

Renormalisation of noncommutative ϕ^4 -theory by multi-scale analysis

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Abstract

In this paper we give a much more efficient proof that the real Euclidean ϕ^4 -model on the four-dimensional Moyal plane is renormalisable to all orders. We prove rigorous bounds on the propagator which complete the previous renormalisation proof based on renormalisation group equations for non-local matrix models. On the other hand, our bounds permit a powerful multi-scale analysis of the resulting ribbon graphs. Here, the dual graphs play a particular rôle because the angular momentum conservation is conveniently represented in the dual picture. Choosing a spanning tree in the dual graph according to the scale attribution, we prove that the summation over the loop angular momenta can be performed at no cost so that the power-counting is reduced to the balance of the number of propagators versus the number of completely inner vertices in subgraphs of the dual graph.

1 Introduction

Field theories on noncommutative spaces became very popular after the discovery that they arise in limiting cases of string theory [1, 2, 3]. Although from string theory's point of view there is no reason that the limit is a well-defined quantum field theory, there has been an enormous activity aiming at renormalisation proofs for noncommutative quantum field theories. Most of the attempts focused at the Moyal plane with the associative and noncommutative product

$$(a \star b)(x) = \int \frac{d^4 k}{(2\pi)^4} \int d^4 y a(x + \frac{1}{2}\theta \cdot k) b(x+y) e^{ik \cdot y} . \quad (1.1)$$

It turned out that the noncommutative analogs of typical field theoretical (in particular four-dimensional) models on the Moyal plane are not renormalisable due to the UV/IR-mixing problem [4]. The construction of dangerous non-planar graphs was made precise in [5] where the problem was traced back to divergences in some of the Hepp sectors which correspond to disconnected ribbon subgraphs wrapping the same handle of a Riemann surface.

Recently, the renormalisation of the noncommutative ϕ_4^4 -model was achieved [6] within a Wilson-Polchinski renormalisation scheme [7, 8] adapted to non-local matrix models [9]. The renormalisable model is defined by the action functional

$$S[\phi] = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{1}{2} \mu_0^2 \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)(x), \quad (1.2)$$

where $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu} x^\nu$ and the Euclidean metric is used.

At first sight, the appearance of the translation invariance breaking harmonic oscillator potential for the ϕ^4 -action (1.2) might appear strange. However, the renormalisation proof shows that there is a marginal interaction which corresponds to that term and as such requires its inclusion in the initial action. Moreover, thanks to the oscillator potential, the action (1.2) transforms covariantly under the Langmann-Szabo duality [10] which exchanges position space and momentum space.

We review the main ideas of the renormalisation proof, in particular the analysis of ribbon graphs, in Section 2. However, it must be underlined that the proof given in [6] relies on a numerical determination of the asymptotic scaling dimensions of the propagator. Our paper fills this gap by computing rigorous bounds on the propagator, at least for large enough Ω . This will be done in Section 3.

On the other hand, our bounds permit another renormalisation strategy which turns out to be much more efficient. See Section 4. The strategy is inspired by constructive methods [11]. The key is a scale decomposition of the propagator and an estimation procedure of the ribbon graphs which takes into account the scale attribution. The proof is carried out for the duals of the ribbon graphs, because the set of independent variables is particularly transparent in dual graphs.

The methods developed in this paper will be crucial to write a constructive version of [9, 6]. Actually the main obstacle to the construction of the usual ϕ^4 model is the non-asymptotic freedom of the theory. For the four-dimensional Moyal plane, the parameter Ω controls the UV/IR mixing. When it reaches 1, the entanglement is maximum, and the β function vanishes [12]. In this view, the Ω -region close to 1, for which we prove analytical estimates, is particularly important.

2 Main ideas of the previous renormalisation proof

In order to make this paper self-contained, we review the main ideas of the renormalisation proof given in [6] for the quantum field theory associated with the action (1.2).

In order to avoid the oscillating phase factors of the \star -product in momentum space, the first step is to pass to the matrix base of the Moyal plane, where the action (1.2)

becomes^a

$$S[\phi] = 4\pi^2\theta_1\theta_2 \sum_{m,n,k,l \in \mathbb{N}^2} \left(\frac{1}{2} \Delta_{m,n;k,l} \phi_{mn} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right). \quad (2.1)$$

As usual, we define the quantum field theory by the partition function, which is expanded into Feynman graphs. As the fields are described by matrices ϕ_{mn} , the resulting Feynman graphs are ribbon graphs build of propagators and vertices,

$$\begin{array}{c} \xrightarrow{n} \\ \xleftarrow{m} \\ \xrightarrow{k} \\ \xleftarrow{l} \end{array} = G_{m,n;k,l}, \quad \begin{array}{c} \nearrow n_4 \\ \nwarrow m_4 \\ \nearrow n_3 \\ \nwarrow m_3 \\ \nearrow n_1 \\ \nwarrow m_1 \\ \nearrow n_2 \\ \nwarrow m_2 \end{array} = \delta_{n_1 m_2} \delta_{n_2 m_3} \delta_{n_3 m_4} \delta_{n_4 m_1}. \quad (2.2)$$

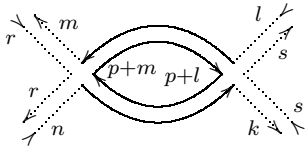
The propagator $G_{mn;kl}$ is the inverse of the kinetic matrix $\Delta_{mn;kl}$ in (2.1). We recall the explicit formula in (3.1) and (3.2). Due to the $SO(2) \times SO(2)$ -symmetry of the action, $G_{mn;kl} \neq 0$ only if $m + k = n + l$. Matrix indices which are not determined by this index conservation or as external indices of the graph are summation indices. The corresponding index summation is possibly divergent and requires a regularisation.

In [9, 6] the regularisation consists in a smooth cut-off of the propagator indices as a function of a renormalisation scale Λ ,

$$Q_{\substack{m^1, n^1, k^1, l^1 \\ m^2, n^2, k^2, l^2}}(\Lambda) = \Lambda \frac{\partial}{\partial \Lambda} \left(\prod_{i \in \{m^1, m^2, \dots, l^1, l^2\}} \chi\left(\frac{i}{\theta \Lambda^2}\right) G_{\substack{m^1, n^1, k^1, l^1 \\ m^2, n^2, k^2, l^2}} \right), \quad (2.3)$$

where $\chi(x)$ is smooth with $\chi(x) = 1$ for $x \leq 1$ and $\chi(x) = 0$ for $x \geq 2$. This implies $Q_{mn;kl}(\Lambda) \neq 0$ only if $\max(m^1, m^2, \dots, l^1, l^2) \in [\theta \Lambda^2, 2\theta \Lambda^2]$. The graph is then realised by the differentiated cut-off propagators $Q(\Lambda_i)$ which regulate the index summations. At the end, the nested integral over $\frac{d\Lambda_i}{\Lambda_i}$ is performed within an interval characterised by mixed boundary conditions [8]. Actually, the graphs are build recursively by adding a new propagator. This allows an inductive proof of the power-counting behaviour. On the other hand, one has to carefully discuss the location of the valence of the graph where one attaches a leg of the additional propagator. This discussion alone extends over 20 pages in [9].

It is time for an example. We consider the (planar) one-loop four-point graph



$$\begin{aligned} &= \left\{ \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda_2}{\Lambda_2} \int_{\Lambda_2}^{\Lambda_0} \frac{d\Lambda_1}{\Lambda_1} \sum_p Q_{m,p+m;p+l,l}(\Lambda_2) Q_{n,p+m;p+l,k}(\Lambda_1) \right\} + \{\Lambda_1 \leftrightarrow \Lambda_2\} \\ &+ A_{rm;ls;sk;nr}(\Lambda_R). \end{aligned} \quad (2.4)$$

^aNote that the symbols Δ and G used in this paper for the (generalised) Laplacian and the Green's function are transposed in [9, 6].

The choice of the boundary conditions is a preliminary one which ensures *convergent integrals at the expense of infinitely many initial data* $A_{rm;ls;sk;nr}(\Lambda_R)$. This will be corrected later in (2.11).

One of the bounds we prove in this paper can be put in the following form

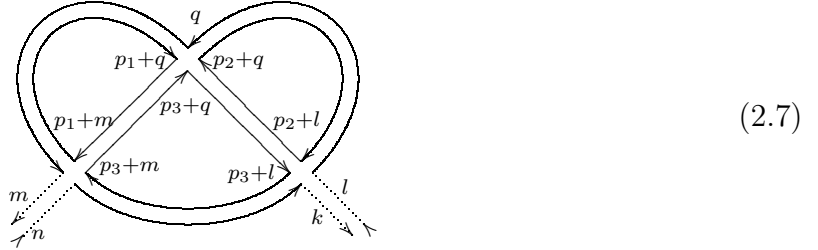
$$G_{\substack{m^1, n^1, k^1, l^1 \\ m^2, n^2, k^2, l^2}} \leq K \int_0^1 d\alpha e^{-c\alpha(m^1+m^2+n^1+n^2+k^1+k^2+l^1+l^2)}. \quad (2.5)$$

From the remarks made after (2.3) on the range of the maximal index we conclude

$$\left| Q_{\substack{m^1, n^1, k^1, l^1 \\ m^2, n^2, k^2, l^2}}(\Lambda) \right| \leq 32K \max_x \chi'(x) \int_0^1 d\alpha e^{-c\theta\Lambda^2\alpha} \leq \frac{32K \max_x \chi'(x)}{c\theta\Lambda^2}. \quad (2.6)$$

We thus estimate the summation over p in (2.4) by the maximum of the propagators Q over p and a volume factor $(2\theta\Lambda_2^2)^2$ from the support of the cut-off function. This shows that the integral (2.4) is estimated by a constant times $\ln \frac{\Lambda}{\Lambda_R}$.

The scaling of (2.6) and the volume of the support of (2.3) with respect to any index seem to suggest that N -point graphs have, as in commutative ϕ_4^4 -theory, a power-counting degree $4 - N$. However, this conclusion is too early: There is a problem in presence of completely inner vertices, which require additional index summations. The graph



entails *four* independent summation indices p_1, p_2, p_3 and q , whereas for the power-counting degree $4 - N$ we should only have three of them. It requires a more careful analysis of the scaling behaviour of the propagator to show that the q -summation can actually be performed at no cost, i.e. without a volume factor. The reason is that the propagators show some sort of quasi-locality which implies that the contribution of a propagator $G_{m,n;k,l}$ to a graph is strongly suppressed if $\|m - l\|$ is large. Thus, taking for given m the entire sum over l does not change the power-counting behaviour,

$$\left| \max_{m^i} \left(\sum_{l^1, l^2} \max_{n^i, k^i} Q_{\substack{m^1, n^1, k^1, l^1 \\ m^2, n^2, k^2, l^2}}(\Lambda) \right) \right| \leq \frac{K'}{\theta\Lambda^2}. \quad (2.8)$$

The two bounds (2.6) and (2.8) together ensure the expected power-counting behaviour for all *planar* ribbon graphs. But (2.8) does even more: it ensures the *irrelevance of all*

non-planar graphs. For instance, in the non-planar graphs

$$\begin{array}{c}
 \left. \begin{array}{c} \text{Left Diagram} \\ \text{External legs: } m_1, m_2, m_3, m_4 \\ \text{Internal indices: } n_1, n_2, n_3, n_4 \\ \text{Topology: } B=2, g=0 \end{array} \right\} q' = n_1 + n_3 - q \\
 \left. \begin{array}{c} \text{Right Diagram} \\ \text{External legs: } m_1, m_2, m_3 \\ \text{Internal indices: } n_1, n_2, n_3 \\ \text{Topology: } B=1, g=1 \end{array} \right\} \begin{array}{l} q' = m_2 + r - q \\ r' = n_2 + r - m_1 \end{array}
 \end{array} \quad (2.9)$$

the summation over q and q, r , respectively, is controlled by (2.8), i.e. the quasi-locality of the propagator, so that the graphs in (2.9) can be estimated without any volume factor.

We recall from [9] that the non-planarity of ribbon graphs is classified by the number B of boundary components and the genus $g = 1 - \frac{1}{2}(F - I + V)$ of the Riemann surface on which the graph is drawn. Here, V and I are the number of vertices and edges (inner double lines) of the graph. To determine the number F of faces we close the external legs, that is, we connect the outgoing arrow labelled m_i of an external leg directly with its incoming arrow n_i . Then, F is the number of closed single lines and B the number of those closed lines which carry external legs. Then, according to [9, 6], the power-counting degree of a N -leg ribbon graph in four dimensions is

$$\omega = (4 - N) - 4(2g + B - 1). \quad (2.10)$$

The left graph in (2.9) has topology $B = 2, g = 0$ and the right graph $B = 1, g = 1$.

As a result, there remain only the planar two- and four-leg graphs which can be relevant and marginal. The quasi-locality of the propagator improves the situation in selecting only

- the planar four-leg graphs with *constant index* along the trajectory as marginal,
- the planar two-leg graphs with *constant index* along the trajectory as relevant,
- the planar two-leg graphs with an *accumulated index jump of 2* along the trajectory as marginal.

We refer to [6] for details. The trajectories are the open single lines of the graph (before the closure which identifies the faces). This leaves still an infinite number of divergent graphs. However, there is a discrete Taylor expansion about vanishing external indices which decomposes these divergent graphs into four relevant and marginal base functions and an irrelevant remainder. For instance, the decomposition for the marginal case $m = l$ and $n = k$ of the graph (2.4) reads

$$\begin{aligned}
 &= \left\{ \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda_2}{\Lambda_2} \int_{\Lambda_2}^{\Lambda_0} \frac{d\Lambda_1}{\Lambda_1} \sum_p (Q_{m,p;p,m}(\Lambda_2) Q_{n,p;p,n}(\Lambda_1) - Q_{0,p;p,0}(\Lambda_2) Q_{0,p;p,0}(\Lambda_1)) \right. \\
 &\quad \left. + \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda_2}{\Lambda_2} \int_{\Lambda_2}^{\Lambda_0} \frac{d\Lambda_1}{\Lambda_1} \sum_p Q_{0,p;p,0}(\Lambda_2) Q_{0,p;p,0}(\Lambda_1) \right\} + \{\Lambda_1 \leftrightarrow \Lambda_2\} + A_{00;00;00;00}(\Lambda_R). \quad (2.11)
 \end{aligned}$$

Thus, this definition necessitates a single initial value $A_{00;00;00;00}(\Lambda_R)$ which represents the normalisation condition for the coupling constant.

Of particular importance are the marginal two-leg graphs with an accumulated index jump by 2, such as

The corresponding initial value represents the normalisation condition for the frequency parameter Ω in the initial action (1.2). Therefore, the harmonic oscillator potential must be present from the beginning in order to obtain a renormalisable model.

3 Bounds for the propagator

3.1 Propagator in the matrix base and cut-offs

The propagator of the noncommutative ϕ^4 -model in the matrix base of the D -dimensional Moyal plane is given by^b a positive sum [6], analogous to the heat-kernel or parametric α -space representation $\frac{1}{p^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)}$ of the ordinary commutative propagator:

$$G_{m,m+h;l+h,l} = \frac{\theta}{8\Omega} \int_0^1 d\alpha \frac{(1-\alpha)^{\frac{u_0^2\theta}{8\Omega} + (\frac{D}{4}-1)}}{(1+C\alpha)^{\frac{D}{2}}} \prod_{s=1}^{\frac{D}{2}} G_{m^s, m^s+h^s; l^s+h^s, l^s}^{(\alpha)}, \quad (3.1)$$

$$G_{m,m+h;l+h,l}^{(\alpha)} = \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l+h} \sum_{u=\max(0,-h)}^{\min(m,l)} \mathcal{A}(m, l, h, u) \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{m+l-2u}, \quad (3.2)$$

where $\mathcal{A}(m, l, h, u) = \sqrt{\binom{m}{m-u} \binom{m+h}{m-u} \binom{l}{l-u} \binom{l+h}{l-u}}$ and C is a function of Ω , namely $C(\Omega) = \frac{(1-\Omega)^2}{4\Omega}$. Indices such as m, l, h and u have $\frac{D}{2}$ non-negative components m^s, l^s, h^s, u^s , one for each symplectic pair of \mathbb{R}^D . However, due to (3.1) it is enough to prove estimations for a single component. We define the norm of an index by $\|m\| = \sum_{s=1}^{D/2} m^s$. A relation $m > n$ means $m^s > n^s$ for all s .

We know that cut-offs in the parametric representation for *commutative* theories are specially convenient both for perturbative and constructive renormalisation. In the same spirit we will divide the integral (3.1) into slices. First we divide it into two different regions, with $M > 1$

- $M^{-1} \leq \alpha \leq 1$ where we expect an exponential decay in $m+l+h$ of order $\mathcal{O}(1)$,

^bOur representation (3.1) and (3.2) corresponds to (A.17) in [6] with $z = 1 - \alpha$. The often used index parameter α in [6] is denoted by h .

- $0 \leq \alpha \leq M^{-1}$. This is the UV/IR region which is further sliced according to a geometric progression. For each slice we expect a scaled exponential decay.

Then, the decomposition

$$\int_0^1 d\alpha = \sum_{i=1}^{\infty} \int_{M^{-i}}^{M^{-i+1}} d\alpha \quad (3.3)$$

leads to the following propagator for the i^{th} slice:

$$G_{m,m+h,l+h,l}^i = \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha \frac{(1-\alpha)^{\frac{\mu_0^2 \theta}{8\Omega} + (\frac{D}{4}-1)}}{(1+C\alpha)^{\frac{D}{2}}} \prod_{s=1}^{\frac{D}{2}} G_{m^s, m^s+h^s, l^s+h^s, l^s}^{(\alpha)}. \quad (3.4)$$

The first slice $i = 1$ is treated separately.

Remark that the factor \mathcal{A} in (3.2) is the only one which prevents us from explicitly performing the u -sum. All the bounds in this paper are obtained by applying to the binomial coefficients in \mathcal{A} the simple overestimate $\binom{n}{q} \leq \frac{n^q}{q!}$. Of course, this bound is sharp only for $q \ll n$. In the regime $n - q \ll n$ one should rather use the symmetric bound $\binom{n}{q} \leq \frac{n^{n-q}}{(n-q)!}$.

For $\alpha = 0$ we see from (3.2) that the propagator vanishes unless $u = l = m$. This suggests to bound \mathcal{A} by

$$\mathcal{A}(m, l, h, u) \leq \frac{\sqrt{m(h+m)}^{m-u} \sqrt{l(h+l)}^{l-u}}{(m-u)!(l-u)!} \leq \frac{(m+h/2)^{m-u} (l+h/2)^{l-u}}{(m-u)!(l-u)!}. \quad (3.5)$$

Hence, for $\alpha \leq M^{-1}$,

$$\begin{aligned} G_{m,m+h;l+h,l}^{(\alpha)} &\leq \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l+h} \\ &\times \sum_{u=\max(0,-h)}^{\min(m,l)} \frac{\left(\frac{\alpha(1-\Omega^2)\sqrt{m(m+h)}}{4\Omega\sqrt{1-\alpha}} \right)^{m-u}}{(m-u)!} \frac{\left(\frac{\alpha(1-\Omega^2)\sqrt{l(l+h)}}{4\Omega\sqrt{1-\alpha}} \right)^{l-u}}{(l-u)!} \\ &\leq e^{-\alpha(C+1/2)(m+l+h)} \sum_{u=\max(0,-h)}^{\min(m,l)} \frac{X^{m-u}}{(m-u)!} \frac{Y^{l-u}}{(l-u)!}, \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} X &= \frac{C(1+\Omega)\alpha(1+\alpha)\sqrt{m(m+h)}}{1-\Omega} = \alpha D(\alpha)\sqrt{m(m+h)}, \\ Y &= \frac{C(1+\Omega)\alpha(1+\alpha)\sqrt{l(l+h)}}{1-\Omega} = \alpha D(\alpha)\sqrt{l(l+h)}, \end{aligned} \quad (3.7)$$

with $D(\alpha) = \frac{1-\Omega^2}{4\Omega}(1+\alpha)$. For the inequality (3.6) we performed a second order Taylor expansion in α and assumed $\Omega \geq 1/2$ and $M \geq \frac{1}{2}(1+\sqrt{5})$.

On the other hand, we observe from (3.2) that for $\alpha = 1$ the propagator vanishes unless $u = h = 0$. In this situation the bound (3.5) is not suitable anymore because $m - u$ and $l - u$ are of order $\mathcal{O}(m)$ and $\mathcal{O}(l)$, respectively. Instead, we can use

$$\mathcal{A}(m, l, h, u) \leq \frac{\sqrt{ml}^u \sqrt{(m+h)(l+h)}^{h+u}}{u!(h+u)!} \leq \frac{((m+l)/2)^u ((m+l+2h)/2)^{h+u}}{u!(h+u)!}. \quad (3.8)$$

Inserting (3.8) into (3.2) we obtain

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq \left(\frac{\alpha(1-\Omega^2)}{4\Omega + (1-\Omega)^2\alpha} \right)^{m+l+h} \\ \times \sum_{u=\max(0,-h)}^{\min(m,l)} \frac{\left(\frac{4\Omega\sqrt{1-\alpha}\sqrt{ml}}{\alpha(1-\Omega^2)} \right)^u}{u!} \frac{\left(\frac{4\Omega\sqrt{1-\alpha}\sqrt{(m+h)(l+h)}}{\alpha(1-\Omega^2)} \right)^{u+h}}{(u+h)!}. \quad (3.9)$$

The further procedure will be to use the estimation

$$\sum_{u=0}^{\min(m,l)} \frac{X^{m-u}}{(m-u)!} \frac{Y^{l-u}}{(l-u)!} \leq e^{X+Y}. \quad (3.10)$$

3.2 Main scaled bounds

The first result is to prove that the propagator shows a scaled exponential decay in any index. This is expressed by

Proposition 1 *For M large enough there exists a constant K such that for $\Omega \in [0.5, 1]$, we have the uniform bound*

$$G_{m,m+h;l+h,l}^i \leq KM^{-i} e^{-\frac{\Omega}{3}M^{-i}\|m+l+h\|}. \quad (3.11)$$

Proof. We give here the proof for $i \geq 2$. The case of the first slice is treated in Lemma 7 and Lemma 8 in the appendix^c. Using $\sqrt{m(m+h)} \leq m+h/2$, $\sqrt{l(l+h)} \leq l+h/2$ and (3.10), the bound (3.6) becomes

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq e^{-\alpha(C+1/2-D(\alpha))(m+l+h)}. \quad (3.12)$$

Then for α small enough (that is M large enough), $C + \frac{1}{2} - D(\alpha) = \Omega/2 - \frac{1-\Omega^2}{4\Omega}\alpha \geq \frac{\Omega}{3}$. We can now estimate (3.4) by

$$G_{m,m+h;l+h,l}^i \leq \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha e^{-\frac{\Omega}{3}\|m+l+h\|\alpha} \quad (3.13)$$

Then, the Proposition follows from (3.13) with $K = \frac{\theta}{8\Omega}(M-1)$. \square

^cWe prove in Lemma 7 and Lemma 8 an exponential decay with $k\|m+l+h\|$ which is possibly smaller than $\frac{\Omega}{3}M^{-1}$. We can ignore this discrepancy, because what counts in the renormalisation proof is the sum over matrix indices such as m , see (4.7). Then, the difference is a simple factor which we absorb into K .

Proposition 2 For M large enough there exists two constants^d K and K_1 such that for all $\Omega \in [0.5, 1]$ we have the uniform bound

$$G_{m,m+h;l+h,l}^i \leq KM^{-i} e^{-\frac{\Omega}{4}M^{-i}\|m+l+h\|} \prod_{s=1}^{\frac{D}{2}} \min \left(1, \left(\frac{K_1 \min(m^s, l^s, m^s + h^s, l^s + h^s)}{M^i} \right)^{\frac{|m^s - l^s|}{2}} \right). \quad (3.14)$$

Proof. Of course, this bound improves (3.11) only when an index component is smaller than M^i/K_1 . As $K_1 \geq M$, there is nothing to prove for the first slice $i = 1$.

Suppose $l \leq m \leq m + h$ and $\delta = m - l$. Instead of (3.10) we use the improved estimation

$$\begin{aligned} \sum_{u=0}^l \frac{X^{m-u}}{(m-u)!} \frac{Y^{l-u}}{(l-u)!} &= \sum_{v=0}^l \frac{X^{m-l+v}}{(m-l+v)!} \frac{Y^v}{v!} \\ &\leq \frac{X^{m-l}}{(m-l)!} \sum_{v=0}^l \left(\frac{1}{v!} \right)^2 (XY)^v \leq \frac{X^{m-l}}{(m-l)!} e^{X+Y}. \end{aligned} \quad (3.15)$$

Then, the propagator (3.6) takes the form

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq e^{-\alpha(C+1/2-D(\alpha))(m+l+h)} \frac{\left(\alpha D(\alpha) \sqrt{m(m+h)} \right)^\delta}{\delta!}. \quad (3.16)$$

By Stirling formula,

$$\frac{1}{\delta!} \leq \left(\frac{e}{\delta} \right)^\delta \quad (3.17)$$

$$\Rightarrow G_{m,m+h;l+h,l}^{(\alpha)} \leq e^{-\frac{\Omega}{3}\alpha(m+l+h)} \left(\frac{6e(D(\alpha))^2\alpha m}{\Omega\delta} \right)^{\delta/2} \sqrt{\frac{\left(\frac{\Omega}{6}\alpha(m+h) \right)^\delta}{\delta!}}. \quad (3.18)$$

For $x \geq 0$ we estimate $\frac{x^\delta}{\delta!} \leq e^x$ and obtain

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq \left(\frac{K_2\alpha m}{\delta} \right)^{\delta/2} e^{-\frac{\Omega}{4}\alpha(m+l+h)}, \quad (3.19)$$

where $K_2 = \frac{3}{8}e(1 + M^{-1})^2(1 - \Omega^2)^2/\Omega^3$.

Now we are left with two cases:

a) $l = 0 \Leftrightarrow m = \delta$:

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq e^{-\frac{\Omega}{4}\alpha(m+l+h)} (K_2\alpha)^{\delta/2}, \quad (3.20)$$

^dIn the following, the K 's will be kinds of "dustbin" constants. It means that their contents changes whereas their names do not.

b) $l \geq 1$ and $\delta \geq 1$: using $l + \delta \leq 2l\delta$,

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq e^{-\frac{\Omega}{4}\alpha(m+l+h)} (2K_2 l \alpha)^{\delta/2} . \quad (3.21)$$

Inserting this into (3.4) and symmetrising with respect to the smallest index we obtain (3.14), with $K_1 = 2MK_2$. This means $K_1 > M$ when all $\Omega \in [0.5, 1]$ are included. \square

Let us now consider a typical graph appearing in the process of renormalisation, that is, with external legs carrying indices *lower* than the internal ones. The bound (3.14) provides a good factor with respect to power-counting unless the index jump $\delta = |m - l|$ is very small, typically $\delta = 0$ or $\delta = 1$. This ensures that if the lower index of a propagator is smaller than the scale we look at, the index is conserved along its trajectory for power-counting relevant and marginal graphs.

Unfortunately, that estimation does not carry any information when the lower index is larger than the scale. It leads to a difficulty for graphs which possess completely inner vertices. Therefore, we have to find estimates for propagators with a sum over the index l . The next section is devoted to these bounds.

3.3 Bounds for sums

Now we want to prove that the summation of the propagator $G_{m,p-l,p,m+l}^{(\alpha)}$ over l , for m and p kept constant, gives the same power-counting as in the previous section. The proof relies on a more accurate estimate of the sum in (3.10).

Proposition 3 *For M large enough there exists a constant K such that for all $\Omega \in [\frac{2}{3}, 1]$, we have the uniform bound*

$$\sum_{l=-m}^p G_{m,p-l,p,m+l}^i \leq KM^{-i} e^{-\frac{\Omega}{4}M^{-i}(\|p\|+\|m\|)} . \quad (3.22)$$

Proof. The first slices, say $i \leq 6$, are trivial to treat. Using (3.11) we have

$$\sum_{l=-m}^p G_{m,p-l,p,m+l}^i \leq \frac{K}{M^i} \prod_{s=1}^2 (m^s + p^s + 1) e^{-\frac{\Omega M^{-i}}{3}(m^s + p^s)} . \quad (3.23)$$

Then, the estimation follows from

$$(x + 1)e^{-\frac{\Omega}{3}M^{-i}x} \leq \left(\frac{12M^i}{\Omega} e^{\frac{\Omega M^{-i}}{12} - 1} \right) e^{-\frac{\Omega}{4}M^{-i}x} . \quad (3.24)$$

This method fails in the limit $i \rightarrow \infty$. Thus, for large i , we have to estimate the propagator (3.6) more carefully, now putting $h \mapsto p - m - l$ and $l \mapsto m + l$. Without loss

of generality we can assume $p \geq m$. We have to divide the range of summation into three parts according to the smallest index. Using (3.15) we estimate

$$\begin{aligned}
& \sum_{l=-m}^p G_{m,p-l;p,m+l}^{(\alpha)} \\
& \leq e^{-\alpha(C+1/2)(m+p)} \left(\sum_{l=-m}^{-1} \sum_{u=0}^{m+l} \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} \right. \\
& \quad \left. + \sum_{l=0}^{p-m-1} \sum_{u=0}^m \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} + \sum_{l=p-m}^p \sum_{v=0}^{p-l} \frac{X^{p-l-v}}{(p-l-v)!} \frac{Y^{p-v}}{(p-v)!} \right) \\
& \leq \left(\sum_{l=-m}^{-1} \frac{X^{|l|}}{|l|!} + \sum_{l=0}^p \frac{Y^l}{l!} \right) e^{-\alpha(C+1/2-D(\alpha))(m+p)} \\
& \leq 2 \underbrace{\sum_{l=0}^p \frac{(\frac{1}{2}\alpha D(\alpha)(m+p+l))^l}{l!}}_Z e^{-\alpha(C+1/2-D(\alpha))(m+p)}, \tag{3.25}
\end{aligned}$$

where $X = \frac{1}{2}\alpha D(\alpha)(m+p-l)$ and $Y = \frac{1}{2}\alpha D(\alpha)(m+p+l)$.

We can now divide the sum over l into two regions corresponding to $l \leq (2\beta-3)(p+m)$ and $l \geq \lceil (2\beta-3)(p+m) \rceil \geq (2\beta-3)(p+m)$, where $\lceil x \rceil$ is the smallest integer which is larger than x and $\beta > \frac{3}{2}$ will be determined later:

$$Z \leq \sum_{l=0}^{(2\beta-3)(p+m)} \frac{((\beta-1)\alpha D(\alpha)(m+p))^l}{l!} + \sum_{l=\lceil (2\beta-3)(p+m) \rceil}^p \frac{\left(\frac{\beta-1}{2\beta-3}\alpha D(\alpha)l\right)^l}{l!}. \tag{3.26}$$

We extend both sums to infinity and use in the second one the identity (3.17):

$$Z \leq e^{(\beta-1)\alpha D(\alpha)(m+p)} + \sum_{l=0}^{\infty} \left(\frac{\beta-1}{2\beta-3}\alpha D(\alpha)e \right)^l. \tag{3.27}$$

For α small enough we have $\frac{\beta-1}{2\beta-3}\alpha D(\alpha)e < 1$. Then the sum gives a constant which we determine later. Combining (3.27) and (3.25), we have to prove that $C+1/2-\beta D(\alpha) \geq \frac{\Omega}{4}$. Let us define D' such that $C + \frac{1}{2} - D' = \frac{\Omega}{4}$. We have $D' = \frac{1}{4\Omega}$:

$$\left\{ \beta D(\alpha) \leq \frac{1}{4\Omega} \right\} \Leftrightarrow \left\{ \beta(1+\alpha)(1-\Omega^2) \leq 1 \right\}. \tag{3.28}$$

For $\alpha \leq \frac{1}{10}$ and $\beta = \frac{35}{22}$, we get $\Omega \geq \frac{2}{3} \Rightarrow \left\{ C + 1/2 - \beta D(\alpha) \geq \frac{\Omega}{4} \right\}$. Under the same conditions we have $\frac{\beta-1}{2\beta-3}\alpha D(\alpha)e < \frac{35}{36}$ so that the second sum in (3.26) is bounded by 36. For $M \geq \frac{1}{2}(\sqrt{5}+1)$ we need $i \geq 6$ in order to reach $\alpha \leq \frac{1}{10}$. This finishes the proof. \square

The previous estimation for the summed propagator is still not enough for the renormalisation proof, because the index sums are entangled in the graph. We have to prove that the exponential decay is still achieved if for given summation variable l we maximise the other index p :

Proposition 4 *For M large enough there exists a constant K such that for all $\Omega \in [\frac{2}{3}, 1]$ we have the uniform bound*

$$\sum_{l=-m}^{\infty} \max_{p \geq \max(l,0)} G_{m,p-l;p,m+l}^i \leq KM^{-i} e^{-\frac{\Omega}{36} M^{-i} \|m\|} . \quad (3.29)$$

Proof. Again, the main estimate (3.11) guarantees the desired behavior (3.29) for the first slices, say $i \leq 16$. For $l \leq 0$, the maximum is attained at $p = 0$ so that we are in the situation of (3.23) and (3.24). For $l > 0$, the maximum is attained at $p = l$ so that the l -sum leads to a geometric series. Here, it is important that i is bounded.

For $i > 16$ we have to proceed differently. We divide the domain of summation according to the smallest index at the propagator:

$$\sum_{l=-m}^{\infty} \max_{p \geq \max(l,0)} = \sum_{l=-m}^{-1} \max_{0 \leq p < m+l} + \sum_{l=-m}^{-1} \max_{m+l \leq p < m} + \sum_{l=-m}^{-1} \max_{p \geq m} + \sum_{l=0}^{\infty} \max_{l \leq p < m+l} + \sum_{l=0}^{\infty} \max_{p \geq m+l} . \quad (3.30)$$

We now obtain from (3.6) and (3.15)

$$\begin{aligned} & \sum_{l=-m}^{\infty} \max_{p \geq \max(l,0)} G_{m,p-l;p,m+l}^{(\alpha)} \\ & \leq \sum_{l=-m}^{-1} \max_{0 \leq p < m+l} e^{-\alpha(C+1/2)(m+p)} \sum_{v=0}^p \frac{X^{p-l-v}}{(p-l-v)!} \frac{Y^{p-v}}{(p-v)!} \\ & \quad + \sum_{l=-m}^{-1} \max_{m+l \leq p < m} e^{-\alpha(C+1/2)(m+p)} \sum_{u=0}^{m+l} \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} \\ & \quad + \sum_{l=-m}^{-1} \max_{p \geq m} e^{-\alpha(C+1/2)(m+p)} \sum_{u=0}^{m+l} \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} \\ & \quad + \sum_{l=0}^{\infty} \max_{l \leq p < m+l} e^{-\alpha(C+1/2)(m+p)} \sum_{v=0}^{p-l} \frac{X^{p-l-v}}{(p-l-v)!} \frac{Y^{p-v}}{(p-v)!} \\ & \quad + \sum_{l=0}^{\infty} \max_{p \geq m+l} e^{-\alpha(C+1/2)(m+p)} \sum_{u=0}^m \frac{X^{m-u}}{(m-u)!} \frac{Y^{m+l-u}}{(m+l-u)!} \end{aligned}$$

$$\leq \sum_{|l|=1}^m \max_{p \geq 0} e^{-\alpha(C+1/2-D(\alpha))(m+p)} \frac{\left(\frac{1}{2}\alpha D(\alpha)(p+m+|l|)\right)^{|l|}}{|l|!} \quad (3.31)$$

$$+ \sum_{l=0}^{\infty} \max_{p \geq l} e^{-\alpha(C+1/2-D(\alpha))(m+p)} \frac{\left(\frac{1}{2}\alpha D(\alpha)(p+m+l)\right)^l}{l!}, \quad (3.32)$$

where $X = \frac{\alpha D(\alpha)}{2}(p+m-l)$ and $Y = \frac{\alpha D(\alpha)}{2}(p+m+l)$.

The function $p \mapsto e^{-\gamma p} \frac{(p+q)^l}{l!}$ attains its maximum $e^{\gamma q-l} \frac{1}{l!} \left(\frac{l}{\gamma}\right)^l$ at $p = \frac{l}{\gamma} - q$. We need this property for $q = m + |l|$ and

$$\gamma := \alpha(C + 1/2 - D(\alpha)). \quad (3.33)$$

However, we have to take into account the range of p . If $\frac{l}{\gamma} - q < 0$ in (3.31), then the function $e^{-\gamma p} \frac{(p+q)^l}{l!}$ is monotonously decreasing for all $p \geq 0$ so that the maximum is at $p = 0$. Otherwise we have to insert the specific maximum. This means that we split the sum over l in (3.31) into two pieces

- $|l| \leq \frac{m\gamma}{1-\gamma}$, where we insert $p = 0$. Actually, we can safely extend this sum from 0 to m , still keeping $p = 0$.
- $|l| \geq \frac{m\gamma}{1-\gamma}$, where we insert $p = \frac{1-\gamma}{\gamma}|l| - m$.

This gives

$$\begin{aligned} & \sum_{|l|=1}^m \max_{p \geq 0} e^{-\alpha(C+1/2-D(\alpha))(m+p)} \frac{\left(\frac{1}{2}\alpha D(\alpha)(p+m+|l|)\right)^{|l|}}{|l|!} \\ & \leq \sum_{l=0}^m e^{-\gamma m} \frac{\left(\frac{1}{2}\alpha D(\alpha)(m+l)\right)^l}{l!} + \sum_{l=\frac{\gamma m}{1-\gamma}}^m e^{(\gamma-1)l} \frac{\left(\frac{\alpha D(\alpha)l}{2\gamma}\right)^l}{l!}. \end{aligned} \quad (3.34)$$

For the first sum on the right hand side we are in the situation of (3.25) with $m \mapsto 0, p \mapsto m$ so that we can bound that sum according to the steps leading to (3.22) by a constant times $e^{-\frac{\Omega}{4}\alpha m}$, for $\Omega \geq 2/3$ and α small enough. In the second sum we use (3.17) so that, for some number $\varepsilon > 0$,

$$\sum_{l=\frac{\gamma m}{1-\gamma}}^m e^{(\gamma-1)l} \frac{\left(\frac{\alpha D(\alpha)l}{2\gamma}\right)^l}{l!} \leq \sum_{l=\frac{\gamma m}{1-\gamma}}^m e^{-\varepsilon l} \left(\frac{\alpha D(\alpha)e^{\gamma+\varepsilon}}{2\gamma}\right)^l \leq e^{-\varepsilon \gamma m} \sum_{l=\frac{\gamma m}{1-\gamma}}^m \left(\frac{\alpha D(\alpha)e^{\gamma+\varepsilon}}{2\gamma}\right)^l. \quad (3.35)$$

In the numerator we can bound γ by $\frac{1}{2}$, otherwise the sum vanishes. Moreover, we choose $\varepsilon = \frac{1}{12}$. With (3.33) it thus remains to prove

$$\frac{D(\alpha)e^{\frac{\gamma}{12}}}{2C + 1 - 2D(\alpha)} < 1. \quad (3.36)$$

For $\alpha \leq \frac{1}{10}$ this means $\frac{11-11\Omega^2}{42\Omega^2-2} e^{\frac{7}{12}} < 1$, which is indeed satisfied for $\Omega \geq \frac{2}{3}$.

We now pass to (3.32). The condition $p \geq l$ leads to a splitting of the l -sum at $\frac{\gamma m}{1-2\gamma}$. For smaller l we have $p = l$:

$$\begin{aligned} & \sum_{l=0}^{\infty} \max_{p \geq l} e^{-\alpha(C+1/2-D(\alpha))(m+p)} \frac{\left(\frac{1}{2}\alpha D(\alpha)(p+m+l)\right)^l}{l!} \\ & \leq \sum_{l=0}^{\infty} e^{-\gamma\alpha(m+l)} \frac{\left(\frac{1}{2}\alpha D(\alpha)(m+2l)\right)^l}{l!} + \sum_{l=\frac{\gamma m}{1-2\gamma}}^{\infty} e^{(\gamma-1)l} \frac{\left(\frac{\alpha D(\alpha)l}{2\gamma}\right)^l}{l!} \end{aligned} \quad (3.37)$$

The second sum on the right hand side is identical to treat as in the previous case (3.31). The first sum on this right hand side is split as in (3.26) at $l = (\beta - \frac{3}{2})m$. Then it works exactly as in (3.27). This finishes the proof. \square

3.4 Composite propagators

This section is devoted to the proofs of bounds on the *composite propagators* introduced in [6]. We define their sliced versions as follows:

$$\mathcal{Q}_{m^1, n^1, n^1, m^1}^{i(0)} = G_{m^1, n^1, n^1, m^1}^i - G_{0, n^1, n^1, 0}^i, \quad (3.38)$$

$$\mathcal{Q}_{m^1, n^1, n^1, m^1}^{i(1)} = \mathcal{Q}_{m^1, n^1, n^1, m^1}^{i(0)} - m^1 \mathcal{Q}_{1, n^1, n^1, 1}^{i(0)} - m^2 \mathcal{Q}_{0, n^1, n^1, 0}^{i(0)}, \quad (3.39)$$

$$\mathcal{Q}_{m^1+1, n^1+1, n^1, m^1}^{i(\frac{1}{2})} = G_{m^1+1, n^1+1, n^1, m^1}^i - \sqrt{m^1+1} G_{1, n^1+1, n^1, 0}^i. \quad (3.40)$$

Proposition 5 *There exist constants K_i such that for $\Omega \in [0.5, 1)$ and $m \leq M^i$, we have the uniform bounds*

$$\left| \mathcal{Q}_{m^1, n^1, n^1, m^1}^{i(0)} \right| \leq K_0 M^{-i} \frac{m^1 + m^2}{M^i} e^{-\frac{\Omega}{3} M^{-i}(n^1+n^2)}, \quad (3.41)$$

$$\left| \mathcal{Q}_{m^1, n^1, n^1, m^1}^{i(1)} \right| \leq K_1 M^{-i} \left(\frac{m^1 + m^2}{M^i} \right)^2 e^{-\frac{\Omega}{3} M^{-i}(n^1+n^2)}, \quad (3.42)$$

$$\left| \mathcal{Q}_{m^1+1, n^1+1, n^1, m^1}^{i(\frac{1}{2})} \right| \leq K_2 M^{-i} \left(\frac{m^1 + m^2 + 1}{M^i} \right)^{3/2} e^{-\frac{\Omega}{4} M^{-i}(n^1+n^2)}. \quad (3.43)$$

Proof. There is no need to discuss the first slice. From (3.4) we have

$$\begin{aligned} \mathcal{Q}_{m^1, n^1, n^1, m^1}^{i(0)} &= \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha \frac{(1-\alpha)^{\frac{\mu_0^2 \theta}{8\Omega}}}{(1+C\alpha)^2} \left((G_{m^1, n^1, n^1, m^1}^{(\alpha)} - G_{0, n^1, n^1, 0}^{(\alpha)}) G_{m^2, n^2, n^2, m^2}^{(\alpha)} \right. \\ & \quad \left. + G_{0, n^1, n^1, 0}^{(\alpha)} (G_{m^2, n^2, n^2, m^2}^{(\alpha)} - G_{0, n^2, n^2, 0}^{(\alpha)}) \right). \end{aligned} \quad (3.44)$$

Taking (3.11) into account, it remains to obtain estimations for $G_{m,n,n,m}^{(\alpha)} - G_{0,n,n,0}^{(\alpha)}$. From (3.2) we have, expressing $m + h$ by n ,

$$\begin{aligned} & |G_{m,n,n,m}^{(\alpha)} - G_{0,n,n,0}^{(\alpha)}| \\ &= \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^n \left| \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^m \sum_{u=0}^{\min(m,n)} \binom{m}{u} \binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{2u} - 1 \right|. \end{aligned} \quad (3.45)$$

The ($u = 0$)-part of the sum and the separate term -1 yield

$$\begin{aligned} 1 - \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^m &= \left(1 - \frac{\sqrt{1-\alpha}}{1+C\alpha} \right) \sum_{j=0}^{m-1} \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^j \\ &\leq m \left(1 - \frac{\sqrt{1-\alpha}}{1+C\alpha} \right) \\ &\leq m\alpha(C+1). \end{aligned} \quad (3.46)$$

Next, using $\binom{m}{v+1} = \frac{m-v}{v+1} \binom{m}{v}$ and our central estimate $\binom{m}{v} \leq \frac{m^v}{v!}$, we have

$$\begin{aligned} & \sum_{u=1}^m \binom{m}{u} \binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{2u} \\ & \leq \sum_{v=0}^{m-1} \frac{m-v}{v+1} \binom{m}{v} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{v+1} \sum_{u=0}^m \binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^u \\ & \leq m \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right) \exp \left(\frac{C\alpha(1+\Omega)(m+n)}{\sqrt{1-\alpha}(1-\Omega)} \right) \\ & \leq m\alpha \frac{(1-\Omega)^2}{4\Omega\sqrt{1-M^{-1}}} e^{\alpha D(\alpha)(m+n)}. \end{aligned} \quad (3.47)$$

Inserting (3.46) and (3.47) back into (3.45) we obtain for $\alpha \leq M^{-1}$ the estimation

$$|G_{m,n,n,m}^{(\alpha)} - G_{0,n,n,0}^{(\alpha)}| \leq \alpha m \left(\frac{(1+\Omega)^2}{4\Omega} + \frac{(1-\Omega)^2}{4\Omega\sqrt{1-M^{-1}}} \right) e^{-\alpha(C+1/2-D(\alpha))n}. \quad (3.48)$$

Comparing (3.48) with (3.12) we obtain in (3.41) the same restrictions to Ω as in Proposition 1.

To approach (3.42) we consider

$$\begin{aligned} \mathcal{Q}_{m^1, n^1, n^1, m^1; m^2, n^2, n^2, m^2}^{i(1)} &= \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha \frac{(1-\alpha)^{\frac{\mu_0^2 \theta}{8\Omega}}}{(1+C\alpha)^2} \\ & \times \left(\left(G_{m^1, n^1, n^1, m^1}^{(\alpha)} - G_{0, n^1, n^1, 0}^{(\alpha)} - m^1 (G_{1, n^1, n^1, 1}^{(\alpha)} - G_{0, n^1, n^1, 0}^{(\alpha)}) \right) G_{m^2, n^2, n^2, m^2}^{(\alpha)} \right. \\ & \quad + m^1 (G_{1, n^1, n^1, 1}^{(\alpha)} - G_{0, n^1, n^1, 0}^{(\alpha)}) (G_{m^2, n^2, n^2, m^2}^{(\alpha)} - G_{0, n^2, n^2, 0}^{(\alpha)}) \\ & \quad \left. + G_{0, n^1, n^1, 0}^{(\alpha)} \left(G_{m^2, n^2, n^2, m^2}^{(\alpha)} - G_{0, n^2, n^2, 0}^{(\alpha)} - m^2 (G_{1, n^2, n^2, 1}^{(\alpha)} - G_{0, n^2, n^2, 0}^{(\alpha)}) \right) \right). \end{aligned} \quad (3.49)$$

The third line is estimated by (3.48). In the other lines we have for $n \geq 1$

$$\begin{aligned}
& |G_{m,n,n,m}^{(\alpha)} - G_{0,n,n,0}^{(\alpha)} - m(G_{1,n,n,1}^{(\alpha)} - G_{0,n,n,0}^{(\alpha)})| \\
&= \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^n \left| \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^m - m \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} - 1 \right) - 1 \right| \\
&+ mn \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{n+1} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^2 \left| \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m-1} - 1 \right| \\
&+ \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+n} \sum_{u=2}^{\min(m,n)} \binom{m}{u} \binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{2u}. \tag{3.50}
\end{aligned}$$

In the third line we use $n \frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \leq \exp(n\alpha D(\alpha))$. The further procedure is as before. Finally, we consider

$$\begin{aligned}
& \mathcal{Q}_{m^1+1, n^1+1, n^1, m^1}^{i(\frac{1}{2})} \\
&= \frac{\theta}{8\Omega} \int_{M^{-i}}^{M^{-i+1}} d\alpha \frac{(1-\alpha)^{\frac{\mu_0^2 \theta}{8\Omega}}}{(1+C\alpha)^2} \left((G_{m^1+1, n^1+1, n^1, m^1}^{(\alpha)} - \sqrt{m^1+1} G_{1, n^1+1, n^1, 0}^{(\alpha)}) G_{m^2, n^2, n^2, m^2}^{(\alpha)} \right. \\
&\quad \left. + \sqrt{m^1+1} G_{1, n^1+1, n^1, 0}^{(\alpha)} (G_{m^2, n^2, n^2, m^2}^{(\alpha)} - G_{0, n^2, n^2, 0}^{(\alpha)}) \right). \tag{3.51}
\end{aligned}$$

The estimation for the third line of (3.51) is immediately obtained from (3.48) and (3.14). In the second line we have

$$\begin{aligned}
& |G_{m+1, n+1, n, m}^{(\alpha)} - \sqrt{m+1} G_{1, n+1, n, 0}^{(\alpha)}| \\
&= \left| \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^m \sum_{u=0}^{\min(m,n)} \frac{\sqrt{(m+1)(n+1)}}{u+1} \binom{m}{u} \binom{n}{u} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right)^{2u+1} \right. \\
&\quad \left. - \sqrt{(m+1)(n+1)} \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} \right) \left| \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{n+1} \right| \right|. \tag{3.52}
\end{aligned}$$

We use the estimation $\sqrt{n \frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)}} \leq \exp(\frac{1}{2}n\alpha D(\alpha))$ and proceed along the same lines as before.

This finishes the proof. \square

4 Perturbative power-counting

4.1 Ribbon graphs

Consider a given ϕ^4 -ribbon graph G with N external legs, V vertices, I inner lines and F faces, hence of genus $g = 1 - \frac{1}{2}(V - I + F)$. There are four indices $\{m, n; k, l\} \in \mathbb{N}^2$ associated to each inner line $\begin{matrix} \xleftarrow{n} & k \\ m & \xrightarrow{l} \end{matrix}$ of the graph and two indices for each external line,

hence $4I + 2N = 8V$ such indices. But at every vertex, the left index of a ribbon coincides with the right one of the next ribbon. This creates $4V$ independent identifications, so we can write the indices of any propagator in terms of a set \mathcal{I} of $4V$ indices, four per vertex, for instance each “left” half-ribbon index.

The amplitude of the graph is then the sum

$$A_G = \sum_{\mathcal{I}} \prod_{\delta \in G} G_{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), l_\delta(\mathcal{I})} \delta_{m_\delta - l_\delta, n_\delta - k_\delta} , \quad (4.1)$$

where the four basic indices of the propagator G for a line δ are functions of \mathcal{I} called $\{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), l_\delta(\mathcal{I})\}$.

The scale decomposition of the propagator being

$$G = \sum_{i=0}^{\infty} G^i , \quad (4.2)$$

we have an associated decomposition of any amplitude of the theory as

$$A_G = \sum_{\mu} A_{G, \mu} , \quad (4.3)$$

$$A_{G, \mu} = \sum_{\mathcal{I}} \prod_{\delta \in G} G_{m_\delta(\mathcal{I}), n_\delta(\mathcal{I}); k_\delta(\mathcal{I}), l_\delta(\mathcal{I})}^{i_\delta} \delta_{m_\delta(\mathcal{I}) - l_\delta(\mathcal{I}), n_\delta(\mathcal{I}) - k_\delta(\mathcal{I})} , \quad (4.4)$$

where $\mu = \{i_\delta\}$ runs over all possible attributions of a positive integer i_δ for each line δ . Such a μ is therefore called a scale attribution.

We recall our two main bounds on the propagator

$$G_{m, n; k, l}^i \leqslant K M^{-i} e^{-cM^{-i}(\|m\| + \|n\| + \|k\| + \|l\|)} , \quad (4.5)$$

$$\sum_l \max_{n, k} G_{m, n; k, l}^i \leqslant K' M^{-i} e^{-c'M^{-i}\|m\|} , \quad (4.6)$$

for some constants K, K' and c, c' .

A considerable fraction of the $4V$ indices initially associated to this graph is determined by external indices of the graph and the δ -function in (4.1). The undetermined indices are summation indices. Perturbative power counting for a graph consists in finding the indices for which the summation costs a factor M^{2i} , and the ones for which it costs only $\mathcal{O}(1)$, thanks to (4.6). The factor M^{2i} follows from (4.5) after summation over some index^e $m \in \mathbb{N}^2$,

$$\sum_{m^1, m^2=0}^{\infty} e^{-cM^{-i}(m^1+m^2)} = \frac{1}{(1 - e^{-cM^{-i}})^2} = \frac{M^{2i}}{c^2} (1 + \mathcal{O}(M^{-i})) . \quad (4.7)$$

^eRemember that there are two symplectic pairs, one for spatial dimensions 1 and 2, and the other for spatial dimensions 3 and 4.

4.2 Dual graphs

We first want to use as much as possible the δ -functions in (4.1) to reduce the set \mathcal{I} to a truly minimal set \mathcal{I}' of independent indices. It is convenient for this task to consider the dual graph for which the problem becomes analogous to an ordinary problem of momentum routing.

The dual graph of a ribbon graph is obtained by associating to each face a vertex and to each vertex a face. Every line bordering two neighbouring faces is replaced by a line joining the two corresponding vertices of the dual graph. Hence, the genus does not change under this duality. We shall write $V' = F$, $F' = V$ for the number of vertices and faces of the dual graph (dual quantities are usually distinguished by a prime). If the initial graph is a ϕ^4 -graph, i.e. has coordination 4 at each vertex, we have $4 = I_{f'} + N_{f'}$ for each face $f' \in F'$, where $I_{f'}$ and $N_{f'}$ denote the numbers of edges and external valences, respectively, which belong to f' . The coordination at the vertices of the dual graph is arbitrary.

The construction of the dual of a graph goes as follows: First, for each oriented face of the original ribbon graph, draw an oriented ribbon vertex by assigning

- to a single line of a propagator of the original graph an internal valence of the dual vertex,
- to an external valence of the original graph an external valence of the dual vertex,

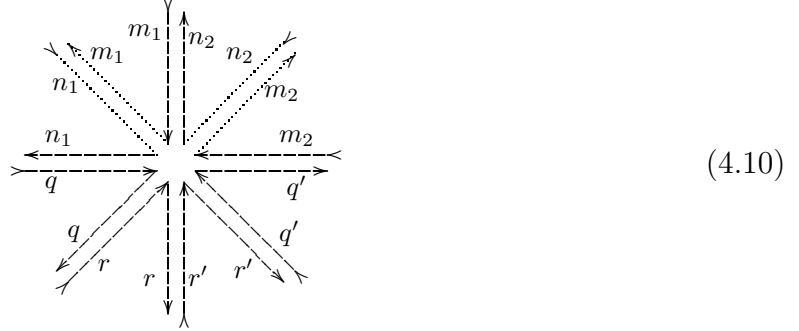
respecting the order according to the arrows on the trajectories. In the second step we connect the valences by the duals of the propagators of the original graph, which is obtained according to

$$\begin{array}{ccc}
 & \text{dual line} & \\
 & \vdots & \\
 \text{m} & \text{---} & \text{l} \\
 \text{---} & \text{---} & \text{---} \text{ line} \\
 \text{n} & \text{---} & \text{k} \\
 & \vdots &
 \end{array} \tag{4.8}$$

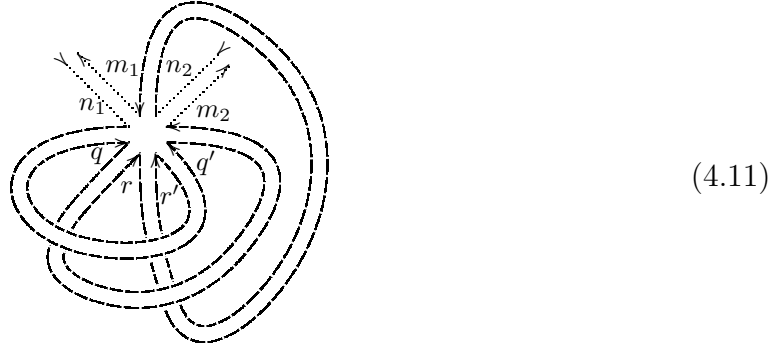
Let us consider the following example of a ribbon graph with a single face:

$$\tag{4.9}$$

The above rules lead to the following dual vertex:



Now we connect the valences by the duals of the propagators of the original graph, i.e. n_1q with $r'q'$, qr with $q'm_2$ and n_2m_1 with rr' :



The dual graph is made of the same propagators as the original graph, only the index assignment is different. Whereas in the original graph we have $G_{mn;kl} = \begin{matrix} \leftarrow m & l \\ n & \rightarrow k \end{matrix}$, the index assignment for propagators in the dual graph is

$$G_{mn;kl} = \begin{matrix} \leftarrow m & n \\ l & \rightarrow k \end{matrix} . \quad (4.12)$$

The conservation rule $\delta_{l-m, -(n-k)}$ in (4.1) now states that the difference between outgoing and incoming indices of the half-propagator attached to a dual vertex, namely $l - m$, is conserved as minus the corresponding difference $n - k$ at the other end of the propagator. Actually, these index differences describe the *angular momentum*, and the conservation of these differences $\ell = l - m$ and $-\ell = n - k$ is nothing but the angular momentum conservation due to the $SO(2) \times SO(2)$ symmetry of the noncommutative ϕ^4 -action. Thus, taking the incoming indices as the reference, the angular momentum through the dual propagator determines the outgoing indices:

$$\begin{matrix} \leftarrow \ell \\ \leftarrow m & n \\ l & \rightarrow k \\ \rightarrow -\ell \end{matrix} \quad l = m + \ell , \quad n = k + (-\ell) . \quad (4.13)$$

In the same way, there are external angular momenta \wp which enter the dual graph through the external valences. We shall also use the incoming arrow as the reference

so that the angular momentum is the index difference between outgoing and incoming arrows. Furthermore, by cyclicity at any vertex, the sum of all incoming differences at a vertex, i.e. the sum of incoming angular momenta, must be zero. Of course, this implies that the total external angular momentum entering a graph is zero, too.

Thus, the angular momentum for the dual graph is exactly like the usual momentum in an ordinary Feynman graph: a momentum that goes out like p at one half-end must go out as $-p$ at the other half-end, and the total momentum is conserved at any vertex. (In Feynman graphs this follows from translation invariance.) It should be stressed that one has to take into account positivity constraints for the angular momenta ℓ : they lie in \mathbb{Z} , but all indices m, n, k, l must be *positive* integers.

We therefore know that the number of *independent* index differences is exactly the number of *loops* L' of the dual graph. For a connected graph, this number is $L' = I - V' + 1$. Furthermore, each index at a vertex is clearly only a function of the differences at a vertex and of a single *reference index* for the dual vertex. If the dual vertex is an external one, we take as the reference index the *outgoing* index at one of the external legs. If the dual vertex is an internal one, we have to make a choice (determined later) for the reference index. These internal vertex reference indices correspond to the loop variable of the original graph. Therefore, after using the conservation rules or δ -functions of each propagator, the number of independent indices to be summed for every graph is simply $V' - B + L' = I + (1 - B)$. Here, $B \geq 0$ is the number of boundary components of the original graph, which coincides with the number of external vertices of the dual graph.

Expressing each index in the graph as a function of a set \mathcal{T}' of such independent indices is therefore identical to the problem called momentum routing in a Feynman graph. It is well-known that the solution is not canonical or unique. A good way to root the momenta is to pick a spanning tree \mathcal{T}_μ of the dual graph, with $V' - 1$ lines, and to use the complement set \mathcal{L}_μ as the set of fundamental independent differences. The subscript μ refers to the choice of the tree which depends on the scale attribution μ in (4.4).

4.3 Choice of the tree

A given scale attribution $\mu = \{i_\delta\}$ defines an order of lines in the dual graph. We define

$$\delta_1 \leq \delta_2 \leq \dots \leq \delta_I \quad \text{if} \quad i_{\delta_1} \leq i_{\delta_2} \leq \dots \leq i_{\delta_I} . \quad (4.14)$$

In case of equality we make any choice. Let $\delta_1^{\mathcal{T}}$ be the smallest line with respect to this order which is not a tadpole, and $G_{m_{\delta_1^{\mathcal{T}}}; n_{\delta_1^{\mathcal{T}}}; k_{\delta_1^{\mathcal{T}}}, l_{\delta_1^{\mathcal{T}}}}^{i_{\delta_1^{\mathcal{T}}}}$ be the corresponding propagator. The line $\delta_1^{\mathcal{T}}$ will then connect two vertices v_1^\pm and forms the first segment of the tree. We let $\mu_1 := \mu \setminus (\delta_1 \cup \dots \cup \delta_1^{\mathcal{T}})$ and $\mathcal{T}_1 = \delta_1^{\mathcal{T}} \cup v_1^+ \cup v_1^-$.

In the remaining set μ_1 of lines we identify the smallest line $\delta_2^{\mathcal{T}}$ of μ_1 which does not form a loop when added to \mathcal{T}_1 . We define $\mu_2 = \mu \setminus (\delta_1 \cup \dots \cup \delta_2^{\mathcal{T}})$ and

- $\mathcal{T}_2 = \mathcal{T}_1 \cup \delta_2^{\mathcal{T}} \cup v_2^+$ if $\delta_2^{\mathcal{T}}$ connects a vertex v_2^+ to \mathcal{T}_1 ,
- $\mathcal{T}_2 = \mathcal{T}_1 \cup \delta_2^{\mathcal{T}} \cup v_2^+ \cup v_2^-$ if $\delta_2^{\mathcal{T}}$ connects two vertices $v_2^\pm \notin \mathcal{T}_1$.

In the n^{th} step, we identify the smallest line δ_n^T of μ_{n-1} which does not form a loop when added to \mathcal{T}_{n-1} . We define $\mu_n = \mu \setminus (\delta_1 \cup \dots \cup \delta_n^T)$ and

- $\mathcal{T}_n = \mathcal{T}_{n-1} \cup \delta_n^T \cup v_n^+$ if δ_n^T connects a vertex v_n^+ to \mathcal{T}_{n-1} ,
- $\mathcal{T}_n = \mathcal{T}_{n-1} \cup \delta_n^T \cup v_n^+ \cup v_n^-$ if δ_n^T connects two vertices $v_n^\pm \notin \mathcal{T}_{n-1}$,
- $\mathcal{T}_n = \mathcal{T}_{n-1} \cup \delta_n^T$ if δ_n^T connects two disjoint subsets of \mathcal{T}_{n-1} .

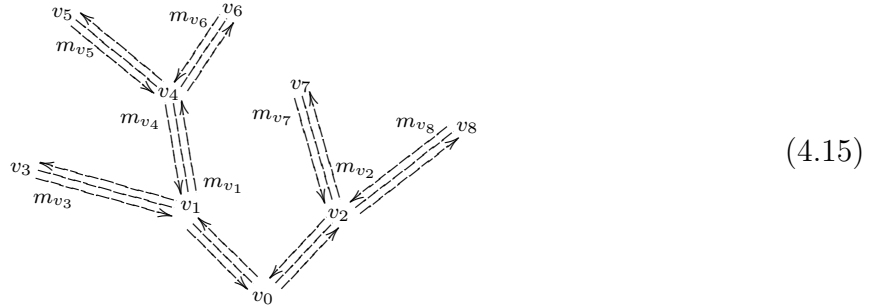
The $(V' - 1)^{\text{th}}$ step leads to the desired tree $\mathcal{T}_\mu = \mathcal{T}_{V'-1}$. The importance of this construction is the fact that any line $\delta_j^L \in \mathcal{L}_\mu$ which connects *different* vertices v_j^\pm of the tree has a scale index $i_{\delta_j^L}$ which is *not smaller* than any scale index of the lines *in the tree* which connect v_j^\pm . Such a tree optimisation for a given scale attribution is one of the most basic tools of constructive field theory [11], so it is an encouraging sign that it appears also here in a natural way.

In the graphical representations we distinguish the tree by triple dashed lines.

4.4 Index assignment for a tree

For the previously constructed tree \mathcal{T}_μ we select one of its $B \geq 1$ external vertices as the root v_0 of the tree. If the graph is a vacuum graph i.e. with $B = 0$ we choose any vertex as the root. We relabel the vertices in the tree such that all vertices in the subtree above a vertex v_n have a label bigger than n .

The order (4.14) of the lines of the graph provides us with a convenient position for the main reference index m at each vertex. If v is an internal vertex, we let δ_v be the smallest line in (4.14) which is attached to v . By construction of the tree we know that either δ_v is a tadpole, or it belongs to the tree. We choose the *outgoing* index (without the arrow when viewed from the vertex) of this particular line δ_v as the main reference index m_v . We let \mathcal{G}_M be the set of lines at which a main reference index resides. It is possible that both outgoing indices of a line $\delta_v = \delta_{v'}$ attached to v and v' are main reference indices. In this case we let δ_v appear twice in \mathcal{G}_M . Thus, \mathcal{G}_M consists of $V' - B$ elements. If v is an external vertex, we take as the “main reference index” the outgoing index at any external leg. The following graph shows a typical situation of a tree and its main reference indices, assuming absence of tadpoles and $B = 1$:

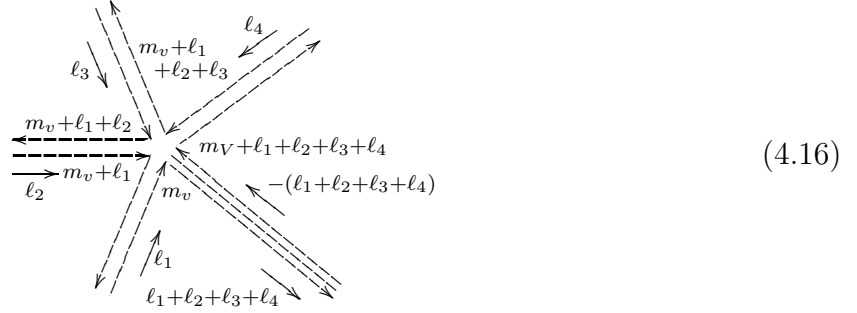


One can now write down every index in a unique way in terms of

- $V' - B$ main reference indices m_v ,
- B main reference indices at external vertices,

- L' internal angular momenta ℓ_δ for the set \mathcal{L}_μ of “loop lines”
- N external angular momenta \wp_ε for the set \mathcal{N} of external lines.

The rule goes as follows. One writes first the indices for all the “leaves” of the tree, that is the vertices (distinct from the root) with coordination 1 in the tree (i.e. vertices v_3, v_5, v_6, v_7 and v_8 in (4.15)). For them, starting from the main index m_v (at the left of the unique line of \mathcal{T}_μ at v that goes towards the root, unless a tadpole at v has the smallest scale), which agrees with the incoming index of the next line at v in clockwise direction, we compute all other indices by turning *clockwise* around the vertex and by adding the angular momenta (internal or external ones) associated according to (4.13) with the loop lines $\delta_1, \dots, \delta_k$ and possibly external lines $\varepsilon_1, \dots, \varepsilon_{k'}$. This gives indices $m_v + \ell_1, m_v + \ell_1 + \ell_2, \dots$ until we arrive at $m_v + \ell_1 + \dots + \ell_{k+k'}$ which is at the right of the unique line at v that goes towards the root. Some of the ℓ_j could be external angular momenta. Then we can compute the angular momentum associated to that line: it is simply $-(\ell_1 + \dots + \ell_{k+k'})$. The following picture shows these assignments for a particular “leaf”:



Having done this for all leaves, we can prune these leaves and consider the next layer of vertices down towards the root (i.e. vertices v_2 and v_4 in (4.15)) and reiterate the argument.

Any summation index at a vertex v is now clearly equal to m_v plus a linear combination of angular momenta ℓ_δ for the set of lines $\mathcal{L}_v \cup \mathcal{N}_v$ hooked to the “subtree above v ”, that is the lines hooked at least at one end to a vertex v' such that the unique path from v' to v_0 passes through v .

We split therefore the set \mathcal{I}' of independent indices to be summed into the two sets:

- the set $\mathcal{M}_\mu = \{m_v\}$ of main reference indices at inner vertices, which consists of $V' - B$ elements,
- the set $\mathcal{J}_\mu = \{\ell_\delta, \delta \in \mathcal{L}_\mu\}$ of angular momenta, which consists of L' elements.

The amplitude of the graph G is now written as:

$$A_G = \sum_{\mu} \sum_{\mathcal{M}_\mu, \mathcal{J}_\mu} \prod_{\delta \in G} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \chi(\mathcal{M}_\mu, \mathcal{J}_\mu), \quad (4.17)$$

where the sum over \mathcal{M}_μ is over positive integers, but the sum over \mathcal{J}_μ is over relative integers and the function $\chi(\mathcal{M}_\mu, \mathcal{J}_\mu)$ is the characteristic function which states that all

the functions $\{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)\}$ are positive. The dependence of A_G on the external indices (B reference indices at external vertices and N external angular momenta) is not made explicit.

4.5 Power-counting

Our goal is now to prove that sums over difference indices can always be performed through (4.6), hence at no cost, *using precisely the propagator G^{i_δ} to perform the sum over the angular momentum ℓ_δ* . However, as the angular momenta are entangled, we need appropriate maximisations of the other $G^{i'_\delta}$ over ℓ_δ . These maximisations require a carefully chosen order. For processing the angular momenta, all main reference indices \mathcal{M}_μ and the external indices are kept constant.

We introduce in the set of loop lines \mathcal{L}_μ another order “ $>$ ” than the previous order “ \leq ” defined in (4.14), now according to the position in the tree. For this purpose we let v_δ be the label of the highest vertex to which $\delta \in \mathcal{L}_\mu$ is hooked. We define $\delta_1 > \delta_2$ if

- $v_{\delta_1} > v_{\delta_2}$, or
- when turning clockwise around $v_{\delta_1} = v_{\delta_2}$ from the main reference index, first δ_1 is encountered, then δ_2 or the tree line which goes down to the root, or
- when turning clockwise around $v_{\delta_1} = v_{\delta_2}$ from the main reference index, first the tree line down to the root is encountered, then δ_2 and finally δ_1 .

We orient the lines $\delta \in \mathcal{L}_\mu$ so that the two indices $m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)$ are hooked to v_δ and their difference is precisely ℓ_δ :

$$l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu) - m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu) = \ell_\delta \quad (4.18)$$

For “tadpole” lines δ of \mathcal{L}_μ , i.e. hooking v_δ to itself, we define the two indices m, l as the ones of the first half-line of δ in the clockwise cyclic ordering between the main reference of v_δ and the tree line down to the root. If the tree line is encountered before both half-lines, we have to select instead the indices on the half-line in anti-clockwise order.

We define $\mathcal{J}_\mu^{\delta+} = \{\ell_{\delta'} \in \mathcal{J}_\mu, \delta' > \delta\}$ and $\mathcal{J}_\mu^{\delta-} = \{\ell_{\delta'} \in \mathcal{J}_\mu, \delta > \delta'\}$. Then, we remark that the two distinguished indices $m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)$ at v_δ are actually functions $m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}), m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}) + \ell_\delta$ which are *independent* of the indices in $\mathcal{J}_\mu^{\delta-}$. We conclude that for $\delta \in \mathcal{L}_\mu$ and fixed indices in \mathcal{M}_μ and $\mathcal{J}_\mu^{\delta+}$,

$$\begin{aligned} & \max_{\ell_{\delta'} \in \mathcal{J}_\mu^{\delta-}} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \\ & \leq \max_{n_\delta, k_\delta} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}), n_\delta; k_\delta, m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}) + \ell_\delta}^{i_\delta} \end{aligned} \quad (4.19)$$

where on the right hand side the max is over *any* indices n_δ, k_δ in \mathbb{N} . It is instructive to look at the example (4.16) where we have $\delta_1 > \delta_2 > \delta_3 > \delta_4$. The indices m_2, l_2 depend on the main reference index m_v and the “bigger” angular momentum $\ell_{\delta_1} \in \mathcal{J}_\mu^{2+}$, but not on the “smaller” angular momenta $\ell_{\delta_3}, \ell_{\delta_4} \in \mathcal{J}_\mu^{2-}$.

For a given scale attribution μ in (4.17) and fixed indices in \mathcal{M}_μ we can now evaluate the telescopic sum over the angular momenta in \mathcal{J}_μ . With respect to the previously defined order “>” let $\delta_{L'} > \delta_{L'-1} > \dots > \delta_2 > \delta_1$ be the loop lines. The summation is performed from the smaller to larger labels:

$$\begin{aligned}
& \sum_{\mathcal{J}_\mu} \prod_{\delta \in G} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \chi(\mathcal{M}_\mu, \mathcal{J}_\mu) \\
& \leq \sum_{\mathcal{J}_\mu^{\delta_1+}} \left\{ \max_{\ell_{\delta_1}} \left(\prod_{\delta \in \mathcal{T}_\mu} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \right) \right. \\
& \times \max_{\ell_{\delta_1}} \left(\prod_{\delta \in \mathcal{L}_\mu, \delta > \delta_1} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \right) \\
& \times \left. \sum_{\ell_{\delta_1}} G_{m_{\delta_1}(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta_1+}), n_{\delta_1}(\mathcal{M}_\mu, \mathcal{J}_\mu); k_{\delta_1}(\mathcal{M}_\mu, \mathcal{J}_\mu), m_{\delta_1}(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta_1+}) + \ell_{\delta_1}}^{i_{\delta_1}} \right\} \chi(\mathcal{M}_\mu, \mathcal{J}_\mu) \\
& \leq \sum_{\mathcal{J}_\mu^{\delta_2+}} \left\{ \max_{\ell_{\delta_1}, \ell_{\delta_2}} \left(\prod_{\delta \in \mathcal{T}_\mu} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \right) \right. \\
& \times \max_{\ell_{\delta_1}, \ell_{\delta_2}} \left(\prod_{\delta \in \mathcal{L}_\mu, \delta > \delta_2} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \right) \\
& \times \sum_{\ell_{\delta_2}} \left(\max_{\ell_{\delta_1}} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta_2+}), n_{\delta_2}(\mathcal{M}_\mu, \mathcal{J}_\mu); k_{\delta_2}(\mathcal{M}_\mu, \mathcal{J}_\mu), m_{\delta_2}(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta_2+}) + \ell_{\delta_2}}^{i_{\delta_2}} \right) \\
& \times \max_{\ell_{\delta_2}} \sum_{\ell_{\delta_1}} \left(\max_{n_{\delta_1}, k_{\delta_1}} G_{m_{\delta_1}(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta_1+}), n_{\delta_1}; k_{\delta_1}, m_{\delta_1}(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta_1+}) + \ell_{\delta_1}}^{i_{\delta_1}} \right) \left. \right\} \chi(\mathcal{M}_\mu, \mathcal{J}_\mu), \quad (4.20)
\end{aligned}$$

and so on. At the end we arrive at

$$\begin{aligned}
& \sum_{\mathcal{J}_\mu} \prod_{\delta \in G} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \chi(\mathcal{M}_\mu, \mathcal{J}_\mu) \\
& \leq \prod_{\delta \in \mathcal{T}_\mu} \left(\max_{\mathcal{J}_\mu} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \right) \\
& \times \prod_{\delta \in \mathcal{L}_\mu} \left\{ \max_{\mathcal{J}_\mu^{\delta+}} \left(\sum_{\ell_\delta \geq -m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+})} \left(\max_{n_\delta, k_\delta} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}), n_\delta; k_\delta, m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}) + \ell_\delta}^{i_\delta} \right) \right) \right\}. \quad (4.21)
\end{aligned}$$

The positivity constraints in χ are used to fix the correct range of sums over ℓ_δ .

The result is a bound for the \mathcal{J}_μ -summation in (4.17). For tree lines $\delta \in \mathcal{T}_\mu$ where all indices depend on \mathcal{J}_μ the bound, due (4.5), is given by

$$\max_{\mathcal{J}_\mu} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \leq KM^{-i_\delta}, \quad \delta \in \mathcal{T}_\mu, \quad \delta \notin \mathcal{G}_\mu. \quad (4.22)$$

If one of the indices of $\delta \in \mathcal{T}_\mu$ is a main reference index at v , we have

$$\max_{\mathcal{J}_\mu} G_{m_v, n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \leqslant K M^{-i_\delta} e^{-cM^{-i_\delta} \|m_v\|}, \quad \delta \in \mathcal{T}_\mu \cap \mathcal{G}_\mu. \quad (4.23)$$

If two indices of δ are main reference indices at v, v' , we have

$$\max_{\mathcal{J}_\mu} G_{m_v, n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); m_{v'}, l_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu)}^{i_\delta} \leqslant K M^{-i_\delta} e^{-cM^{-i_\delta} (\|m_v\| + \|m_{v'}\|)}, \quad \delta \in \mathcal{T}_\mu, \quad \delta \in \mathcal{G}_\mu \setminus \{\delta\}. \quad (4.24)$$

Next, each summed propagator which corresponds to a line $\delta \in \mathcal{L}_i$ delivers according to (4.21) a factor $K M^{-i_\delta}$,

$$\max_{\mathcal{J}_\mu^{\delta+}} \left(\sum_{\ell_\delta} \left(\max_{\mathcal{J}_\mu^{\delta-}} G_{m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}), n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), m_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu^{\delta+}) + \ell_\delta}^{i_\delta} \right) \right) \leqslant K' M^{-i_\delta}, \quad \delta \in \mathcal{L}_\mu, \quad \delta \notin \mathcal{G}_\mu. \quad (4.25)$$

In the case that $\delta \in \mathcal{L}_i$ is a tadpole at v_i which has the smallest scale index among the set of lines at v_i we obtain from (4.6) the bound

$$\max_{\mathcal{J}_\mu^{\delta+}} \left(\sum_{\ell_\delta} \left(\max_{\mathcal{J}_\mu^{\delta-}} G_{m_v, n_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu); k_\delta(\mathcal{M}_\mu, \mathcal{J}_\mu), m_v + \ell_\delta}^{i_\delta} \right) \right) \leqslant K' M^{-i_\delta} e^{-c' M^{-i_\delta} \|m_v\|}, \quad \delta \in \mathcal{L}_\mu \cap \mathcal{G}_\mu. \quad (4.26)$$

Eventually, there will be indices $m_\varepsilon, n_\varepsilon$ which are fixed as external ones. Each one delivers according to (4.5) and (4.6) an additional factor $e^{-cM^{-i_\varepsilon} \|m_\varepsilon\|}$ and $e^{-cM^{-i_\varepsilon} \|n_\varepsilon\|}$, respectively, because these decays cannot be removed by maximising loop momenta. For external indices which are not connected to internal lines we put $c \equiv 0$.

Altogether, the \mathcal{J}_μ -summation in (4.17) can be estimated by

$$A_G \leqslant \sum_{\mu} \sum_{m_1, \dots, m_{V'-B} \in \mathbb{N}^2} \left(\prod_{\delta' \in \mathcal{G}_\mu} e^{-cM^{-i_{\delta'}} \|m_{v(\delta')}\|} \right) \left(\prod_{\delta \in G} K M^{-i_\delta} \right) \times \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon} \|m_\varepsilon\|} \right) \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon} \|n_\varepsilon\|} \right), \quad (4.27)$$

where $m_{v(\delta')}$ is the main reference index at $\delta' \in \mathcal{G}_\mu$. After summation over $m_1, \dots, m_{V'-B}$ we have

$$A_G \leqslant \sum_{\mu} \frac{K^I}{c^{2(V'-B)}} \left(M^{-\sum_{\delta \in G} i_\delta} \right) \left(M^{2 \sum_{\delta' \in \mathcal{G}_\mu} i_{\delta'}} \right) \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon} \|m_\varepsilon\|} \right) \left(\prod_{\varepsilon=1}^N e^{-cM^{-i_\varepsilon} \|n_\varepsilon\|} \right). \quad (4.28)$$

The dangerous region of the sum over the scale attribution is at large scale indices. To identify this region, we associate to the order (4.14) of lines a sequence of subgraphs $G_I \subset G_{I-1} \subset \dots \subset G_1 = G$ of the original ribbon graph by defining G_γ as the possibly disconnected set of lines $\delta_{\gamma'} \geq \delta_\gamma$, together with all vertices attached to them. To G_γ we associate the scale attribution μ_γ which starts from an irrelevant low-scale cut-off $i_{\gamma-1} \leq i_{\gamma'}$. We conclude from (4.28) that the amplitude A_{G_γ} corresponding to the subgraph G_γ converges if

$$\omega_\gamma := 2(V'_\gamma - B_\gamma) - I_\gamma = 2F_\gamma - 2B_\gamma - I_\gamma = (2 - \frac{N_\gamma}{2}) - 2(2g_\gamma + B_\gamma - 1) \quad (4.29)$$

is negative, where N_γ , V_γ , $I_\gamma = 2V_\gamma - \frac{N_\gamma}{2}$, F_γ and B_γ are the numbers of external legs, vertices, edges, faces and external faces of G_γ , respectively, and $g_\gamma = 1 - \frac{1}{2}(V_\gamma - I_\gamma + F_\gamma)$ is its genus. We have thus proven the following

Theorem 6 *The sum over the scale attribution μ in (4.28) converges if for all subgraphs $G_\gamma \subset G$ we have $\omega_\gamma < 0$.*

For the total graph $\gamma = G$ the power-counting degree becomes $\omega = (2 - \frac{N}{2}) - 2(2g + B - 1)$, which reproduces the power-counting degree derived in [9].

4.6 Subtraction procedure for divergent subgraphs

The power-counting theorem 6 implies that planar subgraphs with two or four external legs are the only ones for which the sum over the scale attribution can be divergent. These graphs require a separate analysis. We first see from Proposition 2 that

- only those planar four-leg subgraphs with *constant index* along the trajectory are marginal,
- only those planar two-leg graphs with *constant index* along the trajectory are relevant,
- only those planar two-leg graphs with an *accumulated index jump of 2* along the trajectory are marginal.

For the other types of graphs there is a sufficient power of M^{-i} through the terms $(M^{-i})^\delta$ in (3.21) which makes the sum over the scale attribution convergent.

For the remaining truly divergent graphs one performs similarly as in the BPHZ scheme a Taylor subtraction about vanishing external indices. For instance, a marginal four-leg graph with amplitude $A_{mn;nk;kl;lm}$ is written as

$$A_{mn;nk;kl;lm} = (A_{mn;nk;kl;lm} - A_{00;00;00;00}) + A_{00;00;00;00} . \quad (4.30)$$

The difference of graphs $A_{mn;nk;kl;lm} - A_{00;00;00;00}$ can be expressed as a linear combination involving the *composite propagators* (3.38). See also [6] for more details. Then, the estimation (3.41) provides an additional factor M^{-i} which makes the sum over the scale attribution for the difference $A_{mn;nk;kl;lm} - A_{00;00;00;00}$ convergent.

Eventually, there remain only the four divergent base functions $A_{\substack{00:00:00:00 \\ 00:00:00:00}}$, $A_{\substack{00:00 \\ 00:00}}$, $(A_{\substack{10:01 \\ 00:00}} - A_{\substack{00:00 \\ 00:00}})$ and $A_{\substack{11:00 \\ 00:00}}$, taking into account the symmetry properties of the model. These are normalised to their “experimentally” determined values: the physical coupling constant, the physical mass, the physical field amplitude and the physical frequency of the harmonic oscillator potential, respectively.

At the end, any graph appearing in the noncommutative ϕ^4 -model has an amplitude which is uniquely expressed by four normalisation conditions as well as convergent sums over the scale attribution. Thus, the model is renormalisable to all orders.

5 Conclusion

For many years, noncommutative quantum field theories were supposed to be ill-behaved due to the UV/IR-mixing problem [4]. Meanwhile, it turned out [9, 6] that at least the Euclidean noncommutative ϕ_4^4 -model is as good as its commutative version: it is renormalisable to all orders. In fact, the noncommutative ϕ_4^4 -model is even better than the commutative version with respect to one important issue: the behaviour of the β -function.

It is well-known that the main obstacle to a rigorous construction of the commutative ϕ_4^4 -model is the non-asymptotic freedom of the theory. The noncommutative model is very different: The computation of the β -function [12] shows that the ratio of the bare coupling constant to the square of the bare frequency parameter remains (at the one loop level) constant over all scales, $\frac{\lambda}{\Omega^2} = \text{const.}$ (This was noticed in [14].) As the bare frequency is bounded by 1, this means that the bare coupling constant is *bounded*. For appropriate renormalised values, the coupling constant can be kept arbitrarily small throughout the renormalisation flow. We are, therefore, optimistic that a rigorous construction of the noncommutative ϕ_4^4 -model will be possible.

In this paper we have undertaken the first important steps in this direction. We have formulated the perturbative renormalisation proof in a language which admits a direct extension to constructive methods. More details about our program are given in [15]. Moreover, our new renormalisation proof is much more efficient than the previous one (by a factor of 3 when looking at the number of pages). Eventually, we have established analytical bounds for the asymptotic behaviour of the propagator which before were only established numerically.

A The first slice

Let us now have a look at the first slice, that is the region $M^{-1} \leq \alpha \leq 1$. For technical reasons, we divide it into two subregions, namely $M^{-1} \leq \alpha \leq a$ called the intermediate region and $a \leq \alpha \leq 1$ called the bulk.

A.1 The intermediate region

Let us call $G^{\text{inter}}(a)$ the propagator restricted to the region $M^{-1} \leq \alpha \leq a$. We have then the following Lemma:

Lemma 7 *Let $\Omega_{\min}(a) = 1 + \frac{18}{25} \frac{\sqrt{1-a}}{a} \ln(1-a)$. For each $\Omega \in [\frac{2}{3}, 1]$ there exists an $a \in [0, 1]$ with $\Omega \in [\Omega_{\min}(a), 1]$ and a constant $k > 0$ such that we have the uniform bound*

$$G_{m,m+h;l+h,l}^{\text{inter}}(a) \leq K e^{-k\|m+l+h\|} . \quad (\text{A.1})$$

Proof. Here we can't safely approximate the expressions involving α by their expansions about $\alpha = 0$. But we have, using (3.5) and (3.10)

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right)^{m+l+h} \exp \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} (m+l+h) \right) . \quad (\text{A.2})$$

To obtain the claimed result, we have to prove that

$$f(\alpha, \Omega) = \frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)} + \ln \left(\frac{\sqrt{1-\alpha}}{1+C\alpha} \right) < 0 . \quad (\text{A.3})$$

We have

$$\begin{aligned} f(\alpha, \Omega) &= \frac{1-\Omega^2}{4\Omega} \frac{\alpha}{\sqrt{1-\alpha}} + \ln \sqrt{1-\alpha} + \ln \left(\frac{4\Omega}{4\Omega + (1-\Omega)^2\alpha} \right) \\ &\leq \frac{1-\Omega^2}{4\Omega} \frac{\alpha}{\sqrt{1-\alpha}} + \ln \sqrt{1-\alpha} . \end{aligned} \quad (\text{A.4})$$

In the preceding sections, we bounded Ω from below by $2/3$. Thus,

$$f(\alpha, \Omega) \leq \frac{5}{8}(1-\Omega) \frac{\alpha}{\sqrt{1-\alpha}} + \frac{1}{2} \ln(1-\alpha) . \quad (\text{A.5})$$

Thus $\forall \Omega \geq \Omega_1(\alpha) = 1 + \frac{4}{5} \frac{\sqrt{1-\alpha}}{\alpha} \ln(1-\alpha)$, $h \leq 0$. An expansion of $\frac{\partial \Omega_1}{\partial \alpha}$ about $\alpha = 0$ shows that Ω_1 increases with α . This means that $h(\alpha, \Omega) \leq -k$ for all $\alpha \in [M^{-1}, a]$ and all

$$\Omega \geq \Omega_{\min}(a) = 1 + \frac{18}{25} \frac{\sqrt{1-a}}{a} \ln(1-a) . \quad (\text{A.6})$$

This finishes the proof. □

A.2 The bulk

Let us call $G^{\text{bulk}}(a)$ the propagator restricted to the region $a \leq \alpha \leq 1$. We have then the following Lemma:

Lemma 8 Let $\Omega_{\min}(a) = 1 + \frac{18}{25} \frac{\sqrt{1-a}}{a} \ln(1-a)$ and $\Omega_{\max}(a) = 1 - \frac{2\sqrt{1-a}}{a}$. For each $\Omega \in [\frac{2}{3}, 1)$ there exists an $a \in [\frac{6}{3+\sqrt{10}}, 1]$ with $\Omega \in [\Omega_{\min}(a), \Omega_{\max}(a)]$ and a constant $k > 0$ such that we have the uniform bound

$$G_{m,m+h;l+h,l}^{bulk}(a) \leq K e^{-k\|m+l+h\|}. \quad (\text{A.7})$$

Proof Using (3.9) and (3.10) we obtain

$$G_{m,m+h;l+h,l}^{(\alpha)} \leq \left(\frac{\alpha(1-\Omega^2)}{4\Omega + (1-\Omega)^2\alpha} \right)^{m+l+h} \exp \left(\frac{4\Omega\sqrt{1-\alpha}}{\alpha(1-\Omega^2)}(m+l+h) \right). \quad (\text{A.8})$$

Like in the intermediate region, we have now to study

$$g(\alpha, \Omega) = \frac{4\Omega}{1-\Omega^2} \frac{\sqrt{1-\alpha}}{\alpha} + \ln \left(\frac{(1-\Omega^2)\alpha}{4\Omega + (1-\Omega)^2\alpha} \right) \quad (\text{A.9})$$

in order to know in which (α, Ω) -region it is negative.

$$\begin{aligned} g(\alpha, \Omega) &\leq \frac{2}{1-\Omega} \frac{\sqrt{1-\alpha}}{\alpha} + \ln \left(\frac{1-\Omega^2}{4\Omega} \right) \\ &\leq \frac{2}{1-\Omega} \frac{\sqrt{1-a}}{a} + \ln \left(\frac{5}{8}(1-\Omega) \right) \equiv i(a, \Omega). \end{aligned} \quad (\text{A.10})$$

A simple analysis of i considered as a function of Ω allows to draw

Ω	0	Ω_2	1
$\frac{\partial i}{\partial \Omega}$	-		+
$i(a, \Omega)$	$i(a, 0)$	$i(a, \Omega_2)$	$+\infty$

We have taken into account that M can be as small as $\frac{1}{2}(\sqrt{5} + 1)$ so that $\frac{\partial i}{\partial \Omega}|_{\Omega=0} < 0$. We have $\Omega_2 = 1 - \frac{2\sqrt{1-a}}{a}$. Let us now remember what we are looking for. We want to determine whether there exists a (a, Ω) -region where $i(a, \Omega) \leq 0$. Clearly, if $i(a, 0) \leq 0$, then $i(a, \Omega) < 0$ for all $\Omega \in (0, \Omega_2]$. We have

$$i(a, 0) \leq 0 \iff a \geq \frac{2}{\ln^2 \frac{8}{5}} \left(-1 + \sqrt{1 + \ln^2 \frac{8}{5}} \right) \simeq 0.9502. \quad (\text{A.11})$$

Actually, the derivation of (A.10) used $\Omega \geq \frac{2}{3}$ so that the relevant condition is

$$\Omega_2 \geq \frac{2}{3} \iff a \geq \frac{6}{3 + \sqrt{10}} \simeq 0.973666. \quad (\text{A.12})$$

The study of the behaviour of the propagator in the intermediate region gave us a lower bound (cf. Lemma 7) for Ω which was Ω_{\min} . For compatibility of the intermediate and bulk regions, we need $\Omega_{\min} < \Omega_2$, which is satisfied under the condition (A.12). This provides a uniform exponential decay of the propagator at least for $a \geq \frac{6}{3+\sqrt{10}}$ and $\Omega \in [\Omega_{\min}(a), \Omega_{\max}(a)]$ with $\Omega_{\max}(a) = \Omega_2 = 1 - \frac{2\sqrt{1-a}}{a}$.

The limit $a \rightarrow 1$ allows to study the propagator when Ω is close to 1 thanks to $\lim_{a \rightarrow 1} \Omega_{\max}(a) = 1$. The case $\Omega = 1$ is treated in the next section. \square

B The case $\Omega = 1$

The case $\Omega = 1$ can be directly treated. According to (3.2), only the terms with $u = m = l$ survive:

$$G_{mn;kl}^{\Omega=1} = \frac{\theta}{8} \int_0^1 d\alpha (1-\alpha)^{\frac{\mu_0^2 \theta}{8} + (\frac{D}{4}-1) + \frac{1}{2}(\|m\| + \|k\|)} \delta_{ml} \delta_{nk} \quad (\text{B.1})$$

$$= \frac{\delta_{ml} \delta_{nk}}{\mu_0^2 + \frac{2}{\theta}(\|m\| + \|n\| + \|k\| + \|l\| + \frac{D}{2})}. \quad (\text{B.2})$$

The exponential decay of the propagator in any index is easily obtained from (B.1) for all slices. Moreover, the l -sum is trivial to perform due to the index conservation δ_{ml} at each trajectory.

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