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Quadratic BSDEs Driven by a Continuous Martingale and Application to the Utility Maximization Problem

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Abstract

In this paper, we will study some quadratic Backward Stochastic Differential Equations (BSDEs) in a continuous filtration which arise naturally in the problem of utility maximization with constraints on the portfolio.

In a first part, we will show existence and uniqueness for those BSDEs. Then we will give an application to the utility maximization problem for three different cases : the exponential utility function, the power one and the logarithmic one.

Keywords :

Backward Stochastic Differential Equations (BSDE), continuous filtration, quadratic growth, utility maximization, incomplete market, constraint portfolio.

1 Introduction :

1.1 Motivation :

In this paper, we will study some quadratic BSDEs : these equations arise naturally in the utility maximization problem.

There is a long list of papers dealing with the classical problem of utility maximization and we will mention only a few of them, close to our setting.

The main interest comes from the existence of incomplete markets in which all contingent claims (or random variables depending on the information available at terminal time T) are not attainable. This explains the interest to introduce a new notion of optimality (and especially of optimal strategy).

To this aim, we consider a usual probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathbb{F}_t))$ equipped with a right-continuous and complete filtration $\mathbb{F} = (\mathbb{F}_t)_{t \in [0, T]}$.

Then, we define the utility maximization problem by setting the value process

$V = (V(x_t))_{t \in [0, T]}$ as follows :

$$V(x_t) = \underset{\nu}{\text{esssup}} \mathbb{E}(U(X_T^\nu) | \mathbb{F}_t) = \underset{\nu}{\text{esssup}} \mathbb{E}(U(x_t + \int_t^T \nu'_s \frac{dS_s}{S_s}) | \mathbb{F}_t) \quad (1)$$

where x_t will be a fix \mathbb{F}_t -measurable random variable, S is the price process and the process $: X^\nu = x + \int (\nu' \cdot \frac{dS}{S})$ stands for the wealth process associated to the strategy ν .

The problem (1) is studied extensively in the literature see [13] (or [14]) for a survey on this topic. The convex duality method is largely used for this type of problem, but this method requires the constraint on the portfolio to be convex.

Another method to solve this problem is to apply the BSDE technique see for example [5], [6] or recently [8]. We mention that in [5] the constraint on the portfolio is a convex cone, and that in [8], no constraint is imposed on the portfolio, but the authors study the problem in a general filtration. On the other hand in [6], the authors study the problem (1) in a Brownian filtration and, in particular, they prove the existence of optimal portfolio with a closed (but non necessarily convex) constraint on the portfolio. Because those authors work in a Brownian setting, the results on quadratic BSDEs are available (for this see [2]).

In the present article, we will study the problem (1) using the BSDE technique inspired by [6] but, since we will work on a continuous (but non Brownian) filtration, no results on quadratic BSDE are available. Hence we begin by a study of the problems of existence and uniqueness for those quadratic BSDEs.

Then, in a second part, we will apply those results to find a construction of the utility value process $(V(x_t))_t$. We will compute for three types of utility functions the expression of this value process.

1.2 Theoretical background :

In this part, we are going to introduce the form of the BSDE we will consider in the sequel. We consider as usual a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a right continuous and complete filtration $\mathbb{F} = (\mathbb{F}_t)$ and with a local and continuous d -dimensional martingale M .

In the sequel, all processes will be considered on $[0, T]$ where T is a deterministic time (T is the horizon or maturity time in finance).

All local and \mathbb{R} -valued martingales are supposed to be of the form $: K = Z' \cdot M + L$, where Z is a process taking its values in $\mathbb{R}^{d \times 1}$ and L is a real valued martingale which is strongly orthogonal to M (that is to say that for each i : $\langle M^i, L \rangle = 0$).

We have moreover that each component $d\langle M^i, M^j \rangle$ ($i, j \in [1, d]^2$) of the quadratic variation of M is absolutely continuous with respect to $d\tilde{C}_s = d(\sum_i d\langle M^i \rangle)$.

This is a simple consequence of Kunita-Watanabe's inequality for local continuous martingales. In fact, using the result of Proposition 1.15 (chapter IV in Revuz-Yor, [12]), we have the following controls (for all i, j) :

$$\int_0^t |d\langle M^i, M^j \rangle_s| \leq \sqrt{\langle M^i \rangle_t} \cdot \sqrt{\langle M^j \rangle_t} \leq \frac{1}{2}(\langle M^i \rangle_t + \langle M^j \rangle_t)$$

Since the process \tilde{C} is in general unbounded, we set C as the bounded, real-valued and increasing process defined by $: C_t = \arctan(\tilde{C}_t)$.

It entails that $: dC_t = \frac{1}{1+\tilde{C}_t^2} d\tilde{C}_t$, and as a consequence, each component $d\langle M^i, M^j \rangle$ of the quadratic variation process $d\langle M \rangle$ is absolutely continuous w.r.t dC_t .

For each i, j we have also existence of a random Radon Nikodym density $z^{i,j}$ such that :
 $d \langle M^i, M^j \rangle_s = z^{i,j} dC_s$.

Finally thanks to the fact that $z = (z^{i,j})$ is a non negative and symmetric matrix, it implies that we can write : $d \langle M, M \rangle_s = m'_s m_s dC_s$,

where m is a predictable process taking its values in $\mathbb{R}^{d \times d}$. We will impose furthermore that the matrix $m'_s m_s$ is invertible (for all s).

We are interested in finding a solution to the following BSDE :

$$(1.1) \quad \begin{cases} dY_s = -F(s, Y_s, Z_s) dC_s - \frac{\beta}{2} \cdot d \langle L \rangle_s \\ \quad + Z'_s \cdot dM_s + dL_s \\ Y_T = B \end{cases}$$

A solution to such a BSDE is a triple of processes (Y, Z, L) in the following space :

$S^\infty \times L^2(d \langle M \rangle \times d\mathbb{P}) \times \mathcal{M}^2([0, T])$ equipped with the norms :

$$\|Y\|_{S^\infty} = \text{esssup} \left(\sup_{0 \leq t \leq T} |Y_t| \right)$$

$$\|Z\|_{L^2(d \langle M \rangle \times d\mathbb{P})} = \mathbb{E} \left(\int_0^T (m'_s \cdot Z_s)' (m_s \cdot Z_s) dC_s \right)^{\frac{1}{2}}$$

$$\|L\|_{\mathcal{M}^2([0, T])} = \mathbb{E}(\langle L \rangle_T)^{\frac{1}{2}}$$

In the sequel, we will impose furthermore that we have the following controls on F :

$$\begin{aligned} \exists \alpha \in L^1(dC_s), \int_0^T |\alpha_s| dC_s \leq a \quad (a > 0) \text{ and } : b, \gamma > 0, \text{ such that :} \\ (H_1) \quad |F(s, y, z)| \leq \alpha_s(1 + b|y|) + \frac{\gamma}{2}|mz|^2 \end{aligned} \quad (2)$$

We impose furthermore that : $\gamma \geq \beta$ (β has been introduced in the expression of the BSDE (1.1)), and $\gamma \geq b$.

To obtain the existence, we will, in a first step, impose the restrictive condition on F :

$$(H'_1) \quad -\alpha_s - C_0(|y| + |mz|) \leq F(s, y, z) \leq \alpha_s(1 + b|y|) + \frac{\gamma}{2}(mz)'(mz) \quad (3)$$

We impose here the same assumptions on the parameters α, b and γ , plus the condition : $C_0 > 0$.

The assumption that F has a lower bound which is globally Lipschitz in its variables y and z will be useful in the proof of existence.

In the preceding inequalities, s is fixed ($s \in [0, T]$), the notation $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^{d \times 1}$.

In the proof of existence, we will introduce a second BSDE of the following type :

$$(1.2) \quad \begin{cases} dU_s = -g(s, U_s, V_s) \cdot dC_s + V'_s dM_s + dN_s \\ U_T = e^{\beta \cdot B} \end{cases}$$

We will show that we have a one to one correspondence between the solutions of those two BSDEs when defining $U = e^{\beta \cdot Y}$ (exponential change), where (Y, Z, L) is a solution of (1.1).

2 Results about quadratic BSDEs

In this section, we will begin by giving some a priori estimates on the norms of processes solving BSDE of the form (1.1) and (1.2) : this will be of great interest to prove the results of existence and uniqueness.

Before giving these proofs, we state here the results we are able to obtain :

We state below the result of existence :

Theorem 1 (i) *Suppose that the generator g satisfies the assumption (H'_1) , then the BSDE (1.2) has a solution (U, V, N) in the space $S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0, T])$.*

(ii) *If the generator F satisfies the assumption (H_1) , then there exists a solution (Y, Z, L) in the space $S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0, T])$ to the BSDE (1.1).*

To prove a result of uniqueness, we need another assumption on the increments of the generator F :

we will impose furthermore that there exists two sequences of processes $(\lambda_N)_N$ and $(\kappa_N)_N$ such that :

(i) For each $N \in \mathbb{N}^*$, there exists two processes $(\lambda_N)_N$ and $(\kappa_N)_N = (\kappa_N(z^1, z^2))$ such that :

$$\begin{aligned} \forall z \in \mathbb{R}^d, \forall y^1, y^2, |y^1|, |y^2| \leq N \quad (y^1 - y^2) \cdot (F(s, y^1, z) - F(s, y^2, z)) &\leq \lambda_N |y^1 - y^2|^2 \\ \forall y \in \mathbb{R}, |y| \leq N, \forall z^1, z^2 \in \mathbb{R}^d, \quad F(s, y, z^1) - F(s, y, z^2) &= (m \cdot \kappa_N)' \cdot (m(z^1 - z^2)) \end{aligned} \quad (4)$$

(ii) For each $N \in \mathbb{N}^*$, the process λ_N is integrable w.r.t dC, and the martingale $\kappa_N \cdot M$ is a BMO martingale.

We recall here that M is a BMO martingale if there exists a constant c ($c > 0$) such that, for all stopping time τ of \mathbb{F} , we have :

$$\mathbb{E}(\langle M \rangle_T - \langle M \rangle_\tau | \mathbb{F}_\tau) \leq c$$

Theorem 2 *Under the assumption (H_1) and the conditions (i) and (ii) on the generators and provided the terminal condition is bounded, the BSDEs of the form (1.1) (resp. of the form (1.2)) defined in the section 1.2 has at most one solution (Y, Z, L) in $S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0, T])$ (resp. (U, V, N) in $S^\infty \times L^2(d < M > \times d\mathbb{P}) \times \mathcal{M}^2([0, T])$)*

2.1 A priori estimates

In this section, we will study under the assumption that the generator F satisfies the condition (H_1) or (H'_1) .

One remark is that it suffices here to consider a BSDE of the form (1.1) with a generator satisfying (H_1) (we obtain a BSDE of the second form when : $\beta \equiv 0$).

Proposition 1 *Keeping the same notations as those mentioned in section 1.2, all triple (Y, Z, L) of processes solving the BSDE (1.1) with the process Y bounded (\mathbb{P} -almost surely*

and for all t) satisfies the following assertions : there exists some constants c and C depending only on the parameters of the BSDE (i.e the constants a , b and $|B|_\infty$) and a constant C' depending on the estimate of the norm of Y in S^∞ such that :

$$\mathbb{P} - \text{almost surely } \forall t, c \leq Y_t \leq C \quad (5)$$

$$\exists C' \quad \forall \tau \text{ (}\tau \text{ } \mathbb{F}\text{-stopping time)}$$

$$\mathbb{E}\left(\int_\tau^T Z'_s m'_s m_s Z_s dC_s + \langle L \rangle_T - \langle L \rangle_\tau \mid \mathbb{F}_\tau\right) \leq C' \quad (6)$$

Proof :

One first remark is that the second estimates will give us a control of the BMO norms of the square integrable martingales Z' .M and L .

We suppose in the sequel that we are given a solution (Y, Z, L) of the BSDE (1.1) with a generator satisfying (H_1) and with the process Y bounded.

To prove the estimates given by (5), we introduce the process : $U = e^{K.Y}$.

It can be easily proven, by using Itô's formula, that this process is solution of a BSDE whose generator g is given by the expression :

$$g(s, u, v) = K.u \left(F\left(s, \frac{\ln(u)}{K}, \frac{v}{K.u}\right) - \frac{K}{2} \left(m \cdot \frac{v}{K.u}\right)' \left(m \cdot \frac{v}{K.u}\right) \right)$$

As a simple consequence resulting from the computation of Itô's formula, if we want to give a majoration of g independent of $|m.v|^2$, it entails that we have to take : $K \geq \gamma$. In the sequel, we fix $K = \gamma$.

Since F satisfies the assumption given by (H_1) , it entails that we obtain the following control on g :

$$g(s, u, v) \leq \gamma.\alpha_s u. \left(1 + \frac{b}{\gamma} \cdot |\ln(u)|\right) \quad (7)$$

We proceed hereafter with the same method as the one in Briand and Hu in [11], we will compare the process U to the solution of a differential equation. To this aim, fixing ω and setting : $z = B(\omega)$ (z is real), we consider the solution $\phi(z)$ of the following equation :

$$\phi_t(z) = e^{\gamma.z} + \int_t^T H(\phi_s) dC_s$$

where H is given by :

$$H(x) = \gamma.\alpha_s .x \left(1 + \frac{b}{\gamma} \cdot \ln(x)\right) \mathbf{1}_{x>1} + \gamma.\alpha_s \mathbf{1}_{x \leq 1}$$

We recall here that : $\frac{b}{\gamma} \leq 1$.

It is also easy to check :

$$\forall u > 0, \gamma.\alpha_s u. \left(1 + \frac{b}{\gamma} \cdot |\ln(u)|\right) \leq H(u) \quad (8)$$

As in [11], we remark that H is locally Lipschitz and convex. But, contrary to their paper, we work here in a non Brownian setting (the random process $C = (C_s)$ (which is of finite

variation) replaces the deterministic process (s)).

To give the expression of the solution ϕ , we have to discuss the sign of z , because the expression of H depends on whether or not the function ϕ is greater than one .

(i) if $z \geq 0$:

It is the simplest case, since we can see easily that : $t \rightarrow \phi_t$ is a decreasing function with terminal value equal to $e^{\gamma \cdot z}$ (which is greater than one in this case). The expression of ϕ is given by :

$$\phi_t(z) = \exp\left(\gamma \cdot \frac{e^{\int_t^T b \cdot \alpha_u dC_u} - 1}{b}\right) \exp(\gamma \cdot z \cdot e^{\int_t^T b \cdot \alpha_u dC_u})$$

(ii) if $z \leq 0$ (then : $\phi_T(z) = e^{\gamma \cdot z} \leq 1$) :

(a) If : $e^{\gamma \cdot z} + \int_0^T \gamma \cdot \alpha_u dC_u \leq 1$, then the solution is defined for all t by :

$$\phi_t = e^{\gamma \cdot z} + \gamma \cdot \int_t^T \alpha_u dC_u$$

(b) Otherwise : $\exists S < T$, $e^{\gamma \cdot z} + \gamma \cdot \int_t^T \alpha_u dC_u = 1$,

$$\phi_t = \left(e^{\gamma \cdot z} + \gamma \cdot \int_t^T \alpha_u dC_u\right) \mathbf{1}_{]S, T[}(t) + \exp\left(\gamma \cdot \frac{e^{\int_t^S b \cdot \alpha_u dC_u} - 1}{b}\right) \mathbf{1}_{]0, S[}(t)$$

Then, we introduce the adapted process Φ defined by :

$\forall t$, $\Phi_t = \mathbb{E}_{\mathbb{F}_t}(\phi_t(B))$. ($\mathbb{E}_{\mathbb{F}_t}$ stands for the conditional expectation w.r.t \mathbb{F}_t)

Introducing the following martingale : $K_t = \mathbb{E}_{\mathbb{F}_t}(\phi_T(B) + \int_0^T \mathbb{E}_{\mathbb{F}_s}(H(\phi_s)) dC_s)$,

we claim that Φ is a semimartingale (whose martingale part is K) which satisfies the following BSDE :

$$\Phi_t = e^{\gamma \cdot B} + \int_t^T \mathbb{E}_{\mathbb{F}_s} H(\phi_s(B)) dC_s - (K_T - K_t)$$

Thanks to the inequalities (7) and (8) and applying the same method as that in Briand and Hu in [11], we can conclude that a comparison result holds and that, for all t :

$$U_t = e^{\gamma \cdot Y_t} \leq \Phi_t, \text{ or equivalently : } Y_t \leq \frac{1}{\gamma} \ln(\mathbb{E}_{\mathbb{F}_t}(\phi_t(B)))$$

To obtain the minoration of Y , it is enough to apply the same method to the process $e^{-\gamma \cdot Y}$: this is justified since the BSDE given by the parameters $(F', -B)$ (where F' is defined by : $F'(s, y, z) = -F(s, -y, -z)$) and whose solution is $(-Y, -Z, -L)$ is such that : the generator F' satisfies again the assumption H_1 with the same parameters and so, it implies that we have : $-Y_t \leq \frac{1}{\gamma} \ln(\mathbb{E}_{\mathbb{F}_t}(\phi_t(-B)))$.

We also obtain the estimates given by (5) by setting :

$$c = \operatorname{ess\,inf}_{\omega} \inf_t -\frac{1}{\gamma} \ln(\Phi_t(-B)) , \text{ and } : C = \operatorname{ess\,sup}_{\omega} \sup_t \frac{1}{\gamma} \ln(\Phi_t(B))$$

One important remark is that it is easy to show that it is possible to give estimates of the process Y in S^∞ which are independent of the parameter γ : in the case where : $B > 0$, we recall that the expression of $\phi(B)$ is :

$$\phi_t(B) = \exp\left(\gamma \cdot \frac{e^{\int_t^T b \cdot \alpha_u dC_u} - 1}{b}\right) \exp(\gamma \cdot B \cdot e^{\int_t^T b \cdot \alpha_u dC_u})$$

This entails that : $\mathbb{E}_{\mathbb{F}_t}(\phi_t(B)) \leq e^{\gamma \cdot \frac{e^{b,a}-1}{b}} \cdot e^{\gamma \cdot |B|_\infty e^{b,a}}$.

and consequently, we obtain : $\forall s Y_s \leq \frac{e^{b,a}-1}{b} + |B|_\infty e^{b,a}$ \mathbb{P} - almost surely.

Similarly, we have : $\forall s -Y_s \leq \frac{e^{b,a}-1}{b} + |-B|_\infty e^{b,a}$ \mathbb{P} - almost surely.

Then, to prove the estimates given by (6), we will apply Itô's formula to the process $\psi_K(Y + m)$ (K and m are constants which will be explicitated later).

The expression of ψ_K is given by : $\psi_K(x) = \frac{e^{K \cdot x} - 1 - K \cdot x}{K}$.

The following properties will be useful in the sequel :

$$\begin{aligned} \psi_K'(x) &\geq 0 \text{ if } : x \geq 0. \\ -K \cdot \psi_K' + \psi_K'' &= K. \end{aligned}$$

Moreover, we will use the fact that there exists a constant m such that :

$\forall s \in [0, T], Y_s + m \geq 0$ \mathbb{P} - almost surely.

Since Y is a bounded process, it suffices to choose : $m = -|Y|_\infty$ (norm of the process in S^∞).

Let τ be an arbitrary stopping time of $(\mathbb{F}_t)_{t \in [0, T]}$. Taking then the conditional expectation with respect to \mathbb{F}_τ , it provides us with :

$$\begin{aligned} \underbrace{\psi_K(Y_\tau + m)}_{= \mathbb{E}_{\mathbb{F}_\tau}(\psi_K(Y_\tau + m))} &- \mathbb{E}_{\mathbb{F}_\tau} \psi_K(Y_T + m) \\ &= -\mathbb{E}_{\mathbb{F}_\tau} \left(\int_\tau^T \psi_K'(Y_s + m) (-F(s, Y_s, Z_s) dC_s - \frac{\beta}{2} \cdot d \langle L \rangle_s) \right) \\ &\quad - \mathbb{E}_{\mathbb{F}_\tau} \left(\int_\tau^T \psi_K'(Y_s + m) (Z_s' dM_s + dL_s) \right) \\ &\quad - \mathbb{E}_{\mathbb{F}_\tau} \left(\int_\tau^T \frac{\psi_K''}{2} (Y_s + m) ((m_s \cdot Z_s)' (m_s \cdot Z_s) dC_s + d \langle L \rangle_s) \right) \end{aligned}$$

The second term in the right-hand side is equal to zero, because it is of the form :

$\mathbb{E}_{\mathbb{F}_\tau}(K_T - K_\tau)$, where K is a martingale of the filtration \mathbb{F} .

Then, remembering the majoration on the generator F given by the assumption (H₁) and after simple computations, we obtain :

$$\begin{aligned} \psi_K(Y_\tau + m) &- \mathbb{E}_{\mathbb{F}_\tau} \psi_K(Y_T + m) \\ &\leq \mathbb{E}_{\mathbb{F}_\tau} \int_\tau^T \psi_K'(Y_s + m) (|\alpha_s| (1 + b|Y|)_\infty) dC_s \\ &\quad + \mathbb{E}_{\mathbb{F}_\tau} \int_\tau^T \left(\frac{\beta}{2} \psi_K' - \frac{1}{2} \psi_K'' \right) (Y_s + m) d \langle L \rangle_s \\ &\quad + \mathbb{E}_{\mathbb{F}_\tau} \int_\tau^T \left(\frac{\gamma}{2} \psi_K' - \frac{1}{2} \psi_K'' \right) (Y_s + m) |m \cdot Z_s|^2 dC_s \end{aligned}$$

We have easily that the terms of the left member are bounded (independently of the stopping time τ) and it is the same for the first term of the right member (thanks to the integrability assumption on α).

We put in the left member the two last terms of the second member of the preceding inequality, and then we claim that : fixing K such that : $K = \gamma$, and remembering that : $\gamma \geq \beta$, we have : $\forall x \geq 0, f_1(x) = \frac{1}{2} \cdot (-\gamma \cdot \psi_\gamma' + \psi_\gamma'')(x) = \frac{\gamma}{2} > 0$, on the one hand, and :

$\frac{1}{2}(-\beta.\psi'_\gamma + \psi''_\gamma)(x) = f_1(x) + \frac{1}{2}(-\beta + \gamma)\psi'_\gamma(x) \geq \frac{\gamma}{2}$, on the other hand.

We use those inequalities for $x = Y_s + m$, quantity which is almost surely non negative. It implies that there exists a constant C' (depending only on the parameters γ , a and $|Y|_\infty$) and which is independent of the stopping time τ) such that :

$$\mathbb{E} \left(\int_\tau^T (m_s.Z_s)'(m_s.Z_s)dC_s + (\langle L \rangle_T - \langle L \rangle_\tau) | \mathbb{F}_\tau \right) \leq C'$$

2.2 Uniqueness for the BSDE (1.1)

Proof of Theorem 2 :

We suppose that we are given two solutions (Y^1, Z^1, L^1) and (Y^2, Z^2, L^2) of the BSDE (1.1) with Y^1 and Y^2 bounded. Let N be a constant such that, \mathbb{P} - almost surely: $|Y^1| \leq N$ (respectively $|Y^2| \leq N$).

The existence of such a constant N is justified by the estimates established in the preceding section under the assumptions (H_1) and (4) on the generator F .

To achieve it, we begin by applying Itô's formula to the non negative semimartingale $(\tilde{Y}^{1,2})$ defined as follows :

$$\forall t, \tilde{Y}_t^{1,2} = e^{-\int_t^T 2.\lambda_N dC_s} |Y_t^{1,2}|^2 .$$

It gives us :

$$\begin{aligned} d(\tilde{Y}_s^{1,2}) &= 2.\lambda_N \tilde{Y}_s^{1,2} dC_s \\ &+ e^{-\int_t^T 2.\lambda_N dC_s} 2.Y_s^{1,2} dY_s^{1,2} \\ &+ \frac{1}{2} e^{-\int_t^T 2.\lambda_N dC_s} 2d \langle Y^{1,2} \rangle_s \end{aligned}$$

We recall that we have :

$$dY^{1,2} = -(F(s, Y_s^1, Z_s^1) - F(s, Y_s^2, Z_s^2))dC_s - \frac{\beta}{2} d(\langle L^1 \rangle_s - \langle L^2 \rangle_s) + dK_s,$$

where K stands for the martingale part : $dK = (Z^{1,2})' dM + dL^{1,2}$.

Then, taking the integral between T and t , and remembering that :

$Y_T^{1,2} = 0$ (the processes have the same terminal value B).

$$\begin{aligned} \tilde{Y}_t^{1,2} - \underbrace{\tilde{Y}_T^{1,2}}_{=0} &= - \int_t^T 2.\lambda_N \tilde{Y}_s^{1,2} dC_s \\ &+ \int_t^T e^{-\int_t^T 2.\lambda_N dC_s} (2.Y_s^{1,2} (F(s, Y_s^1, Z_s^1) - F(s, Y_s^2, Z_s^2))) dC_s \\ &+ \int_t^T e^{-\int_t^T 2.\lambda_N dC_s} 2.Y_s^{1,2} \frac{\beta}{2} d(\langle L^{1,2} \rangle_s, L^1 + L^2 \rangle_s) \\ &- \int_t^T e^{-\int_t^T 2.\lambda_N dC_s} 2.Y_s^{1,2} (Z_s^{1,2}.dM + d \langle L^{1,2} \rangle_s) \\ &- \underbrace{\int_t^T e^{-\int_t^T 2.\lambda_N dC_s} \frac{1}{2}.2d \langle Y^{1,2} \rangle_s}_{\leq 0} \end{aligned}$$

We write then more precisely the majoration of the increments in the variables y and z of the generator F (given by (4)) .

$$2.Y_s^{1,2}(F(s, Y_s^1, Z_s^1) - F(s, Y_s^2, Z_s^2)) \leq 2.\lambda_N |Y_s^{1,2}|^2 + 2.Y_s^{1,2} \cdot (m.\kappa_N)'(m.Z_s^{1,2})$$

Rewriting Itô's formula, we obtain :

$$\begin{aligned} \tilde{Y}_t^{1,2} &\leq \int_t^T 2.|Y_s^{1,2}|^2 e^{-\int_t^T 2.\lambda_N dC_s} \cdot (-\lambda_N + \lambda_N) dC_s \\ &\quad + \int_t^T 2.Y_s^{1,2} e^{-\int_t^T 2.\lambda_N dC_s} (m.\kappa_N)'(m.Z_s^{1,2}) dC_s \\ &\quad + \int_t^T 2.Y_s^{1,2} e^{-\int_t^T 2.\lambda_N dC_s} \frac{\beta}{2} \cdot (d \langle L^{1,2}, L^1 + L^2 \rangle_s) \\ &\quad - \int_t^T 2.e^{-\int_t^T 2.\lambda_N dC_s} .Y_s^{1,2} (Z_s^{1,2}) dM - \int_t^T 2.e^{-\int_t^T 2.\lambda_N dC_s} Y_s^{1,2} dL^{1,2} \end{aligned}$$

Considering the following stochastic integrals:

on the one hand : $N = \left(2.e^{-\int_t^T 2.\lambda_N dC_s} Y^{1,2} Z^{1,2} \right)' .M$, and : $\bar{N} = \kappa_N' .M$, and on the other hand : $L = \left(2.Y^{1,2} e^{-\int_t^T 2.\lambda_N dC_s} L^{1,2} \right)$, and : $\bar{L} = \frac{\beta}{2} \cdot (L^1 + L^2)$

It entails that the semimartingale $\tilde{Y}^{1,2}$ satisfies the following inequality :

$$\tilde{Y}_t^{1,2} \leq \int_t^T (d \langle N, \bar{N} \rangle_s - dN_s) + \int_t^T d \langle L, \bar{L} \rangle_s - dL_s$$

Replacing then the measure dM by $d\hat{M} = dM - \kappa_N' .d \langle M \rangle$ and $dL^{1,2}$ by :

$d\hat{L} = dL^{1,2} - \frac{\beta}{2} d \langle L^1 + L^2, L^{1,2} \rangle$, and introducing the following measure :

$d\mathbb{Q} = \mathcal{E}(\kappa_N' .M + \frac{\beta}{2} \cdot (L^1 + L^2)) d\mathbb{P}$ (where $\mathcal{E}(K)$ stands for the exponential of the martingale K),

we can write : $\tilde{Y}_t^{1,2} \leq -M_{t,T}$, where : $M_{t,T}$ is the increment between t and T of a martingale under the equivalent probability measure $d\mathbb{Q}$ of the filtration \mathbb{F} . Its expression is simply :

$$M_{t,T} = \int_t^T 2.Y_s^{1,2} e^{-\int_t^T 2.\lambda_N dC_s} (Z_s^{1,2} .d\hat{M}_s + d\hat{L}_s)$$

The fact that we have an equivalent probability measure is justified by the use of Girsanov's theorem because of the results obtained in the preceding section .

Thanks to the controls of the BMO norms of $Z' .M$ and L , for all solution (Y, Z, L) to the BSDE (1.1) with the process Y bounded and if the process κ_N satisfies the following control :

$$\forall s \quad |(\kappa_N)_s| \leq \hat{C}(1 + |Z_s^1| + |Z_s^2|) \quad \mathbb{P} - \text{almost surely}$$

we can claim that $\mathcal{E}(\kappa_N' .M + \frac{\beta}{2} \cdot (L^1 + L^2))$ is a uniformly integrable martingale (with constant expectation equal to one) : it is a density of probability.

This is true in particular in the financial application given in the second part of this article : in fact, the generator can be written under the following form :

$F(s, y, z) = \hat{C} \cdot (m.z)'(m.z) + F_0(s, y, z)$, where F_0 is a Lipschitz function and \hat{C} a non negative constant.

Taking the conditional expectation with respect to \mathbb{F}_t , we can conclude that : $\tilde{Y}^{1,2} \leq 0$ \mathbb{Q} - almost surely (and so : \mathbb{P} - almost surely since the two measures are equivalent), $Y^{1,2} \equiv 0$.

This achieves the proof of uniqueness.

2.3 Existence

Proof of Theorem 1 :

This proof will be achieved in three main steps and following the method given by Kobylanski in her PhDthesis (see [2])

2.3.1 Building the approximation

Step 1 : Troncation in y

We aim at using the a priori estimates we have proven in Section 2.1 so that we can relax the assumption on the generator and then obtain precise estimates for an intermediate BSDE .

In this step, we will show that it is possible to restrict ourselves to the assumption (H_2) , instead of (H_1) , on our generator F :

$$(H_2) \quad \exists \tilde{\alpha} \int_0^T \tilde{\alpha}_s dC_s \leq a \ (a > 0) , \text{ such that : } |F(s, y, z)| \leq \tilde{\alpha}_s + \frac{\gamma}{2} |m.z|^2$$

Let (Y, Z, U) be a solution to the BSDE (1.1) with Y bounded such that F satisfies (H_1) , and let introduce the following BSDEs (for $K > 0$):

$$dY_s^K = -F^K(s, Y_s^K, Z_s^K) dC_s - \frac{\beta}{2} d \langle L^K \rangle_s + (Z_s^K)' dM + dL_s^K$$

where : $F^K(s, y, z) = F(s, \rho_K(y), z)$, and :

ρ_K is a regular function (at least a C^2 function) verifying the following conditions :

$\rho_K(x) = x$, if : $|x| \leq K$, and : $\rho_K(x) = K + 1$, if : $|x| \geq K + 1$.

We impose besides that : $|\rho_K(x)| \leq |x|$.

Since F satisfies (H_1) , it implies that :

$$\forall y \in \mathbb{R}, z \in \mathbb{R}^d, F^K(s, y, z) \leq \alpha_s(1 + b|\rho_K(y)|) + \frac{\gamma}{2} |m.z|^2$$

Since : $|\rho_K(x)| \leq |x|$, we can conclude that for all K, F^K satisfies (H_1) with the same parameters as F : this implies that there exists a constant l depending only on $|B|_\infty$, a and b and independent of K, and such that : $\forall K |Y^K|_\infty \leq l$ (all bounded solution of the BSDEs characterized by (F^K, B) (or (F, B)) satisfy the same a priori estimates).

Setting : $\forall s, \tilde{\alpha}_s = \alpha_s(1 + b.l)$, we can conclude that both F and F^K satisfies the assumptions (H_2) .

Step 2: an intermediate BSDE

To solve the BSDE (1.1), we begin by setting formally : $U = e^{\beta.Y}$ (exponential change) and by supposing that we have a solution (Y, Z, L) to (1.1) .

Using Itô's formula, we show that this new process U is solution of a BSDE of the following form :

$$(1.2) \begin{cases} dU_s = -g(s, U_s, V_s) dC_s + d(V'_s \cdot M_s) + dN_s \\ U_T = e^{\beta \cdot B} \end{cases}$$

where we have defined the following processes: $V_s = \beta \cdot U_s \cdot Z_s$, and: $dN_s = \beta \cdot U_s \cdot dL_s$, so that the martingale part of U can be written : $V' \cdot M + N$. The finite variation term is independent of $d < N >$ and its expression (in the differentiate form) is simply : $-g(s, U_s, V_s) dC_s$.

This BSDE is characterized by its generator g :

$$g(s, u, v) = \beta \cdot u \cdot F\left(s, \frac{\ln(u)}{\beta}, \frac{v}{\beta \cdot u}\right) - \frac{1}{\beta \cdot u} (m \cdot v)' \cdot (m \cdot v)$$

In this second step, we will use the fact that F satisfies the assumption (H₂) to give precise estimates of all solutions of the BSDE characterized by (g, e^{β·B}). Defining (U^K, V^K, N^K) as a solution of the BSDE characterized by (g^K, e^{β·B}) and with : g^K(s, u, v) = g(s, ρ_K(u), v), we are going to show that we can give estimates which are independent of K (those estimates will again hold true for all bounded solution of the BSDE whose parameters are g and e^{β·B}).

Since F satisfies (H₂), we deduce the following inequalities :

$$\forall K \quad g^K(s, u, v) \leq \beta \cdot |u| (\tilde{\alpha}_s + \frac{\gamma_K}{2} |m \cdot v|^2) \leq \tilde{\alpha}_s (1 + b|u|) + \frac{\tilde{\gamma}_K}{2} |m \cdot v|^2$$

We can see that g^K satisfies (H₁) with the parameters : α = β · α̃ (and : a = |β · α̃|_{L¹(dC_s)}), b = 1 and : γ = γ̃_K. Furthermore, since the BSDEs characterized by (g^K, e^{β·B}) have a positive bounded terminal condition (equal to e^{β·B}), the remark given in section 2.1 about a priori estimates allows us to claim that we have a bound independent of K for all solutions to the BSDEs . In fact, the estimates of the norm in S[∞] of all solution U^K are independent of the parameter γ.

More precisely, we obtain that all solutions (U^K, V^K, N^K) satisfy :

$$\frac{1}{\frac{e^{b \cdot a} - 1}{b} + |e^{-\beta \cdot B}|_\infty e^{b \cdot a}} \leq U^K \leq \frac{e^{b \cdot a} - 1}{b} + |e^{\beta \cdot B}|_\infty e^{b \cdot a}$$

The lower bound (which is strictly positive) is obtained by giving a majoration of the process (U^K)' = $\frac{1}{U^K}$, and by remarking that : $\frac{1}{U^K} = e^{-\beta \cdot Y^K}$. Replacing Y by - Y, we apply the same method as before to obtain a majoration of the process (U^K)' : it satisfies again the same type of BSDE with a generator g' defined by :

$$g'(s, u, v) = \beta \cdot u \cdot \left(-F\left(s, -\frac{\ln(u)}{\beta}, -\frac{v}{\beta \cdot u}\right)\right) - \frac{1}{\beta \cdot u} (m \cdot v)' \cdot (m \cdot v)$$

Thanks to the symetry of the assumptions (H₂) on the generator F, the parameters a, b which have been introduced for the generators g^K remain unchanged .

Finally, let (U, V, N) be a solution of the equation of the form (1.2) characterized by (g, e^{β·B}) we have that the process U is non negative and bounded, and so we can set :

$$Y = \frac{\ln(U)}{\beta}, Z = \frac{V}{\beta \cdot U}, \text{ et: } dL = \frac{dN}{\beta \cdot U},$$

This provides us with a solution of the BSDE (1.1).

Step 2: Approximation

In this part, we aim at building a sequence of processes (U^n, V^n, N^n) which are solutions of BSDEs characterized by (g^n, B) and such that (U^n) is monotone. This will require the construction of a monotone sequence of Lipschitz functions (g^n) which will converge (locally uniformly) to g . Then, analogously to Kobylanski, we will establish the strong convergence of the sequences $((V^n) \cdot M)$ and (N^n) .

In the sequel, $\|\cdot\|$ is an arbitrary norm (on \mathbb{R} or \mathbb{R}^d).

To achieve this aim, we will suppose first that g has a lower bound with at most a linear growth in z (and an upper bound with a quadratic growth) this corresponds to the control given by the inequality (H'_1) .

If it is not the case, it will be enough to consider the solutions $(U^{n,p}, V^{n,p}, N^{n,p})$ of the BSDEs characterized by $(g^{n,p}, e^{\beta \cdot B})$, where $g^{n,p}$ is defined by :

$$g^{n,p}(s, u, v) = \text{ess inf}_{u', v'} (g^+(s, u', v') + n|u - u'| + n|m \cdot (v - v')|) \\ - \text{ess inf}_{u', v'} (g^-(s, u', v') + p|u - u'| + p|m \cdot (v - v')|)$$

To obtain one solution to the BSDE we are interested in, it will be necessary to proceed with two successive passages to the limit : the solution U of the BSDE characterized by $(g, e^{\beta \cdot B})$ will be equal to : $U = \lim_n \nearrow (\lim_p \searrow U^{n,p})$.

It is the same justification that holds true for these two passages to the limit, so, without any restriction, we suppose now that g satisfies the assumption given by (H'_1) .

In this case and analogously to the paper of Briand and Hu ([11]), we define the sequence (g^n) by setting:

$$g^n(s, u, v) = \text{ess inf}_{u', v'} (g(s, u', v') + n|u - u'| + n|m \cdot (v - v')|)$$

The infimum is taken over all u' and v' in $\mathbb{Q} \times \mathbb{Q}^d$, which is a countable (and dense) subset of $\mathbb{R} \times \mathbb{R}^d$: this ensures the measurability of each function g^n .

This is an increasing sequence of Lipschitz functions which converges (locally and uniformly) to g (thanks to Dini's theorem).

The Lipschitz property of the function g^n is a simple consequence of the triangular inequality and of the following result:

$$\inf_u ((\check{g})(u, u^1)) - \inf_u ((\check{g})(u, u^2)) \leq \sup_u |(\check{g})(u, u^1) - (\check{g})(u, u^2)| \quad (9)$$

Besides we have that :

$$|g(s, 0, 0)| \leq |\alpha_s| \Rightarrow (s \rightarrow g(s, 0, 0) \in L^1(dC_s)) \quad (10)$$

Under the conditions (9) and (10) on the generators, we have existence and uniqueness of solutions (U^n, V^n, N^n) to those BSDEs and the usual comparison theorems (firstly established by Pardoug and Peng in [10] and by Manuela Royer in the discontinuous case) allow us to conclude that the sequence of processes (U^n) is increasing.

Moreover, it results from simple computations (we skip here) that all functions of the sequence (g^n) satisfy the assumption (H'_1) with the same parameters C_0 , α , b , and γ as the function g .

The main interest in those estimates of the generators (g^n) is that the associated solutions $(U^n, V^n, N^n)_{n \geq \hat{C}_0}$ satisfy the same a priori estimates which have been established for all bounded solution to the BSDEs characterized by the parameters (g, B) (the generator g satisfies (H'_1)).

Using the fact that (U^n) is increasing, we set $\tilde{U} : \tilde{U} = \lim_n \nearrow (U^n)$.

Moreover, thanks to the boundedness of the martingales $V^n.M$ and N^n in their respective BMO spaces, it is possible to give limits for this sequence in the weak sense :

On the one hand, we introduce \tilde{V} as the weak limit of the sequence (V^n) bounded in $L^2(d < M > \times d\mathbb{P})$, and on the other hand,

we define \tilde{N} as the weak limit of (N^n) : this means that, for each s : \tilde{N}_s is a weak limit of (N^n_s) in $L^2(\mathbb{F}_s)$.

Defining first \tilde{N}_T as the weak limit of $(N^n_T)_n$,

a weak limit of the sequence of processes (N^n) is given by this expression :

$\forall t \tilde{N}_t = \mathbb{E}(\tilde{N}_T | \mathbb{F}_t)$.

2.3.2 Monotone stability

Proposition 2 *Considering the BSDE (1.2) and using the same notations as in the preceding section, if the sequence $(g^n)_n$ is such that:*

- s being fixed ($s \in [0, T]$), the sequence (g^n) converges locally uniformly on $\mathbb{R} \times \mathbb{R}^d$ to g .
- For all n , g^n satisfies the property given by (H'_1) , that is to say :

$$\begin{aligned} \exists \alpha = (\alpha_s) \in L^1(dC_s), C_0 > 0 \\ -\alpha_s - C_0.(|u| + |mv|) \leq g(s, u, v) \leq \alpha_s.(1 + b|u|) + \frac{\gamma}{2}(mv)'(mv) \end{aligned}$$

- the sequence is increasing.

If, besides, we have existence of solutions (U^n, V^n, N^n) to those BSDEs given by the parameters (g^n, B) , B being a bounded \mathbb{F}_T measurable random variable, then :
the triple $(\tilde{U}, \tilde{V}, \tilde{N})$ is solution of the BSDE (1.2).

Proof :

Following the same method as the one used by Kobylanski in her PhDthesis ([2]), we will in a first step begin to prove the strong convergence of the sequences $(V^n.M)_n$ and (N^n) to $\tilde{V}.M$ and \tilde{N} (this will require the a priori estimates established in the section 2.1).

We write Itô's formula for the non negative semimartingale $\Phi_K(U^m - U^p)$ ($m \geq p$), where Φ_K is given by :

$$\Phi_K(x) = \frac{e^{2K.x} - 2K.x - 1}{2.K}$$

K will be determined later.

This is a C^2 function which satisfies on the one hand: $\Phi_K \geq 0$, $\Phi_K(0) = 0$, and on the

other hand : $\Phi'_K(x) \geq 0$, if $x \geq 0$.

Taking the opposite of the integral between 0 and T and the expectation, the martingale part vanishes and it results :

$$\begin{aligned} \mathbb{E}\Phi_K(U_0^m - U_0^p) - \underbrace{\mathbb{E}\Phi_K(U_T^m - U_T^p)}_{=0} &= \\ \mathbb{E} \int_0^T (\Phi'_K(U_s^m - U_s^p))(g^m(s, U_s^m, V_s^m) - (g^p(s, U_s^p, V_s^p)))dC_s & \\ - \mathbb{E} \int_0^T \frac{\Phi''_K}{2}(U_s^m - U_s^p)(m_s \cdot (V_s^m - V_s^p))'(m_s \cdot (V_s^m - V_s^p))dC_s & \\ - \mathbb{E} \int_0^T \frac{\Phi''_K}{2}(U_s^m - U_s^p)d \langle N^m - N^p \rangle_s & \end{aligned}$$

We now aim at controlling the increments of the generators, so that we make appear the norm of the stochastic integral $(V^m - V^p) \cdot M$ before putting it in the left-hand side and passing to the liminf when $p \rightarrow \infty$.

Firstly, we claim that :

$$-g^p(s, U_s^p, V_s^p) \leq \alpha_s + C_0(|U_s^p| + |m \cdot V_s^p|) \leq \bar{\alpha}_s + \bar{C} \cdot |m \cdot V_s^p|^2$$

We write that : $\forall v \in \mathbb{R}^d$, $C_0|m \cdot v| \leq 1 + \bar{C}|m \cdot v|^2$.

Then, we have the following inequalities : those are obtained by remembering the assumptions given by (H_1) on the sequence (g^n) and by using both the convexity and of the bilinearity of $z \rightarrow (m \cdot z)' m \cdot z$:

$$\begin{aligned} g^m(s, U_s^m, V_s^m) &\leq \alpha_s \cdot (1 + b|U_s^m|_{S^\infty}) + \frac{3\cdot\gamma}{2} ((m \cdot (V_s^m - V_s^p))'(m \cdot (V_s^m - V_s^p))) \\ &\quad + \frac{3\cdot\gamma}{2} (m \cdot (V_s^p - \tilde{V}_s))'(m \cdot (V_s^p - \tilde{V}_s)) + \frac{3\cdot\gamma}{2} ((m \cdot \tilde{V}_s)'(m \cdot \tilde{V}_s)) \end{aligned}$$

and :

$$\begin{aligned} -g^p(s, U_s^p, V_s^p) & \\ \leq \bar{\alpha}_s + 2\cdot\bar{C} \cdot \left(m \cdot (V_s^p - \tilde{V}_s))'(m \cdot (V_s^p - \tilde{V}_s)) + (m \cdot \tilde{V}_s)' \cdot (m \cdot \tilde{V}_s) \right) & \end{aligned}$$

We can easily find an integrable bound (w.r.t dC_s) for the following processes : $(\alpha_s \cdot (1 + b|U_s^m|_{S^\infty}))_s$, and : $(\bar{\alpha}_s)_s$, those bounds do not depend on the integer m (or p) because we have uniform estimates of the norm in S^∞ of the processes $(U^m)_{m \geq \hat{C}}$. Putting in the left-hand side all the terms containing $(V^m - V^p)$, we obtain :

$$\begin{aligned} \mathbb{E}\Phi_K(U_0^m - U_0^p) + \mathbb{E} \int_0^T \frac{\Phi''_K}{2}(U_s^m - U_s^p)d \langle N^m - N^p \rangle_s & \\ + \mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3\cdot\gamma}{2} \cdot \Phi'_K \right) (U_s^m - U_s^p)(m \cdot (V_s^m - V_s^p))'(m \cdot (V_s^m - V_s^p)) \right) dC_s & \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{E} \int_0^T \tilde{C}^1 \Phi'_K(U_s^m - U_s^p) \left((m.(V_s^p - \tilde{V}_s))' (m.(V_s^p - \tilde{V}_s)) + (m.\tilde{V}_s)' .(m.\tilde{V}_s) \right) dC_s \\ &+ \mathbb{E} \int_0^T \left(\bar{C}_s . \Phi'_K(U_s^m - U_s^p) dC_s \right) \end{aligned}$$

where we have set :

$$\tilde{C}^1 = \frac{3.\gamma}{2} + 2\bar{C}, \text{ and } : \bar{C}_s = (\alpha_s(1 + b.\sup_m |U^m|^\infty) + \bar{\alpha}_s).$$

\bar{C} is a process (integrable w.r.t the measure dC_s),

Then, we introduce a new constant : $\tilde{C}^2 := \tilde{C}^1 + \frac{3.\gamma}{2}$.

We then fix a value for the parameter K such that :

$$\frac{\Phi''_K}{2} - \tilde{C}^2 \Phi'_K \geq \tilde{C}^2 \quad (11)$$

In the sequel, we fix : $K = \tilde{C}^2$.

This last inequality (11) entails the strict positivity of the last term of the left-hand side.

Besides, thanks to the weak convergence of (V^m) to \tilde{V} , we have :

$$\begin{aligned} \liminf_{m \rightarrow \infty} \mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3.\gamma}{2} . \Phi'_K \right) (U_s^m - U_s^p) (m.(V_s^m - V_s^p))' (m.(V_s^m - V_s^p)) \right) dC_s &\geq \quad (12) \\ \mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3.\gamma}{2} . \Phi'_K \right) (\tilde{U}_s - U_s^p) (m.(\tilde{V}_s - V_s^p))' (m.(\tilde{V}_s - V_s^p)) \right) dC_s & \end{aligned}$$

To prove this inequality, it is necessary to proceed in two steps : we can see that since the real sequence $(\mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3.\gamma}{2} . \Phi'_K \right) (U_s^m - U_s^p) (m.(V_s^m - V_s^p))' (m.(V_s^m - V_s^p)) \right) dC_s)_n$ is bounded , it implies that the lim inf is well defined.

We are going to show the inequality (12) by giving a minoration of the following quantity :

$$\liminf_{l, m \rightarrow \infty} \mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3.\gamma}{2} . \Phi'_K \right) (U_s^l - U_s^p) (m.(V_s^m - V_s^p))' (m.(V_s^m - V_s^p)) \right) dC_s \quad (13)$$

We remark, firstly, that this quantity satisfies the following inequality:

$$\begin{aligned} \forall m \geq l \quad \mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3.\gamma}{2} . \Phi'_K \right) (U_s^l - U_s^p) (m.(V_s^m - V_s^p))' (m.(V_s^m - V_s^p)) \right) dC_s &\leq \\ \mathbb{E} \int_0^T \left(\left(\frac{\Phi''_K}{2} - \frac{3.\gamma}{2} . \Phi'_K \right) (U_s^m - U_s^p) (m.(V_s^m - V_s^p))' (m.(V_s^m - V_s^p)) \right) dC_s & \end{aligned}$$

In a second step, we take the lim inf when m goes to ∞ in both sides of the last inequality.

Finally, we take the lim inf when l goes to ∞ , remarking that, since the right member is

independent of l , it gives a majoration of the quantity given by (13).

To obtain the minoration of the expression given by (13), we have to justify two successive passages to the limit.

If m, l and p are such that : $p \leq l \leq m$, and if m goes to ∞ then thanks to the monotonicity of the sequence (U^m) , we obtain :

$$\begin{aligned} \liminf_{m \rightarrow \infty} \mathbb{E} \int_0^T \left(\left(\frac{\Phi_K''}{2} - \frac{3 \cdot \gamma}{2} \cdot \Phi_K' \right) (U_s^l - U_s^p) (m \cdot (V_s^m - V_s^p))' (m \cdot (V_s^m - V_s^p)) \right) dC_s \\ \geq \mathbb{E} \int_0^T \left(\left(\frac{\Phi_K''}{2} - \frac{3 \cdot \gamma}{2} \cdot \Phi_K' \right) (U_s^l - U_s^p) (m \cdot (\tilde{V}_s - V_s^p))' (m \cdot (\tilde{V}_s - V_s^p)) \right) dC_s \end{aligned}$$

We then pass to the limit when l goes to ∞ (p is always fixed).

This entails that the expression given by (13) is greater than :

$$\mathbb{E} \int_0^T \left(\left(\frac{\Phi_K''}{2} - \frac{3 \cdot \gamma}{2} \cdot \Phi_K' \right) (\tilde{U}_s - U_s^p) (m \cdot (\tilde{V}_s - V_s^p))' (m \cdot (\tilde{V}_s - V_s^p)) \right) dC_s$$

This implies that the inequality (12) is satisfied .

The passage to the limit when $m \rightarrow \infty$ in the right-hand side results from an application of Lebesgue's theorem.

On the one hand, we have almost sure convergence of (U^m) to the process \tilde{U} .

On the other hand, the following quantity :

$\Phi_K'(U_s^m - U_s^p) (|m \cdot (\tilde{V}_s - V_s^p)|^2 + |m \cdot \tilde{V}_s|^2 + \tilde{C}_s)$ is integrable with respect to dC_s , because it is the product of a bounded process and a sum of integrable processes .

Rewriting Itô's formula after passing to the limit when m goes to ∞ , it gives us :

$$\begin{aligned} \mathbb{E} \Phi_K(\tilde{U}_0 - U_0^p) + \mathbb{E} \int_0^T \frac{\Phi_K''}{2} (\tilde{U}_s - U_s^p) d < \tilde{N} - N^p >_s \\ + \mathbb{E} \int_0^T \left(\left(\frac{\Phi_K''}{2} - \frac{3 \cdot \gamma}{2} \cdot \Phi_K' \right) (\tilde{U}_s - U_s^p) (m \cdot (\tilde{V}_s - V_s^p))' (m \cdot (\tilde{V}_s - V_s^p)) \right) dC_s \\ \leq \mathbb{E} \int_0^T \left(\tilde{C} \Phi_K'(\tilde{U}_s - U_s^p) (m \cdot (\tilde{V}_s - V_s^p))' (m \cdot (\tilde{V}_s - V_s^p)) \right) dC_s \\ + \mathbb{E} \int_0^T \left(\tilde{C} \Phi_K'(\tilde{U}_s - U_s^p) (m \cdot \tilde{V}_s)' \cdot (m \cdot \tilde{V}_s) \right) dC_s + \mathbb{E} \int_0^T \left(\tilde{C}_s \cdot \Phi_K'(\tilde{U}_s - U_s^p) \right) dC_s \end{aligned}$$

To justify the passage to the limit when $p \rightarrow \infty$, it remains to put in the left-hand side the terms containing the following quantity $(m \cdot (\tilde{V} - V^p))' (m \cdot (\tilde{V} - V^p))$.

Using both the condition (11) and the inequality : $\frac{\Phi_K''}{2} (\tilde{U}_s - U_s^p) \geq \frac{1}{2}$ (satisfied \mathbb{P} -almost surely and for all s thanks to the strict positivity of $(\tilde{U} - U^p)$), we can conclude that the limit of the left-hand side exists and is strictly larger than :

$$\mathbb{E} \left(\int_0^T \tilde{C} \cdot (m \cdot (\tilde{V}_s - V_s^p))' (m \cdot (\tilde{V}_s - V_s^p)) dC_s + \frac{1}{2} < \tilde{N}_s - N_s^p >_T \right)$$

Concerning the right-hand side, we use the two following results to justify the application of Lebesgue's theorem :

- The sequence (U^p) converges almost surely to \tilde{U} and,

- The condition of domination results from the fact that \tilde{V} is square integrable with respect to the measure $d\langle M \rangle \times d\mathbb{P}$ and that the quantity $\Phi'_K(\tilde{U} - U^p)$ is uniformly bounded, the limit exists and is equal to zero.

Finally we can conclude :

$$\liminf_p \mathbb{E} \left(\int_0^T \hat{C}_U \cdot (\tilde{V}_s - V_s^p)' (m \cdot (\tilde{V}_s - V_s^p)) dC_s + \frac{1}{2} \langle \tilde{N}_s - N_s^p \rangle_T \right) \leq 0$$

This can be expressed under the form :

$V^p.M$ converges to $\tilde{V}.M$ in $L^2(d\langle M \rangle \times d\mathbb{P})$.

N^p converges to \tilde{N} in $\mathcal{M}^2([0, T])$.

2.3.3 Conclusion

To achieve the proof of existence, it remains to justify the passage to the limit in the equation :

$$U_t^n = U_T^n + \int_t^T g^n(s, U_s^n, V_s^n) dC_s + \int_t^T V_s^n \cdot dM_s + N_T^n - N_t^n$$

It is necessary to check the following assertions :

- (i) $\int_0^t V_s^m dM_s \xrightarrow{m \rightarrow \infty} \int_0^t \tilde{V}_s dM_s$ \mathbb{P} -almost surely and for all t .
- (ii) $N_t^n \xrightarrow{n \rightarrow \infty} \tilde{N}$ (\mathbb{P} -almost surely and for all t).
- (iii) $\int_0^t g^n(s, U_s^n, V_s^n) dC_s \xrightarrow{m \rightarrow \infty} \int_0^t g(s, \tilde{U}_s, \tilde{V}_s) dC_s$

Since we have proven the convergence in L^2 of the sequences $(V^n.M)$ and (N^n) we can suppose that the convergence is achieved \mathbb{P} -almost surely (taking a subsequence if necessary).

Concerning the assertion (iii), we can apply Lebesgue's theorem which requires to check the following two conditions:

On the one hand, we have almost sure convergence of $(g^n(s, U_s^n, V_s^n))$ to $g(s, \tilde{U}_s, \tilde{V}_s)$ with respect to the measure $d\langle M \rangle_s \times d\mathbb{P}$.

On the other hand, we have domination of $(g^n(s, U_s^n, V_s^n))_n$ by a quantity which is in the space $L^1(d\langle M \rangle \times d\mathbb{P})$ and this domination holds true uniformly in n : this results from the quadratic control of the sequence (g^n) and from the fact that $\sup_n (V^n)$

is in the space $L^2(d\langle M \rangle \times d\mathbb{P})$.

The triple of processes $(\tilde{U}, \tilde{V}, \tilde{N})$ is a solution of (1.2).

By setting : $Y = \frac{\ln(\tilde{U})}{\beta}$, $Z = \frac{\tilde{V}}{\beta \cdot \tilde{U}}$, and the martingale measure $dL = \frac{d\tilde{N}}{\beta \cdot \tilde{U}}$, we obtain a solution to the BSDE (1.1). \square

3 Applications to finance :

3.1 The case of the exponential utility :

One important interest of this theoretical work is the link between the solution of BSDE with quadratic growth and some problems arising from Mathematical Finance : we will in this section focus our attention on one particular problem dealing with the notion of utility value process as regards to the exponential utility of a portfolio.

We begin here by summing up the main assumptions about modelization in our case of a general continuous filtration (we refer here to Mania et Schweizer in [8]) .

As before, we are given a probability space and a continuous filtration \mathbb{F} .

We set : $S = (S_t^i)$ the semi martingale which takes its values in \mathbb{R}^d and which modelizes the discounted prices of d risky assets. Its evolution S is given by the following equation :

$$\frac{dS_s}{S_s} = dM_s + dA_s \text{ with : } dA = d \langle M \rangle \cdot \lambda$$

We recall here that M is a local martingale of the filtration whose quadratic variation $d \langle M \rangle$ can be written as explained in section 1.2 under the following form :

$$d \langle M \rangle_s = m \cdot m' dC_s.$$

A is a process with bounded variation : we will suppose in the sequel that λ is a \mathbb{R}^d -valued process which is besides almost surely bounded and which satisfies :

$$\mathbb{E} \left(\int_0^T |m \cdot \lambda|^2 dC_s \right) < \infty \quad (H_3)$$

Furthermore, this expression for S provides a justification for the no arbitrage condition : this condition in our non Brownian setting is justified in the recent article [1].

We define then the notion of a portfolio associated to a strategy ν :

Définition 1 A \mathbb{R}^d -valued process ν which is predictable with respect to the filtration \mathbb{F} is called trading strategy if the following stochastic integral : $\int \nu' \frac{dS}{S} = \int \sum_i \frac{\nu^i}{S^i} dS^i$ is well defined .

Each component ν_i of the trading strategy corresponds to the amount of money invested in the i^{th} asset.

The process X^ν given below is called wealth process of an agent having the strategy ν and x represents the initial wealth :

$$\forall t \in [0, T], \quad X_t^\nu = x + \int_0^t \nu'_s \frac{dS_s}{S_s} \quad (14)$$

We suppose besides that we have an incomplete financial market, that is to say that all contingent claims (i.e square integrable variables with respect to \mathbb{F}_T) are not attainable. A contingent claim B is attainable provided it exists a strategy ν (and also a process X^ν) such that : $B = X_T^\nu$, where T represents the horizon (or maturity time) which will be a deterministic time in this work.

We then introduce the utility value process at time t $V_t^B(x_t)$: it is a \mathbb{F}_t random variable defined by :

$$V_t^B(x_t) = \text{esssup}_{\nu \in C} \mathbb{E}(U_\alpha(X_T^\nu - B) | \mathbb{F}_t)$$

U_α whose expression is given by : $U_\alpha(x) = -exp(-\alpha.x)$, stands for the utility function, and the set C represents the set of constraints : it is a set where all strategies take their values (i.e for all t and \mathbb{P} almost surely, $\nu_t(\omega)$ is in C , $C \subset \mathbb{R}^d$). We impose besides that the strategy defined by : $\nu \equiv 0$, is in C .

Then, before explaining how to solve this problem, let us introduce the notion of admissible strategy in our context :

Définition 2 *Let C be a closed (and non necessarily convex set) in $\mathbb{R}^{1 \times d}$. The set \mathcal{A} of admissible strategies consists of all d -dimensional predictable processes $\nu = (\nu_t)_{t \in [0, T]}$ satisfying : $\mathbb{E}(\int_0^T |m.\nu|^2 dC_s) < \infty$, as well as the uniform integrability of the family:*

$$\{exp(-\alpha.X_\tau^\nu), \tau \text{ stopping time with values in } [0, T]\}$$

We aim here at giving an expression of the value process by a dynamical method : this approach will require the theoretical study we have made on quadratic BSDEs. The method applied here is the same as the one used by Imkeller, Hu and Muller in their paper(see [6])

To achieve this, we introduce for all strategies ν the process R^ν by setting :

$$\forall s, R_s^\nu = U_\alpha(X_s^\nu - Y_s).$$

We search to construct Y such that R^ν satisfies :

- (i) $R_T^\nu = -exp(-\alpha(X_T^\nu - F))$, for all strategies ν .
- (ii) $R_t^\nu = R_t = U_\alpha(x_t - Y_t)$ (where x_t is a fix \mathbb{F}_t random variable and such that : for all strategy ν , $X_t^\nu = x_t$).
- (iii) R^ν is a supermartingale for all strategies and there exists ν^* such that R^{ν^*} is a martingale.

Besides, this method allows us to give a positive answer to the existence of an optimal strategy .

3.2 A dynamical way to solve the problem :

In this section, we will show that the process $Y = (Y_t)$ is solution of a BSDE with quadratic growth of the same type as (1.1), whose parameters are β and the generator : $F = F(s, z)$:

$$\begin{cases} dY_s = -F(s, Z_s)dC_s - \frac{\beta}{2}.d \langle L \rangle_s \\ \quad + Z_s \cdot dM_s + dL_s \\ Y_T = B \end{cases}$$

Proposition 3 *Keeping the same notations, the process Y (such that the family of processes (R^ν) satisfies the assumptions (i), (ii) and (iii) introduced when describing the problem in the exponential case) is solution of a BSDE with quadratic growth of the form (1.1) with the following parameters : β is given by : $\beta = \alpha$ (corresponding to the risk*

aversion parameter), and the expression of F is :

$$F(s, Z_s) = \text{ess inf}_{\nu \in C} \left(\frac{\alpha}{2} |m(\nu_s - (Z_s + \frac{\lambda_s}{\alpha}))|^2 \right) - (m.Z_s)'(m.\lambda_s) - \frac{1}{2.\alpha} (m.\lambda_s)'(m.\lambda_s) \quad (15)$$

The expression of the value process at time t is a consequence of the dynamical principle and it is given by :

$$V_t^B(x_t) = \text{esssup}_{\nu} \mathbb{E}(U_{\alpha}(X_T^{\nu} - B) | \mathbb{F}_t) = U_{\alpha}(x_t - Y_t)$$

Besides, there exists an optimal strategy ν^* which satisfies :

$$\nu^* = \underset{\nu \in C}{\text{argmin}} |m.(\nu - (Z_s + \frac{\lambda_s}{\alpha}))|^2$$

Before justifying how we can find the expression of the generator, we can remark that the expression of F given in Proposition 3 satisfies the assumption (H_1) with : $b \equiv 0$ (F is independent of y).

In fact, we have the following control on F :

$$F(s, Z_s) \geq - (m.Z_s)'(m.\lambda_s) - \frac{1}{2.\alpha} (m.\lambda_s)'(m.\lambda_s) \geq - |(m.Z_s)'(m.\lambda_s)| - \frac{1}{2.\alpha} (m.\lambda_s)'(m.\lambda_s)$$

Finally, we obtain :

$$F(s, Z_s) \geq - \left(\frac{\alpha}{2} |m.Z_s|^2 + \frac{1}{\alpha} |m.\lambda_s|^2 \right) \quad (16)$$

Then, if we set : $\forall s, \alpha_s = \frac{1}{\alpha} |m.\lambda_s|^2$, we can claim : α is in $L^1(dC_s)$, thanks to the assumption (H_3) made on the process λ .

The majoration is obtained by remarking that the essential infimum is lower than the value of the expression in : $\nu \equiv 0$, this entails that :

$$F(s, Z_s) \leq \frac{\alpha}{2} |m.Z_s|^2$$

According to the preceding section, this entails that we have existence and uniqueness for the BSDE characterized by its terminal condition B and its generator F .

Proof of Proposition 3 To find the expression of the parameters β and F , we apply Itô's formula to the semimartingale $R^{\nu} = U_{\alpha}(X^{\nu} - Y)$, which is justified since U_{α} is a C^2 function :

$$\begin{aligned} dU_{\alpha}(X_s^{\nu} - Y_s) &= -\alpha.U_{\alpha}(X_s^{\nu} - Y_s)d(X^{\nu} - Y)_s + \frac{\alpha^2}{2}U_{\alpha}(X_s^{\nu} - Y_s)d \langle X^{\nu} - Y \rangle_s \\ &= -\alpha.U_{\alpha}(X_s^{\nu} - Y_s)(\nu' .dM_s - Z' dM - dL + \nu' d \langle M \rangle_s \lambda_s) \\ &\quad - \alpha.U_{\alpha}(X_s^{\nu} - Y_s)(F(s, Z_s)dC_s + \frac{\beta_s}{2}d \langle L \rangle_s) \\ &\quad + \frac{\alpha^2}{2}U_{\alpha}(X_s^{\nu} - Y_s) \left((\nu - Z)' d \langle M \rangle_s (\nu - Z) + d \langle L \rangle_s \right) \end{aligned}$$

We remark that the process R^ν satisfies the following equation :

$$dR_s^\nu = R_s^\nu d\tilde{M}_s^\nu + R_s^\nu dA_s^\nu$$

where A^ν is a process with finite variation and $\tilde{M}^\nu = -\alpha \cdot ((\nu - Z)') \cdot M - L$ is a local martingale thanks to the fact that $\nu \in \mathcal{A}$.

We deduce then that R^ν has the following expression :

$$R_t^\nu = R_0^\nu \cdot \mathcal{E}(\tilde{M}^\nu) \exp(A_t^\nu).$$

As a consequence, a necessary condition for the process R to be a supermartingale for all ν is :

$$\forall s \quad R_s^\nu dA_s^\nu \leq 0 \quad \mathbb{P} - \text{almost surely} \quad (C_1)$$

To prove that (C_1) is sufficient with the assumptions, we can see that there exists a sequence of stopping times $(\tau_n)_n$ such that the process $R_{t \wedge \tau_n}^\nu = R_0^\nu \cdot \mathcal{E}(\tilde{M}_{t \wedge \tau_n}^\nu) \exp(A_{t \wedge \tau_n}^\nu)$ is a supermartingale for each n .

(this results from the fact that, for all ν , \tilde{M}^ν is a local martingale and that e^{A^ν} is a decreasing process). As a consequence for each $A \in \mathbb{F}_s$, we have :

$$\mathbb{E}(R_{t \wedge \tau_n}^\nu \mathbf{1}_A) \leq \mathbb{E}(R_{s \wedge \tau_n}^\nu \cdot \mathbf{1}_A)$$

Thanks to the assumption of uniform integrability given in the definition of admissibility and the boundedness of the process Y (provided we have a generator satisfying the good assumptions), we claim that $(R_{s \wedge \tau_n}^\nu)$ is uniformly integrable and passing to the limit when n goes to ∞ , it implies :

$$\forall A \in \mathbb{F}_s, \quad \mathbb{E}(R_t^\nu \mathbf{1}_A) \leq \mathbb{E}(R_s^\nu \cdot \mathbf{1}_A)$$

Finally, since the process R^ν is non positive and thanks to the preceding computations, we obtain that the condition (C_1) is equivalent to :

$$\begin{cases} -\alpha \cdot \frac{\beta_s}{2} d \langle L \rangle_s + \frac{\alpha^2}{2} d \langle L \rangle_s = 0 \Rightarrow \beta = \alpha \\ -\alpha \cdot (F(s, Z_s) + \nu'_s m'_s m_s \lambda_s) + \frac{\alpha^2}{2} (\nu_s - Z_s)' m'_s m_s (\nu_s - Z_s) \geq 0 \end{cases}$$

where we replace $d \langle M \rangle$ by its following expression : $d \langle M \rangle_s = m'_s m_s dC_s$.
So we obtain as expression for F :

$$F(s, Z_s) = \text{ess inf}_{\nu \in C} \left(\frac{\alpha}{2} (\nu_s - Z_s)' m'_s m_s (\nu_s - Z_s) + \nu'_s m'_s m_s \lambda_s \right)$$

or equivalently after simple computations we obtain the expression given by (15).

3.3 Power and logarithmic utilities

Similarly as Imkeller and Hu in their article [6], we introduce two other utility functions and we study for each the corresponding utility maximization problem : in our special case of a general (and, in particular, non Brownian) continuous filtration, we will use the same dynamical method as in the exponential case by introducing a family of random processes and give an expression of both the value function and the optimal strategy.

The main difference with the exponential case is that for those type of utility functions

we have to impose furthermore that the wealth process is non negative.
Denoting by U the utility function, the problem we are interested in is to compute $V(x_t)$ defined as in [8] :

$$V(x_t) = \underset{\rho}{\text{esssup}} \mathbb{E}(U_\gamma(X_T^\rho) | \mathbb{F}_t)$$

where x_t is a fixed \mathbb{F}_t -measurable random variable such that for all strategies $\rho : X_t^\rho = x_t$.

In the two cases we will study here, we define another notion of strategy by introducing a d-dimensionnal process ρ which denotes the part of the wealth invested in stock i . This will imply we will obtain a very specific expression for the wealth process. Keeping here the same conventions for the price process S as in Section 3.1, we define X^ρ by setting :

$$X_t^\rho = x + \int_0^t X_s^\rho \rho'_s \cdot \frac{dS_s}{S_s} = x + \int_0^t X_s^\rho \rho'_s \cdot dM_s + \int_0^t X_s^\rho \cdot \rho'_s \cdot d \langle M \rangle_s \lambda_s$$

The link between the process π defined in the exponential case and ρ is given by : $\pi = X \cdot \rho$ (where X is the wealth process).

A strategy ρ is said to be admissible when it satisfies : the martingale $\rho' \cdot M$ is a BMO martingale. This implies that the exponential of this martingale (usually denoted by : $\mathcal{E}(\rho' \cdot M)$) is uniformly integrable, and also, that it is a true martingale. In particular, we have : $\mathbb{E} \int_0^T |m \cdot \rho_s|^2 dC_s < \infty$.

Finally, the wealth process has the following expression:

$$\begin{aligned} X_t^\rho &= x \cdot \exp \left(\int_0^t \rho'_s \cdot dM_s - \frac{1}{2} \int_0^t \rho'_s \cdot d \langle M \rangle_s \rho_s + \int_0^t \rho'_s \cdot d \langle M \rangle_s \lambda_s \right) \\ &= x \cdot \mathcal{E}(\rho' \cdot M) \exp \left(\int_0^t \rho'_s \cdot d \langle M \rangle_s \lambda_s \right) \end{aligned}$$

In the sequel, we will study the two following utility functions :

1-The first one is the power utility :

For all real $\gamma \in]0, 1[$, we will consider : $U_\gamma(x) = \frac{1}{\gamma} \cdot x^\gamma$.

We will fix γ and set : $U_\gamma = U^1$.

2-The second one is the logarithmic utility whose definition is : $U^2(x) = \ln(x)$.

We state hereafter the main results we are able to obtain :

Theorem 3 *Keeping the introduced notation, and denoting respectively by V^1 (resp. V^2) the value function of the utility maximization problem related to U^1 (resp U^2) we obtain the following results. On the one hand, by setting V^1 under the form :*

$$V^1(x_t) = x_t^\gamma \cdot \exp(Y_t)$$

we have that Y is defined as the unique solution (Y, Z) of a BSDE of the following form :

$$Y_t = 0 - \int_t^T f^1(s, Z_s) dC_s + \int_t^T \frac{1}{2} d \langle L \rangle_s - \int_t^T Z'_s dM_s - \int_t^T dL_s$$

L is a \mathbb{R} -valued martingale strongly orthogonal to M , and f^1 is given by :

$$f^1(s, Z_s) = \operatorname{ess\,inf}_\rho \frac{\gamma \cdot (1 - \gamma)}{2} \left(|m \cdot (\rho_s - (\frac{Z_s + \lambda_s}{1 - \gamma}))|^2 \right) - \frac{\gamma \cdot (1 - \gamma)}{2} |m \cdot (\frac{Z_s + \lambda_s}{1 - \gamma})|^2 - \frac{1}{2} |m \cdot Z_s|^2 \quad (17)$$

The optimal strategy ρ_1^* is defined by : $\rho_1^* = \operatorname{arg\,min}_\rho |m \cdot (\rho_s - (\frac{Z_s + \lambda_s}{1 - \gamma}))|^2$
On the other hand, if the expression of V^2 is given by : $V^2(x_t) = \ln(x_t) + Y_t$.

then Y is again defined as the solution of the following BSDE (with the terminal condition : $Y_T \equiv 0$):

$$Y_t = 0 - \int_t^T f^2(s, Z_s) dC_s - \int_t^T Z'_s dM_s - \int_t^T dL_s$$

, and f^2 is given by :

$$f^2(s, z) = \operatorname{ess\,inf}_\rho \frac{1}{2} |m \cdot (\rho_s - \lambda_s)|^2 - \frac{1}{2} |m \cdot \lambda_s|^2 \quad (18)$$

The optimal strategy ρ_2^* is defined by : $\rho_2^* = \operatorname{arg\,min}_\rho |m \cdot (\rho_s - \lambda_s)|^2$

Proof of the theorem :

Analogously to the exponential case, we can show that the two generators (f^1 and f^2) satisfy whether the estimates given by (H_1) or (H'_1) :

On the one hand, the generator f^1 satisfies :

$$-\frac{\gamma \cdot (1 - \gamma)}{2} |m \cdot (\frac{z + \lambda_s}{1 - \gamma})|^2 - \frac{1}{2} |m \cdot z|^2 \leq f^1(s, z) \leq -\frac{1}{2} |m \cdot z|^2$$

The majoration is obtained by taking the value when : $\rho \equiv 0$. To obtain the lower bound, we use the fact that the quantity for which we take the essential infimum is positive. It results from easy computations that : $f^1(s, z) \geq -\bar{\gamma} \cdot |m \cdot z|^2 - \bar{\alpha}_s$, $\bar{\alpha} \in L^1(dC_s)$.

So, it implies that the generator f^1 satisfies the assumption (H_2) (and, a fortiori H_1)

On the other hand, we claim that : $-\frac{|m \cdot \lambda|^2}{2} \leq f^2(s) \leq 0$

The lower bound is integrable w.r.t dC_s thanks to the assumption (H_3) on λ , so f^2 satisfies (H_1) .

1. Power utility :

As before, we search to construct a process R^ρ which is to define such that for all ρ it is a supermartingale. According to the form of X , we set :

$$R^\rho = \gamma \cdot U(X^\rho) \cdot \exp(Y) = (X^\rho)^\gamma \cdot e^Y.$$

As in the first part of this section, we apply Itô's formula to the process R^ρ using the fact

that it is a product of two semimartingales, it gives us :

$$dR_s^\rho = d(U(X_s^\rho))e^{Y_s} + U(X_s^\rho)de^{Y_s} + d\langle U(X^\rho), e^Y \rangle_s$$

Moreover we have that, on the one hand :

$$\begin{aligned} d(U(X_s^\rho)) &= (X_s^\rho)^{\gamma-1} \cdot X_s^\rho (\rho_s' dM_s + \rho_s' d\langle M \rangle_s \cdot \lambda_s) \\ &\quad + (\gamma-1)(X_s^\rho)^{\gamma-2} (X_s^\rho)^2 \cdot \frac{1}{2} \rho_s' d\langle M \rangle_s \rho_s \\ &= (X_s^\rho)^\gamma \left(\rho_s' dM_s - \rho_s' d\langle M \rangle_s \cdot \lambda_s + \frac{1}{2}(\gamma-1)\rho_s' d\langle M \rangle_s \rho_s \right) \end{aligned}$$

and on the other hand :

$$\begin{aligned} d(e^{Y_s}) &= e^{Y_s} dY_s + \frac{1}{2} e^{Y_s} d\langle Y \rangle_s \\ &= e^{Y_s} (f^1(s, Z_s) dC_s) - \frac{1}{2} d\langle L \rangle_s + \frac{1}{2} (d\langle L \rangle_s + Z_s' d\langle M \rangle_s Z_s) \\ &\quad + e^{Y_s} (Z_s' dM_s + dL_s) \end{aligned}$$

The quadratic variation process $d\langle U(X^\rho), e^Y \rangle_s$ is obtained by taking the quadratic variation of the martingale part of each semimartingale, this gives us :

$$d\langle U(X^\rho), e^Y \rangle_s = U(X_s^\rho) \cdot e^{Y_s} \gamma \cdot \rho_s d\langle M \rangle_s Z_s.$$

Collecting all those results and replacing $(X^\rho)^\gamma$ by $\gamma \cdot U(X^\rho)$ and $d\langle M \rangle$ by $m' m dC_s$, we obtain the following equation for the process R^ρ :

$$\begin{aligned} dR_s^\rho &= R_s^\rho (\gamma \cdot \rho_s' dM_s + Z_s dM_s) \\ &\quad + R_s^\rho \left(\gamma \cdot (m \cdot \rho_s)' (m \cdot \lambda_s) dC_s + \frac{1}{2} \gamma \cdot (\gamma-1) (m \cdot \rho_s)' (m \cdot \rho_s) dC_s + \gamma \cdot (m \cdot \rho_s)' (m \cdot Z_s) dC_s \right. \\ &\quad \left. + \frac{1}{2} (m \cdot Z_s)' \cdot (m \cdot Z_s) dC_s + f^1(s, Z_s) \right) dC_s \end{aligned}$$

This equation can be rewritten under the form :

$$dR_s^\rho = R_s^\rho d\tilde{M}_s^\rho + R_s^\rho dA_s^\rho \quad (**)$$

, where \tilde{M}^ρ is a uniformly integrable martingale : $d\tilde{M}^\rho = \mathcal{E}(\gamma \cdot \rho + Z)' \cdot dM$, and A^ρ is a process with finite variation .

This condition of uniform integrability is satisfied provided : ρ is admissible (in the sense defined at the beginning of this section) , and provided that $Z' \cdot M$ is a BMO martingale, this will be an immediate consequence of the expression of the generator f^1 of the BSDE. Remembering that the quadratic variation process $d\langle M \rangle_s$ can be written under the form : $m' \cdot m dC_s$ (m is a process taking its values in $\mathbb{R}^{d \times d}$), the expression of dA^ρ is the following :

$$dA_s^\rho = \left(\gamma \cdot (m \cdot \rho_s)' (m \cdot (\lambda_s + Z_s)) + \frac{\gamma \cdot (\gamma-1)}{2} |m \cdot \rho_s|^2 + \frac{1}{2} |m \cdot Z_s|^2 + f^1(s, Z_s) \right) dC_s$$

Thanks to the expression of the process R^ρ given by (**), we can claim that the supermartingale property is equivalent to : $\forall \rho \ dA_s^\rho \leq 0$.

After simple computations, we obtain as expression for f^1 the formula given by (17).

2. Logarithmic utility :

Using the same notations as in the case of the power utility (in particular for the definition of a strategy ρ and for the corresponding process X^ρ), we obtain the following expression for the process $U(X^\rho) = \ln(X^\rho)$:

$$U(X_s^\rho) = \ln(x) + \int_0^t \rho_s dM_s - \int_0^t \frac{1}{2} \cdot \rho_s' d \langle M \rangle_s \rho_s + \int_0^t \rho_s d \langle M \rangle_s \cdot \lambda_s$$

As usual, we search to construct a family of processes R^ρ verifying the supermartingale property, and, in this special case, we define the process by the following expression : $R_s^\rho = \ln(X_s^\rho) + Y_s$.

where Y satisfies a BSDE completely similar as in the case of power utility.

In this case, following the same method, we obtain as expression for the generator f^2 of the BSDE satisfied by the process Y the one given by (18).

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