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# Shock Profiles for Non Equilibrium Radiating Gases

Chunjin LIN<sup>†</sup>, Jean-François COULOMBEL<sup>†‡</sup>, Thierry GOUDON<sup>†‡</sup>

<sup>†</sup> Team SIMPAF–INRIA Futurs & Université Lille 1, Laboratoire Paul Painlevé, UMR CNRS 8524

Cité scientifique, 59655 VILLENEUVE D’ASCQ Cedex, France

<sup>‡</sup> CNRS

E-mails: [chunjin.lin@math.univ-lille1.fr](mailto:chunjin.lin@math.univ-lille1.fr), [jfcoulom@math.univ-lille1.fr](mailto:jfcoulom@math.univ-lille1.fr),

[thierry.goudon@math.univ-lille1.fr](mailto:thierry.goudon@math.univ-lille1.fr)

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## Abstract

We study a model of radiating gases that describes the interaction of an inviscid gas with photons. We show the existence of smooth traveling waves called ‘shock profiles’, when the strength of the shock is small. Moreover, we prove that the regularity of the traveling wave increases when the strength of the shock tends to zero.

## 1 Introduction and main results

We are interested in a system of PDEs describing astrophysical flows, where a gas interacts with radiation through energy exchanges. Similar questions arise in the modeling of reentry problems, or high temperature combustion phenomena. The gas is described by its density  $\rho > 0$ , its bulk velocity  $u \in \mathbb{R}$ , and its specific total energy  $E = e + u^2/2$ , where  $e$  stands for the specific internal energy. (Our analysis is restricted to a one-dimensional framework, but this is not a loss of generality, as shown below.) We consider a situation where the gas is not in thermodynamical equilibrium with the radiations, which are thus described by their own energy  $n$ . The evolution of the gas flow is governed by the system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + P u) = n - \theta^4, \end{cases} \quad (1)$$

where the right-hand side in the last equation accounts for energy exchanges with the radiations,  $P$  being the pressure of the gas, and  $\theta$  its temperature. Throughout the paper, we always assume that the gas obeys the perfect gas pressure law:

$$P = R \rho \theta = (\gamma - 1) \rho e, \quad (2)$$

where  $R$  is the perfect gas constant, and  $\gamma > 1$  is the ratio of the specific heats at constant pressure, and volume. This assumption yields many algebraic simplifications, but we believe that our results still hold for a general pressure law satisfying the usual requirements of thermodynamics. System (1) is completed by considering that radiations are described by a stationary diffusion regime that reads:

$$-\partial_{xx} n = \theta^4 - n. \quad (3)$$

We detail in Appendix A how the system (1), (3) can be formally derived by asymptotics arguments, starting from a more complete system involving a kinetic equation for the specific intensity of radiation.

As a matter of fact, the operator  $(1 - \partial_{xx})$  can be explicitly inverted, and (3) can be recast as a convolution:

$$n(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \theta(t, y)^4 dy. \quad (4)$$

Let us introduce the quantity:

$$q(t, x) := -\partial_x n(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \operatorname{sgn}(x-y) \theta(t, y)^4 dy, \quad (5)$$

where  $\operatorname{sgn}$  is the sign function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

The quantity  $q$  can be interpreted as the radiative heat flux. Then, we can rewrite (1), (3) as follows:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + P u + q) = 0, \end{cases} \quad (6)$$

with  $q$  given by (5). Recall that  $E = e + u^2/2$ , and  $P$  is given by (2).

In this paper, we address the question of the influence of the energy exchanges on the structure of shock waves. More precisely, let us consider given states at infinity  $(\rho_{\pm}, u_{\pm}, e_{\pm})$ , and let us assume that:

$$(\rho, u, e)(t, x) = \begin{cases} (\rho_-, u_-, e_-) & \text{if } x < \sigma t, \\ (\rho_+, u_+, e_+) & \text{if } x > \sigma t, \end{cases} \quad (7)$$

is a shock wave, with speed  $\sigma$ , solution to the standard Euler equations (that is, system (6) with  $q \equiv 0$ ). We refer to [Lax73, Ser99, Smo94] for a detailed study of shock waves for the Euler equations. The question we ask is the following: does there exist a traveling wave  $(\rho, u, e)(x - \sigma t)$  solution to (6), with  $q$  given by (5), that satisfies the asymptotic conditions:

$$\lim_{\xi \rightarrow \pm\infty} (\rho, u, e)(\xi) = (\rho_{\pm}, u_{\pm}, e_{\pm}). \quad (8)$$

In other words, we are concerned with the existence of a shock profile, and a natural expectation (at least for shocks of small amplitude) is that the step shock (7) is smoothed into a continuous profile, due to the dissipation introduced by (3). The analogous problem for the compressible Navier-Stokes system has been treated a long time ago, see [Gil51], without any smallness assumption on the shock wave. Concerning radiative transfer, a formal analysis of shock profiles has been performed in [Hea63], together with rough numerical simulations. (We refer also to [Zel66, Mih84] for the physical background.) The main purpose of this work is to make the analysis of [Hea63] rigorous. Since we are only concerned in this paper with the existence of shock profiles, and not with their stability, the problem is purely one-dimensional (due to the Galilean invariance of the Euler equations). This is why we have directly restricted to the one-dimensional case. However, the formal derivation of Appendix A is made in several space dimensions.

Before stating our main results, let us mention that a simplified version of (6), (5) has been introduced, and studied in [ST92] and later in [Kaw99b, Kaw99a]. This 'baby-model' consists in a Burgers type equation:

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = -\partial_x q,$$

coupled to the diffusion equation:

$$-\partial_{xx} q + q = -\partial_x u.$$

These two equations can be seen as a scalar version of (6), (5) since they can be recast as:

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = Ku - u, \quad (9)$$

where  $K$  is the integral operator already arising in (5):

$$Ku(t, x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} u(t, y) dy.$$

The thorough study of (9) has motivated a lot of works; we mention in particular [Nis00, Lat03, Liu01, Ser03]. Clearly (9) can be seen as a prototype for discussing (6), (5); nevertheless, replacing (6), and (5) by (9) has two important consequences: the equation becomes scalar, and the 'diffusion'  $K - 1$  applies to the unique unknown (while in (6), the 'diffusion' appears through the radiative heat flux  $q$  only in the third equation). Our work is a first attempt to extend the known results for (9) to the more physical model (6), (5).

Let us now state our main results. The first result deals with the existence of smooth shock profiles when the strength of the shock is small:

**Theorem 1.** *Let  $\gamma$  satisfy*

$$1 < \gamma < \frac{\sqrt{7} + 1}{\sqrt{7} - 1} \simeq 2.215,$$

*and let  $(\rho_-, u_-, e_-)$  be fixed. Then there exists a positive constant  $\delta$  (that depends on  $(\rho_-, u_-, e_-)$ , and  $\gamma$ ) such that, for all state  $(\rho_+, u_+, e_+)$  verifying:*

- $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta$ ,
- *the function (7) is a shock wave, with speed  $\sigma$ , for the (standard) Euler equations,*

*then there exists a  $C^2$  traveling wave  $(\rho, u, e)(x - \sigma t)$  solution to (6), (5), (8).*

As in the study of the 'baby-model' (9), the existence of a smooth shock profile is linked to a smallness assumption on the shock strength, see [Kaw99b]. Here the smallness parameter  $\delta$  may depend on the state  $(\rho_-, u_-, e_-)$ , while for (9), the smallness parameter is uniform (and even explicit!).

The restriction on the adiabatic constant  $\gamma$  might be unnecessary, but it simplifies the proof, and it covers the main physical cases  $1 < \gamma \leq 2$ .

Our second result is also in the spirit of [Kaw99b], and deals with the smoothness of the shock profile constructed in the previous Theorem:

**Theorem 2.** *Let  $\gamma$  satisfy*

$$1 < \gamma < \frac{\sqrt{7} + 1}{\sqrt{7} - 1} \simeq 2.215,$$

*and let  $(\rho_-, u_-, e_-)$  be fixed. Then there exists a decreasing sequence of positive numbers  $(\delta_n)_{n \in \mathbb{N}}$  (the sequence depends on  $(\rho_-, u_-, e_-)$ , and  $\gamma$ ) such that, for all  $n \in \mathbb{N}$ , and for all state  $(\rho_+, u_+, e_+)$  verifying:*

- $\|(\rho_+, u_+, e_+) - (\rho_-, u_-, e_-)\| \leq \delta_n$ ,
- *the function (7) is a shock wave, with speed  $\sigma$ , for the (standard) Euler equations,*

*then there exists a  $C^{n+2}$  traveling wave  $(\rho, u, e)(x - \sigma t)$  solution to (6), (5), (8).*

To a large extent, our analysis follows the arguments of [Hea63], [ST92] and [Kaw99b]. The proof of Theorem 1 is presented in Section 2, while Section 3 is devoted to the proof of Theorem 2. The investigation of strong shocks, as well as stability issues will be addressed in a forthcoming work.

## 2 Existence of smooth shock profiles

In this section, we prove Theorem 1. We first recall some basic facts on shock waves for the Euler equations. Then, we make some transformations on the traveling wave equation. Eventually, we prove Theorem 1 by using an auxiliary system of Ordinary Differential Equations, that is introduced and studied in the last paragraph of this section.

### 2.1 Shock wave solutions to the Euler equations

In this paragraph, we recall some basic facts about the (entropic) shock wave solutions to the Euler equations:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0, \\ \partial_t(\rho E) + \partial_x(\rho E u + P u) = 0, \end{cases}$$

where  $P$ , and  $E$  are given as in the introduction. We refer to [Lax73, Ser99, Smo94] for all the details, and omit the calculations. In all what follows, we only consider shock waves that satisfy Lax shock inequalities. We shall thus speak of 1-shock waves, or 3-shock waves.

We consider a fixed 'left' state  $(\rho_-, u_-, e_-)$ . Then the 'right' states  $(\rho_+, u_+, e_+)$  such that  $(\rho_\pm, u_\pm, e_\pm)$  define a 1-shock wave, with speed  $\sigma$ , is a half-curve initiating at  $(\rho_-, u_-, e_-)$ . Introducing the notation  $v_\pm = u_\pm - \sigma$ , the Rankine-Hugoniot jump conditions can be rewritten as:

$$\begin{cases} \rho_+ v_+ = \rho_- v_- =: j, \\ \rho_+ v_+^2 + P_+ = \rho_- v_-^2 + P_- =: j C_1, \\ \rho_+ v_+ (e_+ + \frac{v_+^2}{2}) + P_+ v_+ = \rho_- v_- (e_- + \frac{v_-^2}{2}) + P_- v_- =: j C_2. \end{cases} \quad (10)$$

Observe that  $v_-$  does not only depend on the 'left' state  $(\rho_-, u_-, e_-)$ , but also on  $(\rho_+, u_+, e_+)$ , because  $v_-$  is defined with the help of the shock speed  $\sigma$ . Consequently, the constants  $j$ ,  $C_1$ , and  $C_2$  depend on both  $(\rho_-, u_-, e_-)$ , and  $(\rho_+, u_+, e_+)$ .

For 1-shocks, that is when the inequalities:

$$u_+ - c_+ < \sigma < u_+, \quad \sigma < u_- - c_-,$$

are satisfied ( $c$  denotes the sound speed), all quantities  $j$ ,  $C_1$ , and  $C_2$  are positive. Moreover, when the strength of the shock tends to zero, that is when  $(\rho_+, u_+, e_+)$  tends to  $(\rho_-, u_-, e_-)$ , one has

$$\begin{pmatrix} \sigma \\ j \\ C_1 \\ C_2 \end{pmatrix} \longrightarrow \begin{pmatrix} u_- - c_- \\ \rho_- c_- \\ c_- + (\gamma - 1)e_-/c_- \\ \gamma e_- + c_-^2/2 \end{pmatrix}. \quad (11)$$

Consequently, when the strength of the shock is small, all quantities  $j$ ,  $C_1$ ,  $C_2$  are bounded away from zero. Recall also that 1-shocks are compressive, in the sense that  $\rho_+ > \rho_-$ . This inequality immediately implies that  $0 < v_+ < v_-$ . Eventually, the strength of the shock tends to zero if, and only if  $u_+$  tends to  $u_-$  (in that case, we also have  $\rho_+ \rightarrow \rho_-$ , and  $e_+ \rightarrow e_-$  because of the Rankine-Hugoniot jump conditions).

In all what follows, we limit our discussion to the case of 1-shocks for simplicity, but the extension to 3-shocks is immediate.

### 2.2 Reduction of the traveling wave equation

In this paragraph, we derive, and transform the equation satisfied by traveling wave solutions to (6), (5). A traveling wave solution to (6), (5) with speed  $\sigma$  is a solution  $(\rho, u, e)(x - \sigma t)$ . For such solutions, the radiative heat flux  $q$  also depends on the sole variable  $x - \sigma t$ :

$$q(x - \sigma t) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x - \sigma t - y|} \operatorname{sgn}(x - \sigma t - y) \theta(y)^4 dy,$$

and (6) reads:

$$\begin{cases} (\rho(u - \sigma))' = 0, \\ (\rho u(u - \sigma) + (\gamma - 1)\rho e)' = 0, \\ (\rho(e + \frac{u^2}{2})(u - \sigma) + (\gamma - 1)\rho e u + q)' = 0, \end{cases}$$

where  $'$  denotes differentiation with respect to the variable  $\xi = x - \sigma t$ . Introducing the new unknown  $v = u - \sigma$ , the above system is easily seen to be equivalent to:

$$\begin{cases} (\rho v)' = 0, \\ (\rho v^2 + (\gamma - 1)\rho e)' = 0, \\ (\rho v(e + \frac{v^2}{2}) + (\gamma - 1)\rho v e + q)' = 0. \end{cases} \quad (12)$$

Since we are looking for a shock profile, the traveling wave solution should also satisfy:

$$\lim_{\xi \rightarrow \pm\infty} (\rho, v, e) = (\rho_{\pm}, v_{\pm}, e_{\pm}), \quad (13)$$

where  $v_{\pm} = u_{\pm} - \sigma$ , and  $(\rho_{\pm}, u_{\pm}, e_{\pm})$  defines a 1-shock wave with speed  $\sigma$  for the Euler equations. Notice that the quantity  $a = |u_+ - u_-|/2$ , that measures the strength of the shock, is invariant with respect to our change of velocity, that is  $a = |u_+ - u_-|/2 = |v_+ - v_-|/2$ . Recall also that for 1-shocks, there holds  $v_- > v_+ > 0$ .

Observing that we have:

$$q(\xi) = \frac{1}{2} \int_0^{+\infty} e^{-y} (\theta(\xi - y)^4 - \theta(\xi + y)^4) dy,$$

we conclude that  $q$  tends to zero at  $\pm\infty$  by Lebesgue's Theorem (because  $\theta$  is necessarily bounded since it has finite limits at  $\pm\infty$ ). We can thus integrate the system (12)-(13) once, and (12) reads equivalently:

$$\begin{cases} (\rho v)(\xi) = j, \\ (\rho v^2 + (\gamma - 1)\rho e)(\xi) = j C_1, \\ (\rho v(e + \frac{v^2}{2}) + (\gamma - 1)\rho v e + q)(\xi) = j C_2, \end{cases} \quad (14)$$

where the constants  $j$ ,  $C_1$  and  $C_2$  are given by the Rankine-Hugoniot conditions (10). For small shocks, the positive constants  $j$ ,  $C_1$ ,  $C_2$  have the asymptotic behavior (11).

From the two first equations of (14), we derive the relations:

$$\rho(\xi) = \frac{j}{v(\xi)}, \quad e(\xi) = \frac{(C_1 - v(\xi))v(\xi)}{\gamma - 1}.$$

The third equation of (14) thus reduces to:

$$v(\xi)^2 - \frac{2\gamma C_1}{\gamma + 1} v(\xi) + \frac{2(\gamma - 1)C_2}{\gamma + 1} = \frac{2(\gamma - 1)}{j(\gamma + 1)} q(\xi). \quad (15)$$

Using the equation of state (2), as well as the second equation of (14), we get:

$$\theta(\xi) = \frac{(\gamma - 1)e(\xi)}{R} = \frac{(C_1 - v(\xi))v(\xi)}{R}.$$

Consequently, (15) can be recast as an integral equation with a single unknown function  $v$ :

$$v(\xi)^2 - \frac{2\gamma C_1}{\gamma + 1} v(\xi) + \frac{2(\gamma - 1)C_2}{\gamma + 1} = \frac{(\gamma - 1)}{j(\gamma + 1)R^4} \int_{\mathbb{R}} e^{-|\xi - y|} \operatorname{sgn}(\xi - y) v(y)^4 (C_1 - v(y))^4 dy. \quad (16)$$

We are searching for a solution  $v$  to (16), that satisfies the asymptotic conditions  $v(\xi) \rightarrow v_{\pm}$ , as  $\xi \rightarrow \pm\infty$ .

**Remark 1.** If we find a  $C^2$  solution  $v$  to (16) that does not vanish, and that satisfies  $v(\xi) \rightarrow v_{\pm}$  as  $\xi \rightarrow \pm\infty$ , then we obtain a  $C^2$  shock profile  $(\rho, u, e)$  by simply setting:

$$\rho(\xi) = \frac{j}{v(\xi)}, \quad u(\xi) = v(\xi) + \sigma, \quad e(\xi) = \frac{(C_1 - v(\xi))v(\xi)}{\gamma - 1}.$$

In particular, if  $v(\xi) \in [v_+, v_-]$  for all  $\xi$ , then  $v$  does not vanish.

**Remark 2.** Since the heat flux  $q$  vanishes at  $\pm\infty$ , (16) can be also rewritten as:

$$(v(\xi) - v_-)(v(\xi) - v_+) = \frac{(\gamma - 1)}{j(\gamma + 1)R^4} \int_{\mathbb{R}} e^{-|\xi - y|} \operatorname{sgn}(\xi - y) v(y)^4 (C_1 - v(y))^4 dy.$$

We are going to rewrite (16) as a second order differential equation, that will be easier to study than the integral equation (16). Indeed, assuming that  $v$  is a  $C^2$  function of  $\xi$ , and differentiating twice (16) with respect to  $\xi$ , we get (see [Hea63] for the details of the computations):

$$\left(v - \frac{\gamma C_1}{\gamma + 1}\right)v'' + (v')^2 - \frac{4(\gamma - 1)}{j(\gamma + 1)R^4} (C_1 - v)^3 v^3 (C_1 - 2v)v' - \frac{1}{2}(v - v_-)(v - v_+) = 0. \quad (17)$$

Conversely, if  $v$  is a  $C^2$  solution to (17) that satisfies  $v(\xi) \rightarrow v_{\pm}$  as  $\xi \rightarrow \pm\infty$ , then  $v$  is also a solution to (16). If in addition  $v$  takes its values in the interval  $[v_+, v_-]$ , then we can construct a  $C^2$  shock profile, and thus prove Theorem 1.

The differential equation (17) can be simplified by introducing the new unknown function  $\hat{v} = v - (v_- + v_+)/2$ , and by rewriting the second order differential equation as a first order system:

$$\begin{cases} \hat{v}' &= w, \\ \hat{v}w' &= -w^2 - f(\hat{v})w + \frac{\hat{v}^2 - a^2}{2}, \end{cases} \quad (18)$$

where  $f$  is the following polynomial function:

$$f(\hat{v}) = \frac{4(\gamma - 1)}{jR^4(\gamma + 1)} \left(\frac{C_1}{\gamma + 1} - \hat{v}\right)^3 \left(\hat{v} + \frac{\gamma C_1}{\gamma + 1}\right)^3 \left(2\hat{v} + \frac{(\gamma - 1)C_1}{\gamma + 1}\right). \quad (19)$$

We recall that  $a = |v_- - v_+|/2$ , and that  $a$  measures the strength of the shock.

**Remark 3.** The asymptotic behavior (11) of  $j$ ,  $C_1$ , and  $C_2$  shows that when the strength of the shock tends to zero ( $a \rightarrow 0^+$ ), the limit of  $f(0)$  is given by:

$$f(0) \rightarrow \frac{4\gamma^3(\gamma - 1)^2}{R^4(\gamma + 1)^8} \frac{(c_- + (\gamma - 1)\frac{e_-}{c_-})^7}{\rho_- c_-} > 0.$$

Since  $v_+ < v_-$  for a 1-shock, we are searching for a solution to (18) that is defined on all  $\mathbb{R}$ , and that satisfies:

$$\lim_{\xi \rightarrow -\infty} (\hat{v}, w)(\xi) = (a, 0), \quad \lim_{\xi \rightarrow +\infty} (\hat{v}, w)(\xi) = (-a, 0). \quad (20)$$

To prove Theorem 1, we are thus reduced to showing the existence of a heteroclinic orbit for (18) that connects the stationary solutions  $(\pm a, 0)$ . Due to the previous transformation  $\hat{v} = v - (v_- + v_+)/2$ , if  $\hat{v}$  takes its values in  $[-a, a]$ , then  $v = \hat{v} + (v_- + v_+)/2$  will take its values in the interval  $[v_+, v_-]$ , and therefore will not vanish.

**Remark 4.** The system (18) is 'singular' at  $\hat{v} = 0$ . Nevertheless, we are searching for a smooth solution connecting  $(\pm a, 0)$ , so that  $\hat{v}$  vanishes in at least one point. Because  $w' = \hat{v}''$  should also have a limit at this point, a  $C^2$  shock profile can exist only if the equation:

$$w^2 + f(0)w + \frac{a^2}{2} = 0,$$

has real roots. The corresponding discriminant condition turns out to be much less simple than the one found in [Kaw99b] for the 'baby model' (9). (In particular,  $f(0)$  depends on the shock through the constants  $j$ , and  $C_1$ ). This is a first 'nonexplicit' restriction on the shock strength to derive the existence of a smooth shock profile.

Due to the singular nature of the system (18) at  $\hat{v} = 0$ , it is more convenient to work on an auxiliary system of ODEs, where the singularity has been eliminated (at least formally) thanks to a change of variables. This procedure was already used in [Kaw99b]. In the next paragraph, we shall introduce this auxiliary system, and complete the proof of Theorem 1.

### 2.3 Existence of a heteroclinic orbit

We begin with a result on an auxiliary system of ODEs, where the singularity at  $\hat{v} = 0$  has been eliminated:

**Proposition 1.** *Assume that  $\gamma$  satisfies  $1 < \gamma < (\sqrt{7} + 1)/(\sqrt{7} - 1)$ , and consider the following system of ODEs:*

$$\begin{cases} V' &= V W, \\ W' &= -W^2 - f(V) W + \frac{(V^2 - a^2)}{2}. \end{cases} \quad (21)$$

*There exists a positive constant  $a_0$ , that depends only on  $(\rho_-, u_-, e_-)$ , and  $\gamma$  such that if the shock strength  $a$  satisfies  $a \in (0, a_0]$ , the following properties hold:*

- $f(0)^2 - 2a^2 > 0$ , and we define  $w_0 := (-f(0) + \sqrt{f(0)^2 - 2a^2})/2 < 0$ .
- *There exists a solution  $(V_b, W_b)$  to (21) that is defined on all  $\mathbb{R}$ , and that satisfies*

$$\lim_{\eta \rightarrow -\infty} (V_b, W_b)(\eta) = (a, 0), \quad \lim_{\eta \rightarrow +\infty} (V_b, W_b)(\eta) = (0, w_0).$$

*Furthermore,  $V_b$  is decreasing, and the convergence of  $V_b$  to 0 as  $\eta \rightarrow +\infty$  is exponential.*

- *There exists a solution  $(V_{\sharp}, W_{\sharp})$  to (21) that is defined on all  $\mathbb{R}$ , and that satisfies*

$$\lim_{\eta \rightarrow -\infty} (V_{\sharp}, W_{\sharp})(\eta) = (-a, 0), \quad \lim_{\eta \rightarrow +\infty} (V_{\sharp}, W_{\sharp})(\eta) = (0, w_0).$$

*Furthermore,  $V_{\sharp}$  is increasing, and the convergence of  $V_{\sharp}$  to 0 as  $\eta \rightarrow +\infty$  is exponential.*

Assuming that the result of Proposition 1 holds, the existence of a heteroclinic orbit for (18) connecting  $(\pm a, 0)$  can be derived by following the analysis of [ST92, Kaw99b]. We briefly recall the method. Using the solution  $(V_b, W_b)$ , we introduce the change of variable:

$$\Xi_b(\eta) = - \int_{\eta}^{+\infty} V_b(\zeta) d\zeta.$$

Since  $V_b$  tends to 0 exponentially as  $\eta$  tends to  $+\infty$ ,  $\Xi_b$  is well-defined, and it is an increasing  $C^\infty$  diffeomorphism from  $\mathbb{R}$  to  $(-\infty, 0)$ . Then  $(\hat{v}_b, w_b) := (V_b, W_b) \circ \Xi_b^{-1}$  is a  $C^\infty$  solution to (18) on the interval  $(-\infty, 0)$ , that satisfies:

$$\lim_{\xi \rightarrow -\infty} (\hat{v}_b, w_b)(\xi) = (a, 0), \quad \lim_{\xi \rightarrow 0^-} (\hat{v}_b, w_b)(\xi) = (0, w_0).$$

Similarly, with the help of the solution  $(V_{\sharp}, W_{\sharp})$  we can construct a  $C^\infty$ , decreasing diffeomorphism  $\Xi_{\sharp}$  from  $\mathbb{R}$  to  $(0, +\infty)$ , and a  $C^\infty$  solution  $(\hat{v}_{\sharp}, w_{\sharp})$  to (18) on the interval  $(0, +\infty)$ . This solution  $(\hat{v}_{\sharp}, w_{\sharp})$  connects  $(0, w_0)$  and  $(-a, 0)$ , as  $\xi$  varies from  $0^+$  to  $+\infty$ . Let us now 'glue' the solutions  $(\hat{v}_b, w_b)$ , and  $(\hat{v}_{\sharp}, w_{\sharp})$ , by defining:

$$(\hat{v}, w)(\xi) := \begin{cases} (\hat{v}_b, w_b)(\xi) & \text{if } \xi < 0, \\ (\hat{v}_{\sharp}, w_{\sharp})(\xi) & \text{if } \xi > 0, \end{cases} \quad (22)$$

and extend the functions  $\hat{v}$ , and  $w$  at 0 by setting  $(\hat{v}, w)(0) = (0, w_0)$ . In this way,  $\hat{v}$ , and  $w$  are continuous on  $\mathbb{R}$ , and  $C^\infty$  on  $\mathbb{R} \setminus \{0\}$ . It remains to show that  $\hat{v} \in C^2(\mathbb{R})$ , that  $(\hat{v}, w)$  solves (18) on  $\mathbb{R}$ , and that  $\hat{v}$  takes its values in  $(-a, a)$ .



Observe first of all that  $\hat{v}$  is a decreasing function, because of the monotonicity properties of  $V_b, V_{\#}, \Xi_b, \Xi_{\#}$ . Using the asymptotic behavior of  $V_b, V_{\#}$  at  $-\infty$ , we get that  $\hat{v}(\xi) \in (-a, a)$  for all  $\xi \in \mathbb{R}$ .

Let us now note that the above construction of  $(\hat{v}, w)$  shows that  $(\hat{v}, w)$  is a solution to (18) on  $\mathbb{R} \setminus \{0\}$ . In particular,  $\hat{v}'(\xi) = w(\xi)$  if  $\xi \neq 0$ . Moreover,  $w$  is continuous on  $\mathbb{R}$ , so  $\hat{v} \in C^1(\mathbb{R})$ , and  $\hat{v}'(0) = w(0) = w_0$ . To prove that  $\hat{v} \in C^2(\mathbb{R})$ , it is sufficient to show that  $w \in C^1(\mathbb{R})$ , which is equivalent to showing that  $w'$  has a limit at 0 (because we already know that  $w$  is  $C^\infty$  on  $\mathbb{R} \setminus \{0\}$ ). To prove that  $w'$  has a limit at 0, we are going to study the asymptotic behavior of  $(V_b, W_b)$ , and  $(V_{\#}, W_{\#})$  at  $+\infty$ . More precisely, let us denote  $U(V, W)$  the vector field associated to the ODE (21):

$$U(V, W) = \begin{pmatrix} V W \\ -W^2 - f(V) W + \frac{V^2 - a^2}{2} \end{pmatrix}, \quad (23)$$

where  $f$  is given by (19). The Jacobian matrix of  $U$  at  $(0, w_0)$  is:

$$\begin{pmatrix} w_0 & 0 \\ -f'(0)w_0 & -2w_0 - f(0) \end{pmatrix} = \begin{pmatrix} \lambda_1^{(0)} & 0 \\ b_0 & \lambda_2^{(0)} \end{pmatrix}.$$

For  $a$  sufficiently small, one checks that  $\lambda_2^{(0)} < \lambda_1^{(0)} < 0$  (see Proposition 1 for the definition of  $w_0$ ). The eigenvectors corresponding to the eigenvalues  $\lambda_1^{(0)}$  and  $\lambda_2^{(0)}$  are:

$$e_1^{(0)} = \begin{pmatrix} f(0) + 3w_0 \\ b_0 \end{pmatrix}, \quad e_2^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The standard theory of autonomous ODEs, see e.g. [Pon75], shows that there are exactly two solutions to (21) that tend to  $(0, w_0)$  as  $\eta$  tends to  $+\infty$ , and that are tangent to the straight line  $(0, w_0) + \mathbb{R} e_2^{(0)}$ . Moreover, all the other solutions to (21) that tend to  $(0, w_0)$  as  $\eta$  tends to  $+\infty$  are tangent to the straight line  $(0, w_0) + \mathbb{R} e_1^{(0)}$ . Now, it is rather simple to see that the two solutions to (21) that tend to  $(0, w_0)$  as  $\eta$  tends to  $+\infty$ , and that are tangent to the straight line  $(0, w_0) + \mathbb{R} e_2^{(0)}$ , satisfy  $V \equiv 0$ , and:

$$W' = -W^2 - f(0)W - \frac{a^2}{2}.$$

Because the solutions  $(V_b, W_b)$ , and  $(V_{\#}, W_{\#})$  given by Proposition 1 cannot satisfy  $V_b \equiv 0$ , and  $V_{\#} \equiv 0$ , we can conclude that the solutions  $(V_b, W_b)$ , and  $(V_{\#}, W_{\#})$  are tangent to  $(0, w_0) + \mathbb{R} e_1^{(0)}$  as  $\eta$  tends to  $+\infty$ . In particular, this yields:

$$\lim_{\eta \rightarrow +\infty} \frac{W_b'(\eta)}{V_b'(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{W_{\#}'(\eta)}{V_{\#}'(\eta)} = \frac{-f'(0)w_0}{f(0) + 3w_0}. \quad (24)$$

(A quick verification shows that  $f(0) + 3w_0 > 0$  for small enough  $a$ .) From the construction of the solutions  $(\hat{v}_b, w_b)$ , and  $(\hat{v}_{\#}, w_{\#})$ , we get:

$$\lim_{\xi \rightarrow 0^-} w_b'(\xi) = \lim_{\xi \rightarrow 0^+} w_{\#}'(\xi) = \frac{-f'(0)w_0^2}{f(0) + 3w_0}.$$

As a consequence, when  $a$  is small enough,  $w \in C^1(\mathbb{R})$ , and therefore  $\hat{v} \in C^2(\mathbb{R})$ . Moreover,  $(\hat{v}, w)$  solves (18) on  $\mathbb{R} \setminus \{0\}$ , so by continuity, it solves (18) on  $\mathbb{R}$ . This completes the proof of Theorem 1, provided that the result of Proposition 1 holds.

## 2.4 Proof of Proposition 1

In this paragraph, we prove Proposition 1, which will complete the proof of Theorem 1. At first, we define the set:

$$P = \left\{ (V, W) \mid V \in [-a, a], W^2 + f(V)W - \frac{V^2 - a^2}{2} = 0 \right\},$$

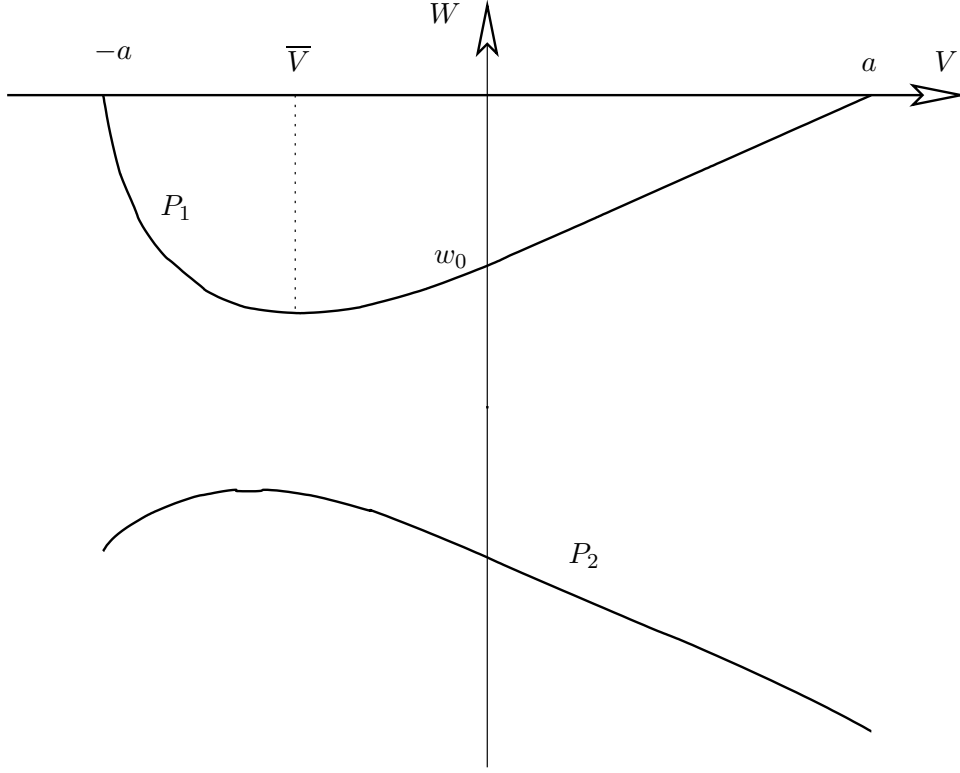


Figure 1: The set  $P = P_1 \cup P_2$ .

so that the points  $(\pm a, 0)$  belong to  $P$ . The following Lemma gives a description of  $P$  for  $a > 0$  small enough. We refer to figure 1 for a schematic picture.

**Lemma 1.** *Assume that  $1 < \gamma < (\sqrt{7} + 1)/(\sqrt{7} - 1)$ . Then there exists a constant  $a_0 > 0$ , that only depends on  $(\rho_-, u_-, e_-)$  and  $\gamma$  such that if the shock strength  $a$  satisfies  $a \in (0, a_0]$ , we have the following results:*

- For all  $V \in [-a, a]$ ,  $f(V)^2 + 2(V^2 - a^2) > 0$ . We can thus define

$$\forall V \in [-a, a], \quad \mathbb{W}_1(V) = \frac{-f(V) + \sqrt{f(V)^2 + 2(V^2 - a^2)}}{2},$$

$$\mathbb{W}_2(V) = \frac{-f(V) - \sqrt{f(V)^2 + 2(V^2 - a^2)}}{2}.$$

- $P = P_1 \cup P_2$ , where  $P_1$  and  $P_2$  are two curves defined by

$$P_1 = \{(V, \mathbb{W}_1(V)) | V \in [-a, a]\}, \quad P_2 = \{(V, \mathbb{W}_2(V)) | V \in [-a, a]\},$$

so that the points  $(\pm a, 0)$ , and  $(0, w_0)$  belong to  $P_1$ .

- There exists a unique point  $\bar{V} \in (-a, 0)$  such that  $\mathbb{W}_1$  is increasing on the interval  $[\bar{V}, a]$ , and  $\mathbb{W}_1$  is decreasing on the interval  $[-a, \bar{V}]$ .
- For all  $V \in [\bar{V}, 0]$ , one has  $\mathbb{W}_2(V) < \mathbb{W}_1(\bar{V})$ .

*Proof.* Let us first define a function  $\Delta$  by setting:

$$\Delta(V) := f(V)^2 + 2(V^2 - a^2).$$

Using (11), for  $a$  small enough, we have:

$$\frac{C_1}{\gamma + 1} - a \geq \kappa > 0, \quad \frac{\gamma C_1}{\gamma + 1} - a \geq \kappa > 0, \quad \frac{(\gamma - 1)C_1}{\gamma + 1} - 2a \geq \kappa > 0,$$

where  $\kappa$  is a positive constant that only depends on  $(\rho_-, u_-, e_-)$  and  $\gamma$ . Moreover, (11) also shows that  $j \geq \kappa$  for  $a \in (0, a_0]$ , up to restricting  $\kappa$ . Consequently, there exist  $a_0 > 0$ , and  $\kappa > 0$  such that for  $a \in (0, a_0]$ , we have  $f(V) \geq \kappa$ , and  $\Delta(V) \geq \kappa$  for all  $V \in [-a, a]$ . This directly shows that the set  $P$  is the union of the two curves  $P_1$ , and  $P_2$ . It is rather clear from the definition of  $P_1$  that  $(\pm a, 0)$ , and  $(0, w_0)$  belong to  $P_1$  (recall that  $w_0$  is defined in Proposition 1). Observe also that  $\mathbb{W}_2(V) < \mathbb{W}_1(V) \leq 0$  for  $V \in [-a, a]$ , and  $\mathbb{W}_1(V) < 0$  if  $V \in (-a, a)$ .

The functions  $\mathbb{W}_1$ , and  $\mathbb{W}_2$  are  $C^\infty$  on  $[-a, a]$ . Moreover, we compute the relation:

$$\forall V \in [-a, a], \quad \sqrt{\Delta(V)} \mathbb{W}'_1(V) = V - \mathbb{W}_1(V) f'(V), \quad (25)$$

and from (19), we also compute

$$f'(V) = \frac{14(\gamma - 1)}{j R^4 (\gamma + 1)} \left( \frac{C_1}{\gamma + 1} - \hat{v} \right)^2 \left( \hat{v} + \frac{\gamma C_1}{\gamma + 1} \right)^2 \left( 2\hat{v} + \frac{(\gamma - 1)C_1}{\gamma + 1} + \frac{C_1}{\sqrt{\bar{\gamma}}} \right) \left( -2\hat{v} - \frac{(\gamma - 1)C_1}{\gamma + 1} + \frac{C_1}{\sqrt{\bar{\gamma}}} \right). \quad (26)$$

As we have done for  $f$ , and  $\Delta$ , a careful analysis shows that for  $1 < \gamma < (\sqrt{\bar{\gamma}} + 1)/(\sqrt{\bar{\gamma}} - 1)$ , and for  $a$  small enough, one has  $f'(V) \geq \kappa > 0$  for all  $V \in [-a, a]$ , because each term in the product (26) is positive. Using this information in (25), we can already conclude that  $\mathbb{W}_1$  is increasing on the interval  $[0, a]$  (see figure 1). Moreover, the relation (25) also shows that  $\mathbb{W}'_1(0) > 0$ , and  $\mathbb{W}'_1(-a) < 0$ . Consequently, there exists some  $\bar{V} \in (-a, 0)$  such that  $\mathbb{W}'_1(\bar{V}) = 0$ . Let us prove that  $\bar{V}$  is the only zero of  $\mathbb{W}'_1$ . We claim that it is sufficient to show the following property:

$$\mathbb{W}'_1(V) = 0 \implies \mathbb{W}''_1(V) > 0. \quad (27)$$

Indeed, if the property (27) holds true, then any point where  $\mathbb{W}'_1$  vanishes is a local strict minimum. If there existed two such local strict minima  $-a < \bar{V}_1 < \bar{V}_2 < a$ , then  $\mathbb{W}_1$  would admit a local maximum  $\bar{V}_3 \in (\bar{V}_1, \bar{V}_2)$ , which is obviously impossible. Therefore let us prove that the property (27) holds true.

Differentiating (25) with respect to  $V$ , we obtain that if  $\mathbb{W}'_1(\bar{V}) = 0$ , then

$$\sqrt{\Delta(\bar{V})} \mathbb{W}''_1(\bar{V}) = 1 - f''(\bar{V}) \mathbb{W}_1(\bar{V}).$$

Observing that

$$|f''(\bar{V}) \mathbb{W}_1(\bar{V})| \leq C |\mathbb{W}_1(\bar{V})| \leq C \frac{a^2 - \bar{V}^2}{f(\bar{V})} \leq \frac{C a^2}{\kappa},$$

for suitable positive constants  $C$ , and  $\kappa$  (that are independent of  $a \in (0, a_0]$ ), we can conclude that  $\mathbb{W}''_1(\bar{V}) > 0$ , provided that  $a$  is small enough. This completes the proof that  $\mathbb{W}'_1$  has a unique zero  $\bar{V} \in (-a, 0)$ , and therefore  $\mathbb{W}_1$  is decreasing on  $[-a, \bar{V}]$ , and is increasing on  $[\bar{V}, a]$ .

For the last point of the lemma, we use the relation:

$$\mathbb{W}'_1(V) + \mathbb{W}'_2(V) = -f'(V) < 0.$$

Because  $\mathbb{W}'_1(V) \geq 0$  for  $V \in [\bar{V}, 0]$ , we get  $\mathbb{W}'_2(V) < 0$  for  $V \in [\bar{V}, 0]$ . Thus for  $V \in [\bar{V}, 0]$ , we have  $\mathbb{W}_2(V) \leq \mathbb{W}_2(\bar{V}) < \mathbb{W}_1(\bar{V})$ , and the proof of the Lemma is complete.  $\square$

Using Lemma 1, we are going to prove Proposition 1. The analysis follows [Gil51].

As we have already seen in the preceding paragraph, the point  $(w_0, 0)$  is a stable node of (21). We now study the nature of the equilibrium points  $(\pm a, 0)$ . Recall that the vector field associated to (21) is denoted  $U$ , see (23). The Jacobian matrix of  $U$  at  $(a, 0)$  is:

$$\begin{pmatrix} 0 & a \\ a & f(a) \end{pmatrix},$$

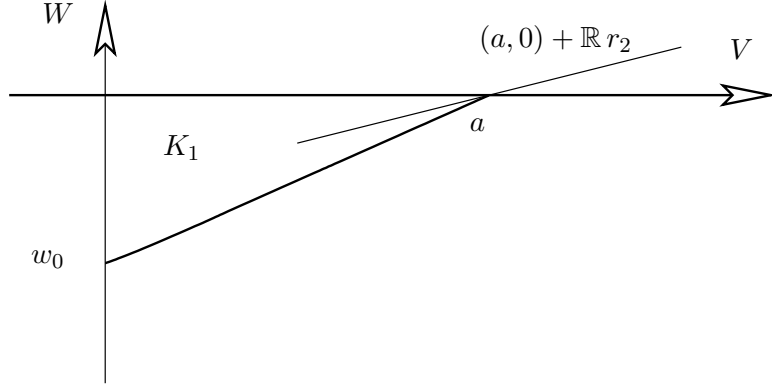


Figure 2: The compact set  $K_1$ .

so it has exactly one negative eigenvalue  $\mu_1$ , and one positive eigenvalue  $\mu_2$  (the equilibrium point  $(a, 0)$  is a saddle point):

$$\mu_1 = \frac{-f(a) - \sqrt{f(a)^2 + 4a^2}}{2}, \quad \mu_2 = \frac{-f(a) + \sqrt{f(a)^2 + 4a^2}}{2}.$$

An eigenvector associated to  $\mu_2$ , and is  $r_2 = (a, \mu_2)$ . Moreover, using the relation (25), we can check that for  $a$  small enough, the following inequality holds:

$$0 < \frac{\mu_2}{a} < \frac{a}{f(a)} = \mathbb{W}'_1(a), \quad (28)$$

where the function  $\mathbb{W}_1$  is defined in Lemma 1. Let us now define a compact set  $K_1$  by:

$$K_1 := \left\{ (V, W) \in [0, a] \times \mathbb{R} \mid \mathbb{W}_1(V) \leq W \leq 0 \right\},$$

Then the inequalities (28) show that for  $s < 0$  small enough, the point  $(a, 0) + s r_2$  belongs to the interior of  $K_1$ . We refer to figure 2 for a detailed picture of the situation.

From the standard theory of autonomous ODEs, see e.g. [Pon75], we know that there exists a maximal solution  $(V_b, W_b)$  to (21) that tends to the saddle point  $(a, 0)$  as  $\eta$  tends to  $-\infty$ , and that is tangent to the half-straight line  $(a, 0) + \mathbb{R}^- r_2$ . This solution is defined on an open interval  $(-\infty, \eta_*)$  (with possibly  $\eta_* = +\infty$ ). For large negative  $\eta$ , the preceding analysis shows that  $(V_b, W_b)(\eta)$  belongs to the interior of  $K_1$ . Moreover,  $(V_b, W_b)$  cannot reach the boundary of  $K_1$ . Indeed  $V_b$  cannot identically vanish so  $(V_b, W_b)(\eta) \notin \partial K_1 \cap \{V = 0\}$ . Similarly, we have  $(V_b, W_b)(\eta) \neq (a, 0)$ . Eventually, on the set:

$$\left\{ (V, 0) \mid V \in (0, a) \right\} \cup \left\{ (V, \mathbb{W}_1(V)) \mid V \in (0, a) \right\},$$

the vector field  $U$  is not zero, and is directed towards the interior of  $K_1$ . Therefore the solution  $(V_b, W_b)$  cannot reach  $\partial K_1$ , so it takes its values in the compact set  $K_1$ . The maximal solution  $(V_b, W_b)$  is thus defined on  $\mathbb{R}$ . It cannot reach the boundary of  $K_1$ , so  $W_b$  takes negative values, which means that  $V_b$  is decreasing (because  $V_b' = V_b W_b$ ). Because  $(V_b, W_b)$  takes values in the interior of  $K_1$ , the function  $W_b$  is also decreasing. This shows that  $(V_b, W_b)(\eta)$  has a limit as  $\eta$  tends to  $+\infty$ , and this limit is necessarily be a stationary solution of (21). The only possibility is that  $(V_b, W_b)(\eta)$  tends to  $(0, w_0)$  as  $\eta$  tends to  $+\infty$ . The convergence is necessarily exponential, because the Jacobian matrix of  $U$  at  $(0, w_0)$  has two negative eigenvalues, see e.g. [Pon75].

To construct the other solution  $(V_{\sharp}, W_{\sharp})$ , we argue similarly by defining a compact set  $K_2$ :

$$K_2 := \left\{ (V, W) \in [-a, \bar{V}] \times \mathbb{R} \mid \mathbb{W}_1(V) \leq W \leq 0 \right\} \cup \left\{ (V, W) \in [\bar{V}, 0] \times \mathbb{R} \mid \mathbb{W}_1(\bar{V}) \leq W \leq 0 \right\},$$

see figure 3. The Jacobian matrix of  $U$  at  $(-a, 0)$  has one negative eigenvalue  $\nu_1$ , and one positive eigenvalue  $\nu_2$ , with:

$$\nu_2 = \frac{-f(-a) + \sqrt{f(-a)^2 + 4a^2}}{2}.$$

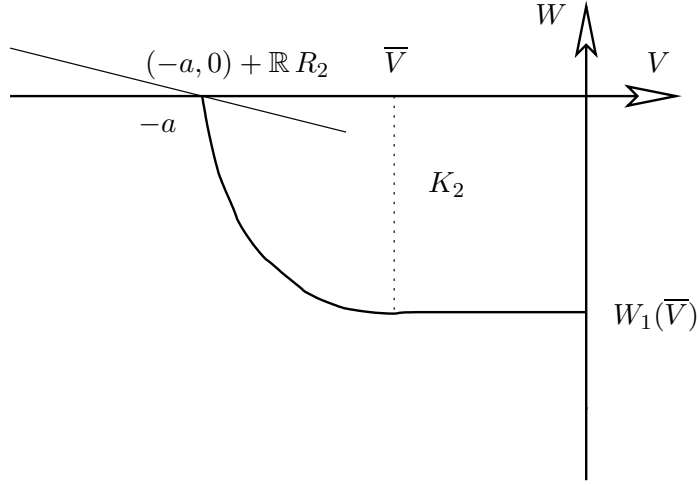


Figure 3: The compact set  $K_2$ .

An eigenvector associated to the eigenvalue  $\nu_2$  is  $R_2 = (-a, \nu_2)$ . As was done earlier, we check that the inequalities:

$$\mathbb{W}'_1(-a) = \frac{-a}{f(-a)} < \frac{\nu_2}{-a} < 0,$$

hold true. Therefore, one can reproduce the above analysis, and show that there exists a solution  $(V_\sharp, W_\sharp)$  to (21) that takes its values in  $K_2$  (and is thus defined on  $\mathbb{R}$ ), and that tends to  $(-a, 0)$  at  $-\infty$ . Moreover,  $(V_\sharp, W_\sharp)$  can not reach the boundary of  $K_2$ , so  $V_\sharp$  is increasing. It only remains to study the monotonicity of  $W_\sharp$ . This is slightly more complicated than for  $W_\flat$ . Observe that  $K_2$  is the union of the sets:

$$\begin{aligned} K_2^1 &:= \left\{ (V, W) \in [-a, 0] \times \mathbb{R} \mid \mathbb{W}_1(V) \leq W \leq 0 \right\}, \\ K_2^2 &:= \left\{ (V, W) \in [\bar{V}, 0] \times \mathbb{R} \mid \mathbb{W}_1(\bar{V}) \leq W \leq \mathbb{W}_1(V) \right\}. \end{aligned}$$

When  $(V_\sharp, W_\sharp)$  takes its values in the interior of  $K_2^1$ , the function  $W_\sharp$  is decreasing (this is the case for large negative  $\eta$ ). At the opposite, when  $(V_\sharp, W_\sharp)$  takes its values in the interior of  $K_2^2$ , the function  $W_\sharp$  is increasing, because thanks to Lemma 1, we have:

$$\begin{aligned} W'_\sharp(\eta) &= -W_\sharp(\eta)^2 - f(V_\sharp(\eta)) W_\sharp(\eta) + \frac{V_\sharp(\eta)^2 - a^2}{2} \\ &= (\mathbb{W}_1(V_\sharp(\eta)) - W_\sharp(\eta)) (W_\sharp(\eta) - \mathbb{W}_2(V_\sharp(\eta))) \\ &\geq (\mathbb{W}_1(V_\sharp(\eta)) - W_\sharp(\eta)) (\mathbb{W}_1(\bar{V}) - \mathbb{W}_2(V_\sharp(\eta))) > 0. \end{aligned}$$

Moreover, if  $(V_\sharp, W_\sharp)(\eta_0)$  belongs to the interior of  $K_2^2$  for some  $\eta_0 \in \mathbb{R}$ , then  $(V_\sharp, W_\sharp)(\eta)$  belongs to the interior of  $K_2^2$  for all  $\eta \geq \eta_0$  (because it cannot reach the boundary of  $K_2^2$  for  $\eta \geq \eta_0$ ). Summing up, either  $(V_\sharp, W_\sharp)(\eta)$  belongs to  $K_2^1$  for all  $\eta$ , and  $W_\sharp$  is monotonic on  $\mathbb{R}$ , either  $(V_\sharp, W_\sharp)(\eta)$  belongs to  $K_2^2$  for all  $\eta$  greater than some  $\eta_0$ , and  $W_\sharp$  is monotonic on  $[\eta_0, +\infty)$ . In any case, the function  $W_\sharp$  is monotonic on a neighborhood of  $+\infty$ , and thus has a limit at  $+\infty$ . This shows that  $(V_\sharp, W_\sharp)(\eta)$  tends to  $(0, w_0)$  as  $\eta$  tends to  $+\infty$ , and the convergence is exponential. As a matter of fact, we have seen in the preceding paragraph that  $(V_\sharp, W_\sharp)$  is tangent to the straight line  $(0, w_0) + \mathbb{R} e_1^{(0)}$  as  $\eta$  tends to  $+\infty$ , so one can check that  $(V_\sharp, W_\sharp)(\eta)$  belongs to the interior of  $K_2^2$  for large positive  $\eta$ . This means that  $W_\sharp$  is decreasing on some interval  $(-\infty, \eta_0)$ , and increasing on  $[\eta_0, +\infty)$ . The proof of Proposition 1 is now complete.

### 3 Additional regularity of shock profiles

As should be clear from the preceding section, the key point in the construction of a shock profile is Proposition 1 that gives the existence of two heteroclinic orbits for the system (21). To

prove Theorem 2, we are going to study the behavior of the derivatives of  $(V_b, W_b)$ , and  $(V_{\sharp}, W_{\sharp})$  near  $+\infty$ . The proof of Theorem 2 follows from an induction argument. To make the arguments clear, we deal with the first case separately. In all what follows,  $(V_b, W_b)$ , and  $(V_{\sharp}, W_{\sharp})$  are the solutions to (21) that are defined in Proposition 1, and  $(\hat{v}, w)$  denotes the solution to (18) that is defined by (22). We have the following:

**Proposition 2.** *Under the assumptions of Proposition 1, there exists a positive constant  $a_1 \leq a_0$  (that depends on  $(\rho_-, u_-, e_-)$ , and  $\gamma$ ), such that for all  $a \in (0, a_1]$ , one has  $w \in C^2(\mathbb{R})$ ,  $\hat{v} \in C^3(\mathbb{R})$ , and:*

$$w(\xi) = w_0 + w_1 \hat{v}(\xi) + w_2 \hat{v}(\xi)^2 + o(\hat{v}(\xi)^2), \quad \text{as } \xi \rightarrow 0,$$

for some suitable constants  $w_1, w_2$  ( $w_0$  has already been defined in Proposition 1).

*Proof.* Recall that  $V_b$ , and  $V_{\sharp}$  do not vanish on  $\mathbb{R}$ , so we can introduce some  $C^\infty$  functions  $W_{b,1}$ , and  $W_{\sharp,1}$  that are defined by:

$$W_b = w_0 + V_b W_{b,1}, \quad W_{\sharp} = w_0 + V_{\sharp} W_{\sharp,1}.$$

Substituting in (21) shows that  $(V_b, W_{b,1})$ , and  $(V_{\sharp}, W_{\sharp,1})$  are solutions to the system:

$$\begin{cases} V' = V(w_0 + V W_1), \\ W_1' = -w_0 \frac{f(V) - f(0)}{V} - 2V W_1^2 - 3w_0 W_1 - f(V) W_1 + \frac{V}{2}. \end{cases} \quad (29)$$

Moreover, we already know from Proposition 1, and (24) that:

$$\lim_{\eta \rightarrow +\infty} (V_b, W_{b,1})(\eta) = \lim_{\eta \rightarrow +\infty} (V_{\sharp}, W_{\sharp,1})(\eta) = \left(0, \frac{-f'(0) w_0}{f(0) + 3w_0}\right).$$

We denote  $U_1(V, W_1)$  the vector field associated with (29):

$$U_1(V, W_1) := \begin{pmatrix} V(w_0 + V W_1) \\ -w_0 \frac{f(V) - f(0)}{V} - 2V W_1^2 - 3w_0 W_1 - f(V) W_1 + \frac{V}{2} \end{pmatrix}.$$

Recall that  $f$  is a polynomial function of degree 7, see (19), thus  $F(V) := (f(V) - f(0))/V$  is a polynomial function of degree 6, and we have  $F(0) = f'(0)$ ,  $F'(0) = f''(0)/2$ . Obviously the system of ODEs (29) admits the equilibrium point  $(0, w_1)$ , where:

$$w_1 := \frac{-F(0) w_0}{f(0) + 3w_0} = \frac{-f'(0) w_0}{f(0) + 3w_0}.$$

We are now going to study the nature of the equilibrium point  $(0, w_1)$ , and show that for  $a$  small enough, this equilibrium point is a stable node for (29). Then we shall show that  $w \in C^2(\mathbb{R})$ , and  $\hat{v} \in C^3(\mathbb{R})$ . In the end, we shall derive the asymptotic expansion near  $\xi = 0$ .

Step 1: the Jacobian matrix of  $U_1$  at  $(0, w_1)$  is:

$$\begin{pmatrix} w_0 & 0 \\ \frac{1}{2} - 2w_1^2 - f'(0)w_1 - \frac{f''(0)}{2}w_0 & -f(0) - 3w_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} & 0 \\ b_1 & \lambda_2^{(1)} \end{pmatrix}.$$

Using Remark 3, we can conclude that for sufficiently small  $a$ , that is  $a \in (0, a_1]$  for some positive number  $a_1$  less than  $a_0$ , one has  $\lambda_2^{(1)} < \lambda_1^{(1)} < 0$ , that is,  $f(0) + 4w_0 > 0$ . Moreover, the eigenvectors corresponding to the eigenvalues  $\lambda_1^{(1)}$ , and  $\lambda_2^{(1)}$  are:

$$e_1^{(1)} = \begin{pmatrix} f(0) + 4w_0 \\ b_1 \end{pmatrix}, \quad e_2^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consequently,  $(0, w_1)$  is a stable node of (29), and there are exactly two solutions to (29) that tend to  $(0, w_1)$  as  $\eta$  tends to  $+\infty$ , and that are tangent to the straight line  $(0, w_1) + \mathbb{R} e_2^{(1)}$ . All the other solutions to (29) that tend to  $(0, w_1)$  as  $\eta$  tends to  $+\infty$  are tangent to the straight line  $(0, w_1) + \mathbb{R} e_1^{(1)}$ . As in the preceding section, we can thus conclude that:

$$\lim_{\eta \rightarrow +\infty} \frac{W'_{b,1}(\eta)}{V'_b(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{W'_{\sharp,1}(\eta)}{V'_{\sharp}(\eta)} = \frac{b_1}{f(0) + 4w_0}. \quad (30)$$

Step 2: if we let  $g_1$  denote the second coordinate of the vector field  $U_1$ , we have  $W'_{b,1} = g_1(V_b, W_{b,1})$ , and  $W'_{\sharp,1} = g_1(V_{\sharp}, W_{\sharp,1})$ . Differentiating once with respect to  $\eta$ , and using (30), we end up with:

$$\lim_{\eta \rightarrow +\infty} \frac{W''_{b,1}(\eta)}{V'_b(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{W''_{\sharp,1}(\eta)}{V'_{\sharp}(\eta)} = \ell_1, \quad (31)$$

where the real number  $\ell_1$  can be explicitly computed (but its exact expression is of no use). Following the analysis of the preceding section, we define some functions  $w_{b,1} := W_{b,1} \circ \Xi_b^{-1}$ , and  $w_{\sharp,1} := W_{\sharp,1} \circ \Xi_{\sharp}^{-1}$ . First of all, (30) yields:

$$\lim_{\xi \rightarrow 0^-} w'_{b,1}(\xi) = \lim_{\xi \rightarrow 0^+} w'_{\sharp,1}(\xi) = \frac{b_1 w_0}{f(0) + 4w_0}. \quad (32)$$

Observe now that we have the relations:

$$\hat{v}_b w'_{b,1} = W'_{b,1} \circ \Xi_b^{-1}, \quad \hat{v}_{\sharp} w'_{\sharp,1} = W'_{\sharp,1} \circ \Xi_{\sharp}^{-1},$$

and combining with (31), we get:

$$\begin{aligned} \lim_{\xi \rightarrow 0^-} (\hat{v}_b w'_{b,1})'(\xi) &= \lim_{\eta \rightarrow +\infty} \frac{W''_{b,1}(\eta)}{V'_b(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{W''_{b,1}(\eta) W_b(\eta)}{V'_b(\eta)} = \ell_1 w_0, \\ \lim_{\xi \rightarrow 0^+} (\hat{v}_{\sharp} w'_{\sharp,1})'(\xi) &= \lim_{\eta \rightarrow +\infty} \frac{W''_{\sharp,1}(\eta)}{V'_{\sharp}(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{W''_{\sharp,1}(\eta) W_{\sharp}(\eta)}{V'_{\sharp}(\eta)} = \ell_1 w_0. \end{aligned} \quad (33)$$

Differentiating twice the relations  $w_b = w_0 + \hat{v}_b w_{b,1}$ , and  $w_{\sharp} = w_0 + \hat{v}_{\sharp} w_{\sharp,1}$ , we obtain:

$$\begin{aligned} w''_b &= \hat{v}_b'' w_{b,1} + \hat{v}_b' w'_{b,1} + (\hat{v}_b w'_{b,1})' = w'_b w_{b,1} + w_b w'_{b,1} + (\hat{v}_b w'_{b,1})', \\ w''_{\sharp} &= w'_{\sharp} w_{\sharp,1} + w_{\sharp} w'_{\sharp,1} + (\hat{v}_{\sharp} w'_{\sharp,1})'. \end{aligned}$$

Using (32), and (33), we get  $w''_b(0^-) = w''_{\sharp}(0^+)$ . Using the definition (22), this shows that  $w \in C^2(\mathbb{R})$ , and using  $\hat{v}' = w$ , we obtain  $\hat{v} \in C^3(\mathbb{R})$ .

Step 3: note that we have the following expansions near  $\xi = 0$ :

$$\begin{aligned} w(\xi) &= w(0) + w'(0) \xi + \frac{w''(0)}{2} \xi^2 + o(\xi^2), \\ \hat{v}(\xi) &= w(0) \xi + \frac{w'(0)}{2} \xi^2 + o(\xi^2), \end{aligned}$$

with  $w(0) = w_0 < 0$ . We can thus combine these expansions, and derive:

$$w(\xi) = w_0 + \alpha \hat{v}(\xi) + \beta \hat{v}(\xi)^2 + o(\hat{v}(\xi)^2),$$

for some appropriate real numbers  $\alpha$ , and  $\beta$ , that we are going to determine. From the relation  $w_b(\xi) = w_0 + \hat{v}_b(\xi) w_{b,1}(\xi)$ , and using that  $w_{b,1}(\xi)$  tends to  $w_1$  as  $\xi$  tends to  $0^-$ , we first get  $\alpha = w_1$ . Then from (32), and from the relation  $\hat{v}'_b(0^-) = w_0$ , we can obtain:

$$w_{b,1}(\xi) = w_1 + \frac{b_1}{f(0) + 4w_0} \hat{v}_b(\xi) + o(\hat{v}_b(\xi)), \quad \text{as } \xi \rightarrow 0^-.$$

We thus obtain  $\beta = b_1/(f(0) + 4w_0)$ , which yields:

$$w(\xi) = w_0 + w_1 \hat{v}(\xi) + w_2 \hat{v}(\xi)^2 + o(\hat{v}(\xi)^2),$$

where  $w_2 := b_1/(f(0) + 4w_0)$ . This latter expansion will be generalized to any order in what follows.  $\square$

We now turn to the proof of Theorem 2. More precisely, we are going to prove the following result, that is a refined version of Theorem 2:

**Theorem 3.** *Let the assumptions of Proposition 1 be satisfied. Then there exists a nonincreasing sequence of positive numbers  $(a_n)_{n \in \mathbb{N}}$  such that, for all integer  $n$ , if  $a \in (0, a_n]$ , then  $w \in C^{n+1}(\mathbb{R})$ , and  $\hat{v} \in C^{n+2}(\mathbb{R})$ . Moreover,  $w$  admits the following asymptotic expansion near  $\xi = 0$ :*

$$w(\xi) = w_0 + w_1 \hat{v}(\xi) + \cdots + w_{n+1} \hat{v}(\xi)^{n+1} + o(\hat{v}(\xi)^{n+1}), \quad (34)$$

where the real numbers  $w_0, \dots, w_{n+1}$  are defined by:

$$\begin{cases} w_0 = \frac{-f(0) + \sqrt{f(0)^2 - 2a^2}}{2}, \\ w_k = \frac{b_{k-1}}{f(0) + (k+2)w_0}, \end{cases} \quad \text{for } k = 1, \dots, n+1,$$

and the real numbers  $b_0, \dots, b_n$  are given by:

$$\begin{cases} b_0 = -f'(0)w_0, \\ b_1 = \frac{1}{2} - 2w_1^2 - f'(0)w_1 - \frac{f''(0)}{2}w_0, \\ b_k = -\sum_{i=1}^{k+1} \frac{f^{(i)}(0)}{i!} w_{k+1-i} - \sum_{i=1}^k (i+1)w_i w_{k+1-i}, \end{cases} \quad \text{for } k = 2, \dots, n.$$

*Proof.* The case  $n = 0$  has been proved in the preceding section, while the case  $n = 1$  is proved in Proposition 2. (The reader can check that the definition of  $w_0, w_1, w_2, b_0$ , and  $b_1$  coincide with our previous notations.) We prove the general case by using an induction with respect to  $n$ , and we thus assume that the result of Theorem 3 holds up to the order  $n \geq 1$ . We are going to construct  $a_{n+1}$  so that the conclusion of Theorem 3 holds for  $a \in (0, a_{n+1}]$ . In particular, the real numbers  $w_0, \dots, w_{n+1}$ , and  $b_0, \dots, b_n$  are given as in Theorem 3, and we can already define the real number  $b_{n+1}$  by the formula:

$$b_{n+1} := -\sum_{i=1}^{n+2} \frac{f^{(i)}(0)}{i!} w_{n+2-i} - \sum_{i=1}^{n+1} (i+1)w_i w_{n+2-i}.$$

(Observe indeed that this definition only involves  $w_0, \dots, w_{n+1}$ , and not  $w_{n+2}$ .)

Step 1: because  $V_b$ , and  $V_{\#}$  do not vanish, we can introduce some functions  $W_{b,n+1}$ , and  $W_{\#,n+1}$  by the relations:

$$W_b = w_0 + w_1 V_b + \cdots + w_n V_b^n + W_{b,n+1} V_b^{n+1}, \quad W_{\#} = w_0 + w_1 V_{\#} + \cdots + w_n V_{\#}^n + W_{\#,n+1} V_{\#}^{n+1}.$$

Thanks to Taylor's formula, we can write the polynomial function  $f$  as:

$$f(V) = f(0) + f'(0)V + \frac{f''(0)}{2}V^2 + \cdots + \frac{f^{(n)}(0)}{n!}V^n + V^{n+1}F_{n+1}(V),$$

where  $F_{n+1}$  is a polynomial function such that:

$$F_{n+1}(0) = \frac{f^{(n+1)}(0)}{(n+1)!}, \quad F'_{n+1}(0) = \frac{f^{(n+2)}(0)}{(n+2)!}.$$

Substituting the expression of  $W_b$ , and  $W_{\#}$  in (21) shows (after a tedious computation!) that  $(V_b, W_{b,n+1})$ , and  $(V_{\#}, W_{\#,n+1})$  are solutions to the following system of ODEs:

$$\begin{cases} V' = V(w_0 + w_1 V + \cdots + w_n V^n + W_{n+1} V^{n+1}), \\ W'_{n+1} = g_{n+1}(V, W_{n+1}), \end{cases} \quad (35)$$



where the function  $g_{n+1}$  is given by:

$$g_{n+1}(V, W_{n+1}) := - (n+2) W_{n+1} \left( \sum_{k=0}^n w_k V^k + V^{n+1} W_{n+1} \right) - W_{n+1} \sum_{k=0}^n (k+1) w_k V^k \\ - W_{n+1} f(V) - F_{n+1}(V) \sum_{k=0}^n w_k V^k + b_n + \frac{f^{(n+1)}(0)}{(n+1)!} + V Q_{n+1}(V), \quad (36)$$

and  $Q_{n+1}$  is a polynomial function that satisfies:

$$Q_{n+1}(0) = b_{n+1} + (n+4) w_1 w_{n+1} + f'(0) w_{n+1} + \frac{f^{(n+1)}(0)}{(n+1)!} w_1 + \frac{f^{(n+2)}(0)}{(n+2)!} w_0.$$

When  $n = 1$ , one has  $Q_2 \equiv Q_2(0) = 0$  (see the above definition for  $b_2$ ). Using the expansion (34), which is part of the induction assumption, we also know that:

$$\lim_{\eta \rightarrow +\infty} (V_b, W_{b,n+1})(\eta) = \lim_{\eta \rightarrow +\infty} (V_{\#}, W_{\#,n+1})(\eta) = (0, w_{n+1}) = \left( 0, \frac{b_n}{f(0) + (n+2) w_0} \right).$$

With the above definitions for  $g_{n+1}$ , and  $Q_{n+1}$ , we can check that  $(0, w_{n+1})$  is a stationary solution to (35). (Recall that  $w_{n+1}$  is defined as in Theorem 3 by the induction assumption.) We can also evaluate the Jacobian matrix of the vector field associated with the system of ODEs (35):

$$\begin{pmatrix} w_0 & 0 \\ b_{n+1} & -f(0) - (n+3) w_0 \end{pmatrix} = \begin{pmatrix} \lambda_1^{(n+1)} & 0 \\ b_{n+1} & \lambda_2^{(n+2)} \end{pmatrix}.$$

There exists a positive number  $a_{n+1} \leq a_n$  such that for all  $a \in (0, a_{n+1}]$ , one has  $\lambda_2^{(n+2)} < \lambda_1^{(n+2)} < 0$ , or equivalently  $f(0) + (n+4) w_0 > 0$ . In that case, the eigenvectors corresponding to the eigenvalues  $\lambda_1^{(n+1)}$  and  $\lambda_2^{(n+1)}$  are:

$$e_1^{(n+1)} = \begin{pmatrix} f(0) + (n+4) w_0 \\ b_{n+1} \end{pmatrix}, \quad e_2^{(n+1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using the same argument as in the proof of Proposition 2, we can conclude that the solutions  $(V_b, W_{b,n+1})$ , and  $(V_{\#}, W_{\#,n+1})$  of (35) are tangent to the straight line  $(0, w_{n+1}) + \mathbb{R} e_1^{(n+1)}$  as  $\eta$  tends to  $+\infty$ . In particular, this yields:

$$\lim_{\eta \rightarrow +\infty} \frac{W'_{b,n+1}(\eta)}{V'_b(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{W'_{\#,n+1}(\eta)}{V'_{\#}(\eta)} = \frac{b_{n+1}}{f(0) + (n+4) w_0} =: w_{n+2}. \quad (37)$$

Step 2: let us define the function  $\tilde{w}_{n+1}$  by the formula:

$$\tilde{w}_{n+1}(\xi) := \begin{cases} W_{b,n+1} \circ \Xi_b^{-1}(\xi) & \text{if } \xi < 0, \\ w_{n+1} & \text{if } \xi = 0, \\ W_{\#,n+1} \circ \Xi_{\#}^{-1}(\xi) & \text{if } \xi > 0. \end{cases}$$

With this definition,  $\tilde{w}_{n+1}$  is continuous, and we have the relation:

$$w = w_0 + w_1 \hat{v} + \dots + w_n \hat{v}^n + \tilde{w}_{n+1} \hat{v}^{n+1}. \quad (38)$$

Moreover, using (37), we obtain:

$$\lim_{\xi \rightarrow 0^-} \frac{\tilde{w}'_{n+1}(\xi)}{\hat{v}'(\xi)} = \lim_{\xi \rightarrow 0^+} \frac{\tilde{w}'_{n+1}(\xi)}{\hat{v}'(\xi)} = w_{n+2}, \quad (39)$$

which yields  $\tilde{w}'_{n+1}(0^+) = \tilde{w}'_{n+1}(0^-)$ . Therefore, we have  $\tilde{w}_{n+1} \in C^1(\mathbb{R})$ . Moreover, using (35), we can compute:

$$\tilde{w}'_{n+1} \hat{v} = g_{n+1}(\hat{v}, \tilde{w}_{n+1}), \quad (40)$$

so we get  $\tilde{w}'_{n+1} \hat{v} \in C^1(\mathbb{R})$ .

Step 3: we use an induction argument to show that  $w \in C^{n+2}(\mathbb{R})$  (which will imply immediately  $\hat{v} \in C^{n+3}(\mathbb{R})$ ). More precisely, we assume that for some  $k \in \{0, \dots, n\}$ , we have:

$$\tilde{w}_{n+1} \hat{v}^k \in C^{k+1}(\mathbb{R}), \quad w \in C^{k+1}(\mathbb{R}), \quad \tilde{w}'_{n+1} \hat{v}^{k+1} \in C^{k+1}(\mathbb{R}). \quad (41)$$

We are going to show that this property implies the same property with  $k$  replaced by  $k+1$ . (Observe that step 2 above shows that the property (41) holds for  $k=0$ .)

We note that  $\hat{v} \in C^{k+2}(\mathbb{R})$ , because  $\hat{v}' = w \in C^{k+1}(\mathbb{R})$ . Moreover, we have  $\tilde{w}_{n+1} \hat{v}^{k+1} = (\tilde{w}_{n+1} \hat{v}^k) \hat{v} \in C^{k+1}(\mathbb{R})$ , and we also have:

$$(\tilde{w}_{n+1} \hat{v}^{k+1})' = \tilde{w}'_{n+1} \hat{v}^{k+1} + (k+1) (\tilde{w}_{n+1} \hat{v}^k) w \in C^{k+1}(\mathbb{R}).$$

Therefore, we get  $\tilde{w}_{n+1} \hat{v}^{k+1} \in C^{k+2}(\mathbb{R})$ .

Using the relation (38), we immediately obtain  $w \in C^{k+2}(\mathbb{R})$ .

We have  $\tilde{w}'_{n+1} \hat{v}^{k+2} = (\tilde{w}'_{n+1} \hat{v}^{k+1}) \hat{v} \in C^{k+1}(\mathbb{R})$ , and using (40), we derive:

$$\begin{aligned} (\tilde{w}'_{n+1} \hat{v}^{k+2})' &= (g_{n+1}(\hat{v}, \tilde{w}_{n+1}) \hat{v}^{k+1})' \\ &= (\partial_1 g_{n+1})(\hat{v}, \tilde{w}_{n+1}) \hat{v}^{k+1} w + (\partial_2 g_{n+1})(\hat{v}, \tilde{w}_{n+1}) \tilde{w}'_{n+1} \hat{v}^{k+1} + (k+1) g_{n+1}(\hat{v}, \tilde{w}_{n+1}) \hat{v}^k w, \end{aligned} \quad (42)$$

where  $\partial_1 g_{n+1}$  (resp.  $\partial_2 g_{n+1}$ ) denotes the partial derivative of  $g_{n+1}$  with respect to its first (resp. second) variable. From the definition (36), we see that  $g_{n+1}(\hat{v}, \tilde{w}_{n+1})$  can be decomposed as follows:

$$g_{n+1}(\hat{v}, \tilde{w}_{n+1}) = -(n+2) \tilde{w}_{n+1}^2 \hat{v}^{n+1} + \tilde{w}_{n+1} P_1(\hat{v}) + P_0(\hat{v}),$$

where  $P_0$ , and  $P_1$  are polynomial functions. Using this decomposition, and the induction assumption (41), we can show that each term of the sum in the right-hand side of (42) belongs to  $C^{k+1}(\mathbb{R})$ . Consequently  $\tilde{w}'_{n+1} \hat{v}^{k+2}$  belongs to  $C^{k+2}(\mathbb{R})$ , and (41) holds with  $k$  replaced by  $k+1$ . Because (41) holds for  $k=0$ , we get that (41) holds for  $k=n+1$ , so we have proved  $w \in C^{n+2}(\mathbb{R})$ , and  $\hat{v} \in C^{n+3}(\mathbb{R})$ .

Step 4: it remains to show that  $w$  satisfies the asymptotic expansion (34) at the order  $n+1$ . Using (39), and  $\tilde{w}_{n+1} \in C^1(\mathbb{R})$ , we obtain:

$$\tilde{w}_{n+1}(\xi) - w_{n+1} = w_{n+2} \hat{v}(\xi) + o(\hat{v}(\xi)), \quad \text{as } \xi \rightarrow 0.$$

Plugging this expansion in (38), we obtain (34) at the order  $n+1$ , so the proof of the induction is complete.  $\square$

Once we know that the function  $\hat{v}$  belongs to  $C^{n+2}(\mathbb{R})$ , for  $a \in (0, a_n]$ , then  $v = \hat{v} + (v_- + v_+)/2$  also belongs to  $C^{n+2}(\mathbb{R})$ , and we have already seen in the previous section that  $v$  does not vanish because  $v(\xi) > v_+ > 0$  for all  $\xi$ . Moreover, the components  $(\rho, u, e)$  of the shock profile are given by:

$$\rho(\xi) = \frac{j}{v(\xi)}, \quad u(\xi) = v(\xi) + \sigma, \quad e(\xi) = \frac{(C_1 - v(\xi)) v(\xi)}{\gamma - 1},$$

so one has  $(\rho, u, e) \in C^{n+2}(\mathbb{R})$ , and the proof of Theorem 2 is complete. (Recall that the strength of the shock tends to zero if, and only if  $a = |u_+ - u_-|/2$  tends to zero.)

## A Formal derivation of the model

It is worth describing how the model (1), (3) can be obtained from a more complete physical system. The derivation we propose below remains formal – a rigorous proof being certainly delicate and beyond the scope of this work – and we refer to [Bue04, Gou06, LMH99, Mih84] for further details. Let us introduce the specific intensity of radiation  $f(t, x, v)$ , that depends on a time variable  $t \geq 0$ , a space variable  $x \in \mathbb{R}^N$ , and a direction  $v \in \mathbb{S}^{N-1}$ . We make the ‘grey assumption’, which means that the frequency dependence is ignored (all photons have the same frequency). Photons are subject to two main interaction phenomena:

- scattering produces changes in the direction of the photons,
- absorption/emission where photons are lost/produced through a transfer mechanism with the surrounding gas.

The scattering phenomenon is described by the operator:

$$Q_s(f)(t, x, v) = \sigma_s \left( \int_{\mathbb{S}^{N-1}} f(t, x, v') dv' - f(t, x, v) \right),$$

(with  $dv$  the normalized Lebesgue measure on  $\mathbb{S}^{N-1}$ ), and the absorption/emission phenomenon is described by the operator:

$$Q_a(f)(t, x, v) = \sigma_a \left( \frac{\sigma}{\pi} \theta(t, x)^4 - f(t, x, v) \right),$$

where we used the Stefan-Boltzmann emission law,  $\theta$  being the temperature of the gas, and  $\sigma$  the Stefan-Boltzmann constant. In these definitions, the coefficients  $\sigma_{s,a}$  are given positive quantities. These phenomena are both characterized by a typical mean free path, denoted  $\ell_s, \ell_a$  respectively. Therefore, the evolution of the specific intensity is driven by:

$$\frac{1}{c} \partial_t f + v \cdot \nabla_x f = \frac{1}{\ell_s} Q_s(f) + \frac{1}{\ell_a} Q_a(f) = Q(f), \quad (43)$$

where  $c$  stands for the speed of light. The equation (43) is coupled to the Euler system describing the evolution of the fluid:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x P = -\frac{1}{c} \int_{\mathbb{S}^{N-1}} v Q(f) dv, \\ \partial_t(\rho E) + \nabla_x \cdot (\rho E u + P u) = - \int_{\mathbb{S}^{N-1}} Q(f) dv. \end{cases} \quad (44)$$

The equations (43), (44) are thus coupled by the exchanges of both momentum and energy, and by the Stefan-Boltzmann emission law. Observe that only the emission/absorption operator enters into the energy equation since the scattering operator is conservative (this would be different if Doppler corrections were taken into account). Note also that the total energy:

$$\frac{1}{c} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} f dv dx + \int_{\mathbb{R}^N} \rho E dx,$$

is (formally) conserved. Writing the system (44), and the kinetic equation (43) in the dimensionless form, we can make four dimensionless parameters appear:

- $\mathcal{C}$ , the ratio of the speed of light over the typical sound speed of the gas,
- $\mathcal{L}_s$ , the Knudsen number associated to the scattering,
- $\mathcal{L}_a$ , the Knudsen number associated to the absorption/emission,
- $\mathcal{P}$ , which compares the typical energy of radiation and the typical energy of the gas.

We thus obtain the rescaled equations:

$$\begin{cases} \frac{1}{\mathcal{C}} \partial_t f + v \cdot \nabla_x f = \frac{1}{\mathcal{L}_s} Q_s(f) + \frac{1}{\mathcal{L}_a} Q_a(f), \\ \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x P = \frac{\mathcal{P}}{\mathcal{L}_s} \sigma_s \int_{\mathbb{S}^{N-1}} v f(v) dv, \\ \partial_t(\rho E) + \nabla_x \cdot (\rho E u + P u) = -\frac{\mathcal{P}}{\mathcal{L}_a} \sigma_a \left( \theta^4 - \int_{\mathbb{S}^{N-1}} f(v) dv \right). \end{cases} \quad (45)$$

System (1), (3) is then obtained in two steps. First of all, we assume  $\mathcal{C} \gg 1$ . Next, we keep  $\mathcal{P}$  of order 1, and we are concerned here with a regime where scattering is the leading phenomenon: the mean free paths are rescaled according to:

$$\mathcal{L}_s \simeq \frac{1}{\mathcal{C}}, \quad \mathcal{L}_a \simeq \mathcal{C}.$$

The asymptotics can be readily understood by means of the Hilbert expansion:

$$f = f^{(0)} + \frac{1}{\mathcal{C}} f^{(1)} + \frac{1}{\mathcal{C}^2} f^{(2)} + \dots$$

Identifying the terms arising with the same power of  $1/\mathcal{C}$ , we get:

- at the leading order,  $f^{(0)}$  belongs to the kernel of the scattering operator, so that it does not depend on the microscopic variable  $v$ :  $f^{(0)}(t, x, v) = n(t, x)$ ,
- the relation  $Q_s(f^{(1)}) = v \cdot \nabla_x f^{(0)}$  then leads to:  $f^{(1)}(t, x, v) = -\frac{1}{\sigma_s} v \cdot \nabla_x n(t, x)$ ,
- integrating the equation for  $f^{(2)}$  over the sphere yields:

$$\partial_t n - \frac{1}{N \sigma_s} \Delta_x n = \sigma_a (\theta^4 - n).$$

Note also that in the momentum equation, we have:

$$\frac{\sigma_s}{\mathcal{L}_s} \int_{\mathbb{S}^{N-1}} v f(v) dv \simeq \sigma_s \int_{\mathbb{S}^{N-1}} v f^{(1)}(v) dv = -\frac{1}{N} \nabla_x n.$$

Finally, we obtain the limit system:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x P = -\frac{\mathcal{P}}{N} \nabla_x n, \\ \partial_t (\rho E) + \nabla_x \cdot (\rho E u + P u) = -\mathcal{P} \sigma_a (\theta^4 - n), \\ \partial_t n - \frac{1}{N \sigma_s} \Delta_x n = \sigma_a (\theta^4 - n). \end{cases} \quad (46)$$

The system (46) describes a nonequilibrium regime, where the material and the radiations have different temperatures ( $\theta \neq n^{1/4}$ ); the equilibrium regime would correspond to assuming that the emission/absorption is the leading contribution.

After this first asymptotics, we perform a second asymptotics where we set:

$$\mathcal{P} \ll 1, \quad \mathcal{P} \sigma_a = 1, \quad N \sigma_s = 1/\sigma_a.$$

This leads to (1), (3). Of course, one might wonder how this second approximation modifies the shock profiles compared to (46), in particular when we get rid of the radiative pressure in the momentum equation. We refer to [Mih84, page 579] for some aspects of this problem.

## References

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