



HAL
open science

Strong localization and macroscopic atoms for directed polymers

Vincent Vargas

► **To cite this version:**

Vincent Vargas. Strong localization and macroscopic atoms for directed polymers. 2006. hal-00020421

HAL Id: hal-00020421

<https://hal.science/hal-00020421>

Preprint submitted on 9 Mar 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STRONG LOCALIZATION AND MACROSCOPIC ATOMS FOR DIRECTED POLYMERS

VINCENT VARGAS

Université Paris 7,
Mathématiques, case 7012,
2, place Jussieu, 75251 Paris, France
e-mail: vargas@math.jussieu.fr

ABSTRACT. In this article, we derive strong localization results for directed polymers in random environment. We show that at "low temperature" the polymer measure is asymptotically concentrated at a few points of macroscopic mass (we call these points ϵ -atoms). These results are derived assuming weak conditions on the tail decay of the random environment.

MSC: 60K37;82B44

Keywords: Directed polymers in random environment.

1. INTRODUCTION

In this article, we consider a model of directed polymers in random environment introduced by Huse and Henley in 1985 ([10]) to modelize impurity-induced domain-wall roughening in the 2D-Ising model. This model relates to many physical models of growing random surfaces including the well known Kardar-Parisi-Zhang equation driven by gaussian noise (we refer to [14] for an account on these models and their relations). In [21], Zhang proposed to replace the gaussian noise in the KPZ equation by a noise with power-law tail to describe fluid flows. Since then, this model has been used to describe fire fronts, bacterial colonies, etc.... In the field of polymers, the authors of [9],[15] study the random energy landscape of zero temperature directed polymers in power-law environment distributions.

The first mathematical study of directed polymers at positive temperature was undertaken by Imbrie, Spencer in 1988 ([11]) and carried out by numerous authors ([1],[2],[3],[4],[18],[19]); for an overview of the achieved results, we refer to [5]. In [3] and [4], the authors show, using martingale techniques, that the quenched free energy is strictly less than the annealed one if and only if a localization theorem for the polymer's favorite point holds. In all these previous mathematical articles, the authors assume that the environment has exponential moments of all order. When considering a temperature where the moment generating function of the environment is infinite, no martingale technique can be used, making the usual strategy

Partially supported by CNRS (UMR 7599 "Probabilités et Modèles Aléatoires").

irrelevant. Hence, a natural question in this case is: what is left from the localization picture? In this paper, our approach is more general than the martingale approach used in the above references and we obtain our localization results under much weaker conditions on the distribution of the environment (including the power tail distributions studied in [9],[15], exponential distributions...). The case of exponentially distributed environments is of particular interest in view of the exact results derived by Johansson in [12] for directed last passage percolation with i.i.d. exponential variables in dimension $d = 1$. Since directed last passage percolation can be recovered from directed polymers by letting the temperature go to 0, one can view the polymer measure as an interpolation between the directed percolation model and the simple random walk.

In this article, we go a step further than favorite point localization and derive localization results in terms of ϵ -atoms by using bounds on the free energy. We call ϵ -atoms, atoms of the polymer measure of mass at least ϵ . Roughly, we show, under certain assumptions on the environment, that the whole mass of the polymer measure at "low temperature" is essentially carried by ϵ -atoms (cf. theorems 3.2 and 3.7 below). Our method of proof relies mainly on a simple inequality (cf. lemma 5.1 below) and on an upper bound on greedy lattice animals established in [16]. Using lemma 5.3, we also give a different proof for localization in terms of the polymer's favorite point if the quenched free energy is strictly less than the annealed one (cf. theorem 3.6 below).

The article is organized as follows: in section 2, we introduce the model and the definition of ϵ -atoms. In section 3, we state an existence theorem for the free energy and our localization theorems. In section 4, we give the proofs.

2. THE MODEL AND DEFINITION OF ϵ -ATOMS

2.1. The model. The model we consider in this paper consists of a simple random walk under a random Gibbs measure depending on the temperature. More precisely,

Let $((\omega_n)_{n \in \mathbb{N}}, P)$ denote the simple random walk starting from 0 on the d -dimensional integer lattice \mathbb{Z}^d , defined on a measurable space (Ω, \mathcal{F}) ; more precisely, under the measure P , $(\omega_n - \omega_{n-1})_{n \geq 1}$ are independent and

$$P(\omega_0 = 0) = 1, \quad P(\omega_n - \omega_{n-1} = \pm \delta_j) = \frac{1}{2d}, \quad j = 1, \dots, d,$$

where $(\delta_j)_{1 \leq j \leq d}$ is the j -th vector of the canonical basis of \mathbb{Z}^d .

The random environment on each lattice site is a sequence $\eta = (\eta(n, x))_{(n, x) \in \mathbb{N} \times \mathbb{Z}^d}$ of real valued, non-constant and i.i.d. random variables defined on a probability space (H, \mathcal{G}, Q) . We denote by F the common distribution function of the sequence $(\eta(n, x))_{(n, x) \in \mathbb{N} \times \mathbb{Z}^d}$. In the whole paper, we will suppose the following:

Assumptions:

$$\int_0^\infty (1 - F(x))^{\frac{1}{d+1}} dx < \infty \tag{2.1}$$

and

$$Q(|\eta(n, x)|) < \infty. \tag{2.2}$$

Let λ be the logarithmic moment generating function of $\eta(n, x)$:

$$\forall \beta \in \mathbb{R}_+ \quad \lambda(\beta) \stackrel{\text{def.}}{=} \ln Q(e^{\beta \eta(n, x)}) \leq \infty.$$

For any $n > 0$, we define the (Q-random) polymer measure μ_n on (Ω, \mathcal{F}) by:

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp(\beta H_n(\omega)) P(d\omega)$$

where $\beta \in \mathbb{R}_+$ is the inverse temperature,

$$H_n(\omega) \stackrel{\text{def.}}{=} \sum_{j=1}^n \eta(j, \omega_j)$$

is the hamiltonian and

$$Z_n = P(\exp(\beta H_n(\omega)))$$

is the partition function.

The above definition shows that the polymer is attracted to sites where the environment is large and positive and repelled by sites where the environment is large and negative; as the inverse temperature β increases, the influence of the environment increases and tends to push the random walk in a few "corridors" where the environment takes high positive values: we will see in the next sections quantitative statements of these heuristics.

2.2. Definition of ϵ -atoms. The purpose of this article is to study where the polymer measure $(\mu_{j-1}(\omega_j = x))_{x \in \mathbb{Z}^d}$ is concentrated for large j ; under assumptions on the environment η and on the inverse temperature β (typically β "large"), we show in some sense that the mass carried by a few points is "significant". To give a quantitative statement of this phenomenon, we are naturally lead to introduce the notion of ϵ -atoms. More precisely, let $\epsilon > 0$ be some positive real number; we define $\mathcal{A}_j^{\epsilon, \beta}$ the set of ϵ -atoms to be the points of \mathbb{Z}^d which carry a mass of at least ϵ :

$$\mathcal{A}_j^{\epsilon, \beta} = \{x \in \mathbb{Z}^d : \mu_{j-1}(\omega_j = x) > \epsilon\}.$$

For $\delta < 1$, we define the event $A_j^{\epsilon, \delta, \beta}$ to be the environments for which $\mathcal{A}_j^{\epsilon, \beta}$ has a mass of at least δ :

$$A_j^{\epsilon, \delta, \beta} = \{\eta : \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon, \beta}) \geq \delta\},$$

and $A_j^{\epsilon, \beta}$ to be the environments for which $\mathcal{A}_j^{\epsilon, \beta}$ has at least one element:

$$A_j^{\epsilon, \beta} = \{\eta : \max_{x \in \mathbb{Z}^d} \mu_{j-1}(\omega_j = x) > \epsilon\}.$$

In terms of ϵ -atoms, we state the following localization result derived in [4] under the assumption $\lambda(\beta) < \infty$ ($\forall \beta$) (cf. corollary 2.2 therein):

$$p(\beta) < \lambda(\beta) \Leftrightarrow \exists \epsilon > 0, \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon, \beta}) \geq \epsilon \quad Q - a.s.$$

This equivalence asserts that the quenched free energy is strictly less than the annealed one if and only if there exists $\epsilon > 0$ such that the mass carried by the

ϵ -atoms is for large n (in the sense of Césaro) bounded below by some positive constant.

We recall that if for all β in \mathbb{R} the moment generating function $\lambda(\beta)$ is finite then for all β different from 0 the strict inequality $p(\beta) < \lambda(\beta)$ holds for dimension $d = 1$ (theorem 1.1 in [7]). For dimension $d = 2$, this problem is still open.

Finally, we introduce the following definition for $(\mu_{j-1}(\omega_j \in \cdot))_{j \geq 1}$:

Definition 2.1. *The sequence $(\mu_{j-1}(\omega_j \in \cdot))_{j \geq 1}$ is asymptotically purely atomic (in Césaro mean) if for all sequence $(\epsilon_j)_{j \geq 1}$ tending to 0 as j goes to infinity the following convergence holds:*

$$\frac{1}{n} \sum_{j=1}^n \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon_j, \beta}) \xrightarrow[n \rightarrow \infty]{} 1 \text{ in } Q\text{-Probab.}$$

Less formally, $(\mu_{j-1}(\omega_j \in \cdot))_{j \geq 1}$ is asymptotically purely atomic if, for large j , the polymer measure concentrates on few atoms.

3. RESULTS

3.1. Existence of the free energy. First, we establish the existence of the free energy for all β in \mathbb{R}_+ . We recall that condition (2.1) implies that $Q[(\eta(n, x)_+)^{d+1}] < \infty$ but is implied by the existence of some $\epsilon > 0$ such that $Q[(\eta(n, x)_+)^{d+1+\epsilon}] < \infty$; in particular, it is much weaker than the existence of exponential moments. We denote by $\vec{\Pi}_n$ the oriented paths of the simple random walk up to time n :

$$\vec{\Pi}_n = \{(j, \omega_j)_{1 \leq j \leq n}; \quad \forall j, \quad |\omega_{j+1} - \omega_j| = 1\}$$

We define $\vec{N}(n)$ as the maximum of the environment η along the paths of $\vec{\Pi}_n$:

$$\vec{N}(n) \stackrel{def.}{=} \max_{(j, \omega_j)_{j \in \vec{\Pi}_n}} \sum_{j=1}^n \eta(j, \omega_j).$$

As a consequence of condition (2.1) and proposition 3.4 in [17], we can define α in the following way:

$$\alpha \stackrel{def.}{=} \sup_{n \geq 1} Q\left(\frac{\vec{N}(n)}{n}\right) < \infty.$$

Since $Q(|\eta(n, x)|) < \infty$, by Kingman's subadditive ergodic theorem, we get that:

$$\frac{\vec{N}(n)}{n} \xrightarrow[n \rightarrow \infty]{} \alpha \quad Q - a.s. \text{ and in } L^1(Q).$$

For all β in \mathbb{R}_+ , the obvious bound $\ln Z_n \leq \beta \vec{N}(n)$ and condition (2.2) ensure the existence in $[Q(\eta(n, x)), \alpha]$ of

$$p(\beta) = \sup_{n \geq 1} Q\left(\frac{\ln Z_n}{n}\right).$$

To get a strong convergence result, we introduce the following condition:

$$\int_{-\infty}^0 F(x)^{\frac{1}{d+1}} dx < \infty. \quad (3.1)$$

Theorem 3.1. *The averaged free energy exists in the following weak sense:*

$$\frac{Q(\ln Z_n)}{n} \xrightarrow{n \rightarrow \infty} p(\beta).$$

We have the following bound on the free energy:

$$p(\beta) \leq \alpha\beta \wedge \lambda(\beta). \quad (3.2)$$

If, in addition, the environment satisfies condition (3.1), one gets the following stronger result:

$$\frac{\ln Z_n}{n} \xrightarrow{n \rightarrow \infty} p(\beta) \quad Q - a.s. \text{ and in } L^1(Q).$$

However, not much is known on the limit p : p is convex and $p(\beta)/\beta \rightarrow \alpha$ as $\beta \rightarrow \infty$. In subsection 3.3, we will tackle the question of the comparison of p with its annealed bound λ .

3.2. Strong localization in probability. In this subsection, we fix the inverse temperature $\beta > 0$ and we suppose that:

$$\lambda(\beta) = \infty.$$

Intuitively, when $\lambda(\beta) = \infty$, the environment can take large values and one expects the polymer measure to concentrate in those regions of high environment. A quantitative statement of this is the following theorem:

Theorem 3.2. *Suppose that $\lambda(\beta) = \infty$. Then, for all $\delta < 1$, there exists $\epsilon(\delta) > 0$ such that:*

$$\liminf_{n \rightarrow \infty} Q\left(\frac{1}{n} \sum_{j=1}^n \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon(\delta), \beta})\right) \geq \delta. \quad (3.3)$$

An immediate corollary of the above theorem is the following convergence result:

Corollary 3.3. *The sequence $(\mu_{j-1}(\omega_j \in \cdot))_{j \geq 1}$ is asymptotically purely atomic.*

3.3. Almost sure strong localization. In order to get almost sure localization results, we must suppose that the environment has non trivial exponential moments. More precisely, let $R = \sup\{\beta \in \mathbb{R}^+ : \lambda(\beta) < \infty\}$. In this subsection, we will suppose that $R > 0$ (possibly $R = \infty$). On the interval $]0, R[$, we want to compare p to its annealed bound λ , a standard procedure in statistical physics. Roughly, we have the following conjectured picture for directed polymers:

- (1) when $p(\beta) = \lambda(\beta)$, μ_n spreads out uniformly.
- (2) when $p(\beta) < \lambda(\beta)$, μ_n has macroscopic atoms which may concentrate the whole mass.

When $d \geq 3$ and β satisfies:

$$\lambda(2\beta) - 2\lambda(\beta) < \ln(1/P(\exists n \geq 1, \omega_n = 0))$$

(this condition implies $p(\beta) = \lambda(\beta)$), the situation is well understood: the polymer is diffusive in the sense that the measure $\mu_n(\omega_n/\sqrt{n} \in \cdot)$ converges weakly to a gaussian law ([2],[11],[19]) and satisfies a local limit theorem ([18],[20]). For case (2), we refer to theorem 3.6 below.

First, we give a preliminary lemma which states that there is a phase transition between case (1) and (2) under some assumptions on the environment.

Lemma 3.4. *The function $p - \lambda$ is nonincreasing on the interval $[0, R[$. Suppose that one of the two following conditions is satisfied:*

- $R < \infty$ and $\alpha < \frac{\lambda(R)}{R}$.
- $R = \infty$ and defining $L = \text{esssup}(\eta(n, x))$, we have

$$Q(\eta(n, x) = L) < \vec{p}_c(d),$$

where $\vec{p}_c(d)$ denotes the site percolation threshold for the oriented graph induced on $\mathbb{N} \times \mathbb{Z}^d$ by the simple random walk.

Then there exists $\beta_c < R$ such that:

$$\begin{aligned} \beta \in [0, \beta_c[&\Rightarrow p(\beta) = \lambda(\beta). \\ \beta \in]\beta_c, R[&\Rightarrow p(\beta) < \lambda(\beta). \end{aligned}$$

Proof. One can adapt the proof of lemma 3.3 in [6] to prove that $p - \lambda$ is nonincreasing on the interval $[0, R[$.

If $R < \infty$ and $\alpha < \frac{\lambda(R)}{R}$ then

$$\limsup_{\beta \rightarrow R} (p(\beta) - \lambda(\beta)) < 0,$$

and the existence of β_c follows.

If $R = \infty$ and $Q(\eta(n, x) = L) < \vec{p}_c(d)$, then by standard branching process arguments (cf. theorem 6.1 in [13]), one can show that $\alpha < L$. Therefore,

$$p(\beta) - \lambda(\beta) \underset{\beta \rightarrow \infty}{\sim} \beta(\alpha - L) \xrightarrow{\beta \rightarrow \infty} -\infty$$

and the existence of β_c follows. □

Remark 3.5. *In lemma 3.4, one can have $\beta_c = 0$. It is believed that this is the case in dimension $d = 1$ and $d = 2$.*

In particular, lemma 3.4 gives sufficient conditions for the existence of β in $]0, R[$ such that the strict inequality $p(\beta) < \lambda(\beta)$ holds. Now, we state our first almost sure localization result which generalizes corollary 2.2 in [4]:

Theorem 3.6. *Suppose that the environment satisfies condition (3.1). Then for all β in $]0, R[$, we have the following implication:*

$$p(\beta) < \lambda(\beta) \quad \Rightarrow \quad \exists \epsilon > 0, \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon, \beta}) \geq \epsilon. \quad Q - a.s.$$

In the next theorem, we will make the assumption that η "explodes" at R :

$$\lambda(R)/R = \infty, \quad (3.4)$$

where we set $\lambda(R)/R = \text{essup}(\eta(n, x))$ if $R = \infty$ and that

$$\exists \theta > 1, \quad Q(|\eta(n, x)|^\theta) < \infty. \quad (3.5)$$

Theorem 3.7. *Suppose that the environment satisfies conditions (3.4), (3.5). Then for all $\delta < 1$, there exists $\epsilon(\delta) > 0$ and $\beta(\delta)$ in $]0, R[$ such that:*

$$\forall \beta \in [\beta(\delta), R[\quad \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon(\delta), \beta}) \geq \delta \quad Q - a.s.$$

The above theorem can be seen as a continuity result in view of theorem 3.2.

4. PROOF OF THEOREM 3.1

Proof of theorem 3.1 Let $L \in \mathbb{N}^* \cup \{\infty\}$. We define $Y_{n,L}$ by

$$Y_{n,L} = \frac{1}{n} \ln P(e^{\beta H_n^L(\omega)}).$$

where

$$H_n^L(\omega) = \sum_{j=1}^n \eta(j, \omega_j) \wedge L \vee -L.$$

Similarly, we define

$$p_L(\beta) = \sup_{n \geq 1} Q(Y_{n,L}).$$

With these notations, $Y_{n,\infty} = \frac{1}{n} \ln Z_n$ and $p_\infty(\beta) = p(\beta)$. It is well known that the sequence $(Q(\ln Z_n))_{n \geq 1}$ is superadditive and so we have the following limit:

$$\lim_{n \rightarrow \infty} Q(Y_{n,\infty}) = p(\beta).$$

The obvious bound $\ln Z_n \leq \beta \vec{N}(n)$ ensures $p(\beta) \leq \alpha\beta$ and an application of Jensen's inequality to \ln ensures that $p(\beta) \leq \lambda(\beta)$, giving the first two parts of theorem 3.1. From now on, we suppose the environment satisfies condition (3.1).

For all $L \in \mathbb{N}^*$, it is known (cf. proposition 2.5 in [4]) that

$$Y_{n,L} \xrightarrow[n \rightarrow \infty]{} p_L(\beta) \quad Q\text{-a.s. and in } L^1(Q).$$

One has the following bounds:

$$\begin{aligned} \forall n, L \geq 1, \quad |Y_{n,\infty} - Y_{n,L}| &= \left| \frac{1}{n} \ln P(e^{\beta H_n(\omega) - H_n^L(\omega)}) \right| \\ &\leq \frac{\beta}{n} \max_{\omega \in \bar{\Pi}_n} |H_n(\omega) - H_n^L(\omega)| \\ &= \frac{\beta}{n} \max_{(j, \omega_j)_{j \in \bar{\Pi}_n}} \sum_{j=1}^n (|\eta|(j, \omega_j) - L)_+. \end{aligned}$$

Therefore proposition 3.4 in [17] ensures the existence of some constant $c < \infty$ such that the following estimates hold

$$\begin{aligned} \limsup_{n \rightarrow \infty} |Y_{n,\infty} - Y_{n,L}| &\leq c\beta \int_L^\infty (1 - F(x) + F(-x))^{\frac{1}{d+1}} dx \\ &\leq c\beta \int_L^\infty (1 - F(x))^{\frac{1}{d+1}} dx + c\beta \int_{-\infty}^{-L} F(x)^{\frac{1}{d+1}} dx \quad Q - a.s. \end{aligned}$$

and similarly

$$\limsup_{n \rightarrow \infty} Q(|Y_{n,\infty} - Y_{n,L}|) \leq c\beta \int_L^\infty (1 - F(x))^{\frac{1}{d+1}} dx + c\beta \int_{-\infty}^{-L} F(x)^{\frac{1}{d+1}} dx.$$

Therefore we have

$$\begin{aligned} |Y_{n,\infty} - Q(Y_{n,\infty})| &\leq |Y_{n,\infty} - Y_{n,L}| + |Y_{n,L} - Q(Y_{n,L})| + |Q(Y_{n,L}) - Q(Y_{n,\infty})| \\ &\leq |Y_{n,\infty} - Y_{n,L}| + |Y_{n,L} - Q(Y_{n,L})| + Q(|Y_{n,L} - Y_{n,\infty}|). \end{aligned}$$

By letting $n \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} |Y_{n,\infty} - p(\beta)| \leq 2c\beta \int_L^\infty (1 - F(x))^{\frac{1}{d+1}} dx + 2c\beta \int_{-\infty}^{-L} (F(x))^{\frac{1}{d+1}} dx \quad Q - a.s.$$

By letting $L \rightarrow \infty$ above, we conclude

$$Y_{n,\infty} \xrightarrow[n \rightarrow \infty]{Q-a.s.} p(\beta).$$

Similarly, we obtain

$$Y_{n,\infty} \xrightarrow[n \rightarrow \infty]{L^1(Q)} p(\beta).$$

□

5. PROOF OF THEOREMS 3.2, 3.6, 3.7

5.1. Some preliminary lemmas. We first introduce a few notations we will use in the following two lemmas. For n a positive integer, we define \mathcal{P}_n to be the standard probability simplex in \mathbb{R}^n :

$$\mathcal{P}_n = \{(\lambda_i)_{1 \leq i \leq n} \in \mathbb{R}_+^n; \sum_{i=1}^n \lambda_i = 1\}.$$

For $\epsilon, \delta \in]0, 1[$, we define

$$\mathcal{P}_n^{\epsilon, \delta} = \{(\lambda_i)_{1 \leq i \leq n} \in \mathcal{P}_n; \sum_{i=1}^n \lambda_i 1_{\lambda_i > \epsilon} \leq \delta\}$$

In this section, $(X_i)_{i \geq 1}$ will denote an i.i.d. sequence of positive random variables on a probability space (H, \mathcal{H}, P) such that:

$$E[|\ln X_1|] < \infty.$$

Lemma 5.1. *Let $\delta \in]\frac{1}{2}, 1[$ and $\epsilon \in]0, 1 - \delta[$ be such that $\frac{(1-\delta)}{\epsilon}$ is a positive integer. We have for all $n \geq \frac{(1-\delta)}{\epsilon} + 1$:*

$$\inf_{(\lambda_i)_{1 \leq i \leq n} \in \mathcal{P}_n^{\epsilon, \delta}} E[\ln(\sum_{i=1}^n \lambda_i X_i)] = E[\ln(\epsilon \sum_{i=1}^{\frac{(1-\delta)}{\epsilon}} X_i + \delta X_{\frac{(1-\delta)}{\epsilon} + 1})].$$

Proof. We can suppose that X_1 is non constant. We first establish an auxiliary result we will use intensively in the rest of the proof. Let k be a integer greater than or equal to 2 and $(\lambda_i)_{1 \leq i \leq k}$ an element of \mathcal{P}_k such that $0 < \lambda_2 \leq \lambda_1 < 1$. One can therefore consider the function $\phi : [0, (1 - \lambda_1) \wedge \lambda_2] \rightarrow \mathbb{R}$ defined by:

$$\forall \rho \in [0, (1 - \lambda_1) \wedge \lambda_2] \quad \phi(\rho) = E[\ln((\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_1 + \sum_{i=3}^k \lambda_i X_i)].$$

One can compute the derivative of ϕ and we get $\forall \rho \in]0, (1 - \lambda_1) \wedge \lambda_2[$:

$$\begin{aligned} \phi'(\rho) &= E\left[\frac{X_1 - X_2}{(\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_2 + \sum_{i=3}^k \lambda_i X_i}\right] \\ &= E\left[\frac{X_1 - X_2}{(\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_2 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1 > X_2}\right] \\ &\quad + E\left[\frac{X_1 - X_2}{(\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_2 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1 < X_2}\right] \\ &= E\left[\frac{X_1 - X_2}{(\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_2 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1 > X_2}\right] \\ &\quad - E\left[\frac{X_1 - X_2}{(\lambda_1 + \rho)X_2 + (\lambda_2 - \rho)X_1 + \sum_{i=3}^k \lambda_i X_i} 1_{X_1 > X_2}\right] \\ &< 0 \end{aligned}$$

where the last inequality comes from the fact that $x \rightarrow \frac{1}{x}$ is decreasing. Therefore, ϕ is a decreasing function and thus one can conclude that $\forall \rho \in]0, (1 - \lambda_1) \wedge \lambda_2[$:

$$E[\ln((\lambda_1 + \rho)X_1 + (\lambda_2 - \rho)X_1 + \sum_{i=3}^k \lambda_i X_i)] < E[\ln(\sum_{i=1}^k \lambda_i X_i)]. \quad (5.1)$$

Let $n \geq \frac{(1-\delta)}{\epsilon} + 1$ be a fixed integer and consider the application $f : \mathcal{P}_n^{\epsilon, \delta} \rightarrow \mathbb{R}$ defined by

$$\forall (\lambda_i) \in \mathcal{P}_n^{\epsilon, \delta} \quad f((\lambda_i)) = E[\ln(\sum_{i=1}^n \lambda_i X_i)].$$

Since f is continuous on the compact set $\mathcal{P}_n^{\epsilon, \delta}$, there exists $(\lambda_i^*) \in \mathcal{P}_n^{\epsilon, \delta}$ such that:

$$\inf_{(\lambda_i) \in \mathcal{P}_n^{\epsilon, \delta}} f((\lambda_i)) = f((\lambda_i^*)) \quad (5.2)$$

Let $p = \#\{i; \lambda_i^* > 0\}$. Since f is symmetric, we can suppose that $\lambda_1^* \geq \lambda_2^* \geq \dots \lambda_p^* > 0$ and that $\lambda_i^* = 0$ for $i > p$. We introduce the following set:

$$F_\epsilon = \{i; \lambda_i^* > \epsilon\}$$

Let $k = \#F_\epsilon$; we have the following identity:

$${}^c F_\epsilon = [|k + 1, p|].$$

If $k \geq 2$, for $\rho > 0$ sufficiently small, we have $(\lambda_1^* + \rho, \lambda_2^* - \rho, \lambda_3^*, \dots, \lambda_p^*, 0, \dots, 0) \in A_n^{\epsilon, \delta}$ and by inequality (5.1), we get

$$f(\lambda_1^* + \rho, \lambda_2^* - \rho, \lambda_3^*, \dots, \lambda_p^*, 0, \dots, 0) < f((\lambda_i^*)),$$

which contradicts (5.2). Therefore, $k \leq 1$. If $\lambda_{p-1}^* < \epsilon$ then for $\rho > 0$ sufficiently small, $(\lambda_1^*, \dots, \lambda_{p-2}^*, \lambda_{p-1}^* + \rho, \lambda_p^* - \rho, 0, \dots, 0) \in \mathcal{P}_n^{\epsilon, \delta}$ and by inequality (5.1) we get

$$f((\lambda_1^*, \dots, \lambda_{p-2}^*, \lambda_{p-1}^* + \rho, \lambda_p^* - \rho, 0, \dots, 0)) < f((\lambda_i^*)),$$

which contradicts (5.2). Therefore, $(\lambda_i^*) = (\lambda_1^*, \epsilon, \dots, \epsilon, \lambda_p^*, 0, \dots, 0)$. If $\lambda_1^* < \delta$, then for $\rho > 0$ sufficiently small, $(\lambda_1^* + \rho, \lambda_2^* - \rho, \lambda_3^*, \dots, \lambda_p^*, 0, \dots, 0) \in \mathcal{P}_n^{\epsilon, \delta}$ and by inequality (5.1) we get

$$f((\lambda_1^* + \rho, \lambda_2^* - \rho, \lambda_3^*, \dots, \lambda_p^*, 0, \dots, 0)) < f((\lambda_i^*)),$$

which contradicts (5.2). Thus $\lambda_1^* = \delta$ and since $\sum_{i=1}^p \lambda_i^* = 1$, we get $p = 1 + \frac{1-\delta}{\epsilon}$ and $\lambda_p^* = \epsilon$. We can conclude

$$\inf_{(\lambda_i) \in \mathcal{P}_n^{\epsilon, \delta}} f((\lambda_i)) = f((\lambda_i^*)) = E[\ln(\epsilon \sum_{i=1}^{\frac{(1-\delta)}{\epsilon}} X_i + \delta X_{\frac{(1-\delta)}{\epsilon} + 1})].$$

□

Remark 5.2. *Under suitable integrability assumptions, the same result holds when one considers a general concave function instead of \ln .*

In the same spirit than the above lemma, we state the following lemma without proving it.

Lemma 5.3. *Let k be some positive integer and $\epsilon = \frac{1}{k}$. Then we have for all $n \geq k$:*

$$\inf_{\substack{(\lambda_i)_{1 \leq i \leq n} \in \mathcal{P}_n \\ \max(\lambda_i) \leq \epsilon}} E[\ln(\sum_{i=1}^n \lambda_i X_i)] = E[\ln(\epsilon \sum_{i=1}^{1/\epsilon} X_i)].$$

Finally, we state the following convergence result:

Lemma 5.4. *Let $a, b > 0$ be two positive numbers such that $a < b$. We have the following convergence:*

$$\inf_{\beta \in [a, b]} E[\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta)] \xrightarrow{n \rightarrow \infty} \inf_{\beta \in [a, b]} \ln E[X_1^\beta].$$

Proof. The fact that the left hand side is less than or equal to the right hand side is a consequence of Jensen's inequality.

Let $L > 0$ be such that $-\frac{L}{a} < E[\ln(X_1)]$. Then for all β in $[a, b]$ we have:

$$\begin{aligned} E[\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta)] &= E[\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) 1_{\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) \geq -L}] \\ &\quad + E[\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) 1_{\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) < -L}] \\ &\geq E[\ln(\frac{1}{n} \sum_{i=1}^n (X_i \wedge L)^\beta) 1_{\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) \geq -L}] \\ &\quad + \beta E[\frac{1}{n} \sum_{i=1}^n \ln X_i 1_{\frac{1}{n} \sum_{i=1}^n \ln X_i < -\frac{L}{\beta}}] \\ &\geq E[\ln(\frac{1}{n} \sum_{i=1}^n (X_i \wedge L)^\beta) 1_{\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) \geq -L}] \\ &\quad + \beta E[\ln(X_1) 1_{\frac{1}{n} \sum_{i=1}^n \ln(X_i) < -\frac{L}{\beta}}] \\ &\geq E[\ln(\frac{1}{n} \sum_{i=1}^n (X_i \wedge L)^\beta) 1_{\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta) \geq -L}] \\ &\quad - b E[|\ln(X_1)| 1_{\frac{1}{n} \sum_{i=1}^n \ln(X_i) < -\frac{L}{a}}]. \end{aligned}$$

By taking the infimum over all $\beta \in [a, b]$ and using the bounded convergence theorem, we conclude that:

$$\liminf_{n \rightarrow \infty} \inf_{\beta \in [a, b]} E[\ln(\frac{1}{n} \sum_{i=1}^n X_i^\beta)] \geq \inf_{\beta \in [a, b]} \ln E[(X_1 \wedge L)^\beta].$$

We obtain the result by letting $L \rightarrow \infty$ in the above inequality. \square

5.2. Proof of theorem 3.2. Following the notations of lemma 5.1, we consider an i.i.d. sequence $(X_i)_{i \geq 1}$ defined on some probability space (H, \mathcal{H}, P) and such that $X_1 \stackrel{law}{=} e^{\eta(n, x)}$. Let $\delta < 1$ and $c(\delta)$ be some integer we will choose at the end of the proof. Finally, we set $\epsilon = \frac{1-\delta}{c(\delta)}$ (for notational convenience, we write ϵ instead of $\epsilon(\delta)$).

We have the following computation:

$$\begin{aligned}
\frac{Q(\ln Z_n)}{n} &= \frac{1}{n} \sum_{j=1}^n Q\left(\ln\left(\frac{Z_j}{Z_{j-1}}\right)\right) \\
&= \frac{1}{n} \sum_{j=1}^n Q\left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right)\right). \\
&\stackrel{(Jensen)}{\geq} \frac{1}{n} \sum_{j=1}^n Q\left(1_{cA_j^{\epsilon,\delta,\beta}} \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right)\right) + \beta E[\ln X_1] Q(A_j^{\epsilon,\delta,\beta})
\end{aligned}$$

Thus, we get the following inequality:

$$\begin{aligned}
\frac{Q(\ln Z_n)}{n} - \beta E[\ln X_1] &\geq \sum_{j=1}^n Q\left(1_{cA_j^{\epsilon,\delta,\beta}} \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right)\right) \\
&\quad - \beta E[\ln X_1] Q(A_j^{\epsilon,\delta,\beta})
\end{aligned}$$

By applying lemma 5.1 to the family $(e^{\beta\eta(j,x)})_{x \in \mathbb{Z}^d}$ under the conditional measure $Q(\cdot | \mathcal{G}_{j-1})$, we get:

$$\frac{Q(\ln(Z_n))}{n} \geq \left(\frac{1}{n} \sum_{j=1}^n Q(cA_j^{\epsilon,\delta,\beta})\right) \left(E\left[\ln\left((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta\right)\right] - \beta E[\ln X_1]\right).$$

Therefore, using (3.2) and letting n go to infinity, we get

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{j=1}^n Q(cA_j^{\epsilon,\delta,\beta})\right) &\leq \frac{\alpha\beta - \beta E[\ln X_1]}{E\left[\ln\left((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta\right)\right] - \beta E[\ln X_1]}. \\
\limsup_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{j=1}^n Q(cA_j^{\epsilon,\delta,\beta})\right) &\leq \frac{\alpha\beta - \beta E[\ln X_1]}{E\left[\ln\left((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta\right)\right] - \beta E[\ln X_1]}.
\end{aligned}$$

Since $\lambda(\beta) = \infty$, by lemma 5.4, one can choose $c(\delta)$ such that

$$\frac{\alpha\beta - \beta E[\ln X_1]}{E\left[\ln\left((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta\right)\right] - \beta E[\ln X_1]} \leq 1 - \delta.$$

Since $\mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon,\beta}) \geq \delta 1_{A_j^{\epsilon,\delta,\beta}}$, we get the desired result. \square

5.3. Proof of theorems 3.6, 3.7. Both theorems are based on lemma 5.1 or lemma 5.1 and on the law of large numbers for martingales. Following the notations of lemma 5.1, we consider an i.i.d. sequence $(X_i)_{i \geq 1}$ defined on some probability space (H, \mathcal{H}, P) and such that $X_1 \stackrel{law}{=} e^{\eta(n,x)}$. We start by proving theorem 3.7.

Proof of theorem 3.7. Let $\delta < 1$ and $c(\delta)$ be some integer we will choose at the end of the proof. Finally, we set $\epsilon = \frac{1-\delta}{c(\delta)}$ (for notational convenience, we write ϵ instead of $\epsilon(\delta)$). We have the following computation:

$$\begin{aligned}
 \frac{\ln Z_n}{n} &= \frac{1}{n} \sum_{j=1}^n \ln\left(\frac{Z_j}{Z_{j-1}}\right) \\
 &= \frac{1}{n} \sum_{j=1}^n \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right). \\
 &= \frac{1}{n} \sum_{j=1}^n 1_{cA_j^{\epsilon,\delta,\beta}} \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \\
 &\quad + \frac{1}{n} \sum_{j=1}^n 1_{A_j^{\epsilon,\delta,\beta}} \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right). \tag{5.3}
 \end{aligned}$$

$$\tag{5.4}$$

Consider the (\mathcal{G}_n) -martingale M_n defined by:

$$M_n = \sum_{j=1}^n 1_{cA_j^{\epsilon,\delta,\beta}} \left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) - Q\left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \middle| \mathcal{G}_{j-1}\right) \right).$$

By definition of M_n and by applying lemma 5.1 to the family $(e^{\beta\eta(j,x)})_{x \in \mathbb{Z}^d}$ under the conditional measure $Q(\cdot | \mathcal{G}_{j-1})$, we get:

$$\begin{aligned}
 &\frac{1}{n} \sum_{j=1}^n 1_{cA_j^{\epsilon,\delta,\beta}} \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \\
 &= M_n + \sum_{j=1}^n 1_{cA_j^{\epsilon,\delta,\beta}} Q\left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \middle| \mathcal{G}_{j-1}\right) \\
 &\geq M_n + \left(\frac{1}{n} \sum_{j=1}^n 1_{cA_j^{\epsilon,\delta,\beta}}\right) E\left[\ln\left((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta\right)\right]
 \end{aligned}$$

Similarly, consider the (\mathcal{G}_n) -martingale N_n defined by:

$$N_n = \sum_{j=1}^n 1_{A_j^{\epsilon,\delta,\beta}} \left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) - Q\left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \middle| \mathcal{G}_{j-1}\right) \right).$$

By concavity of \ln , we get

$$\begin{aligned}
 &\frac{1}{n} \sum_{j=1}^n 1_{A_j^{\epsilon,\delta,\beta}} \ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \\
 &= N_n + \sum_{j=1}^n 1_{A_j^{\epsilon,\delta,\beta}} Q\left(\ln\left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta\eta(j,x)}\right) \middle| \mathcal{G}_{j-1}\right) \\
 &\geq N_n + \beta E[\ln X_1] \left(\frac{1}{n} \sum_{j=1}^n 1_{A_j^{\epsilon,\delta,\beta}}\right)
 \end{aligned}$$

Plugging the two above inequalities in inequality (5.3), we get:

$$\begin{aligned} & \frac{\ln Z_n}{n} - \frac{M_n}{n} - \frac{N_n}{n} - \beta E[\ln X_1] \geq \\ & \left(\frac{1}{n} \sum_{j=1}^n 1_{cA_j^{\epsilon, \delta, \beta}} \right) (E[\ln((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]) \end{aligned} \quad (5.5)$$

There exists some constant $C > 0$ such that for all j :

$$\beta \sum_x \mu_{j-1}(\omega_j = x) \eta(j, x) \leq \ln \left(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta \eta(j, x)} \right) \leq C \left| \sum_x \mu_{j-1}(\omega_j = x) e^{\beta \eta(j, x)} \right|^{\frac{1}{\theta}}$$

Thus there exists some constant $C' > 0$ such that for all j :

$$\begin{aligned} Q(|\ln(\sum_x \mu_{j-1}(\omega_j = x) e^{\beta \eta(j, x)})|^\theta) & \leq C' Q(|\sum_x \mu_{j-1}(\omega_j = x) e^{\beta \eta(j, x)}|) \\ & \quad + C' Q(|\sum_x \mu_{j-1}(\omega_j = x) \eta(j, x)|^\theta) \\ & \leq C' (\lambda(\beta) + Q(|\eta(j, x)|^\theta)). \end{aligned}$$

By using theorem 2.19 in [8], we conclude that:

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = \lim_{n \rightarrow \infty} \frac{N_n}{n} = 0 \quad Q - a.s.$$

By letting n go to infinity in inequality (5.5), we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n 1_{cA_j^{\epsilon, \delta, \beta}} \leq \frac{\alpha\beta - \beta E[\ln X_1]}{E[\ln((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]}$$

By using lemma 5.4, one can choose $c(\delta)$ and $\beta(\delta)$ in $]0, R[$ such that:

$$\forall \beta \in [\beta(\delta), R[\quad \frac{\alpha\beta - \beta E[\ln X_1]}{E[\ln((1-\delta) \sum_{k=1}^{c(\delta)} \frac{1}{c(\delta)} X_k^\beta + \delta X_{c(\delta)+1}^\beta)] - \beta E[\ln X_1]} \leq 1 - \delta,$$

which implies the result since $\mu_{j-1}(\omega_j \in \mathcal{A}_j^{\epsilon, \beta}) \geq \delta 1_{A_j^{\epsilon, \delta, \beta}}$. □

The proof of theorem 3.6 follows a similar strategy to the proof of theorem 3.7. Therefore, we only give a sketch of the proof.

Proof of theorem 3.6.

Suppose that β is such that $p(\beta) < \lambda(\beta)$. Then one can chose a positive integer k sufficiently large for the following inequality to hold with $\epsilon = \frac{1}{k}$:

$$p(\beta) < E[\ln(\epsilon \sum_{i=1}^{1/\epsilon} X_i^\beta)].$$

By the same strategy than for the proof of theorem 3.7 (using lemma 5.3 instead of lemma 5.1), we get

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n 1_{cA_j^{\epsilon, \beta}} \leq \frac{p(\beta) - \beta E[\ln X_1]}{E[\ln(\epsilon \sum_{i=1}^{\frac{1}{\epsilon}} X_i^\beta)] - \beta E[\ln X_1]} < 1. \quad (5.6)$$

This implies easily the desired result. \square

Acknowledgements: I would like to thank my Ph.D. supervisor Francis Comets for his help and suggestions.

REFERENCES

- [1] Albeverio, S., Zhou, X.: A martingale approach to directed polymers in a random environment, *J. Theoret. Probab.*, **9** (1996), 171-189.
- [2] Bolthausen, E.: A note on diffusion of directed polymers in a random environment, *Commun. Math. Phys.*, **123** (1989) 529-534.
- [3] Carmona, P., Hu, Y.: On the partition function of a directed polymer in a random environment, *Probab. Theory. Related Fields*, **124**, (2002), 431-457.
- [4] Comets, F., Shiga, T., Yoshida, N.: Directed polymers in a random environment: path localization and strong disorder, *Bernoulli*, **9**(4), 2003, 705-723.
- [5] Comets, F., Shiga, T., Yoshida, N.: Probabilistic analysis of directed polymers in random environment. Stochastic analysis on scale interacting systems, *Advanced Studies in Pure Mathematics*, 39, Math. Soc. Japan, Tokyo, (2004), 115-142.
- [6] Comets, F., Yoshida, N.: Directed polymers in random environment are diffusive at weak disorder, arXiv:math.PR/0411223, to appear in *Annals of Probability*.
- [7] Comets, F., Vargas, V.: Majorizing multiplicative cascades for directed polymers in random media, arXiv:math.PR/0510525.
- [8] Hall, P., Heyde, C.C.: Martingale limit theory and its Application, *Probability and Mathematical Statistics* (1980).
- [9] Halpin-Healy, T., Pang N.-N.: Discerning differences among anomalously wandering directed polymers, *Physical Review E*, **47**(2), (1993), 784-786.
- [10] Huse, D.A., Henley, C.L.: Pinning and roughening of domain wall in Ising systems due to random impurities, *Phys. Rev. Lett.*, **54** (1985), 2708-2711.
- [11] Imbrie, J.Z., Spencer, T.: Diffusion of directed polymer in a random environment, *J. Stat. Phys.*, **52** no. 3/4 (1988), 609-626.
- [12] Johansson, K.: Shape fluctuations and random matrices, *Communications in Mathematical Physics*, **209** no. 2 (2000), 437-476.
- [13] Kesten, H.: Aspects of first passage percolation, *Ecole d'Eté de Probabilités de Saint-Flour*, XIV-1984, 125-264.
- [14] Krug, H., Spohn, H.: Kinetic roughening of growing surfaces, *Solids Far from Equilibrium*, C. Godrèche ed., Cambridge University Press (1991).
- [15] Marconi, U.M.B., Zhang, Y.C.: Novel scaling behavior of directed polymers: disorder distribution with long tails, *J. Stat. Phys.*, **61** (1990), 885-889.
- [16] Martin, J.B.: Linear growth for greedy lattice animals, *Stochastic Processes and their Applications*, **98** (2002), 43-66.
- [17] Martin, J.B.: Limiting shape for directed percolation models, *Annals of Probability*, **32** no. 4 (2004), 2908-2937.
- [18] Sinai, Y.: A remark concerning random walks with random potentials, *Fund. Math.*, **147** (1995), 173-180.

- [19] Song, R., Zhou, X.Y.: A remark on diffusion of directed polymers in random environment, *J. Stat. Phys.*, **85** no. 1/2 (1996), 277-289.
- [20] Vargas, V.: A Local Limit Theorem for directed polymers in random media: the continuous and the discrete case, arXiv:math.PR/0503596, to appear in *Annales de l'I.H.P.*.
- [21] Zhang, Y.-C.: *J. Phys.* (Paris), **51**, (1990), 2129.