ by putting

$$d\sigma(X)(\nabla + V) = d\pi^\nabla(X) \circ (\nabla + V) - (\nabla + V) \circ d\pi^{(3,0)}(X),$$

but the action is simply linear this time, since $d\pi^\nabla \circ \nabla = \nabla \circ d\pi^{(3,0)}(X)$. So this action is not very interesting and doesn't give anything new.

4 Cartan's prolongation and generalized modules of tensor densities.

4.1 The Lie algebra \mathfrak{sv} as a Cartan prolongation

As in the case of vector fields on the circle, it is natural, starting from the representation $d\tilde{\pi}$ of \mathfrak{sv} given by formula (1.18), to consider the subalgebra $\mathfrak{fsv} \subset \mathfrak{sv}$ made up of the vector fields with polynomial coefficients. Recall from Definition 1.6 that the outer derivation δ_2 of \mathfrak{sv} is defined by

$$\delta_2(L_n) = n, \quad \delta_2(Y_m) = m - \frac{1}{2}, \quad \delta_2(M_n) = n - 1 \quad (n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}). \quad (4.1)$$

and that δ_2 is simply obtained from the Lie action of the Euler operator $t\partial_t + r\partial_r + \zeta\partial_\zeta$ in the representation $d\tilde{\pi}$. The Lie subalgebra \mathfrak{fsv} is given more abstractly, using δ_2 , as

$$\mathfrak{fsv} = \bigoplus_{k=-1}^{+\infty} \mathfrak{sv}_k \quad (4.2)$$

where $\mathfrak{sv}_k = \{X \in \mathfrak{sv} \mid \delta_2(X) = kX\} = \langle L_k, Y_{k+\frac{1}{2}}, M_{k+1} \rangle$ is the eigenspace of δ_2 corresponding to the eigenvalue $k \in \mathbb{Z}$.

Note in particular that $\mathfrak{sv}_{-1} = \langle L_{-1}, Y_{-\frac{1}{2}}, M_0 \rangle$ is commutative, generated by the infinitesimal translations $\partial_t, \partial_r, \partial_\zeta$ in the vector field representation, and that $\mathfrak{g}_0 = \langle L_0, Y_{\frac{1}{2}}, M_1 \rangle = \langle L_0 \rangle \ltimes \langle Y_{\frac{1}{2}}, M_1 \rangle$ is solvable.

Theorem 4.1.

The Lie algebra \mathfrak{sv} is isomorphic to the Cartan prolongation of $\mathfrak{sv}_{-1} \oplus \mathfrak{sv}_0$ where $\mathfrak{sv}_{-1} \simeq \langle X_{-1}, Y_{-\frac{1}{2}}, M_0 \rangle$ and $\mathfrak{sv}_0 \simeq \langle X_0, Y_{\frac{1}{2}}, M_1 \rangle$.

Proof.

Let \mathfrak{sv}_n ($n = 1, 2, \dots$) the n -th level vector space obtained from Cartan's construction, so that the Cartan prolongation of $\mathfrak{sv}_{-1} \oplus \mathfrak{sv}_0$ is equal to the Lie algebra $\mathfrak{sv}_{-1} \oplus \mathfrak{sv}_0 \oplus \bigoplus_{n \geq 1} \mathfrak{g}_n$. It will be enough, to establish the required isomorphism, to prove the following. Consider the representation $d\tilde{\pi}$ of \mathfrak{sv} . Then the space \mathfrak{h}_n defined through induction on n by

$$\mathfrak{h}_{-1} = \pi(\mathfrak{sv}_{-1}) = \langle \partial_t, \partial_r, \partial_\zeta \rangle \quad (4.3)$$

$$\mathfrak{h}_0 = \pi(\mathfrak{sv}_0) = \langle t\partial_t + \frac{1}{2}r\partial_r, t\partial_r + r\partial_\zeta, t\partial_\zeta \rangle \quad (4.4)$$

$$\mathfrak{h}_{k+1} = \{X \in \mathcal{X}_{k+1} \mid [X, \mathfrak{h}_{-1}] \subset \mathfrak{h}_k\}, \quad (k \geq 0) \quad (4.5)$$

(where \mathcal{X}_k is the space of vector fields with polynomial coefficients of degree k) is equal to $\pi(\mathfrak{sv}_n)$ for any $n \geq 1$.

So assume that $X = f(t, r, \zeta)\partial_t + g(t, r, \zeta)\partial_r + h(t, r, \zeta)\partial_\zeta$ satisfies

$$[X, \mathfrak{h}_{-1}] \subset \pi(\mathfrak{sv}_n) = \langle t^{n+1}\partial_t + \frac{1}{2}(n+1)t^n r\partial_r + \frac{1}{4}(n+1)nt^{n-1}r^2\partial_\zeta, t^{n+1}\partial_r + (n+1)t^n r\partial_\zeta, t^{n+1}\partial_\zeta \rangle. \quad (4.6)$$

In the following lines, C_1, C_2, C_3 are undetermined constants. Then (by comparing the coefficients of ∂_t)

$$f(t, r, \zeta) = C_1 t^{n+2}.$$

By inspection of the coefficients of ∂_r , one gets then

$$\partial_t g(t, r, \zeta) = \frac{C_1}{2}(n+1)(n+2)t^n r + C_2(n+2)t^{n+1}$$

so

$$g(t, r, \zeta) = \frac{C_1}{2}(n+2)t^{n+1}r + C_2 t^{n+2} + G(r, \zeta)$$

with an unknown polynomial $G(r, \zeta)$. But

$$[X, Y_{-\frac{1}{2}}] = \left(\frac{C_1}{2}(n+2)t^{n+1} + \partial_r G(r, \zeta) \right) \partial_r \text{ mod } \partial_\zeta$$

so $\partial_r G(r, \zeta) = 0$.

Finally, by comparing the coefficients of ∂_ζ , one gets

$$[X, L_{-1}] = (n+2)C_1[t^{n+1}\partial_t + \frac{1}{2}(n+1)t^n r\partial_r] + C_2(n+2)t^{n+1}\partial_r + \partial_t h \partial_\zeta$$

so

$$\partial_t h(t, r, \zeta) = \frac{C_1}{4}(n+2)(n+1)nt^{n-1}r^2 + C_2(n+2)(n+1)t^n r + C_3(n+2)t^{n+1}$$

whence

$$h(t, r, \zeta) = \frac{C_1}{4}(n+2)(n+1)t^n r^2 + C_2(n+2)t^{n+1}r + C_3 t^{n+2} + H(r, \zeta)$$

where $H(r, \zeta)$ is an unknown polynomial. Also

$$[X, Y_{-\frac{1}{2}}] = \frac{C_1}{2}(n+2)t^{n+1}\partial_r + \frac{C_1}{2}(n+2)(n+1)t^n r\partial_\zeta + C_2(n+2)t^{n+1}\partial_\zeta + \partial_r H(r, \zeta)\partial_\zeta,$$

so $H = H(\zeta)$ does not depend on r ; finally

$$[X, M_0] = \frac{dG(\zeta)}{d\zeta} \partial_r + \frac{dH(\zeta)}{d\zeta} \partial_\zeta$$

so $G = H = 0$.

□

Remark : by modifying slightly the definition of \mathfrak{sv}_0 , one gets related Lie algebras. For instance, substituting $L_0^\varepsilon := -t\partial_t - (1 + \varepsilon)r\partial_r - (1 + 2\varepsilon)\zeta\partial_\zeta$ for L_0 leads to the 'polynomial part' of $\mathfrak{sv}_{1+2\varepsilon}$ (see Theorem 3.11 for an explicit realization of $\mathfrak{sv}_{1+2\varepsilon}$).

4.2 Coinduced representations of \mathfrak{sv}

In order to classify 'reasonable' representations of the Virasoro algebra, V. G. Kac made the following conjecture: the Harish-Chandra representations, those for which ℓ_0 acts semi-simply with finite-dimensional eigenspaces, are either higher- (or lower-) weight modules, or tensor density modules. As proved in [22] and [23], one has essentially two types of Harish-Chandra representations of the Virasoro algebra :

- Verma modules which are *induced* to \mathfrak{vir} from a character of $\mathfrak{vir}_+ = \langle L_0, L_1, \dots \rangle$, zero on the subalgebra $\mathfrak{vir}_{\geq 1} = \langle L_1, \dots \rangle$, and quotients of degenerate Verma modules (see Section 6 for a generalization in our case);

- tensor modules of formal densities which are *coinduced* to the subalgebra of formal or polynomial vector fields $\text{Vect}(S^1)_{\geq -1} = \langle L_{-1}, L_0, \dots \rangle$ from a character of $\text{Vect}(S^1)_{\geq 0}$ that is zero on the subalgebra $\text{Vect}(S^1)_{\geq 1}$. These modules extend naturally to representations of $\text{Vect}(S^1)$.

We shall generalize in this paragraph this second type of representations to the case of \mathfrak{sv} . Note that although we have two natural graduations on \mathfrak{sv} , the one given by the structure of Cartan prolongation is most adapted here since \mathfrak{sv}_{-1} is commutative (see [1]).

Let $d\rho$ be a representation of $\mathfrak{sv}_0 = \langle L_0, Y_{\frac{1}{2}}, M_1 \rangle$ into a vector space \mathcal{H}_ρ . Then $d\rho$ can be trivially extended to $\mathfrak{sv}_+ = \bigoplus_{i \geq 0} \mathfrak{sv}_i$ by setting $d\rho(\sum_{i > 0} \mathfrak{sv}_i) = 0$. Let $\mathfrak{fsv} = \bigoplus_{i \geq -1} \mathfrak{sv}_i \subset \mathfrak{sv}$ be the subalgebra of 'formal' vector fields: in the representation $d\pi$, the image of \mathfrak{fsv} is the subset of vector fields that are polynomial in the time coordinate.

Let us now define the representation of \mathfrak{fsv} coinduced from $d\rho$.

Definition 4.1.

The ρ -formal density module $(\tilde{\mathcal{H}}_\rho, d\tilde{\rho})$ is the coinduced module

$$\tilde{\mathcal{H}}_\rho = \text{Hom}_{\mathcal{U}(\mathfrak{sv}_+)}(\mathcal{U}(\mathfrak{fsv}), \mathcal{H}_\rho) \tag{4.7}$$

$$= \{ \phi : \mathcal{U}(\mathfrak{fsv}) \rightarrow \mathcal{H}_\rho \text{ linear} \mid \phi(U_0 V) = d\rho(U_0) \cdot \phi(V), \quad U_0 \in \mathcal{U}_{\mathfrak{sv}_+}, V \in \mathcal{U}(\mathfrak{fsv}) \} \tag{4.8}$$

with the natural action of $\mathcal{U}(\mathfrak{fsv})$ on the right

$$(d\tilde{\rho}(U) \cdot \phi)(V) = \phi(VU), \quad U, V \in \mathcal{U}(\mathfrak{fsv}). \tag{4.9}$$

By Poincaré-Birkhoff-Witt's theorem, this space can be identified with

$$\text{Hom}(\mathcal{U}(\mathfrak{sv}_+) \setminus \mathcal{U}(\mathfrak{fsv}), \mathcal{H}_\rho) \simeq \text{Hom}(\text{Sym}(\mathfrak{sv}_{-1}), \mathcal{H}_\rho)$$

(linear applications from the symmetric algebra on \mathfrak{sv}_{-1} into \mathcal{H}_ρ), and this last space is in turn isomorphic with the space $\mathcal{H}_\rho \otimes \mathbb{R}[[t, r, \zeta]]$ of \mathcal{H}_ρ -valued functions of t, r, ζ , through the application

$$\mathcal{H}_\rho \otimes \mathbb{R}[[t, r, \zeta]] \longrightarrow \text{Hom}(\text{Sym}(\mathfrak{sv}_{-1}), \mathcal{H}_\rho) \quad (4.10)$$

$$F(t, r, \zeta) \longrightarrow \phi_F : (U \rightarrow \partial_U F|_{t=0, r=0, \zeta=0}) \quad (4.11)$$

where ∂_U stands for the product derivative $\partial_{L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l} = (-\partial_t)^j (-\partial_r)^k (-\partial_\zeta)^l$ (note our choice of signs!).

We shall really be interested in the action of \mathfrak{fsv} on functions $F(t, r, \zeta)$ that we shall denote by $d\sigma_\rho$, or $d\sigma$ for short.

The above morphisms allow one to compute the action of \mathfrak{fsv} on monomials through the equality

$$\left(\frac{\partial_t^j}{j!} \frac{\partial_r^k}{k!} \frac{\partial_\zeta^l}{l!} \right) \Big|_{t=0, r=0, \zeta=0} (d\sigma(X).F) = \frac{(-1)^{j+k+l}}{j!k!l!} (d\tilde{\rho}(X). \phi_F)(L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l) \quad (4.12)$$

$$= \frac{(-1)^{j+k+l}}{j!k!l!} \phi_F(L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l X), \quad X \in \mathfrak{fsv}. \quad (4.13)$$

In particular,

$$\partial_t^j \partial_r^k \partial_\zeta^l \Big|_{t=0, r=0, \zeta=0} (d\sigma(L_{-1}).F) = -\partial_t^{j+1} \partial_r^k \partial_\zeta^l \Big|_{t=0, r=0, \zeta=0} F$$

so $d\sigma(L_{-1}).F = -\partial_t F$; similarly, $d\sigma(Y_{-\frac{1}{2}}).F = -\partial_r F$ and $d\sigma(M_0).F = -\partial_\zeta F$.

So one may assume that $X \in \mathfrak{sv}_+$: by Poincaré-Birkhoff-Witt's theorem, $L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l X$ can be rewritten as $U + V$ with

$$U \in \mathfrak{sv}_{>0} \mathcal{U}(\mathfrak{fsv})$$

and

$$V = V_1 V_2, \quad V_1 \in \mathcal{U}(\mathfrak{sv}_0), V_2 \in \mathcal{U}(\mathfrak{sv}_{-1}).$$

Then $\phi_F(U) = 0$ by definition of $\tilde{\mathcal{H}}_\rho$, and $\phi_F(V)$ may easily be computed as $\phi_F(V) = d\rho(V_1) \otimes \partial_{V_2} \Big|_{t=0, r=0, \zeta=0} F$.

Theorem 4.2.

Let $f \in \mathbb{R}[t]$, the coinduced representation $d\tilde{\rho}$ is given by the action of the following matrix differential operators on functions:

$$d\tilde{\rho}(L_f) = \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}f''(t)r^2\partial_\zeta \right) \otimes \text{Id}_{\mathcal{H}_\rho} + f'(t)d\rho(L_0) + \frac{1}{2}f''(t)r d\rho(Y_{\frac{1}{2}}) + \frac{1}{4}f'''(t)r^2 d\rho(M_1); \quad (4.14)$$

$$d\tilde{\rho}(Y_f) = (-f(t)\partial_r - f'(t)r\partial_\zeta) \otimes \text{Id}_{\mathcal{H}_\rho} + f'(t)d\rho(Y_{\frac{1}{2}}) + f''(t)r d\rho(M_1); \quad (4.15)$$

$$d\tilde{\rho}(M_f) = -f(t)\partial_\zeta \otimes \text{Id}_{\mathcal{H}_\rho} + f'(t) d\rho(M_1). \quad (4.16)$$

Proof.

One easily checks that these formulas define a representation of $\mathfrak{fs}\mathfrak{v}$. Since $(L_{-1}, Y_{-\frac{1}{2}}, M_0, L_0, L_1, L_2)$ generated $\mathfrak{fs}\mathfrak{v}$ as a Lie algebra, it is sufficient to check the above formulas for L_0, L_1, L_2 (they are obviously correct for $L_{-1}, Y_{-\frac{1}{2}}, M_0$).

Note first that M_0 is central in $\mathfrak{fs}\mathfrak{v}$, so

$$\partial_\zeta^l(d\sigma(X).F) = d\sigma(X).(\partial_\zeta^l F).$$

Hence it will be enough to compute the action on monomials of the form $t^j r^l \otimes v$, $v \in \mathcal{H}_\rho$.

We shall give a detailed proof since the computations in $\mathcal{U}(\mathfrak{fs}\mathfrak{v})$ are rather involved.

Let us first compute $d\sigma(L_0)$: one has

$$\begin{aligned} (-\partial_t)^j (-\partial_r)^k |_{t=0, r=0}(d\sigma(L_0).F) &= \phi_F(L_{-1}^j Y_{-\frac{1}{2}}^k L_0) \\ &= \phi_F(L_{-1}^j L_0 Y_{-\frac{1}{2}}^k - \frac{k}{2} L_{-1}^j Y_{-\frac{1}{2}}^k) \\ &= \phi_F(L_0 L_{-1}^j Y_{-\frac{1}{2}}^k - (j + \frac{k}{2}) L_{-1}^j Y_{-\frac{1}{2}}^k) \\ &= \left[d\rho(L_0)(-\partial_t)^j (-\partial_r)^k - (j + \frac{k}{2})(-\partial_t)^j (-\partial_r)^k \right] F(0) \end{aligned}$$

so

$$d\sigma(L_0) = -t\partial_t - \frac{1}{2}r\partial_r + d\rho(L_0).$$

Next,

$$\begin{aligned} \phi(L_{-1}^j Y_{-\frac{1}{2}}^k L_1) &= \phi(L_{-1}^j L_1 Y_{-\frac{1}{2}}^k - k L_{-1}^j Y_{\frac{1}{2}} Y_{-\frac{1}{2}}^{k-1} + \frac{k(k-1)}{2} L_{-1}^j Y_{-\frac{1}{2}}^{k-2} M_0) \\ &= \phi((-2j L_0 L_{-1}^{j-1} + j(j-1) L_{-1}^{j-1}) Y_{-\frac{1}{2}}^k - k \phi(Y_{\frac{1}{2}} L_{-1}^j Y_{-\frac{1}{2}}^{k-1})) \\ &\quad + jk \phi(L_{-1}^{j-1} Y_{-\frac{1}{2}}^k) + \frac{k(k-1)}{2} \phi(L_{-1}^j Y_{-\frac{1}{2}}^{k-2} M_0) \\ &= [(-2j)(-\partial_r)^k (-\partial_t)^{j-1} d\rho(L_0) + j(j-1)(-\partial_r)^k (-\partial_t)^{j-1} - k(-\partial_t)^j (-\partial_r)^{k-1} d\rho(Y_{\frac{1}{2}}) \\ &\quad + jk(-\partial_t)^{j-1} (-\partial_r)^k - \frac{k(k-1)}{2} \partial_\zeta(-\partial_t)^j (-\partial_r)^{k-2}] F(0) \end{aligned}$$

hence the result for $d\sigma(L_1)$.

Finally,

$$\begin{aligned} \phi(L_{-1}^j Y_{-\frac{1}{2}}^k L_2) &= \phi(L_{-1}^j L_2 Y_{-\frac{1}{2}}^k - \frac{3}{2} k L_{-1}^j Y_{\frac{3}{2}} Y_{-\frac{1}{2}}^{k-1} + 3 \frac{k(k-1)}{2} L_{-1}^j M_1 Y_{-\frac{1}{2}}^{k-2}) \\ &= \phi((-3j L_1 L_{-1}^{j-1} + 3j(j-1) L_0 L_{-1}^{j-2} - j(j-1)(j-2) L_{-1}^{j-3}) Y_{-\frac{1}{2}}^k) \\ &\quad - \frac{3}{2} k \phi(-2j Y_{\frac{1}{2}} L_{-1}^{j-1} Y_{-\frac{1}{2}}^{k-1} + j(j-1) L_{-1}^{j-2} Y_{-\frac{1}{2}}^k) + \frac{3}{2} k(k-1) \phi(M_1 L_{-1}^j Y_{-\frac{1}{2}}^{k-2} - j M_0 L_{-1}^{j-1} Y_{-\frac{1}{2}}^{k-2}) \\ &= [(3j(j-1) d\rho(L_0)(-\partial_t)^{j-2} - j(j-1)(j-2)(-\partial_t)^{j-2})(-\partial_r)^k \\ &\quad - \frac{3}{2} k(-2j d\rho(Y_{\frac{1}{2}})(-\partial_t)^{j-1} (-\partial_r)^{k-1} + j(j-1)(-\partial_t)^{j-2} (-\partial_r)^k) \\ &\quad + \frac{3}{2} k(k-1)(d\rho(M_1)(-\partial_t)^j (-\partial_r)^{k-2} + j \partial_\zeta(-\partial_t)^{j-1} (-\partial_r)^{k-2})] F(0). \end{aligned}$$

Hence

$$d\sigma(L_2) = -t^3\partial_t - \frac{3}{2}t^2r\partial_r - \frac{3}{2}tr^2\partial_\zeta + 3t^2d\rho(L_0) + \frac{3}{2}t^2r d\rho(Y_{\frac{1}{2}}) + \frac{3}{2}r^2 d\rho(M_1).$$

□

Let us see how all actions defined in Chapter 3 (except for the coadjoint action!) derive from this construction.

Example 1. Take $\mathcal{H}_{\rho_\lambda} = \mathbb{R}$, $d\rho_\lambda(L_0) = -\lambda$, $d\rho_\lambda(Y_{\frac{1}{2}}) = d\rho_\lambda(M_1) = 0$ ($\lambda \in \mathbb{R}$). Then $d\tilde{\rho}_\lambda = d\tilde{\pi}_\lambda$ (see second remark following Proposition 1.6 for a definition of $d\tilde{\pi}_\lambda$).

Example 2. The linear part of the infinitesimal action on the affine space of Schrödinger operators (see Proposition 3.4) is given by the restriction of $d\tilde{\rho}_{-1}$ to functions of the type $g_0(t)r^2 + g_1(t)r + g_2(t)$.

Example 3. Take $\mathcal{H}_{\rho_\lambda} = \mathbb{R}^2$, $d\rho_\lambda(L_0) = \begin{pmatrix} 1/4 & \\ & -1/4 \end{pmatrix} - \lambda\text{Id}$, $d\rho_\lambda(Y_{\frac{1}{2}}) = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $d\rho_\lambda(M_1) = 0$. Then the infinitesimal representation $d\pi_\lambda^\sigma$ of Definition 3.3 (associated with the action on Dirac operators) is equal to $d\tilde{\rho}_\lambda$ (up to a Laplace transform in the mass).

Example 4. (action on multi-diagonal matrix differential operators) Take $\mathcal{H}_\rho = \mathbb{R}^3$, $d\rho(L_0) = \begin{pmatrix} -1 & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix}$, $d\rho(Y_{\frac{1}{2}}) = d\rho(M_1) = 0$ on the one hand;

$$\mathcal{H}_\sigma = \mathbb{R}^3, d\rho(L_0) = \begin{pmatrix} -1 & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix}, d\rho(Y_{\frac{1}{2}}) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$

and $d\rho(M_1) = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ on the other. Then $d\pi^{(3,0)} = d\tilde{\rho}$ and $d\pi^\nabla = d\tilde{\sigma}$ (see Proposition 3.11 in

Section 3.4).

The fact that the coadjoint action cannot be obtained by this construction follows easily by comparing the formula for the action of the Y and M generators of Theorem 3.2 and Theorem 4.2: the second derivative f'' does not appear in $\text{ad}^*(Y_f)$, while it does in $d\tilde{\rho}(Y_f)$ for any representation ρ such that $d\rho(M_1) \neq 0$; if $d\rho(M_1) = 0$, then, on the contrary, there's no way to account for the first derivative f' in $\text{ad}^*(M_f)$.

Remark: The problem of classifying all coinduced representations is hence reduced to the problem of classifying the representations $d\rho$ of the Lie algebra $\langle X_0, Y_{\frac{1}{2}}, M_1 \rangle$. This is *a priori* an untractable problem (due to the non-semi-simplicity of this Lie algebra), even if one is satisfied with finite-dimensional representations. An interesting class of examples (to which examples 1 through 4 belong) is provided by extending a (finite-dimensional, say) representation $d\rho$ of the $(ax + b)$ -type Lie algebra $\langle X_0, Y_{\frac{1}{2}} \rangle$ to $\langle X_0, Y_{\frac{1}{2}}, M_1 \rangle$ by putting $d\rho(M_1) = d\rho(Y_{\frac{1}{2}})^2$. In particular, one may consider the spin s -representation $d\sigma$ of $\mathfrak{sl}(2, \mathbb{R})$, restrict it to the Borel subalgebra considered as $\langle X_0, Y_{\frac{1}{2}} \rangle$, 'twist' it by putting $d\sigma^\lambda := d\sigma + \lambda\text{Id}$ and extend it to $\langle X_0, Y_{\frac{1}{2}}, M_1 \rangle$ as we just explained.

5 Cohomology of \mathfrak{sv} and \mathfrak{tsv} and applications to central extensions and deformations

Cohomological computations for Lie algebras are mainly motivated by the search for deformations and central extensions. We concentrate on \mathfrak{tsv} in the first three paragraphs of this section, because the generators of \mathfrak{tsv} bear integer indices, which is more natural for computations. The main theorem is Theorem 5.1 in paragraph 5.1, which classifies all deformations of \mathfrak{tsv} ; Theorem 5.5 shows that all the infinitesimal deformations obtained in paragraph 5.1 give rise to genuine deformations. One particularly interesting family of deformations is provided by the Lie algebras \mathfrak{tsv}_λ ($\lambda \in \mathbb{R}$), which were introduced in Definition 1.7. We compute their central extensions in paragraph 5.2, and compute in paragraph 5.3 their deformations in the particular case $\lambda = 1$, for which \mathfrak{tsv}_1 is the tensor product of $\text{Vect}(S^1)$ with a nilpotent associative and commutative algebra. Finally, in paragraph 5.4, we come back to the original Schrödinger-Virasoro algebra and compute its deformations, as well as the central extensions of the family of deformed Lie algebras \mathfrak{sv}_λ .

5.1 Classifying deformations of \mathfrak{tsv}

We shall be interested in the classification of all formal deformations of \mathfrak{tsv} , following the now classical scheme of Nijenhuis and Richardson: deformation of a Lie algebra \mathcal{G} means that one has a formal family of Lie brackets on \mathcal{G} , denoted $[\cdot, \cdot]_t$, inducing a Lie algebra structure on the extended Lie algebra $\mathcal{G} \otimes_k k[[t]] = \mathcal{G}[[t]]$. As well-known, one has to study the cohomology of \mathcal{G} with coefficients in the adjoint representation; degree-two cohomology $H^2(\mathcal{G}, \mathcal{G})$ classifies the infinitesimal deformations (the terms of order one in the expected formal deformations) and $H^3(\mathcal{G}, \mathcal{G})$ contains the potential obstructions to a further prolongation of the deformations. So we shall naturally begin with the computation of $H^2(\mathfrak{tsv}, \mathfrak{tsv})$ (as usual, we shall consider only local cochains, equivalently given by differential operators, or polynomial in the modes):

Theorem 5.1

One has $\dim H^2(\mathfrak{tsv}, \mathfrak{tsv}) = 3$. A set of generators is provided by the cohomology classes of the cocycles c_1, c_2 and c_3 , defined as follows in terms of modes (the missing components of the cocycles are meant to vanish):

$$\begin{aligned} c_1(L_n, Y_m) &= -\frac{n}{2}Y_{n+m}, & c_1(L_n, M_n) &= -nM_{n+m} \\ c_2(L_n, Y_m) &= Y_{n+m} & c_2(L_n, M_m) &= 2M_{n+m} \\ c_3(L_n, L_m) &= (m-n)M_{n+m} \end{aligned}$$

Remarks:

1. The cocycle c_1 gives rise to the family of Lie algebras \mathfrak{tsv}_ϵ described in Definition 1.7.
2. The cocycle c_3 can be described globally as $c_3 : \text{Vect}(S^1) \times \text{Vect}(S^1) \longrightarrow \mathcal{F}_0$ given by

$$c_3(f\partial, g\partial) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

This cocycle appeared in [10] and has been used in a different context in [12].

Before entering the technicalities of the proof, we shall indicate precisely, for the comfort of the

reader, some cohomological results on $\mathcal{G} = \text{Vect}(S^1)$ which will be extensively used in the sequel.

Proposition 5.2. (see [10], or [12], chap. IV for a more elementary approach)

- (1) $\text{Inv}_{\mathcal{G}}(\mathcal{F}_{\lambda} \otimes \mathcal{F}_{\mu}) = 0$ unless $\mu = -1 - \lambda$ and $\mathcal{F}_{\mu} = \mathcal{F}_{\lambda}^*$; then $\text{Inv}_{\mathcal{G}}(\mathcal{F}_{\lambda} \otimes \mathcal{F}_{\lambda}^*)$ is one-dimensional, generated by the identity mapping.
- (2) $H^i(\mathcal{G}, \mathcal{F}_{\lambda} \otimes \mathcal{F}_{\mu}) \equiv 0$ if $\lambda \neq 1 - \mu$ and λ or μ are not integers.
(1) and (2) can be immediately deduced from [10], theorem 2.3.5 p. 136-137.
- (3) Let W_1 be the Lie algebra of formal vector fields on the line, its cohomology represents the algebraic part of the cohomology of $\mathcal{G} = \text{Vect}(S^1)$ (see again [10], theorem 2.4.12). Then $H^1(W_1, \text{Hom}(\mathcal{F}_{\lambda}, \mathcal{F}_{\lambda}))$ is one-dimensional, generated by the cocycle $(f\partial, \text{adx}^{-\lambda}) \longrightarrow f' \text{adx}^{-\lambda}$ (cocycle I_{λ} in [10], p. 138).
- (4) Invariant antisymmetric bilinear operators $\mathcal{F}_{\lambda} \times \mathcal{F}_{\mu} \longrightarrow \mathcal{F}_{\nu}$ between densities have been determined by P. Grozman (see [11], p. 280).
They are of the following type:

(a) the Poisson bracket for $\nu = \lambda + \mu - 1$, defined by

$$\{f dx^{-\lambda}, g dx^{-\mu}\} = (\lambda f g' - \mu f' g) dx^{-(\lambda+\mu-1)}$$

(b) the following three exceptional brackets :

$$\mathcal{F}_{1/2} \times \mathcal{F}_{1/2} \rightarrow \mathcal{F}_{-1} \text{ given by } (f\partial^{1/2}, g\partial^{1/2}) \rightarrow \frac{1}{2}(fg'' - gf'')dx;$$

$$\mathcal{F}_0 \times \mathcal{F}_0 \rightarrow \mathcal{F}_{-3} \text{ given by } (f, g) \rightarrow (f''g' - g''f')dx^3;$$

and an operator $\mathcal{F}_{2/3} \times \mathcal{F}_{2/3} \rightarrow \mathcal{F}_{-\frac{5}{3}}$ called the Grozman bracket (see [11], p 274).

Proof of Theorem 5.1.

We shall use standard techniques in Lie algebra cohomology; the proof will be rather technical, but without specific difficulties. Let us fix the notations: set $\mathfrak{tsv} = \mathcal{G} \ltimes \mathfrak{h}$ where $\mathcal{G} = \text{Vect}(S^1)$ and \mathfrak{h} is the nilpotent part of \mathfrak{tsv} .

One can consider the exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathcal{G} \ltimes \mathfrak{h} \longrightarrow \mathcal{G} \longrightarrow 0 \quad (5.1)$$

as a short exact sequence of $\mathcal{G} \ltimes \mathfrak{h}$ modules, thus inducing a long exact sequence in cohomology:

$$\dots \longrightarrow H^1(\mathfrak{tsv}, \mathcal{G}) \longrightarrow H^2(\mathfrak{tsv}, \mathfrak{h}) \longrightarrow H^2(\mathfrak{tsv}, \mathfrak{tsv}) \longrightarrow H^2(\mathfrak{tsv}, \mathcal{G}) \longrightarrow H^3(\mathfrak{tsv}, \mathfrak{h}) \longrightarrow \dots \quad (5.2)$$

So we shall consider $H^*(\mathfrak{tsv}, \mathcal{G})$ and $H^*(\mathfrak{tsv}, \mathfrak{h})$ separately.

Lemma 5.3:

$H^*(\mathfrak{tsv}, \mathcal{G}) = 0$ for $*$ = 0, 1, 2.

Proof of Lemma 5.3:

One uses the Hochschild-Serre spectral sequence associated with the exact sequence (5.1). Let us remark first that $H^*(\mathcal{G}, H^*(\mathfrak{h}, \mathcal{G})) = H^*(\mathcal{G}, H^*(\mathfrak{h}) \otimes \mathcal{G})$ since \mathfrak{h} acts trivially on \mathcal{G} . So one has to

understand $H^*(\mathfrak{h})$ in low dimensions; let us consider the exact sequence $0 \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{h} \longrightarrow y \longrightarrow 0$, where $\mathfrak{n} = [\mathfrak{h}, \mathfrak{h}]$. As \mathcal{G} -modules, these algebras are density modules, more precisely $\mathfrak{n} = \mathcal{F}_0$ and $y = \mathcal{F}_{1/2}$. So $H^1(\mathfrak{h}) = y^* = \mathcal{F}_{-3/2}$ as a \mathcal{G} -module. Let us recall that, as a module on itself, $\mathcal{G} = \mathcal{F}_1$. One gets: $E_2^{p,0} = H^p(\mathcal{G}, \mathcal{G}) = 0$ as well-known (see [10]),

$$E_2^{1,1} = H^1(\mathcal{G}, H^1(\mathfrak{h}) \otimes \mathcal{G}) = H^1(\mathcal{G}, \mathcal{F}_{-3/2} \otimes \mathcal{F}_1)$$

The determination of cohomologies of $\text{Vect}(S^1)$ with coefficients in tensor products of modules of densities has been done by Fuks (see [10], chap. 2, thm 2.3.5, or Proposition 5.2 (2) above), in this case everything vanishes and $E_2^{1,1} = 0$.

One has now to compute $H^2(\mathfrak{h})$ in order to get $E_2^{0,2} = \text{Inv}_{\mathcal{G}}(H^2(\mathfrak{h}) \otimes \mathcal{G})$. We shall use the decomposition of cochains on \mathfrak{h} induced by its splitting into vector subspaces: $\mathfrak{h} = \mathfrak{n} \oplus y$. So $C^1(\mathfrak{h}) = \mathfrak{n}^* \oplus y^*$ and $C^2(\mathfrak{h}) = \Lambda^2 \mathfrak{n}^* \oplus \Lambda^2 y^* \oplus y^* \wedge \mathfrak{n}^*$. The coboundary ∂ is induced by the only non-vanishing part $\partial : \mathfrak{n}^* \longrightarrow \Lambda^2 y^*$ which is dual to the bracket $\Lambda^2 y \longrightarrow \mathfrak{n}$. So the cohomological complex splits into three subcomplexes and one deduces the following exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathfrak{n}^* \xrightarrow{\partial} \Lambda^2 y^* \longrightarrow M_1 \longrightarrow 0 \\ 0 &\longrightarrow M_2 \longrightarrow \Lambda^2 \mathfrak{n}^* \xrightarrow{\partial} \Lambda^2 y^* \otimes \mathfrak{n}^* \\ 0 &\longrightarrow M_3 \longrightarrow y^* \wedge \mathfrak{n}^* \xrightarrow{\partial} \Lambda^3 y^* \end{aligned}$$

and $H^2(\mathfrak{h}) = M_1 \oplus M_2 \oplus M_3$. One can then easily deduce the invariants $\text{Inv}_{\mathcal{G}}(H^2(\mathfrak{h}) \otimes \mathcal{G}) = \bigoplus_{i=1}^3 \text{Inv}_{\mathcal{G}}(M_i \otimes \mathcal{G})$ from the cohomological exact sequences associated with the above short exact sequences. One has:

$$0 \longrightarrow \text{Inv}_{\mathcal{G}}(M_2 \otimes \mathcal{G}) \longrightarrow \text{Inv}_{\mathcal{G}}(\Lambda^2 \mathfrak{n}^* \otimes \mathcal{G}) = 0$$

and

$$0 \longrightarrow \text{Inv}_{\mathcal{G}}(M_3 \otimes \mathcal{G}) \longrightarrow \text{Inv}_{\mathcal{G}}(y^* \wedge \mathfrak{n}^* \otimes \mathcal{G}) = 0$$

from Proposition 5.2;

$$\dots \longrightarrow \text{Inv}_{\mathcal{G}}(\Lambda^2 y^* \otimes \mathcal{G}) \longrightarrow \text{Inv}_{\mathcal{G}}(M_1 \otimes \mathcal{G}) \longrightarrow H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathcal{G}) \xrightarrow{\partial_*} H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathcal{G}) \longrightarrow \dots$$

From the same proposition, one gets $\text{Inv}_{\mathcal{G}}(\Lambda^2 y^* \otimes \mathcal{G}) = 0$ and we shall see later (see the last part of the proof) that ∂_* is an isomorphism. So $\text{Inv}_{\mathcal{G}}(H^2(\mathfrak{h}) \otimes \mathcal{G}) = 0$ and $E_2^{0,2} = 0$. The same argument shows that $E_2^{0,1} = 0$, which ends the proof of the lemma.

□

From the long exact sequence (5.2) one has now: $H^*(\mathfrak{tsv}, \mathfrak{tsv}) = H^*(\mathfrak{tsv}, \mathfrak{h})$ for $* = 0, 1, 2$. We shall compute $H^*(\mathfrak{tsv}, \mathfrak{h})$ by using the Hochschild-Serre spectral sequence once more; there are three terms to compute.

1. First $E_2^{2,0} = H^2(\mathcal{G}, H^0(\mathfrak{h}, \mathfrak{h}))$, but $H^0(\mathfrak{h}, \mathfrak{h}) = Z(\mathfrak{h}) = \mathfrak{n} = \mathcal{F}_0$ as \mathcal{G} -module. So $E_2^{2,0} = H^2(\mathcal{G}, \mathcal{F}_0)$ which is one-dimensional, given by $c_3(f\partial, g\partial) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$, or in terms of modes $c_3(L_n, L_m) = (m - n)M_{n+m}$. Hence we have found one of the classes announced in the theorem.

2. One must now compute $E_2^{1,1} = H^1(\mathcal{G}, H^1(\mathfrak{h}, \mathfrak{h}))$. The following lemma will be useful for this purpose, and also for the last part of the proof.

Lemma 5.4 (identification of $H^1(\mathfrak{h}, \mathfrak{h})$ as a \mathcal{G} -module).

The space $H^1(\mathfrak{h}, \mathfrak{h})$ splits into the direct sum of two \mathcal{G} -modules $H^1(\mathfrak{h}, \mathfrak{h}) = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that

1. $\text{Inv}_{\mathcal{G}}\mathcal{H}_2 = 0$, $H^1(\mathcal{G}, \mathcal{H}_2) = 0$;

2. $\text{Inv}_{\mathcal{G}}\mathcal{H}_1$ is one-dimensional, generated by the 'constant multiplication' cocycle l defined by

$$l(Y_n) = Y_n, \quad l(M_n) = 2M_n \quad (5.3)$$

3. $H^1(\mathcal{G}, \mathcal{H}_1)$ is two-dimensional, generated by two cocycles c_1, c_2 defined by

$$c_1(f\partial, g\partial^{1/2}) = f'g\partial^{1/2}, \quad c_1(f\partial, g) = 2f'g$$

and

$$c_2(f\partial, g\partial^{1/2}) = fg\partial^{1/2}, \quad c_2(f\partial, g) = 2fg.$$

4. $H^2(\mathcal{G}, \mathcal{H}_1)$ is one-dimensional, generated by the cocycle c_{12} defined by

$$c_{12}(f\partial, g\partial, h\partial^{1/2}) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h\partial^{1/2}, \quad c_{12}(f\partial, g\partial, h) = 2 \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h$$

Proof of lemma 5.4:

We shall split the cochains according to the decomposition $\mathfrak{h} = \mathfrak{y} \oplus \mathfrak{n}$. Set $C^1(\mathfrak{h}, \mathfrak{h}) = C_1 \oplus C_2$, where:

$$C_1 = (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (\mathfrak{y}^* \otimes \mathfrak{y}) \quad C_2 = (\mathfrak{n}^* \otimes \mathfrak{y}) \oplus (\mathfrak{y}^* \otimes \mathfrak{n}).$$

So one readily obtains the splitting $H^1(\mathfrak{h}, \mathfrak{h}) = \mathcal{H}_1 \oplus \mathcal{H}_2$ where

$$\begin{aligned} 0 &\longrightarrow \mathcal{H}_1 \longrightarrow (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (\mathfrak{y}^* \otimes \mathfrak{y}) \xrightarrow{\partial} \Lambda^2 \mathfrak{y}^* \otimes \mathfrak{n} \\ 0 &\longrightarrow \mathfrak{y} \xrightarrow{\partial} \mathfrak{y}^* \otimes \mathfrak{n} \longrightarrow \mathcal{H}_2 \longrightarrow 0 \end{aligned}$$

∂ being the coboundary on the space of cochains on \mathfrak{h} with coefficients into itself. Its non vanishing pieces in degrees 0, 1 and 2 are the following ones: $\mathfrak{y} \xrightarrow{\partial} \mathfrak{y}^* \otimes \mathfrak{n}$, $\mathfrak{n}^* \otimes \mathfrak{n} \xrightarrow{\partial} \Lambda^2 \mathfrak{y}^* \otimes \mathfrak{n}$, $\mathfrak{y}^* \otimes \mathfrak{y} \xrightarrow{\partial} \Lambda^2 \mathfrak{y}^* \otimes \mathfrak{n}$. We can now describe the second exact sequence in terms of densities as follows:

$$0 \longrightarrow \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_{-3/2} \otimes \mathcal{F}_0 \longrightarrow \mathcal{H}_2 \longrightarrow 0 \quad (5.4)$$

From Proposition 5.2, one has $\text{Inv}_{\mathcal{G}}(\mathcal{F}_{-3/2} \otimes \mathcal{F}_0) = 0$ as well as $H^i(\mathcal{G}, \mathcal{F}_{1/2}) = 0$, for $i = 0, 1, 2$, and $H^1(\mathcal{G}, \mathcal{F}_{-3/2} \otimes \mathcal{F}_0) = 0$. So the long exact sequence in cohomology associated with (5.3) gives $\text{Inv}_{\mathcal{G}}(\mathcal{H}_2) = 0$ and $H^1(\mathcal{G}, \mathcal{H}_2) = 0$.

For \mathcal{H}_1 , one has to analyse the cocycles by direct computation. So let $l \in C_1$ given by $l(Y_n) = a_n(k)Y_{n+k}$, $l(M_n) = b_n(k)M_{n+k}$. The cocycle conditions are given by:

$$\partial l(Y_n, Y_m) = l((m-n)M_{n+m}) - Y_n \cdot l(Y_m) + Y_m \cdot l(Y_n) = 0$$

for all $(n, m) \in \mathbb{Z}^2$. So identifying the term in M_{n+m+k} , one obtains:

$$(m-n)b_{n+m}(k) = (m-n+k)a_m(k) - (n-m+k)a_n(k)$$

so $b_{n+m}(k) = a_m(k) + a_n(k) + \frac{k}{m-n}(a_m(k) - a_n(k)) = f(n, m, k)$.

One can now determine the $a_n(k)$, remarking that the function $f(n, m, k)$ depends only on k and $(n+m)$.

One then obtains that $a_n(k)$ must be affine in n so:

$$a_n(k) = n\lambda(k) + \mu(k)$$

$$b_n(k) = n\lambda(k) + k\lambda(k) + 2\mu(k)$$

So, as a vector space \mathcal{H}_1 is isomorphic to $\bigoplus_k \mathbb{C}(\lambda(k)) \bigoplus_k \mathbb{C}(\mu(k))$, two copies of an infinite direct sum of a numerable family of one-dimensional vector spaces.

Now we have to compute the action of \mathcal{G} on \mathcal{H}_1 ; let $L_p \in \mathcal{G}$, one has

$$\begin{aligned} (L_p \cdot l)(Y_n) &= ((n - \frac{p}{2})a_{n+p}(k) - (n + k - \frac{p}{2})a_n(k))Y_{n+p+k} \\ &= \left(n(p - k)\lambda(k) - (\frac{p^2}{2}\lambda(k) + k\mu(k)) \right) Y_{n+p+k} \end{aligned}$$

So if one sets $(L_p \cdot l)(Y_n) = (n(L_p \cdot \lambda)(k + p) + (L_p \cdot \mu)(k + p))Y_{n+p+k}$ one obtains:

$$\begin{aligned} (L_p \cdot \lambda)(k + p) &= (p - k)\lambda(k) \\ (L_p \cdot \mu)(k + p) &= -\frac{p^2}{2}\lambda(k) + k\mu(k) \end{aligned}$$

Finally, \mathcal{H}_1 appears as an extension of modules of densities of the following type:

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{H}_1 \longrightarrow \mathcal{F}_1 \longrightarrow 0, \text{ in which } \mathcal{F}_0 \text{ corresponds to } \bigoplus_k \mathbb{C}(\mu(k)) \text{ and } \mathcal{F}_1 \text{ to } \bigoplus_k \mathbb{C}(\lambda(k)).$$

There is a non-trivial extension cocycle γ in $Ext_{\mathcal{G}}^1(\mathcal{F}_1, \mathcal{F}_0) = H^1(\mathcal{G}, Hom(\mathcal{F}_1, \mathcal{F}_0))$, given by $\gamma(f\partial)(g\partial) = f''g$; this cocycle corresponds to the term in p^2 in the above formula. In any case one has a long exact sequence in cohomology

$$\dots \longrightarrow H^i(\mathcal{G}, \mathcal{F}_0) \longrightarrow H^i(\mathcal{G}, \mathcal{H}_1) \longrightarrow H^i(\mathcal{G}, \mathcal{F}_1) \longrightarrow H^{i+1}(\mathcal{G}, \mathcal{F}_0) \longrightarrow \dots$$

As well-known, $H^*(\mathcal{G}, \mathcal{F}_1) = H^*(\mathcal{G}, \mathcal{G})$ is trivial, and finally $H^i(\mathcal{G}, \mathcal{F}_0)$ is isomorphic to $H^i(\mathcal{G}, \mathcal{H}_1)$. In particular $H^0(\mathcal{G}, \mathcal{H}_1) = H^0(\mathcal{G}, \mathcal{F}_0)$ is one-dimensional, given by the constants; a scalar μ induces an invariant cocycle as $l(Y_n) = \mu Y_n$, $l(M_n) = 2\mu M_n$.

Moreover, $H^1(\mathcal{G}, \mathcal{F}_0)$ has dimension 2: it is generated by the cocycles \bar{c}_1 and \bar{c}_2 , defined by $\bar{c}_1(f\partial) = f'$ and $\bar{c}_2(f\partial) = f$ respectively. So one obtains two generators of $H^1(\mathcal{G}, \mathcal{H}_1)$ given by

$$c_1(f\partial, g\partial^{1/2}) = f'g\partial^{1/2}, \quad c_1(f\partial, g) = 2f'g$$

and

$$c_2(f\partial, g\partial^{1/2}) = fg\partial^{1/2}, \quad c_2(f\partial, g) = 2fg$$

respectively.

Finally $H^2(\mathcal{G}, \mathcal{F}_0)$ is one-dimensional, with the cup-product \bar{c}_{12} of \bar{c}_1 and \bar{c}_2 as generator (see [10], p. 177), so $\bar{c}_{12}(f\partial, g\partial) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$, and one deduces the formula for the corresponding cocycle c_{12} in $H^2(\mathcal{G}, \mathcal{H}_1)$:

$$c_{12}(f\partial, g\partial, h\partial^{1/2}) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h\partial^{1/2}, \quad c_{12}(f\partial, g\partial, h) = 2 \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h$$

This finishes the proof of Lemma 5.4. \square

So, from Lemma 5.4, we have computed $E_2^{1,1} = H^1(\mathcal{G}, H^1(\mathfrak{h}, \mathfrak{h}))$; it is two-dimensional, generated by c_1 and c_2 , while earlier we had $H^2(\mathcal{G}, H^0(\mathfrak{h}, \mathfrak{h})) = E_2^{2,0}$, a one-dimensional vector space generated by c_3 . We have to check now that these cohomology classes shall not disappear in the spectral sequence; the only potentially non-vanishing differentials are $E_2^{0,1} \rightarrow E_2^{2,0}$ and $E_2^{1,1} \rightarrow E_2^{3,0}$. One has $E_2^{3,0} = H^3(\mathcal{G}, \mathfrak{h}) = H^3(\mathcal{G}, \mathfrak{n}) = H^3(\mathcal{G}, \mathcal{F}_0) = 0$ (see [10] p. 177); here we consider only local cohomology), then $E_2^{0,1}$ is one-dimensional determined by the constant multiplication (see above) and direct verification shows that $E_2^{0,1} \rightarrow E_2^{2,0}$ vanishes. So we have just proved that the cocycles c_1 , c_2 and c_3 defined in the theorem represent genuinely non-trivial cohomology classes in $H^2(\mathfrak{tsv}, \mathfrak{tsv})$.

3. In order to finish the proof, we still have to prove that there does not exist any other non-trivial class in the last piece of the Hochschild-Serre spectral sequence. We shall thus prove that $E_2^{0,2} = \text{Inv}_{\mathcal{G}} H^2(\mathfrak{h}, \mathfrak{h}) = 0$. As in the proofs of the previous lemmas, we shall use decompositions of the cohomological complex of \mathfrak{h} with coefficients into itself as sums of \mathcal{G} -modules.

The space of adjoint cochains $C^2(\mathfrak{h}, \mathfrak{h})$ will split into six subspaces according to the vector space decomposition $\mathfrak{h} = \mathfrak{y} \oplus \mathfrak{n}$. So we can as well split the cohomological complex

$$C^1(\mathfrak{h}, \mathfrak{h}) \xrightarrow{\partial} C^2(\mathfrak{h}, \mathfrak{h}) \xrightarrow{\partial} C^3(\mathfrak{h}, \mathfrak{h})$$

into its components, and the coboundary operators will as well split into different components, as we already explained. So one obtains the following families of exact sequences of \mathcal{G} -modules:

$$(\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (\mathfrak{y}^* \otimes \mathfrak{y}) \xrightarrow{\partial} \Lambda^2 \mathfrak{y}^* \otimes \mathfrak{n} \longrightarrow A_1 \longrightarrow 0 \quad (5.5)$$

$$\begin{aligned} 0 \longrightarrow K \longrightarrow (\Lambda^2 \mathfrak{y}^* \otimes \mathfrak{y}) \oplus (\mathfrak{n}^* \wedge \mathfrak{y}^* \otimes \mathfrak{y}) &\xrightarrow{\partial} \Lambda^3 \mathfrak{y}^* \otimes \mathfrak{n} \\ 0 \longrightarrow \mathfrak{n}^* \otimes \mathfrak{y} \xrightarrow{\partial} K \longrightarrow A_2 &\longrightarrow 0 \end{aligned} \quad (5.6)$$

$$\begin{aligned} 0 \longrightarrow A_3 \longrightarrow \mathfrak{n}^* \wedge \mathfrak{y}^* \otimes \mathfrak{y} \xrightarrow{\partial} (\Lambda^3 \mathfrak{y}^* \otimes \mathfrak{y}) \oplus (\mathfrak{n}^* \wedge \Lambda^2 \mathfrak{y}^*) \otimes \mathfrak{n} \\ 0 \longrightarrow A_4 \longrightarrow \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n} \xrightarrow{\partial} (\mathfrak{n}^* \wedge \Lambda^2 \mathfrak{y}^*) \otimes \mathfrak{n} \\ 0 \longrightarrow A_5 \longrightarrow \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{y} \xrightarrow{\partial} (\mathfrak{n}^* \wedge \Lambda^2 \mathfrak{y}^*) \otimes \mathfrak{y} \oplus (\Lambda^2 \mathfrak{n}^* \wedge \mathfrak{y}^*) \otimes \mathfrak{n} \end{aligned}$$

The restrictions of coboundary operators are still denoted by ∂ , and the other arrows are either inclusions of subspaces or projections onto quotients. So one has $H^2(\mathfrak{h}, \mathfrak{h}) = \bigoplus_{i=1}^5 A_i$, and our result will follow from $\text{Inv}_{\mathcal{G}} A_i = 0$, $i = 1, \dots, 5$. For the last three sequences, the result follows immediately from the cohomology long exact sequence by using $\text{Inv}_{\mathcal{G}}(\mathfrak{n}^* \wedge \mathfrak{y}^* \otimes \mathfrak{y}) = 0$, $\text{Inv}_{\mathcal{G}} \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n} = 0$, $\text{Inv}_{\mathcal{G}} \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{y} = 0$: there results are deduced from those of Grozman, recalled in Proposition 5.2. (Note that the obviously \mathcal{G} -invariant maps $\mathfrak{n} \otimes \mathfrak{n} \rightarrow \mathfrak{n}$ and $\mathfrak{n} \otimes \mathfrak{y} \rightarrow \mathfrak{y}$ are not antisymmetric!) So one has $\text{Inv}_{\mathcal{G}} A_i = 0$ for $i = 3, 4, 5$.

An analogous argument will work for K , since $\text{Inv}_{\mathcal{G}} \Lambda^2 \mathfrak{y}^* \otimes \mathfrak{y} = 0$ and $\text{Inv}_{\mathcal{G}}(\mathfrak{n}^* \wedge \mathfrak{y}^*) \otimes \mathfrak{y} = 0$ from the same results. So the long exact sequence associated with the short sequence (5.5) above will give:

$$0 \longrightarrow \text{Inv}_{\mathcal{G}}(K) \longrightarrow \text{Inv}_{\mathcal{G}}(A_2) \longrightarrow H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{y})$$

One has $H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{y}) = H^1(\mathcal{G}, \mathcal{F}_{-1} \otimes \mathcal{F}_{1/2}) = 0$ (see Proposition 5.2). So $\text{Inv}_{\mathcal{G}}(A_2) = 0$.

For A_1 , we shall require a much more subtle argument. First of all, the sequence (5.4) can be split into two short exact sequences:

$$\begin{aligned} 0 \longrightarrow \mathcal{H}_1 \longrightarrow (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y) \longrightarrow B \longrightarrow 0 \\ 0 \longrightarrow B \longrightarrow \Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow A_1 \longrightarrow 0. \end{aligned}$$

Let us consider the long exact sequence associated with the first one:

$$\begin{aligned} 0 \longrightarrow \text{Inv}_{\mathcal{G}} \mathcal{H}_1 \longrightarrow \text{Inv}_{\mathcal{G}}(\mathfrak{n}^* \otimes \mathfrak{n}) \oplus \text{Inv}_{\mathcal{G}}(y^* \otimes y) \longrightarrow \text{Inv}_{\mathcal{G}} B \longrightarrow \dots \\ \dots \hookrightarrow H^1(\mathcal{G}, \mathcal{H}_1) \longrightarrow H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) \oplus H^1(\mathcal{G}, y^* \otimes y) \longrightarrow H^1(\mathcal{G}, B) \longrightarrow \dots \\ \dots \longrightarrow H^2(\mathcal{G}, \mathcal{H}_1) \longrightarrow H^2(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) \oplus H^2(\mathcal{G}, y^* \otimes y) \longrightarrow H^2(\mathcal{G}, B) \longrightarrow \dots \end{aligned}$$

The case of $H^i(\mathcal{G}, \mathcal{H}_1)$, $i = 0, 1, 2$ has been treated in Lemma 5.4, and analogous techniques can be used to study $H^i(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n})$ and $H^i(\mathcal{G}, y^* \otimes y)$ for $i = 0, 1, 2$. The cohomology classes come from the inclusion $\mathcal{F}_0 \subset \mathfrak{n}^* \otimes \mathfrak{n}$, $y^* \otimes y$ or \mathcal{H}_1 , and from the well-known computation of $H^*(\mathcal{G}, \mathcal{F}_0)$ (Remark: using the results of Fuks [10], chap 2, one should keep in mind the fact that he computes cohomologies for W_1 , the formal part of $\mathcal{G} = \text{Vect}(S^1)$). To get the cohomologies for $\text{Vect}(S^1)$ one has to add the classes of differentiable order 0 (or "topological" classes), this is the reason for the occurrence of c_2 in Lemma 5.4).

So $H^i(\mathcal{G}, \mathcal{H}_1) = H^i(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) = H^i(\mathcal{G}, y^* \otimes y)$, $i = 0, 1, 2$, and the maps on the modules are naturally defined through the injection $\mathcal{H}_1 \longrightarrow (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y)$: each generator of $H^i(\mathcal{G}, \mathcal{H}_1)$, $i = 0, 1, 2$, say e , will give $(e, -e)$ in the corresponding component of $H^i(\mathcal{G}, (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y))$. So $\text{Inv}_{\mathcal{G}} B$ and $H^2(\mathcal{G}, B)$ are one-dimensional and $H^1(\mathcal{G}, B)$ is two-dimensional.

Now we can examine the long exact sequence associated with:

$$0 \longrightarrow B \xrightarrow{\partial} \Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow A_1 \longrightarrow 0,$$

which is:

$$0 \longrightarrow \text{Inv}_{\mathcal{G}} B \xrightarrow{\partial^*} \text{Inv}_{\mathcal{G}} \Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow \text{Inv}_{\mathcal{G}} A_1 \longrightarrow H^1(\mathcal{G}, B) \longrightarrow H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathfrak{n}) \longrightarrow \dots$$

The generator of $\text{Inv}_{\mathcal{G}} B$ comes from the identity map $\mathfrak{n} \longrightarrow \mathfrak{n}$, and $\text{Inv}_{\mathcal{G}} \Lambda^2 y^* \otimes \mathfrak{n}$ is generated by the bracket $y \wedge y \longrightarrow \mathfrak{n}$, so ∂^* is an isomorphism in this case. So one has

$$0 \longrightarrow \text{Inv}_{\mathcal{G}} A_1 \longrightarrow H^1(\mathcal{G}, B) \xrightarrow{\partial^*} H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathfrak{n})$$

The result will follow from the fact that this ∂^* is also an isomorphism. The two generators in $H^1(\mathcal{G}, B)$ come from the corresponding ones in $H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) \oplus H^1(\mathcal{G}, y^* \otimes y)$, modulo the classes coming from $H^1(\mathcal{G}, \mathcal{H}_1)$; so these generators can be described in terms of Yoneda extensions, since $H^1(\mathcal{G}, \mathcal{F}_0^* \otimes \mathcal{F}_0) = \text{Ext}_{\mathcal{G}}^1(\mathcal{F}_0, \mathcal{F}_0)$, as well as $H^1(\mathcal{G}, \mathcal{F}_{1/2}^* \otimes \mathcal{F}_{1/2}) = \text{Ext}_{\mathcal{G}}^1(\mathcal{F}_{1/2}, \mathcal{F}_{1/2})$.

Let us write this extension as $0 \longrightarrow \mathcal{F}_0 \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$; the action on E can be given in terms of modes as follows:

$$e_n^1(f_a, g_b) = (af_{n+a} + ng_{b+n}, bg_{b+n})$$

or

$$e_n^2(f_a, g_b) = (af_{n+a} + g_{b+n}, bg_{b+n}).$$

The images of these classes in $H^1(\mathcal{G}, \Lambda^2 \mathfrak{y}^* \otimes \mathfrak{n})$ are represented by the extensions obtained through a pull-back

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & E & \longrightarrow & \mathcal{F}_0 \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow [\cdot, \cdot] \\ 0 & \longrightarrow & \mathcal{F}_0 & \dashrightarrow & E' & \dashrightarrow & \Lambda^2 \mathcal{F}_{1/2} \longrightarrow 0 \end{array}$$

where $[\cdot, \cdot]$ denotes the mapping given by the Lie bracket $\Lambda^2 \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_0$. One can easily check that these extensions are non-trivial, so finally ∂^* is injective and $Inv_{\mathcal{G}}(A_1) = 0$, which finishes the proof of $E_2^{0,2} = Inv_{\mathcal{G}} H^2(\mathfrak{h}, \mathfrak{h}) = 0$ and the proof of Theorem 5.1. \square

Theorem 5.1 implies that we have three independent infinitesimal deformations of \mathfrak{tsv} , defined by the cocycles c_1, c_2 and c_3 , so the most general infinitesimal deformation of \mathfrak{tsv} is of the following form:

$$[\cdot, \cdot]_{\lambda, \mu, \nu} = [\cdot, \cdot] + \lambda c_1 + \mu c_2 + \nu c_3.$$

In order to study further deformations of this bracket, one has to compute the Richardson-Nijenhuis brackets of c_1, c_2 and c_3 in $H^3(\mathfrak{tsv}, \mathfrak{tsv})$. One can compute directly using our explicit formulas and finds $[c_i, c_j] = 0$ in $H^3(\mathfrak{tsv}, \mathfrak{tsv})$ for $i, j = 1, 2, 3$; and even better, the bracket of the cocycles themselves vanish, not only their cohomology classes. So one has the

Theorem 5.5.

The bracket $[\cdot, \cdot]_{\lambda, \mu, \nu} = [\cdot, \cdot] + \lambda c_1 + \mu c_2 + \nu c_3$ where $[\cdot, \cdot]$ is the Lie bracket on \mathfrak{tsv} and $c_i, i = 1, 2, 3$ the cocycles given in Theorem 5.1, defines a three-parameter family of Lie algebra brackets on \mathfrak{tsv} . For the sake of completeness, we give below the full formulas in terms of modes:

$$[L_n, L_m]_{\lambda, \mu, \nu} = (m - n)L_{n+m} + \nu(m - n)M_{n+m}$$

$$[L_n, Y_m]_{\lambda, \mu, \nu} = (m - \frac{n}{2} - \frac{\lambda n}{2} + \mu)Y_{n+m}$$

$$[L_n, M_m]_{\lambda, \mu, \nu} = (m - \lambda n + 2\mu)M_{n+m}$$

$$[Y_n, Y_m] = (n - m)M_{n+m}$$

All other terms are vanishing.

The term with cocycle c_3 has already been considered in a slightly different context in [12]. The term with c_2 induces only a small change in the action on \mathfrak{h} : the modules $\mathcal{F}_{1/2}$ and \mathcal{F}_0 are changed into $\mathcal{F}_{1/2, \mu}$ and $\mathcal{F}_{0, \mu}$ (see [10], p.127), the bracket on \mathfrak{h} being fixed. This is nothing but a reparametrization of the generators in the module, and for integer values of μ , the Lie algebra given by $[\cdot, \cdot]_{0, \mu, 0}$ is isomorphic to the original one.

We shall focus in the sequel on the term proportional to c_1 , and denote by \mathfrak{tsv}_{λ} the one-parameter family of Lie algebra structures on \mathfrak{tsv} given by $[\cdot, \cdot]_{\lambda} = [\cdot, \cdot]_{\lambda, 0, 0}$, in coherence with Definition 1.7.

Inspection of the above formulas shows that \mathfrak{tsv}_λ is a semi-direct product $\text{Vect}(S^1) \ltimes \mathfrak{h}_\lambda$ where \mathfrak{h}_λ is a deformation of \mathfrak{h} as a $\text{Vect}(S^1)$ -module; one has $\mathfrak{h}_\lambda = \mathcal{F}_{\frac{1+\lambda}{2}} \oplus \mathcal{F}_\lambda$, and the bracket $\mathcal{F}_{\frac{1+\lambda}{2}} \times \mathcal{F}_{\frac{1+\lambda}{2}} \longrightarrow \mathcal{F}_\lambda$ is the usual one, induced by the Poisson bracket on the torus.

Now, as a by-product of the above computations, we shall determine explicitly $H^1(\mathfrak{tsv}, \mathfrak{tsv})$.

Theorem 5.6.

$H^1(\mathfrak{tsv}, \mathfrak{tsv})$ is three-dimensional, generated by the following cocycles, given in terms of modes by:

$$\begin{aligned} c_1(L_n) &= M_n & c_2(L_n) &= nM_n \\ l(Y_n) &= Y_n & l(M_n) &= 2M_n. \end{aligned}$$

The cocycle l already appeared in Chapter one, when we discussed the derivations of \mathfrak{tsv} ; with the notations of Definition 1.6 one has $l = 2(\delta_2 - \delta_1)$.

Proof:

From Lemma 5.3 above, one has $H^1(\mathfrak{tsv}, \mathcal{G}) = 0$, and so $H^1(\mathfrak{tsv}, \mathfrak{tsv}) = H^1(\mathfrak{tsv}, \mathfrak{h})$. One is led to compute the H^1 of a semi-direct product, as already done in paragraph 3.3. The space $H^1(\mathfrak{tsv}, \mathfrak{h})$ is made from two parts $H^1(\mathcal{G}, \mathfrak{h})$ and $H^1(\mathfrak{h}, \mathfrak{h})$ satisfying the compatibility condition as in Theorem 3.8:

$$c([X, \alpha]) - [X, c(\alpha)] = -[\alpha, c(X)]$$

for $X \in \mathcal{G}$ and $\alpha \in \mathfrak{h}$.

The result is then easily deduced from the previous computations : $H^1(\mathcal{G}, \mathfrak{h}) = H^1(\mathcal{G}, \mathfrak{n})$ is generated by $f\partial \longrightarrow f$ and $f\partial \longrightarrow f'$, which correspond in the mode decomposition to the cocycles c_1 and c_2 . As a corollary, one has $[\alpha, c(X)] = 0$ for $X \in \mathcal{G}$ and $\alpha \in \mathfrak{h}$. Hence the compatibility condition reduces to $c([X, \alpha]) = [X, c(\alpha)]$ and thus $c \in \text{Inv}_{\mathcal{G}} H^1(\mathfrak{h}, \mathfrak{h})$. It can now be deduced from Lemma 5.4 above, that the latter space is one-dimensional, generated by l .

We shall now determine the central charges of \mathfrak{tsv}_λ ; the computation will shed light on some exceptional values of λ , corresponding to interesting particular cases.

5.2 Computation of $H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$

We shall again make use of the exact sequence decomposition $0 \longrightarrow \mathfrak{h}_\lambda \longrightarrow \mathfrak{tsv}_\lambda \xrightarrow{\pi} \mathcal{G} \longrightarrow 0$, and classify the cocycles with respect to their "type" along this decomposition; trivial coefficients will make computations much easier than in the above case. First of all, $0 \longrightarrow H^2(\mathcal{G}, \mathbb{R}) \xrightarrow{\pi^*} H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$ is an injection. So the Virasoro class $c \in H^2(\text{Vect}(S^1), \mathbb{R})$ always survives in $H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$.

For \mathfrak{h}_λ , let us use once again the decomposition $0 \longrightarrow \mathfrak{n}_\lambda \longrightarrow \mathfrak{h}_\lambda \longrightarrow \mathfrak{y}_\lambda \longrightarrow 0$ where $\mathfrak{n}_\lambda = [\mathfrak{h}_\lambda, \mathfrak{h}_\lambda]$. One has: $H^1(\mathcal{G}, H^1(\mathfrak{h}_\lambda)) = H^1(\mathcal{G}, \mathfrak{y}_\lambda^*) = H^1(\mathcal{G}, \mathcal{F}_{\frac{1+\lambda}{2}}^*) = H^1(\mathcal{G}, \mathcal{F}_{-(\frac{3+\lambda}{2})})$.

The cohomologies of degree one of $\text{Vect}(S^1)$ with coefficients in densities are known (see [10], Theorem 2.4.12): the space $H^1(\mathcal{G}, \mathcal{F}_{-(\frac{3+\lambda}{2})})$ is trivial, except for the three exceptional cases $\lambda = -3, -1, 1$:

$H^1(\mathcal{G}, \mathcal{F}_0)$ is generated by the cocycles $f\partial \longrightarrow f$ and $f\partial \longrightarrow f'$;

$H^1(\mathcal{G}, \mathcal{F}_{-1})$ is generated by the cocycle $f\partial \longrightarrow f'' dx$;

$H^1(\mathcal{G}, \mathcal{F}_{-2})$ is generated by the cocycle $f\partial \longrightarrow f''' (dx)^2$, corresponding to the "Souriau cocycle" associated to the central charge of the Virasoro algebra (see [12], chapter IV).

In terms of modes, the corresponding cocycles are given by:

$$\begin{aligned} c_1(L_n, Y_m) &= \delta_{n+m}^0, & c_2(L_n, Y_m) &= n\delta_{n+m}^0 \text{ for } \lambda = -3; \\ c(L_n, Y_m) &= n^2\delta_{n+m}^0 \text{ for } \lambda = -1; \\ c(L_n, Y_m) &= n^3\delta_{n+m}^0 \text{ for } \lambda = 1. \end{aligned}$$

The most delicate part is the investigation of the term $E_2^{0,2} = \text{Inv}_{\mathcal{G}}H^2(\mathfrak{h}_\lambda)$ (this refers of course to the Hochschild-Serre spectral sequence associated to the above decomposition). For $H^2(\mathfrak{h}_\lambda)$ we shall use the same short exact sequences as for \mathfrak{h} in the proof of Lemma 5.3:

$$\begin{aligned} 0 &\longrightarrow \text{Ker}\partial \longrightarrow \Lambda^2\mathfrak{n}_\lambda^* \xrightarrow{\partial} \Lambda^2y_\lambda^* \wedge \mathfrak{n}_\lambda^* \\ 0 &\longrightarrow \text{Ker}\partial \longrightarrow y_\lambda^* \wedge \mathfrak{n}_\lambda^* \xrightarrow{\partial} \Lambda^3y_\lambda^* \\ 0 &\longrightarrow \underline{\mathfrak{n}}_\lambda^* \longrightarrow \Lambda^2y_\lambda^* \longrightarrow \text{Coker}\partial \longrightarrow 0 \end{aligned}$$

(where $\underline{\mathfrak{n}}_\lambda$ stands for \mathfrak{n}_λ divided out by the space of constant functions). One readily shows that for the first two sequences one has $\text{Inv}_{\mathcal{G}}\text{Ker}\partial = 0$. The third one is more complicated; the cohomology exact sequence yields:

$$0 \longrightarrow \text{Inv}_{\mathcal{G}}\text{Coker}\partial \longrightarrow H^1(\mathcal{G}, \underline{\mathfrak{n}}_\lambda^*) \longrightarrow H^1(\mathcal{G}, \Lambda^2y_\lambda^*) \longrightarrow \dots$$

The same result as above (see [10], Theorem 2.4.12) shows that:

$$H^1(\mathcal{G}, \mathfrak{n}_\lambda^*) = H^1(\mathcal{G}, \mathcal{F}_{(-1-\lambda)}) = 0 \text{ unless } \lambda = 1, -1, 0,$$

and one then has to investigate case by case; set $c(Y_p, Y_q) = a_p\delta_{p+q}^0$ for the potential cochains on y_λ . For each n one has the relation:

$$(ad_{L_n}c)(Y_p, Y_q) - (q-p)\gamma(L_n)(M_{p+q}) = 0 \quad (5.6)$$

for some 1-cocycle $\gamma : \mathcal{G} \longrightarrow \mathfrak{n}_\lambda^*$, and for all (p, q) such that $n + p + q = 0$; if $\gamma(L_n)(M_k) = b_n\delta_{n+m+k}^0$, one obtains in terms of modes, using $a_p = -a_{-p}$:

$$(p+q) \left(\frac{1+\lambda}{2} \right) (a_p - a_q) + qa_q - pa_q - (q-p)b_{-(p+q)} = 0$$

Let us now check the different cases of non-vanishing terms in $H^1(\mathcal{G}, \mathfrak{n}_\lambda^*)$.

For $\lambda = 1$ one has $b_n = n^3$, and one deduces $a_p = p^3$.

For $\lambda = -1$ there are two possible cases $b_n = n$ or $b_n = 1$, the above equation gives

$$qa_p - pa_q = (q-p)(\alpha(p+q) + \beta);$$

the only possible solution would be to set a_p constant, but this is not consistent with $a_p = -a_{-p}$. For $\lambda = 0$, one gets $b_n = n^2$ and the equation gives

$$\left(\frac{p+q}{2} \right) (a_p + a_q) + qa_p - pa_q - (q-p)(p+q)^2 = 0$$

One easily checks that there are no solutions.

Finally, one gets a new cocycle generating an independent class in $H^2(\mathfrak{tsv}_1, \mathbb{R})$, given by the formulas:

$$\begin{aligned} c(Y_n, Y_m) &= n^3 \delta_{n+m}, \\ c(L_n, M_m) &= n^3 \delta_{n+m} \end{aligned}$$

Let us summarize our results:

Theorem 5.7.

For $\lambda \neq -3, -1, 1$, $H^2(\mathfrak{tsv}_\lambda, \mathbb{R}) \simeq \mathbb{R}$ is generated by the Virasoro cocycle.

For $\lambda = -3, -1$, $H^2(\mathfrak{tsv}_\lambda, \mathbb{R}) \simeq \mathbb{R}^2$ is generated by the Virasoro cocycle and an independent cocycle of the form $c(L_n, Y_m) = \delta_{n+m}^0$ for $\lambda = -3$ or $c(L_n, Y_m) = n^2 \delta_{n+m}^0$ for $\lambda = -1$.

For $\lambda = 1$, $H^2(\mathfrak{tsv}_1, \mathbb{R}) \simeq \mathbb{R}^3$ is generated by the Virasoro cocycle and the two independent cocycles c_1 and c_2 defined by (all other components vanishing)

$$\begin{aligned} c_1(L_n, Y_m) &= n^3 \delta_{n+m}^0; \\ c_2(L_n, M_m) &= c_2(Y_n, Y_m) = n^3 \delta_{n+m}^0 \end{aligned}$$

Remark : The isomorphism $H^2(\mathfrak{sv}_0, \mathbb{R}) \simeq \mathbb{R}$ has already been proved in [14]. As we shall see in paragraph 5.4, generally speaking, local cocycles may be carried over from \mathfrak{tsv} to \mathfrak{sv} or from \mathfrak{sv} to \mathfrak{tsv} without any difficulty.

Let us look more carefully at the $\lambda = 1$ case. One has that $\mathfrak{h}_1 = \mathcal{F}_1 \oplus \mathcal{F}_1$ with the obvious bracket $\mathcal{F}_1 \times \mathcal{F}_1 \longrightarrow \mathcal{F}_1$; so, algebraically, $\mathfrak{h}_1 = \text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$. One deduces immediately that $\mathfrak{tsv}_1 = \text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$; so the cohomological result for \mathfrak{tsv}_1 can be easily reinterpreted. Let $f_\varepsilon \partial$ and $g_\varepsilon \partial$ be two elements in $\text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$, and compute the Virasoro cocycle $c(f_\varepsilon \partial, g_\varepsilon \partial) = \int_{S^1} f_\varepsilon''' g_\varepsilon dt$ as a truncated polynomial in ε ; one has $f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2$ and $g_\varepsilon = g_0 + \varepsilon g_1 + \varepsilon^2 g_2$ so finally:

$$c(f_\varepsilon \partial, g_\varepsilon \partial) = \int_{S^1} f_0''' g_0 dt + \varepsilon \int_{S^1} (f_0''' g_1 + f_1''' g_0) dt + \varepsilon^2 \int_{S^1} (f_0''' g_2 + f_1''' g_1 + f_2''' g_0) dt.$$

In other terms: $c(f_\varepsilon \partial, g_\varepsilon \partial) = c_0(f_\varepsilon \partial, g_\varepsilon \partial) + \varepsilon c_1(f_\varepsilon \partial, g_\varepsilon \partial) + \varepsilon^2 c_2(f_\varepsilon \partial, g_\varepsilon \partial)$. One can easily identify the c_i , $i = 0, 1, 2$ with the cocycles defined in the above theorem, using a decomposition into modes. This situation can be described by a universal central extension

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow \widehat{\mathfrak{tsv}}_1 \longrightarrow \text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0) \longrightarrow 0$$

and the formulas of the cocycles show that $\widehat{\mathfrak{tsv}}_1$ is isomorphic to $\text{Vir} \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$

Remarks:

1. Cohomologies of Lie algebra of type $\text{Vect}(S^1) \underset{\mathbb{R}}{\otimes} A$, where A is an associative and commutative algebra (the Lie bracket being as usual given by $[f\partial \otimes a, g\partial \otimes b] = (fg' - gf')\partial \otimes ab$), have been studied by C. Sah and collaborators (see [29]). Their result is: $H^2 \left(\text{Vect}(S^1) \underset{\mathbb{R}}{\otimes} A \right) = A'$ where $A' = \text{Hom}_{\mathbb{R}}(A, \mathbb{R})$; all cocycles are given by the Virasoro cocycle composed with the linear form on A . The isomorphism $H^2(\mathfrak{tsv}_1, \mathbb{R}) \simeq \mathbb{R}^3$ (see Theorem 5.7) could have been deduced from this general theorem.

2. One can obtain various generalisations of our algebra \mathfrak{h} as nilpotent Lie algebras with $\text{Vect}(S^1)$ -like brackets, such as

$$[Y_n, Y_m] = (m - n)M_{n+m} \quad (5.7)$$

by using the same scheme. Let A be an artinian ring quotient of some polynomial ring $\mathbb{R}[t_1, \dots, t_n]$ and $A_0 \subset A$ its maximal ideal; then $\text{Vect}(S^1) \otimes_{\mathbb{R}} A_0$ is a nilpotent Lie algebra whose successive brackets are of the same type (5.7). One could speak of a "virasorization" of nilpotent Lie algebras. Explicit examples are provided in the subsection 3.5 about multi-diagonal operators of the present article.

3. It is interesting in itself to look at how the dimension of $H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$ varies under deformations. For generic values of λ , this dimension is equal to one, and it increases for some exceptional values of λ ; one can consider this as an example of so called "Fuks principle" in infinite dimension: deformations can decrease the rank of cohomologies but never increase it.
4. Analogous Lie algebra structures, of the "Virasoro-tensorized" kind, have been considered in a quite different context in algebraic topology by Tamanoi, see [30].

5.3 About deformations of \mathfrak{tsv}_1

We must consider the local cochains $C_{loc}^*(\mathfrak{tsv}_1, \mathfrak{tsv}_1)$. The Lie algebra \mathfrak{tsv}_1 admits a graduation mod 3 by the degree of polynomial in ε , the Lie bracket obviously respects this graduation; this graduation induces on the space of local cochains a graduation by weight, and $C_{loc}^*(\mathfrak{tsv}_1, \mathfrak{tsv}_1)$ splits into direct sum of sub-complexes of homogeneous weight denoted by $C_{loc}^*(\mathfrak{tsv}_1, \mathfrak{tsv}_1)_{(p)}$. Moreover, as classical in computations for Virasoro algebra, one can use the adjoint action of the zero mode L_0 (corresponding geometrically to the Euler field $z \frac{\partial}{\partial z}$) to reduce cohomology computations to the subcomplexes $C_{loc}^*(\mathfrak{tsv}_1, \mathfrak{tsv}_1)_{(p)(0)}$ of cochains which are homogeneous of weight 0 with respect to $\text{ad } L_0$ (see e.g. [12], chapter IV).

We can use the graduation in ε and consider homogeneous cochains with respect to that graduation. Here is what one gets, according to the weight:

- weight 1: one has cocycles of the form

$$c(L_n, Y_m) = (m - n)M_{n+m}, \quad c(L_n, L_m) = (m - n)Y_{n+m}$$

but if $b(L_n) = Y_n$, then $c = \partial b$.

- weight 0: all cocycles are coboundaries, using the well-known result $H^*(\text{Vect}(S^1), \text{Vect}(S^1)) = 0$.
- weight -1: one has to consider cochains of the following form

$$\begin{aligned} c(Y_n, M_m) &= a(m - n)M_{n+m} \\ c(Y_n, Y_m) &= b(m - n)Y_{n+m} \\ c(L_n, M_m) &= e(m - n)Y_{n+m} \\ c(L_n, Y_m) &= d(m - n)L_{n+m} \end{aligned}$$

and check that $\partial c = 0$. It readily gives $e = d = 0$.

If one sets $\tilde{c}(Y_n) = \alpha L_n$ and $\tilde{c}(M_n) = \beta Y_n$, then

$$\partial \tilde{c}(Y_n, M_m) = (\alpha + \beta)(m - n)M_{n+m}$$

$$\partial\tilde{c}(Y_n, M_m) = (2\alpha - \beta)(m - n)Y_{n+m}$$

So all these cocycles are cohomologically trivial,

- weight -2: set

$$c(Y_n, M_m) = \alpha(m - n)Y_{n+m}$$

$$c(M_n, M_m) = \beta(m - n)M_{n+m}$$

$$c(L_n, M_m) = \gamma(m - n)L_{n+m}$$

Coboundary conditions give $\gamma = \alpha$ and $\beta = \gamma + \alpha$, but if $\bar{c}(M_n) = L_n$, then

$$\partial\bar{c}(M_n, M_m) = (m - n)Y_{n+m}$$

$$\partial\bar{c}(Y_n, M_m) = 2(m - n)M_{n+m}$$

$$\partial\bar{c}(L_n, M_m) = (m - n)L_{n+m}$$

- weight -3: we find for this case the only surviving cocycle.

One readily checks that $C \in C_{loc}^*(\mathfrak{tsv}_1, \mathfrak{tsv}_1)_{(-3)(0)}$ defined by

$$C(Y_n, M_m) = (m - n)L_{n+m}$$

$$C(M_n, M_m) = (m - n)Y_{n+m}$$

is a cocycle and cannot be a coboundary.

We can describe the cocycle C above more pleasantly by a global formula: let $f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2$ and $g_\varepsilon = g_0 + \varepsilon g_1 + \varepsilon^2 g_2$, with f_i, g_i elements of $\text{Vect}(S^1)$. The bracket $[,]$ of \mathfrak{tsv}_1 is then the following:

$$[f_\varepsilon, g_\varepsilon] = \sum_{k=0}^2 \sum_{i+j=k} [f_i, g_j] \quad \text{or} \quad (f_\varepsilon g'_\varepsilon - g_\varepsilon f'_\varepsilon)|_{\varepsilon^3=0},$$

and the deformed bracket $[,] + \mu C$ will be $[f_\varepsilon, g_\varepsilon]_\mu = (f_\varepsilon g'_\varepsilon - g_\varepsilon f'_\varepsilon)|_{\varepsilon^3=\mu}$. So we have found that $\dim H^2(\mathfrak{tsv}_1, \mathfrak{tsv}_1) = 1$.

In order to construct deformations, we still have to check for the Nijenhuis-Richardson bracket $[C, C]$ in $C^3(\mathfrak{tsv}_1, \mathfrak{tsv}_1)$. The only possibly non-vanishing term is:

$$\begin{aligned} [C, C](M_n, M_n, M_p) &= \sum_{(cycl)} C(C(M_n, M_m,), M_p) = \sum_{(cycl)} (m - n)C(Y_{n+m}, M_p) \\ &= \sum_{(cycl)} (pm - pn + n^2 - m^2)L_{n+m+p} = 0 \end{aligned}$$

So there does not exist any obstruction and we have obtained a genuine deformation. We summarize all these results in the following

Proposition 5.8.

There exists a one-parameter deformation of the Lie algebra \mathfrak{tsv}_1 , as $\mathfrak{tsv}_{1,\mu} = \text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3=\mu)$. This deformation is the only one possible, up to isomorphism.

If one is interested in central charges, the above-mentioned theorem of C. Sah and al., see [29], shows that $\dim H^2(\mathfrak{tsv}_{1,\mu}, \mathbb{R}) = \mathbb{R}^3$ and the universal central extension $\widehat{\mathfrak{tsv}}_{1,\mu}$ is isomorphic to $\text{Vir} \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = \mu)$. We did not do the computations, but we conjecture that $\mathfrak{tsv}_{1,\mu}$ is rigid, the ring $\mathbb{R}[\varepsilon]/(\varepsilon^3 = \mu)$ being more generic than $\mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$. More generally, it could be interesting to study systematically Lie algebras of type $\text{Vect}(S^1) \otimes A$ where A is a commutative ring, their geometric interpretation being "Virasoro current algebras".

5.4 Coming back to the original Schrödinger-Virasoro algebra

The previous results concern the twisted Schrödinger-Virasoro algebra generated by the modes (L_n, Y_m, M_p) for $(n, m, p) \in \mathbb{Z}^3$, which make computations easier and allows direct application of Fuks' techniques. The "actual" Schrödinger-Virasoro algebra is generated by the modes (L_n, Y_m, M_p) for $(n, p) \in \mathbb{Z}^2$ but $m \in \mathbb{Z} + \frac{1}{2}$. Yet Theorem 5.1 and Theorem 5.5 on deformations of \mathfrak{tsv} are also valid for \mathfrak{sv} : one has $\dim H^2(\mathfrak{sv}, \mathfrak{sv}) = 3$ with the same cocycles c_1, c_2, c_3 , since these do not allow 'parity-changing' terms such as $L \times Y \rightarrow M$ or $Y \times Y \rightarrow Y$ for instance (the (L, M) -generators being considered as 'even' and the Y -generators as 'odd').

But the computation of $H^2(\mathfrak{sv}_\lambda, \mathbb{R})$ will yield very different results compared to Theorem 5.7, since 'parity' is not conserved for all the cocycles we found, so we shall start all over again. Let us use the adjoint action of L_0 to simplify computations : all cohomologies are generated by cocycles c such that $ad_{L_0} \cdot c = 0$, i.e. such that $c(A_k, B_l) = 0$ for $k + l \neq 0$, A and B being L, Y or M . So, for non-trivial cocycles, one must have $c(Y_n, L_p) = 0, c(Y_n, M_p) = 0$ for all Y_n, L_p, M_p ; in $H^2(\mathfrak{sv}_\lambda, \mathbb{R})$, terms of the type $H^1(\mathcal{G}, H^1(\mathfrak{h}_\lambda))$ will automatically vanish. The Virasoro class in $H^2(\mathcal{G}, \mathbb{R})$ will always survive, and one has to check what happens with the terms of type $Inv_{\mathcal{G}} H^2(\mathfrak{h}_\lambda)$. As in the proof of Lemma 5.3, the only possibilities come from the short exact sequence:

$$0 \longrightarrow \mathfrak{n}_\lambda^* \longrightarrow \Lambda^2 y_\lambda^* \longrightarrow \text{Coker} \partial \longrightarrow 0$$

which induces: $0 \longrightarrow Inv_{\mathcal{G}} \text{Coker} \partial \longrightarrow H^1(\mathcal{G}, \mathfrak{n}_\lambda^*) \longrightarrow H^1(\mathcal{G}, \Lambda^2 y_\lambda^*)$ and one obtains the same equation (5.6) as above:

$$(ad_{L_n} c)(Y_p, Y_q) + (q - p)\gamma(L_n)(M_{p+q}) = 0$$

If $c(Y_p, Y_q) = a_p \delta_{p+q}^0$, the equation gives:

$$-a_p \left(p + \frac{\lambda + 3}{2} n \right) + a_{p+n} \left(p - \frac{\lambda + 1}{2} n \right) - (p - q)\gamma(L_n)(M_{p+q}) = 0$$

One finds two exceptional cases with non-trivial solutions:

- for $\lambda = 1$, $a_p = p^3$ and $c(L_n, M_m) = n^3 \delta_{n+m}^0$ gives a two-cocycle, very much analogous to the $\text{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$ case, except that one has no term in $c(L_n, Y_p)$.
- for $\lambda = -3$, if $\gamma \equiv 0$, the above equation gives $pa_p = (p + n)a_{p+n}$ for every p and n . So $a_p = \frac{1}{p}$ is a solution, and one sees why this solution was not available in the twisted case.

Let us summarize:

Proposition 5.9. *The space $H^2(\mathfrak{sv}_\lambda, \mathbb{R})$ is one-dimensional, generated by the Virasoro cocycle, save for two exceptional values of λ , for which one has one more independent cocycle, denoted by c_1 , with the following non-vanishing components:*

- for $\lambda = 1$: $c_1(Y_p, Y_q) = p^3 \delta_{p+q}^0$ and $c_1(L_p, M_q) = p^3 \delta_{p+q}^0$,
- for $\lambda = -3$: $c_1(Y_p, Y_q) = \frac{\delta_{p+q}^0}{p}$.

Remark: The latter case is the most surprising one, since it contradicts the well-established dogma asserting that only local classes are interesting. This principle of locality has its roots in quantum field theory (see e.g [20] for basic principles of axiomatic field theory); its mathematical status has its foundations in the famous theorem of J. Peetre, asserting that local mappings are given by differential operators, so – in terms of modes – the coefficients are polynomial in n . Moreover, there is a general theorem in the theory of cohomology of Lie algebras of vector fields (see [10]) which states that continuous cohomology is in general multiplicatively generated by local cochains, called diagonal in [10]. Here our cocycle contains an anti-derivative, so there could be applications in integrable systems, considered as Hamiltonian systems, the symplectic manifold given by the dual of (usually centrally extended) infinite dimensional Lie algebras (see for example [12], Chapters VI and X).

6 Verma modules of \mathfrak{sv} and Kac determinants

6.1 Introduction

There are a priori infinitely many ways to define Verma modules on \mathfrak{sv} , corresponding to the two natural gradations: one of them (called *degree* and denote by *deg* in the following) corresponds to the adjoint action of L_0 , so that $\deg(X_n) = -n$ for $X = L, Y$ or M ; it is given by $-\delta_1$ in the notation of Definition 1.6. The other one, corresponding to the gradation of the Cartan prolongation (see Section 4.1), is given by the outer derivation δ_2 . The action of both gradations is diagonal on the generators (X_n); the subalgebra of weight 0 is two-dimensional abelian, generated by L_0 and M_0 , in the former case, and three-dimensional solvable, generated by $L_0, Y_{\frac{1}{2}}, M_1$ in the latter case.

Since Verma modules are usually defined by inducing a character of an abelian subalgebra to the whole Lie algebra (although this is by no means necessary), we shall forget altogether the gradation given by δ_2 in this section and consider representations of \mathfrak{sv} that are induced from $\langle L_0, M_0 \rangle$.

Let $\mathfrak{sv}_{(n)} = \{Z \in \mathfrak{sv} \mid ad_{L_0}.Z = nZ\}$ ($n \in \frac{1}{2}\mathbb{Z}$), $\mathfrak{sv}_{>0} = \bigoplus_{n>0} \mathfrak{sv}_{(n)}$, $\mathfrak{sv}_{<0} = \bigoplus_{n<0} \mathfrak{sv}_{(n)}$, $\mathfrak{sv}_{\leq 0} = \mathfrak{sv}_{<0} \oplus \mathfrak{sv}_{(0)}$. Define $\mathbb{C}_{h,\mu} = \mathbb{C}\psi$ ($h, \mu \in \mathbb{C}$) to be the character of $\mathfrak{sv}_{(0)} = \langle L_0, M_0 \rangle$ such that $L_0\psi = h\psi$, $M_0\psi = \mu\psi$. Following the usual definition of Verma modules (see [19] or [24]), we extend $\mathbb{C}_{h,\mu}$ trivially to $\mathfrak{sv}_{\leq 0}$ by putting $\mathfrak{sv}_{<0}.\psi = 0$ and call $\mathcal{V}_{h,\mu}$ the induction of the representation $\mathbb{C}_{h,\mu}$ to \mathfrak{sv} :

$$\mathcal{V}_{h,\mu} = \mathcal{U}(\mathfrak{sv}) \otimes_{\mathcal{U}(\mathfrak{sv}_{\leq 0})} \mathbb{C}_{h,\mu}. \quad (6.1)$$

Take notice that with this choice of signs, *negative* degree elements X_n, Y_n, M_n ($n > 0$) applied to ψ yield zero.

The Verma module $\mathcal{V}_{h,\mu}$ is positively graded through the natural extension of *deg* from \mathfrak{g} to $\mathcal{U}(\mathfrak{g})$, namely, we put $(\mathcal{V}_{h,\mu})_{(n)} = \mathcal{V}_{(n)} = \mathcal{U}(\mathfrak{sv}_{>0})_{(n)} \otimes \mathbb{C}_{h,\mu}$.

There exists exactly one bilinear form $\langle \mid \rangle$ (called the *contravariant Hermitian form* or, as we shall also say, 'scalar product', although it is neither necessarily positive nor even necessarily non-degenerate) on $\mathcal{V}_{h,\mu}$ such that $\langle \psi \mid \psi \rangle = 1$ and $X_n^* = X_{-n}$, $n \in \frac{1}{2}\mathbb{Z}$ ($X = L, Y$ or M), where the star means taking the adjoint with respect to the bilinear form (see [19]). For the contravariant Hermitian

form, $\langle \mathcal{V}_{(j)} | \mathcal{V}_{(k)} \rangle = 0$ if $j \neq k$. It is well-known that the module $\mathcal{V}_{h,\mu}$ is indecomposable and possesses a unique maximal proper sub-representation $\mathcal{K}_{h,\mu}$, which is actually the kernel of Hermitian form, and such that the quotient module $\mathcal{V}_{h,\mu}/\mathcal{K}_{h,\mu}$ is irreducible.

Hence, in order to determine if $\mathcal{V}_{h,\mu}$ is irreducible, and find the irreducible quotient representation if it is not, one is naturally led to the computation of the *Kac determinants*, by which we mean the determinants of the Hermitian form restricted to $\mathcal{V}_{(n)} \times \mathcal{V}_{(n)}$ for each n .

Let us introduce some useful notations for *partitions*.

Definition 6.1. *A partition $A = (a^1, a^2, \dots)$ of degree $n = \deg(A) \in \mathbb{N}^*$ is an ordered set a^1, a^2, \dots of non-negative integers such that $\sum_{i \geq 1} ia^i = n$.*

A partition can be represented as a Young tableau: one associates to A a set of vertical stacks of boxes put side by side, with (from left to right) a^1 stacks of height 1, a^2 stacks of height 2, and so on.

The *width* $\text{wid}(A)$ of the tableau is equal to $\sum_{i \geq 1} a^i$.

By convention, we shall say that there is exactly one partition of degree 0, denoted by \emptyset , and such that $\text{wid}(\emptyset) = 0$.

Now any partition A defines elements of $\mathcal{U}(\mathfrak{sv})$, namely, let us put $X^{-A} = X_{-1}^{A_1} X_{-2}^{A_2} \dots$ (X stands here for L or M) and $Y^{-A} = Y_{-\frac{1}{2}}^{A_1} Y_{-\frac{3}{2}}^{A_2} \dots$, so that $\deg(X^{-A}) = \deg(A)$ and

$$\deg(Y^{-A}) = \sum_{i \geq 1} (i - \frac{1}{2}) A^i = \deg(A) - \frac{1}{2} \text{wid}(A)$$

(we shall also call this expression the *shifted degree* of A , and write it $\widetilde{\deg}(A)$).

Definition 6.2. *We denote by $\mathcal{P}(n)$ (resp. $\tilde{\mathcal{P}}(n)$) the set of partitions of degree (resp. shifted degree) n .*

By Poincaré-Birkhoff-Witt's theorem (PBW for short), $\mathcal{V}_{(n)}$ is generated by the vectors $Z = X^{-A} Y^{-B} M^{-C} \psi$ where A, B, C range among all partitions such that $\deg(A) + \widetilde{\deg}(B) + \deg(C) = n$. On this basis of $\mathcal{V}_{(n)}$, that we shall call in the sequel the *PBW basis at degree n* , it is possible to define three partial graduations, namely, $\deg_L(Z) = \deg(A)$, $\widetilde{\deg}_Y(Z) = \widetilde{\deg}(B)$ and $\deg_M(Z) = \deg(C)$.

It is then of course easy to express the dimension of $\mathcal{V}_{(n)}$ as a (finite) sum of products of the partition function p of number theory, but we do not know how to simplify this (rather complicated) expression, so it is of practically no use. Let us rather write the set of above generators for degree $n = 0, \frac{1}{2}, 1, \frac{3}{2}$ and 2:

$$\begin{aligned} \mathcal{V}_{(0)} &= \langle \psi \rangle \\ \mathcal{V}_{(\frac{1}{2})} &= \langle Y_{-\frac{1}{2}} \psi \rangle \\ \mathcal{V}_{(1)} &= \langle (M_{-1} \psi), (Y_{-\frac{1}{2}}^2 \psi, L_{-1} \psi) \rangle \\ \mathcal{V}_{(\frac{3}{2})} &= \langle (M_{-1} Y_{-\frac{1}{2}} \psi), (Y_{-\frac{1}{2}}^3 \psi, Y_{-\frac{3}{2}} \psi, L_{-1} Y_{-\frac{1}{2}} \psi) \rangle \\ \mathcal{V}_{(2)} &= \langle (M_{-1}^2 \psi, M_{-2} \psi), (M_{-1} Y_{-\frac{1}{2}}^2 \psi, M_{-1} L_{-1} \psi), (Y_{-\frac{1}{2}}^4 \psi, Y_{-\frac{1}{2}} Y_{-\frac{3}{2}} \psi, X_{-1} Y_{-\frac{1}{2}}^2 \psi, X_{-2} \psi, X_{-1}^2 \psi) \rangle. \end{aligned} \quad (6.2)$$

The elements of these Poincaré-Birkhoff-Witt bases have been written in the *M-order* and separated into *blocks* (see below paragraph 6.3 for a definition of these terms).

The Kac determinants are quite easy to compute in the above bases at degree $0, \frac{1}{2}, 1$. If $\{x_1, \dots, x_{\dim(\mathcal{V}_n)}\}$ is the PBW basis at degree n , put

$$\Delta_n^{\mathfrak{sv}} = \det(\langle x_i | x_j \rangle)_{i,j=1,\dots,\dim(\mathcal{V}_n)}.$$

Note that $\Delta_n^{\mathfrak{sv}}$ does not depend on the ordering of the elements of the basis $\{x_1, \dots, x_{\dim(\mathcal{V}_n)}\}$.

Then

$$\begin{aligned} \Delta_0^{\mathfrak{sv}} &= 1 \\ \Delta_{\frac{1}{2}}^{\mathfrak{sv}} &= \langle Y_{-\frac{1}{2}}\psi, Y_{-\frac{1}{2}}\psi \rangle = \mu \\ \Delta_1^{\mathfrak{sv}} &= \det \begin{pmatrix} 0 & 0 & \mu \\ 0 & 2\mu^2 & \mu \\ \mu & \mu & 2h \end{pmatrix} = -2\mu^4. \end{aligned}$$

For higher degrees, straightforward computations become quickly dull: even at degree 2, one gets a 9×9 determinant, to be compared with the simple 2×2 -determinant that one gets when computing the Kac determinant of the Virasoro algebra at level 2.

The essential idea for calculating this determinant at degree n is to find two permutations σ, τ of the set of elements of the basis, $x_1, \dots, x_{\dim(\mathcal{V}_n)}$, in such a way that the matrix $(\langle x_{\sigma_i} | x_{\tau_j} \rangle)_{i,j=1,\dots,p}$ be upper-triangular. Then the Kac determinant $\Delta_n^{\mathfrak{sv}}$, as computed in the basis $\{x_1, \dots, x_{\dim(\mathcal{V}_n)}\}$, is equal (up to a sign) to the product of diagonal elements of that matrix, which leads finally to the following theorem.

Theorem 6.2.

The Kac determinant $\Delta_n^{\mathfrak{sv}}$ is given (up to a non-zero constant) by

$$\Delta_n^{\mathfrak{sv}} = \mu^{\sum_{0 \leq j \leq n} \sum_{B \in \tilde{\mathcal{P}}(j)} (\text{wid}(B) + 2 \sum_{0 \leq i \leq n-j} p(n-j-i) (\sum_{A \in \mathcal{P}(i)} \text{wid}(A)))} \quad (6.3)$$

where $p(k) := \#\mathcal{P}(k)$ is the usual partition function.

The same formula holds for the central extension of \mathfrak{sv} .

The proof is technical but conceptually easy, depending essentially on the fact that \mathfrak{h} contains a central subalgebra, namely $\langle M_n \rangle_{n \in \mathbb{Z}}$, that is a module of tensor densities for the action of the Virasoro algebra. This fact implies that, in the course of the computations, the (L, M) -generators decouple from the other generators (see Lemma 6.6).

As a matter of fact, we shall need on our way to compute the Kac determinants for the subalgebra $\langle L_n, M_n \rangle_{n \in \mathbb{Z}} \simeq \text{Vect}(S^1) \rtimes \mathcal{F}_0$ or $\text{Vir} \rtimes \mathcal{F}_0$. The result is very similar and encaptures, so we think, the main characteristics of the Kac determinants of Lie algebras $\text{Vect}(S^1) \rtimes \mathfrak{k}$ or $\text{Vir} \rtimes \mathfrak{k}$ such that \mathfrak{k} contains in its center a module of tensor densities. A contrario, the very first computations for the deformations and central extensions of $\text{Vect}(S^1) \rtimes \mathcal{F}_0$ obtained through the cohomology spaces $H^2(\text{Vect}(S^1), \mathcal{F}_0)$ and $H^2(\text{Vect}(S^1) \tilde{\rtimes} \mathcal{F}_0, \mathbb{C})$ (denoting by $\tilde{\rtimes}$ any deformed product) show that the Kac determinants look completely different as soon as the image of $\langle M_n \rangle_{n \in \mathbb{Z}}$ is not central any more in \mathfrak{k} .

We state the result for $\text{Vir} \rtimes \mathcal{F}_0$ as follows.

Theorem 6.1 *Let Vir_c be the Virasoro algebra with central charge $C \in \mathbb{R}$ and $\mathcal{V}' = \mathcal{V}'_{h,\mu} = \mathcal{U}((\text{Vir}_c \rtimes \mathcal{F}_0)_0) \otimes_{\mathcal{U}((\text{Vir}_c \rtimes \mathcal{F}_0)_{\leq 0})} \mathbb{C}_{h,\mu} \subset \mathcal{V}_{h,\mu}$ be the Verma module representation of $\text{Vir}_c \rtimes \mathcal{F}_0$ induced from the character $\mathbb{C}_{h,\mu}$, with the graduation naturally inherited from that of $\mathcal{V}_{h,\mu}$.*

Then the Kac determinant (computed in the PBW bases) $\Delta_n^{\text{Vir} \times \mathcal{F}_0}$ of $\mathcal{V}'_{h,\mu}$ at degree n is equal (up to a positive constant) to

$$\Delta_n^{\text{Vir} \times \mathcal{F}_0} = (-1)^{\dim(\mathcal{V}'_\mu)} \mu^{2 \sum_{0 \leq i \leq n} p(n-i) (\sum_{A \in \mathcal{P}(i)} \text{wid}(A))}. \quad (6.4)$$

It does not depend on C .

6.2 Kac determinant formula for $\text{Vir} \times \mathcal{F}_0$

We first need to introduce a few notations and define two different orderings for the PBW bases of $\text{Vir} \otimes \mathcal{F}_0$.

Definition 6.3. Let A, B be two partitions of $n \in \mathbb{N}^*$. One says that A is finer than B , and write $A \preceq B$, if A can be gotten from B by a finite number of transformations $B \rightarrow \dots \rightarrow D \rightarrow D' \rightarrow \dots \rightarrow A$ where

$$D^i = D^i + C^i \quad (i \neq p), \quad D^p = D^i - 1,$$

$C = (C^i)$ being a non-trivial partition of degree p .

Graphically, this means that the Young tableau of A is obtained from the Young tableau of B by splitting some of the stacks of boxes into several stacks.

The relation \preceq gives a *partial* order on the set of partitions of fixed degree n , with smallest element $(1, 1, \dots, 1)$ and largest element the trivial partition (n) . One chooses arbitrarily, for every degree n , a total ordering \leq of $\mathcal{P}(n)$ compatible with \preceq , i.e. such that $(A \preceq B) \Rightarrow (A \leq B)$.

Definition 6.4. If A is a partition, then $X^A := (X^{-A})^*$ is given by $X^A = \dots X_2^{A_2} X_1^{A_1}$ (where X stands for L or M).

Let $n \in \mathbb{N}$. By Poincaré-Birkhoff-Witt's theorem, the $L^{-A}M^{-C}$ (A partition of degree p , C partition of degree q , $p, q \geq 0$, $p + q = n$) form a basis \mathcal{B}' of \mathcal{V}'_n , the subspace of \mathcal{V}' made up of the vectors of degree n .

We now give two different orderings of the set \mathcal{B}' , that we call *horizontal ordering* (or *M-ordering*) and *vertical ordering* (or *L-ordering*). For the *horizontal ordering*, we proceed as follows:

-we split \mathcal{B}' into $(n + 1)$ blocks $\mathcal{B}'_0, \dots, \mathcal{B}'_n$ such that

$$\mathcal{B}'_j = \{L^{-A}M^{-C}\psi \mid \deg(A) = j, \deg(C) = n - j\}; \quad (6.5)$$

-we split each block \mathcal{B}'_j into *sub-blocks* $\mathcal{B}'_{j,C}$ (also called *j-sub-blocks* if one wants to be more explicit) such that C runs among the set of partitions of $n - j$ in the *increasing* order chosen above, and

$$\mathcal{B}'_{j,C} = \{L^{-A}M^{-C}\psi \mid \deg(A) = j\}.$$

-finally, inside each sub-block $\mathcal{B}'_{j,C}$, we take the elements $L^{-A}M^{-C}$, $A \in \mathcal{P}_j$, in the *decreasing order*.

For the *vertical ordering*, we split \mathcal{B}' into blocks

$$\tilde{\mathcal{B}}'_j = \{L^{-A}M^{-C}\psi \mid \deg(A) = n - j, \deg(C) = j\},$$

each of these blocks into sub-blocks $\tilde{\mathcal{B}}'_{j,A}$ where A runs among all partitions of $n - j$ in the increasing order, and take the elements $L^{-A}M^{-C} \in \tilde{\mathcal{B}}'_{j,A}$ according to the decreasing order of the partitions C of degree j .

As one easily checks, $L^{-C}M^{-A}\psi$ is at the same place in the vertical ordering as $L^{-A}M^{-C}$ in the horizontal ordering. Note that the vertical ordering can also be obtained by reversing the horizontal ordering. Yet we maintain the separate definitions both for clarity and because these definitions will be extended in the next paragraph to the case of \mathfrak{sv} , where there is no simple relation between the two orderings.

Roughly speaking, one may say that, for the horizontal ordering, the degree in M decreases from one block to the next one, the $(M^C)_{C \in \mathcal{P}(n-j)}$ are chosen in the increasing order inside each block, and then the L^A are chosen in the decreasing order inside each sub-block; the vertical ordering is defined in exactly the same way, except that L and M are exchanged.

We shall compute the Kac determinant $\Delta'_n := \Delta_n^{\text{Vir} \times \mathcal{F}_0}$ relative to \mathcal{B}' by representing it as

$$\Delta'_n = \pm \det \mathcal{A}'_n, \quad \mathcal{A}'_n = (\langle H_j | V_i \rangle)_{i,j} \quad (6.6)$$

where the $(H_j)_j \in \mathcal{B}'$ are chosen in the horizontal order and the $(V_i)_i \in \mathcal{B}'$ in the vertical order.

The following facts are clear from the above definitions:

- the horizontal and vertical blocks $\mathcal{B}'_j, \tilde{\mathcal{B}}'_j$ (j fixed) and sub-blocks $\mathcal{B}'_{j,C}, \tilde{\mathcal{B}}'_{j,C}$ (j, C fixed) have same size, so one has matrix diagonal blocks and sub-blocks;
- diagonal elements are of the form $\langle L^{-A}M^{-C}\psi | L^{-C}M^{-A}\psi \rangle$;
- define *sub-diagonal* elements to be the $\langle H|V \rangle$, $H \in \mathcal{B}'_j, V \in \mathcal{B}'_i$ such that $i > j$; then $\deg_L(V) < \deg_M(H)$;
- define *j -sub-sub-diagonal* elements (or simply sub-sub-diagonal elements if one doesn't need to be very definite) to be the $\langle H|V \rangle$, $H \in \mathcal{B}'_{j,C_1}, V \in \mathcal{B}'_{j,C_2}$ with $C_2 > C_1$. Then $\deg_L(V) = \deg_M(H) = n - j$ and $H = L^{-A_1}M^{-C_1}, V = L^{-C_2}M^{-A_2}$ for certain partitions A_1, A_2 of degree j ;
- define *(j, C) -sub³-diagonal* elements to be the $\langle H|V \rangle$, $H, V \in \mathcal{B}'_{j,C}$, such that $H = L^{-A_1}M^{-C}, V = L^{-C}M^{-A_2}$ with $A_1 > A_2$.

Then the set of sub-diagonal elements is the union of the matrix blocks situated under the diagonal; the set of j -sub-sub-diagonal elements is the union of the matrix sub-blocks situated under the diagonal of the j -th matrix diagonal blocks; the set of (j, C) -sub³-diagonal elements is the union of the elements situated under the diagonal of the (j, C) -diagonal sub-block. All these elements together form the set of lower-diagonal elements of the matrix \mathcal{A}'_n .

Elementary computations show that

$$\mathcal{A}'_1 = \begin{pmatrix} \mu & 2h \\ 0 & \mu \end{pmatrix}$$

with horizontal ordering $(M^{-1}\psi, L^{-1}\psi)$;

$$\mathcal{A}'_2 = \begin{pmatrix} 2\mu^2 & 2\mu & 2\mu(1+2h) & 6h & 4h(2h+1) \\ 0 & 2\mu & 3\mu & 4h+c/2 & 6h \\ 0 & 0 & \mu^2 & 3\mu & 2\mu(1+2h) \\ 0 & 0 & 0 & 2\mu & 2\mu \\ 0 & 0 & 0 & 0 & 2\mu^2 \end{pmatrix}$$

with horizontal ordering $((M_{-1}^2\psi, M_{-2}\psi), (L_{-1}M_{-1}\psi), (L_{-2}\psi, L_{-1}^2\psi))$ (the blocks being separated by parentheses). Note that the Kac determinant of Vir_c at level 2 appears as the top rightmost 2×2 upper-diagonal block of \mathcal{A}'_2 , and hence does not play any role in the computation of Δ'_n .

So $\mathcal{A}'_1, \mathcal{A}'_2$ are upper-diagonal matrices, with diagonal elements that are (up to a coefficient) simply powers of μ . More specifically, $\Delta'_1 = -\det \mathcal{A}'_1 = -\mu^2$ and $\Delta'_2 = \det \mathcal{A}'_2 = 16\mu^8$.

The essential technical lemmas for the proof of Theorem 6.1 and Theorem 6.2 are Lemma 6.1 and Lemma 6.2, which show, roughly speaking, how to move the M 's through the L 's.

Lemma 6.1

Let A, C be two partitions of degree n . Then:

(i) If $A \not\leq C$, then $\langle L^{-A}\psi \mid M^{-C}\psi \rangle = 0$.

(ii) If $A \leq C$, then

$$\langle L^{-A}\psi \mid M^{-C}\psi \rangle = a \cdot \mu^{\text{wid}(C)} \tag{6.7}$$

for a certain positive constant a depending only on A and C .

Proof of Lemma 6.1

We use induction on n . Take $A = (a_i), C = (c_i)$ of degree n , then

$$\langle L^{-A}\psi \mid M^{-C}\psi \rangle = \langle \psi \mid \left(\prod_{i=\infty}^1 L_i^{a_i} \right) \left(\prod_{j=1}^{\infty} M_{-j}^{c_j} \right) \psi \rangle.$$

We shall compute $\langle L^{-A}\psi \mid M^{-C}\psi \rangle$ by moving successively to the left the M_{-1} 's, then the M_{-2} 's and so on, and taking care of the commutators that show up in the process.

- Suppose $c_1 > 0$. By commuting M_{-1} with the L 's, there appear terms of two types (modulo positive constants) :
 - either of type

$$\langle \psi \mid \left(\prod_{i=\infty}^{k+1} L_i^{a_i} \right) (L_k^{a'_k} M_{k-1} L_k^{a''_k}) \left(\prod_{i=k-1}^1 L_i^{a_i} \right) M_{-1}^{c_1-1} \left(\prod_{i=2}^{\infty} M_{-i}^{c_i} \right) \psi \rangle,$$

with $a'_k + a''_k = a_k - 1$ (by commuting with $L_k, k \geq 2$). But this is zero since transferring M_{k-1} (of negative degree $1 - k$) to the right through the L 's can only lower its degree.

- or of type

$$\langle \psi \mid \left(\prod_{i=\infty}^2 L_i^{a_i} \right) (L_1^{a'_1} M_0 L_1^{a''_1}) M_{-1}^{c_1-1} \left(\prod_{i=2}^{\infty} M_{-i}^{c_i} \right) \psi \rangle,$$

with $a'_1 + a''_1 = a_1 - 1$ (by commuting with L_1). Since the central element $M_0 \equiv \mu$ can be taken out of the brackets, we may compute this as μ times a scalar product between two elements of degree $n - 1$. Removing one M_{-1} and one L_{-1} means removing the leftmost column of the Young tableaux C and A . Call C', A' the new tableaux: it is clear that $A \not\leq C \Rightarrow A' \not\leq C'$. So we may conclude by induction.

- Suppose that all $c_1 = \dots = c_{j-1} = 0$ and $c_j > 0$. Then, by similar arguments, one sees that all potentially non-zero terms appearing on the way (while moving M_{-j} to the left) are of the form

$$\alpha \mu \cdot \langle \psi \mid \left(\prod_{i=\infty}^1 L_i^{a'_i} \right) M_{-j}^{c_j-1} \left(\prod_{i_j} M_{-i}^{c_i} \right) \psi \rangle,$$

$\sum_{i=\infty}^1 i(a_i - a'_i) = j$, with α defined by

$$\left(\prod_{i=\infty}^1 (adL_i)^{a_i - a'_i} \right) \cdot M_{-j} = \alpha M_0.$$

Without computing α explicitly, it is clear that $\alpha > 0$. On the Young tableaux, this corresponds to removing one stack of height j from C and $(a_i - a'_i)$ stacks of height i ($i = \infty, \dots, 1$) from A . Once again, $A \not\leq C \Rightarrow A' \not\leq C'$. So one may conclude by induction.

□

Lemma 6.2.

Let A_1, A_2, C_1, C_2 be partitions such that $\deg A_1 + \deg C_1 = \deg A_2 + \deg C_2$. Then :

(i)

$$\langle L^{-C_2} M^{-A_2} \psi \mid L^{-A_1} M^{-C_1} \psi \rangle = 0$$

if $\deg A_1 < \deg A_2$.

(ii) If $\deg A_1 = \deg A_2$, then

$$\langle L^{-C_2} M^{-A_2} \psi \mid L^{-A_1} M^{-C_1} \psi \rangle = \langle L^{-C_2} \psi \mid M^{-C_1} \psi \rangle \langle M^{-A_2} \psi \mid L^{-A_1} \psi \rangle. \quad (6.8)$$

Remark. The central argument in this Lemma can be trivially generalized (just by using the fact that the M 's are central in \mathfrak{h}) in a form that will be used again and again in the next section : namely,

$$\langle \psi \mid UL^{C_2} VL^{-A} WM^{-C_1} \psi \rangle = 0 \quad (U, V, W \in \mathcal{U}(\mathfrak{h}))$$

if $\deg(C_1) > \deg(C_2)$.

Proof.

Putting all generators on one side, one gets

$$\langle L^{-C_2} M^{-A_2} \psi \mid L^{-A_1} M^{-C_1} \psi \rangle = \langle \psi \mid M^{A_2} L^{C_2} L^{-A_1} M^{-C_1} \psi \rangle.$$

Assume that $\deg(A_1) \leq \deg(A_2)$ (so that $\deg(C_2) \leq \deg(C_1)$). Let us move M^{-C_1} to the left and consider the successive commutators with the L 's: it is easy to see that one gets (apart from the trivially commuted term $M^{-C_1} L^{-A_1}$) a sum of terms of the type $M^{-C'_1} L^{-A'_1}$ with

$$\deg(C'_1) + \deg(A'_1) = \deg(C_1) + \deg(A_1), \quad \deg(C'_1) > \deg(C_1), \quad \deg(A'_1) < \deg(A_1).$$

Now comes the central argument : let us commute $M^{-C'_1}$ through L^{C_2} . It yields terms of the type $M_0^\alpha M^{-C''_1} M^{C'''_1} L^{C'_2}$ with $\deg(C_2) - \deg(C'_1) = \deg(C'_2) + \deg(C'''_1) - \deg(C''_1)$.

If $C''_1 \neq \emptyset$, then $M^{-C''_1}$ commutes with M^{A_2} and gives 0 when set against $\langle \psi$; so one may assume $C''_1 = \emptyset$. But this is impossible, since it would imply

$$\deg(C'''_1) = \deg(C_2) - \deg(C'_1) - \deg(C'_2) \quad (6.9)$$

$$\leq \deg(C_1) - \deg(C'_1) \quad (\text{even} < \text{ if } \deg(C_2) < \deg(C_1)) \quad (6.10)$$

$$< 0. \quad (6.11)$$

So

$$\langle L^{-C_2} M^{-A_2} \psi \mid L^{-A_1} M^{-C_1} \psi \rangle = \langle \psi \mid M^{A_2} L^{C_2} M^{-C_1} L^{-A_1} \psi \rangle.$$

What's more, the above argument applied to M^{-C_1} instead of $M^{-C_1'}$, shows that

$$\langle \psi \mid M^{A_2} (L^{C_2} M^{-C_1}) L^{-A_1} \psi \rangle = \langle \psi \mid M^{A_2} (M^{-C_1} L^{C_2}) L^{-A_1} \psi \rangle = 0$$

if $\deg(C_2) < \deg(C_1)$. So (i) holds and one may assume that $\deg(A_1) = \deg(A_2)$, $\deg(C_1) = \deg(C_2)$ in the sequel.

Now move M^{A_2} to the right: the same argument proves that

$$\langle \psi \mid M^{A_2} L^{C_2} M^{-C_1} L^{-A_1} \psi \rangle = \langle \psi \mid (L^{C_2} M^{-C_1}) (M^{A_2} X^{-A_1}) \psi \rangle.$$

But $M^{A_2} L^{-A_1}$ has degree 0, so

$$\langle \psi \mid (L^{C_2} M^{-C_1}) (M^{A_2} L^{-A_1}) \psi \rangle = \langle \psi \mid L^{C_2} M^{-C_1} \psi \rangle \langle \psi \mid M^{A_2} L^{-A_1} \psi \rangle \quad (6.12)$$

$$= \langle L^{-C_2} \psi \mid M^{-C_1} \psi \rangle \langle M^{-A_2} \psi \mid L^{-A_1} \psi \rangle. \quad (6.13)$$

□

Corollary 6.3.

The matrix \mathcal{A}'_n is upper-diagonal.

Proof.

Lower-diagonal elements come in three classes: let us give an argument for each class.

By Lemma 6.2, (i), sub-diagonal elements are zero.

Consider a j -sub-sub-diagonal element $\langle L^{-C_2} M^{-A_2} \psi \mid L^{-A_1} M^{-C_1} \psi \rangle$ with $\deg(C_1) = \deg(C_2) = n-j$, $\deg(A_1) = \deg(A_2)$, $C_2 > C_1$. Then, by Lemma 6.2., (ii), and Lemma 6.1, (i),

$$\langle L^{-C_2} M^{-A_2} \psi \mid L^{-A_1} M^{-C_1} \psi \rangle = \langle L^{-C_2} \psi \mid M^{-C_1} \psi \rangle \langle M^{-A_2} \psi \mid L^{-A_1} \psi \rangle = 0.$$

Finally, consider a (C, j) -sub³-diagonal element $\langle L^{-C} M^{-A_2} \psi \mid L^{-A_1} M^{-C} \psi \rangle$ with $A_1 > A_2$. By the same arguments, this is zero. □

Proof of Theorem 6.1

By Corollary 6.3, one has

$$\Delta'_n = (-1)^{\dim(\mathcal{V}'_n)} \prod_{i=1}^{\dim(\mathcal{V}'_n)} (\mathcal{A}'_n)_{ii}.$$

By Lemma 6.2 (ii) and Lemma 6.1 (ii), the diagonal elements of \mathcal{A}'_n are equal (up to a positive constant) to μ to a certain power. Now the total power of μ is equal to

$$\sum_{0 \leq j \leq n} \sum_{A \in \mathcal{P}(j)} \sum_{C \in \mathcal{P}(n-j)} (\text{wid}(A) + \text{wid}(C)) = 2 \sum_{0 \leq j \leq n} p(n-j) \left(\sum_{A \in \mathcal{P}(j)} \text{wid}(A) \right). \quad (6.14)$$

□

6.3 Kac determinant formula for \mathfrak{sp}

Let $n \in \frac{1}{2}\mathbb{N}$. We shall define in this case also a *horizontal ordering* (also called *M-ordering*) and a *vertical ordering* (also called *L-ordering*) of the Poincaré-Birkhoff-Witt basis $\mathcal{B} = \{L^{-A}Y^{-B}M^{-C}\psi \mid \deg(A) + \widetilde{\deg}(B) + \deg(C) = n\}$ of $\mathcal{V}_{(n)}$.

The *M-ordering* is defined as follows (note that blocks and sub-blocks are defined more or less as in the preceding sub-section):

- split \mathcal{B} into $(n + 1)$ -blocks $\mathcal{B}_0, \dots, \mathcal{B}_n$ such that $\mathcal{B}_j = \{L^{-A}Y^{-B}M^{-C}\psi \in \mathcal{B} \mid \deg(C) = n - j\}$;
- split each block \mathcal{B}_j into sub-blocks \mathcal{B}_{jC} such that $\mathcal{B}_{jC} = \{L^{-A}Y^{-B}M^{-C}\psi \in \mathcal{B}\}$, with C running among the set of partitions of $n - j$ in the *increasing* order;
- split each sub-block \mathcal{B}_{jC} into sub²-blocks $\mathcal{B}_{jC,\kappa}$ (for *decreasing* κ) with $L^{-A}Y^{-B}M^{-C} \in \mathcal{B}_{jC,\kappa} \Leftrightarrow L^{-A}Y^{-B}M^{-C} \in \mathcal{B}_{jC}$ and $\widetilde{\deg}(B) = \kappa$;
- split each sub²-block $\mathcal{B}_{jC,\kappa}$ into sub³-blocks $\mathcal{B}_{jC,\kappa B}$ where B runs among all partitions of shifted degree κ (in any randomly chosen order);
- finally, order the elements $L^{-A}Y^{-B}M^{-C}\psi$ of $\mathcal{B}_{jC,\kappa B}$ ($A \in \mathcal{P}(n - \deg(C) - \widetilde{\deg}(B)) = \mathcal{P}(j - \kappa)$) so that the A 's appear in the *decreasing* order.

Then the *L-ordering* is chosen in such a way that $L^{-C}Y^{-B}M^{-A}$ appears vertically in the same place as $L^{-A}Y^{-B}M^{-C}$ in the *M-ordering*. From formula (6.2) giving the *M-ordering* of $\mathcal{V}_{(2)}$, it is clear that the *L-ordering* is *not* the opposite of the *M-ordering*.

These two orderings define as in paragraph 6.2 a block matrix \mathcal{A}_n whose determinant is equal to $\pm \Delta_n^{\mathfrak{sp}}$.

We shall need three preliminary lemmas.

Lemma 6.4.

Let $B = (b_j), B' = (b'_j)$ be two partitions with same shifted degree: then

$$\langle Y^{-B}\psi \mid Y^{-B'}\psi \rangle = \delta_{B,B'} \left[\prod_{j \geq 1} (b_j)! (2j - 1)^{b_j} \right] \cdot \mu^{\text{wid}(B)}. \quad (6.15)$$

Proof.

Consider the sub-module $\mathcal{W} \subset \mathcal{V}$ generated by the $Y^{-P}M^{-Q}\psi$ (P, Q partitions). Then, inside this module, and as long as vacuum expectation values $\langle \psi \mid Y_{\pm(j_1 - \frac{1}{2})} \cdots Y_{\pm(j_k - \frac{1}{2})} \psi \rangle$ are concerned, the $(Y_{-(j - \frac{1}{2})}, Y_{j - \frac{1}{2}})$ can be considered as independent couples of creation/annihilation operators with $[Y_{-(j - \frac{1}{2})}, Y_{j - \frac{1}{2}}] = (2j - 1)M_0$. Namely, other commutators $[Y_k, Y_l]$ with $k + l \neq 0$ yield $(k - l)M_{k+l}$ which commutes with all other generators Y 's and M 's and gives 0 when applied to $\psi \rangle$ or $\psi \langle$ according to the sign of $k + l$. The result now follows for instance by an easy application of Wick's theorem, or by induction. \square

Lemma 6.5.

Let B_1, B_2, C_1, C_2 be partitions such that $C_1 \neq \emptyset$ or $C_2 \neq \emptyset$. Then

$$\langle Y^{-B_1}M^{-C_1}\psi \mid Y^{-B_2}M^{-C_2}\psi \rangle = 0.$$

Proof. Obvious (the $M^{\pm C}$'s are central in the subalgebra $\mathfrak{h} \subset \mathfrak{sv}$ and can thus be commuted freely with the Y 's and the M 's; set against $\langle \psi \text{ or } \psi \rangle$ according to the sign of their degree, they give zero). \square

Lemma 6.6.

Let $A_1, A_2, B_1, B_2, C_1, C_2$ be partitions such that $\deg(A_1) + \deg(B_1) + \deg(C_1) = \deg(A_2) + \deg(B_2) + \deg(C_2)$. Then

(i) If $\deg(A_1) < \deg(A_2)$ or $\deg(C_2) < \deg(C_1)$, then

$$\langle L^{-C_2} Y^{-B_2} M^{-A_2} \psi \mid L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle = 0.$$

(ii) If $\deg(A_1) = \deg(A_2)$ and $\deg(C_1) = \deg(C_2)$, then

$$\langle L^{-C_2} Y^{-B_2} M^{-A_2} \psi \mid L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle = \langle Y^{-B_2} \psi \mid Y^{-B_1} \psi \rangle \langle M^{-A_2} \psi \mid L^{-A_1} \psi \rangle \langle L^{-C_2} \psi \mid M^{-C_1} \psi \rangle.$$

Proof of Lemma 6.6.

(i) is a direct consequence of the remark following Lemma 6.2. So we may assume that $\deg(A_1) = \deg(A_2)$ and $\deg(C_1) = \deg(C_2)$. Using the hypothesis $\deg(A_1) = \deg(A_2)$, we may choose this time to move L^{-A_1} to the left in the expression $\langle \psi \mid M^{A_2} Y^{B_2} L^{C_2} L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle$. Commuting L^{-A_1} through L^{C_2} and Y^{B_2} , one obtains terms of the type

$$\langle \psi \mid M^{A_2} (L^{-A_1} Y^{B_2} L^{C_2'}) Y^{-B_1} M^{-C_1} \psi \rangle,$$

with $\deg(A_1') \leq \deg(A_1)$, and $(\deg(A_1') = \deg(A_1)) \Rightarrow (A_1' = A_1, B_2' = B_2, C_2' = C_2)$. So, by the Remark following Lemma 6.2,

$$\langle L^{-C_2} Y^{-B_2} M^{-A_2} \psi \mid L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle = \langle \psi \mid M^{A_2} Y^{B_2} L^{C_2} L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle \quad (6.16)$$

$$= \langle \psi \mid Y^{B_2} L^{C_2} M^{A_2} L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle \quad (6.17)$$

$$= \langle \psi \mid M^{A_2} L^{-A_1} \psi \rangle \langle \psi \mid Y^{B_2} L^{C_2} Y^{-B_1} M^{-C_1} \psi \rangle. \quad (6.18)$$

Moving L^{C_2} to the right in the same way leads to (ii), thanks to the hypothesis $\deg(C_1) = \deg(C_2)$ this time. \square

Corollary 6.7.

The matrix \mathcal{A}_n is upper-diagonal.

Proof.

Lower-diagonal elements $f = \langle L^{-C_2} Y^{-B_2} M^{-A_2} \psi \mid L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle$ come this time in five classes. Let us treat each class separately.

By Lemma 6.6 (i), sub-diagonal elements (characterized by $\deg(C_2) < \deg(C_1)$) are zero.

Next, j -sub²-diagonal elements (characterized by $\deg(C_2) = \deg(C_1) = n - j$, $C_2 > C_1$) are also zero: namely, the hypothesis $\deg(C_1) = \deg(C_2)$ gives as in the proof of Lemma 6.6.

$$f = \langle \psi \mid M^{A_2} Y^{B_2} L^{C_2} L^{-A_1} Y^{-B_1} M^{-C_1} \psi \rangle \quad (6.19)$$

$$= \langle \psi \mid L^{C_2} M^{-C_1} \psi \rangle \langle \psi \mid M^{A_2} Y^{B_2} L^{-A_1} Y^{-B_1} \psi \rangle \quad (6.20)$$

$$= 0 \text{ (by Lemma 6.1).} \quad (6.21)$$

Now (j, C) -sub³-diagonal elements (characterized by $C_1 = C_2 := C$, $\widetilde{\deg}(B_2) < \widetilde{\deg}(B_1)$) are again zero because, as we have just proved,

$$f = \langle \psi | L^{C_2} M^{-C_1} \psi \rangle \langle \psi | M^{A_2} Y^{B_2} L^{-A_1} Y^{-B_1} \psi \rangle$$

and this time $\langle \psi | M^{A_2} Y^{B_2} L^{-A_1} Y^{-B_1} \psi \rangle = 0$ since

$$\deg(A_1) = j - \widetilde{\deg}(B_1) < j - \widetilde{\deg}(B_2) = \deg(A_2).$$

Finally, (jC, κ) -sub⁴-diagonal elements (with $C_1 = C_2 = C$, $\deg B_1 = \deg B_2$, $\deg A_1 = \deg A_2$, $B_1 \neq B_2$) are 0 by Lemma 6.6 (ii) and Lemma 6.4 since $\langle Y^{-B_2} \psi | Y^{-B_1} \psi \rangle = 0$, and $(jC, \kappa B)$ -sub⁵-diagonal elements (such that $B_1 = B_2$ but $A_2 < A_1$) are 0 by Lemma 6.6 (ii) and Lemma 6.1. (i) since $\langle M^{-A_2} \psi | L^{-A_1} \psi \rangle = 0$. \square

Proof of Theorem 6.2

By Corollary 6.7., $\Delta_n^{\text{sv}} = \pm \prod_{i=1}^{\dim(\mathcal{V}_n)} (\mathcal{A}_n)_{ii}$. By Lemma 6.6. (ii), Lemma 6.1 (ii) and Lemma 6.4., the diagonal element $\langle L^{-C} Y^{-B} M^{-A} \psi | L^{-A} Y^{-B} M^{-C} \psi \rangle$ is equal (up to a positive constant) to $\mu^{\text{wid}(A)+\text{wid}(B)+\text{wid}(C)}$, so (see proof of Theorem 6.1) $\Delta_n^{\text{sv}} = c_n \mu^{a_n}$ (c_n non-zero constant, $a_n \in \mathbb{N}$) with

$$a_n = \sum_{0 \leq j \leq n} \sum_{B \in \tilde{\mathcal{P}}(j)} (\text{wid}(B) + a'_{n-j}),$$

where a'_{n-j} is the power of μ appearing in $\Delta_{n-j}^{\text{Vir} \times \mathcal{F}_0}$. Hence the final result. \square

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