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Abstract: A new class of random partial differential equations of parabolic type is considered where the stochastic term consists in an irregular noisy drift, not necessarily Gaussian, for which a suitable interpretation is provided. After freezing a realization of the drift (stochastic process), we study existence and uniqueness (in some appropriate sense) of the associated parabolic equation and a probabilistic interpretation is investigated.

Key words: Singular drifted PDEs, Dirichlet processes, martingale problem, stochastic partial differential equations, distributional drift.

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1 Introduction

This paper focuses on a random partial differential equation consisting in a parabolic PDE with irregular noise in the drift. Formulation, existence (with uniqueness in a certain sense) and double probabilistic representation are discussed. The equation itself is motivated by *random irregular media models*.

Let $T > 0$, $\sigma : \mathbb{R} \to \mathbb{R}$ be a continuous function and $\dot{\eta}(x)$ a generalized random field playing the role of a noise. Let $u^0 : \mathbb{R} \to \mathbb{R}, \lambda : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous. Consider the problem

$$\begin{cases} -\partial_t v(t,x) + \frac{\sigma^2(x)}{2}\partial^2_{xx} v(t,x) + \dot{\eta}(x)\partial_x v(t,x) &= \lambda(T-t,x), \\ v(0,x) &= u^0(x), \end{cases} \tag{1.1}$$

where $\dot{\eta}$ is the derivative in the sense of distributions of a continuous process. Among examples of possible η, one has in mind different possibilities of continuous processes as classical Wiener process, (multi)fractional Brownian motion, but also non-Gaussian processes. The derivative in the sense of distributions $\dot{\eta}(x)$ will be the associated noise. (1.1) is a new type of SPDE, not yet really studied even when η is a classical Brownian When $\dot{\eta}(x)$ is replaced by a space-time white noise $\dot{\eta}(t,x)$, some relevant work was done by Nualart and Viens, see e.g. [17]. In this article, dependence in time is useful for the corresponding stochastic integration.

(1.1) is equivalent to the following *dual* problem:

$$\begin{cases} \partial_t u(t,x) + \frac{\sigma^2(x)}{2}\partial^2_{xx} u(t,x) + \dot{\eta}(x)\partial_x u(t,x) &= \lambda(t,x), \\ u(T,x) &= u^0(x). \end{cases} \tag{1.2}$$

Formally speaking, setting $u(t,x) = v(T-t,x)$, v solves (1.1) if and only if u solves (1.2). This is rigorously confirmed at section 9 so that, at this stage, the choice whether to work with equation (1.1) or (1.2) is arbitrary. We decided to concentrate on equation (1.2), because it corresponds to the standard form for probabilistic representation.

The idea of the paper is to freeze first the realization ω, to set $b(x) = \eta(x)(\omega)$, and then to consider the deterministic Cauchy problem associated to (1.2)

$$\begin{cases} \partial_t u(t,x) + \frac{\sigma^2(x)}{2}\partial^2_{xx} u(t,x) + b'(x)\partial_x u(t,x) &= \lambda(t,x), \\ u(T,x) &= u^0(x), \end{cases} \tag{1.3}$$

where b' is the derivative of the continuous function b.

Since the product of a distribution and a continuous function is not defined in the theory of Schwartz distributions, one has to elaborate some substitution tools. Ideally, one would like to represent the parabolic PDE probabilistically through a diffusion which is the solution of the following stochastic differential equation (SDE)

$$dX_t = \sigma(X_t)dW_t + b'(X_t)dt, \tag{1.4}$$

with generalized drift. We will give a meaning to (1.4) at three different levels.

- The level of a martingale problem.
- The level of a stochastic differential equation in the sense of probability laws.
- The level of stochastic differential equation in the strong sense.

For each of these levels we shall provide conditions for equation (1.4), with given initial data, to be well-posed. Later on, the notion of C_b^0-solution to the generalized parabolic PDE (1.3) will be defined: to this effect, existence, uniqueness and probabilistic representation will be shown.

When η is a strong finite cubic variation process and $\sigma = 1$, then the solutions to (1.3) obtained for $b = \eta(\omega)$ provide solutions to the SPDE (1.1). This is shown in the last part of the paper. A typical example of a strong zero cubic variation process is the fractional Brownian motion with Hurst index $H \geq \frac{1}{3}$. Equation (1.3) will be understood in some *weak distributional* sense that we can formally reconstruct as follows. We freeze $b = \eta(\omega)$ as a realization and formally integrate equation (1.1) from 0 to t in time against a smooth test function α with compact support in space. The result is

$$\begin{aligned} -\int_{\mathbb{R}} dx\alpha(x)u(t,x) \quad &+ \quad \int_{\mathbb{R}} dx\alpha(x)u^0(x) - \int_0^t ds\frac{1}{2}\int_{\mathbb{R}} dx\alpha'(x)\partial_x u(s,x) \\ &+ \quad \int_0^t ds\int_{\mathbb{R}} b(dx)\alpha(x)\partial_x u(s,x) \\ &= \quad \int_0^t ds\int_{\mathbb{R}} dx\alpha(x)\lambda(T-s,x). \end{aligned} \tag{1.5}$$

The integral $\int_{\mathbb{R}} \alpha(x)\partial_x u(s,x)b(dx)$ needs interpretation since b is not generally of bounded variation and it involves the product of the distribution b' and the function $\partial_x u(s,\cdot)$; in general, this function is unfortunately only continuous. As foreseen, this operation is deterministically undefined, unless one uses a generalized functions theory. However, since b is a frozen realization of a stochastic process η, we can hope to justify that integral in a stochastic sense. Note that it cannot be of Itô type even if η were a semimartingale, since $\partial_x u(s,\cdot)$ is not necessarily adapted to some corresponding filtration. We will in fact interpret the stochastic integral element $b(dx)$ or $\eta(dx)$ as a symmetric (Stratonovich) integral $d^0\eta$ of regularization type, see section 3.

Definition 1.1 *A continuous random field $(v(t,x), t \in [0,T], x \in \mathbb{R})$, a.s. in $C^{0,1}(]0,T[\times\mathbb{R})$, is said to be a* **(weak) solution** *to the SPDE (1.1) if*

$$\begin{aligned} -\int_{\mathbb{R}} dx\alpha(x)v(t,x) \quad &+ \quad \int_{\mathbb{R}} dx\alpha(x)v^0(x) - \int_0^t ds\frac{1}{2}\int_{\mathbb{R}} dx\alpha'(x)\partial_x v(s,x) \\ &+ \quad \int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\left(\int_0^t ds\partial_x v(s,x)\right) \\ &= \quad \int_0^t ds\int_{\mathbb{R}} dx\alpha(x)\lambda(T-s,x), \end{aligned} \tag{1.6}$$

for every smooth function with compact support α.

If we integrate equation (1.2) from t to T in time against a smooth test function α with compact support in space, we are naturally led to the following.

Definition 1.2 *A continuous random field $(u(t,x), t \in [0,T], x \in \mathbb{R})$, a.s. in $C^{0,1}(]0,T[\times\mathbb{R})$, is said to be a* **(weak) solution** *to the SPDE (1.2) if*

$$\begin{aligned} -\int_{\mathbb{R}} dx\alpha(x)u(t,x) \quad &+ \quad \int_{\mathbb{R}} dx\alpha(x)u^0(x) - \int_t^T ds\frac{1}{2}\int_{\mathbb{R}} dx\alpha'(x)\partial_x u(s,x) \\ &+ \quad \int_{\mathbb{R}} d^\circ\eta(x)\alpha(x)\left(\int_t^T ds\partial_x u(s,x)\right) \\ &= \quad \int_t^T ds\int_{\mathbb{R}} dx\alpha(x)\lambda(s,x), \end{aligned} \tag{1.7}$$

for every smooth function with compact support α.

We will show that the *probabilistic* solutions that we construct through stochastic equation (1.4) will in fact solve (1.5).

Diffusions in the generalized sense were studied by several authors beginning with, at least in our knowledge [19]; later on, many authors considered special cases of stochastic differential equations with generalized coefficients, it is difficult to quote them all: in particular, we refer to the case when b is a measure: [7], [16], [18]. In all these cases solutions were semimartingales. More recently, [8] considered special cases of non-semimartingales solving stochastic differential equations with generalized drift; those cases include examples coming from Bessel processes.

[10] and [11] treated well-posedness of the martingale problem, Itô formula under weak conditions, semimartingale characterization and Lyons-Zheng decomposition. The only assumption was the strict positivity of σ and the existence of the function $\Sigma(x) = 2\int_0^x \frac{b'}{\sigma^2}dy$ with appropriate regularizations. Bass and Chen [2] were also interested in (1.4) and they provided a well stated framework when σ is $\frac{1}{2}$-Hölder continuous and b is γ-Hölder continuous, $\gamma > \frac{1}{2}$.

Besides the martingale problem, in the present paper we shall put emphasis on the formulation of (1.4) as a stochastic differential equation which can be solved adding more assumptions on the coefficients. Several examples are provided for the case of weak and strong solutions of (1.4).

The paper is organized as follows. Section 2 is devoted to basic preliminaries, including definitions and properties related to Young integrals. Section 3 is devoted to some useful remainder in stochastic calculus via regularization. In Section 4, we introduce the formal *elliptic* operator L and we recall the concept of C^1-generalized solution of $Lf = \dot{\ell}$ for continuous real functions $\dot{\ell}$. We further introduce a fundamental hypothesis on L for the sequel, called Technical Assumption $\mathcal{A}(\nu_0)$ and we illustrate several examples where it is verified. In Section 5 we discuss different notions of martingale problems. Section 6 provides notions of solutions to *stochastic differential equations with distributional drift* and its connections with martingale problems. The notion of solution is coupled with a property of *extended local time regularity*. This concept of solution is new even when the drift is an ordinary function.

Section 7 presents the notion of C_b^0-solution for a parabolic equation $\mathcal{L}u = \lambda$ where λ is bounded and continuous with $\mathcal{L} = \partial_t + L$. We also provide existence, uniqueness and probabilistic representation of C_b^0-solutions to $\mathcal{L}u = \lambda$. Section 8 discusses *mild* solutions to the previous parabolic PDE and useful integrability properties for its solutions. In Section 9, we finally show that the C_b^0-solutions provide in fact true *weak* solutions to the SPDE (1.1) if $\sigma = 1$.

2 Preliminaries

In this paper T will be a fixed horizon time, unless something else is specified. A function f defined on $[0,T]$ (resp. $\mathbb{R}_+$) is extended without mention setting $f(t) = f(0)$ for $t \leq 0$ and $f(T)$ for $t \geq T$ (resp. $f(0)$ for $t \leq 0$).

$C^0(\mathbb{R})$ will indicate the set of continuous functions defined on $\mathbb{R}$, $C^p(\mathbb{R})$, the space of real functions with class of differentiability C^p. We denote by $C_0^0(\mathbb{R})$ (resp. $C_0^1(\mathbb{R})$) the space of continuous (continuous differentiable) functions vanishing at zero. When there will be no confusion we will also simply use the symbols C^0, C^p, C_0^0, C_0^1. We denote by $C_b^0([0,T]\times\mathbb{R})$ the space of real continuous bounded functions defined on $[0,T]\times\mathbb{R}$. $C_b^0(\mathbb{R})$ or simply C_b^0 indicates the continuous bounded functions defined on $\mathbb{R}$.

The vector spaces $C^0(\mathbb{R})$ and $C^p(\mathbb{R})$ are topological Fréchet spaces or F-space according to the terminology of [5] chapter 1.2. They are equipped with the following natural topology. A sequence f_n belonging to $C^0(\mathbb{R})$ (resp. $C^p(\mathbb{R})$) is said to converge to f in the $C^0(\mathbb{R})$ (resp. $C^p(\mathbb{R})$) sense if f_n (resp. f_n and all the derivatives up to order p) converges (resp. converge) to f (resp. to f and all its derivatives) uniformly on each compact of $\mathbb{R}$.

We will consider functions $u : [0,T]\times\mathbb{R} \to \mathbb{R}$, which are bounded continuous. A sequence (u_n) in $C_b^0([0,T]\times\mathbb{R})$ will be said to converge in **a bounded way** to u if

- $\lim_{n\to\infty} u_n(t,x) = u(t,x), \quad \forall (t,x) \in [0,T]\times\mathbb{R},$

- there is a constant $c > 0$, independent of the sequence such that

$$\sup_{t \leq T, x \in \mathbb{R}} |u_n(t,x)| \leq c, \forall n \in \mathbb{N}. \tag{2.1}$$

If the sequence (u_n) does not depend on t we define similarly the convergence of $(u_n) \in C_b^0(\mathbb{R})$ to $u \in C_b^0(\mathbb{R})$ in a bounded way.

Given two functions $u_1, u_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$; the composition notation $u_1 \circ u_2$, means $(u_1 \circ u_2)(t,x) = u_1(t, u_2(t,x))$.

For positive integers m, k, $C^{m,k}$ will indicate functions in the corresponding class of differentiability. For instance, $C^{1,2}([0,T[\times\mathbb{R})$ will be the space of $(t,x) \mapsto u(t,x)$ functions which are C^1 on $[0,T[\times\mathbb{R}$ (i.e. one times continuously differentiable) and such that $\partial^2_{xx} u$ exists and is continuous.

$C_b^{m,k}$ will indicate the set of functions $C^{m,k}$ such that the partial derivatives of all order are bounded.

If I is a real compact interval and $\gamma \in]0,1[$, we denote by $C^\gamma(I)$ the vector space of real functions defined on I being Hölder with parameter γ. We denote by $C^\gamma(\mathbb{R})$, or simply C^γ, the space of locally Hölder functions, i.e. Hölder on each real compact interval.

Suppose $I = [\tau, T]$, τ, T being two real numbers such that $\tau < T$. Here T does not need to be necessarily positive. Recall that $f : I \mapsto \mathbb{R}$ belongs to $C^\gamma(I)$ if

$$N_\gamma(f) := \sup_{\tau \leq s,t \leq T} \frac{|f(t) - f(s)|}{|t-s|^\gamma} < \infty.$$

Clearly $f \mapsto |f(\tau)| + N_\gamma(f)$ defines a norm on $C^\gamma(I)$ which makes it a Banach space. $C^\gamma(\mathbb{R})$ is an F-space, if equipped with the topology of convergence related to $C^\gamma(I)$ for each compact interval I. A sequence (f_n) in $C^\gamma(\mathbb{R})$ converges to f if it converges according to $C^\gamma(I)$ for every compact interval I.

We will also provide some reminder about the so called **Young integrals**, see [27], remaining however in a simplified framework as in [9] or [23]. We recall the essential inequality, stated for instance in [9].

Let $\gamma, \beta > 0$, such that $\gamma + \beta > 1$. If $f, g \in C^1(I)$, then

$$\left| \int_a^b (f(x) - f(a)) dg(x) \right| \leq C_\rho (b-a)^{1+\rho} N_\gamma(f) N_\beta(g), \tag{2.2}$$

for any $[a,b] \subset I$ and $\rho \in]0, \gamma+\beta-1[$, where C_ρ constant not depending on f, g. The bilinear map sending (f,g) to $\int_0^\cdot f dg$ can be continuously extended to $C^\gamma(I) \times C^\beta(I)$ with values in $C^0(I)$. By definition that object will be called the **Young integral** of f with respect to g on I. We also denote it $\int_\tau^\cdot f d^{(y)} g$.

By additivity we set for $a, b \in [\tau, T]$,

$$\int_a^b f d^{(y)} g = \int_\tau^b f d^{(y)} g - \int_\tau^a f d^{(y)} g.$$

Moreover, the bilinear map defined on $C^1(\mathbb{R}) \times C^1(\mathbb{R})$, by $(f,g) \to \int_0^\cdot f dg$ extends continuously to $C^\gamma(\mathbb{R}) \times C^\beta(\mathbb{R})$ onto $C^0(\mathbb{R})$. Again, that object, defined on the whole real line, will be called **Young integral** of f with respect to g and it will again be denoted by $\int_0^\cdot f d^{(y)} g$.

Remark 2.1 *Inequality (2.2) remains true for $f \in C^\gamma(I), g \in C^\beta(I)$. In particular $t \mapsto \int_\tau^t f d^{(y)} g$ belongs to $C^\beta(I)$. In fact*

$$\left| \int_a^b f dg \right| \leq \left| \int_a^b (f - f(a)) dg \right| + |f(a)(g(b) - g(a))| .$$

Through the extension of the bilinear operator sending (f,g) to $\int_0^\cdot f dg$ it is possible to get the following chain rule for Young integrals.

Proposition 2.2 *Let $f, g, F : I \to \mathbb{R}, I = [\tau, T]$. We suppose $g \in C^\beta(I), f \in C^\gamma(I), F \in C^\delta(I)$ with $\gamma + \beta > 1, \quad \delta + \beta > 1$. We define $G(t) = \int_\tau^t f d^{(y)} g$. Then*

$$\int_\tau^t F d^{(y)} G = \int_\tau^t F f d^{(y)} g.$$

Proof. If $g \in C^1(I)$ then the result is obvious. We remark that $G \in C^\gamma(I)$. Using successively inequality (2.2) one can show that the two linear maps $g \mapsto \int_\tau^\cdot F d^{(y)} G$ and $g \mapsto \int_\tau^\cdot F f d^{(y)} g$ are continuous from $C^\delta(I)$ to $C^0(I)$. This concludes the proof of the Proposition. ■

By a **mollifier**, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ (i.e. a C^∞-function such that itself and all its derivatives decrease to zero faster than any power of $|x|^{-1}$ as $|x| \to \infty$) with $\int \Phi(x)\, dx = 1$. We set $\Phi_n(x) := n\Phi(nx)$.

The result below shows that mollifications of a Hölder function f converge to f with respect to the Hölder topology.

Proposition 2.3 *Let Φ be a mollifier and $f \in C^{\gamma'}(I)$; we denote $f_n = \Phi_n * f$. Then $f_n \to f$ in the $C^{\gamma}(I)$ topology for any $0 < \gamma < \gamma'$.*

Proof. We need to show that $N_\gamma(f - f_n)$ converges to zero. We set $\Delta_n(t) = (f - f_n)(t)$. Let $a, b \in I$. We will establish that

$$|\Delta_n(b) - \Delta_n(a)| \leq \text{const}|b-a|^\gamma \left(\frac{1}{n}\right)^{\gamma'-\gamma}. \tag{2.3}$$

Without restriction of generality we can suppose $a < b$. We distinguish two cases.

Case $a < a + \frac{1}{n} < b$.

We have

$$\begin{aligned} |\Delta_n(b) - \Delta_n(a)| &\leq \left| \int (f(b - \frac{y}{n}) - f(b))\Phi(y)dy \right| \\ &+ \left| \int (f(a - \frac{y}{n}) - f(a))\Phi(y)dy \right| \\ &\leq 2 \int \left|\frac{y}{n}\right|^{\gamma'} |\Phi(y)|dy \\ &\leq 2 \int |\Phi(y)||y|^{\gamma'} dy \, (b-a)^\gamma \left(\frac{1}{n}\right)^{\gamma'-\gamma} \end{aligned}$$

Case $a < b \leq a + \frac{1}{n}$.

In this case we have

$$\begin{aligned} |\Delta_n(b) - \Delta_n(a)| &\leq \int |f(b) - f(a)||\Phi(y)|dy + \int |f(b + \frac{y}{n}) - f(a + \frac{y}{n})||\Phi(y)|dy \\ &\leq 2(b-a)^{\gamma'} \int |\Phi(y)|dy \leq 2 \int |\Phi(y)|dy \, (b-a)^\gamma \left(\frac{1}{n}\right)^{\gamma'-\gamma} \end{aligned}$$

Therefore (2.3) is verified with $const = 2 \int |\Phi(y)|(1 + |y|^{\gamma'})dy$. This implies

$$N_\gamma(f - f_n) \leq \text{const} \left(\frac{1}{n}\right)^{\gamma'-\gamma},$$

which allows to conclude. ■

For convenience we introduce the topological vector space defined by

$$D^\gamma = \bigcup_{\gamma' > \gamma} C^{\gamma'}(\mathbb{R}).$$

It is also a **vector algebra** i.e. D^γ is a vector space and an algebra with respect to the *sum and product of functions*.

Next Corollary is a consequence of the definition of Young integral and Remark 2.1.

Corollary 2.4 *Let $f \in D^\gamma, g \in D^\beta$ with $\gamma + \beta \geq 1$. Then $t \mapsto \int_0^t f d^{(y)} g$ is well defined and it belongs to D^β.*

D^γ is not a metric space but an inductive limit of the F-spaces C^γ and the weak version of Banach-Steinhaus theorem for F-spaces can be adapted.

In fact, a direct consequence of the Banach-Steinhaus theorem of [5] section 2.1 is the following.

Theorem 2.5 *Let $E = \bigcup_n E_n$ be an inductive limit of F-spaces E_n and F another F-space. Let (T_n) be a sequence of linear continuous operators $T_n : E \to F$. Suppose that $Tf := \lim_{n\to\infty} T_n f$ exists for any $f \in E$. Then $T : E \to F$ is again a continuous (linear) operator.*

3 Previous results in stochastic calculus via regularization

We recall here a few notions related to stochastic calculus via regularization, a theory which was started in [21]. We refer to a recent survey paper [23].

The considered stochastic processes may be defined on $[0, T], \mathbb{R}_+$ or $\mathbb{R}$. Let $X = (X_t, t \in \mathbb{R})$ be a continuous process and $Y = (Y_t, t \in \mathbb{R})$ be a process with paths in L^1_{loc}. For the paths of process Y with parameter on $[0, T]$ (resp. $\mathbb{R}_+$) we apply the same convention as, at the beginning of previous section, for functions. So we extend without other mention setting Y_0 for $t \leq 0$ and Y_T for $t \geq T$ (resp. Y_0 for $t \leq 0$). $\mathcal{C}$ will denote the vector algebra of continuous processes. It is an F-space if equipped with the topology of the ucp (uniform convergence in probability) convergence.

We recall in the sequel the most useful rules of calculus, see for instance [23] or also [22].

The forward, symmetric integrals and the covariation process are defined by the following limits in the ucp sense whenever they exist

$$\int_0^t Y\, d^- X := \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_s}{\varepsilon}\, ds \tag{3.4}$$

$$\int_0^t Y_s\, d^\circ X_s := \lim_{\varepsilon\to 0+} \int_0^t Y_s \frac{X_{s+\varepsilon} - X_{s-\varepsilon}}{2\varepsilon}\, ds \tag{3.5}$$

$$[X,Y]_t := \lim_{\varepsilon\to 0+} C^\varepsilon(X,Y)_t, \tag{3.6}$$

where

$$C^\varepsilon(X,Y)_t := \frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s)\, ds.$$

All stochastic integrals and covariation processes will be of course elements of $\mathcal{C}$. If $[X,Y]$, $[X,X]$, $[Y,Y]$ exist we say that (X,Y) **has all its mutual covariations**.

Remark 3.1 *If X is (locally) of bounded variation, we have*

- $\int_0^t X\, d^- Y = \int_0^t X_s\, d^\circ Y_s = \int_0^t X_s\, dY_s$ *where the third integral is meant in the Lebesgue-Stieltjes sense.*
- $[X,Y] \equiv 0$.

Remark 3.2 *a)* $\int_0^t Y_s\, d^\circ X_s = \int_0^t Y_s\, d^- X_s + \frac{1}{2}[X,Y]$ *provided that two among the three integrals or covariations exist.*

b) $X_t Y_t = X_0 Y_0 + \int_0^t Y_s\, d^- X_s + \int_0^t X_s\, d^- Y_s + [X,Y]_t$ *provided that two of the three integrals or covariations exist.*

c) $X_t Y_t = X_0 Y_0 + \int_0^t Y\, d^\circ X + \int_0^t X_s\, d^\circ Y_s$ *provided that one of the two integrals exists.*

Remark 3.3 *a) If $[X,X]$ exists then it is always an increasing process and X is called a* **finite quadratic variation process**. *If $[X,X] = 0$ then X is said to be a* **zero quadratic variation process**.

b) Let X, Y be continuous processes such that (X,Y) has all its mutual covariations. Then $[X,Y]$ has locally bounded variation. If $f,g \in C^1$ then

$$[f(X),g(Y)]_t = \int_0^t f'(X)g'(Y)\,d[X,Y].$$

c) If A is a zero quadratic variation process and X is a finite quadratic variation process then $[X,A] \equiv 0$.

d) A bounded variation process is a zero quadratic variation process.

*e) (***Classical Itô formula***) If $f \in C^2$ then $\int_0^{\cdot} f'(X)\,d^-X$ exists and is equal to*

$$f(X) - f(X_0) - \frac{1}{2}\int_0^{\cdot} f''(X)\,d[X,X].$$

f) If $g \in C^1$ and $f \in C^2$ then the forward integral $\int_0^{\cdot} g(X)d^-f(X)$ is well defined.

In this paper all filtrations are supposed to fulfill the usual conditions. If $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration, X an $\mathbb{F}$-semimartingale, Y is $\mathbb{F}$-adapted cadlag process, then $\int_0^{\cdot} Y\,d^-X$ is the usual Itô integral. If Y is $\mathbb{F}$-semimartingale then $\int_0^{\cdot} Y\,d^0X$ is the classical Fisk-Stratonovich integral and $[X,Y]$ the usual covariation process $\langle X,Y\rangle$.

We introduce now the notion of Dirichlet process which were essentially introduced by H. Föllmer [12] and considered by many authors, see for instance [3, 24] for classical properties.

In the present section, (W_t) will denote a classical $(\mathcal{F}_t)$-Brownian motion.

Definition 3.4 *An $(\mathcal{F}_t)$-adapted (continuous) process is said $(\mathcal{F}_t)$-***Dirichlet process** *if it is the sum of an $(\mathcal{F}_t)$- local martingale M plus a zero quadratic variation process A. For simplicity we will suppose $A_0 = 0$ p.s.*

Remark 3.5 *(i) Process (A_t) in the previous decomposition is an $(\mathcal{F}_t)$-adapted process.*

(ii) An $(\mathcal{F}_t)$-semimartingale is an $(\mathcal{F}_t)$-Dirichlet process.

(iii) The decomposition $M + A$ is unique.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ of class C^1, X an $(\mathcal{F}_t)$-Dirichlet process. Then $f(X)$ is again a $(\mathcal{F}_t)$-Dirichlet process with local martingale part $M_t^f = f(X_0) + \int_0^t f'(X)dM$.

The class of semimartingales with respect to a given filtration is known to be stable with respect to C^2 transformations. Remark 3.3 b) says that finite quadratic variation processes are stable through C^1 transformations. The last point of previous remark states that C^1 stability also holds for Dirichlet processes.

Young integrals introduced in the Preliminaries can be connected with the forward and symmetric integrals via regularization appearing before Remark 3.1. The next proposition has been proved in [23].

Proposition 3.6 *Let X, Y be processes whose paths are respectively in C^γ and in C^β, with $\gamma > 0, \beta > 0$ and $\gamma + \beta > 1$.*
For any symbol $\star \in \{-, \circ\}$ the integral $\int_0^\cdot Y d^\star X$ coincides with the Young integral $\int_0^\cdot Y d^{(y)} X$.

Remark 3.7 *Suppose that X and Y verifies the conditions of Proposition 3.6. Then Remark 3.2 a) implies that $[X, Y] = 0$.*

We need an extension of stochastic calculus via regularization in the direction of higher $n-$variation. The properties concerning higher variation than 2 can be found for instance in [6].

We set

$$[X, X, X]_t^\varepsilon = \frac{1}{\varepsilon} \int_0^t (X_{s+\varepsilon} - X_s)^3 \, ds.$$

We define also

$$\left\| [X, X, X]^\varepsilon \right\|_t = \frac{1}{\varepsilon} \int_0^t |X_{s+\varepsilon} - X_s|^3 \, ds.$$

If the limit when $\varepsilon \to 0$ in probability of $[X, X, X]_t^\varepsilon$ exists for any t, we denote it by $[X, X, X]_t$. If the limiting process $[X, X, X]$ has a continuous version, we say that X is a **finite cubic variation process**.

If moreover, there is a positive sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero such that

$$\sup_{\varepsilon_n} \left\| [X,X,X]^{\varepsilon_n} \right\|_T < +\infty, \tag{3.7}$$

we will say that it X is a **(strong) finite cubic variation process**. If X is a (strong) finite cubic variation process such that $[X,X,X]=0$, X will be said (strong) zero **finite cubic variation process**.

For instance, if $X = B^H$, a fractional Brownian motion with Hurst index then X is a finite quadratic variation process if and only if $H \geq \frac{1}{2}$, see [22]. It is a strong zero cubic variation process if and only if $H \geq \frac{1}{3}$, see [6]. On the other hand B^H is a zero cubic variation process if and only if $H > \frac{1}{6}$, see [13].

It is clear that a finite quadratic variation process is a strong zero cubic variation process. On the other hand, processes whose paths are Hölder continuous with parameter greater than $\frac{1}{3}$ are strong zero cubic variation processes.

As for finite quadratic variation and Dirichlet processes, the C^1-stability also holds for finite cubic variation processes. The next Proposition is a particular case of a result contained in [6].

Proposition 3.8 *Let X be a strong finite cubic variation process, V a locally bounded variation process and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ of class C^1. Then $Z = f(V,X)$ is again a strong finite cubic variation process and*

$$[Z,Z,Z]_t = \int_0^t \partial_x f(V_s,X_s)^3 d[X,X,X]_s.$$

Moreover a Itô chain rule property holds as follows.

Proposition 3.9 *Let X be a strong finite cubic variation process, V a bounded variation process and a cadlag process Y. Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$*

of class $C^{1,3}$.

$$\begin{aligned}\int_0^t Y d^\circ f(V,X) &= \int_0^t Y\partial_v f(V_s,X_s)dV_s \\ &+ \int_0^t Y\partial_x f(V_s,X_s)d^\circ X_s \\ &- \frac{1}{12}\int_0^t Y\partial^3_{xxx} f(V_s,X_s)d[X,X,X]_s\end{aligned}$$

We deduce in particular that a C^1 transformation of a strong zero cubic variation process is again a strong zero cubic variation process.

We conclude the section introducing a concept of **definite integral** via regularization. If processes X, Y are indexed by the whole real line, a.s. with compact support, we define

$$\int_{\mathbb{R}} Y\, d^- X := \lim_{\varepsilon\to 0+}\int_{\mathbb{R}} Y_s \frac{X_{s+\varepsilon}-X_s}{\varepsilon}\, ds \tag{3.8}$$

$$\int_{\mathbb{R}} Y_s\, d^\circ X_s := \lim_{\varepsilon\to 0+}\int_{\mathbb{R}} Y_s \frac{X_{s+\varepsilon}-X_{s-\varepsilon}}{2\varepsilon}\, ds, \tag{3.9}$$

where the limit is understood in probability. Integration by parts (Remark 3.2) c)), Proposition 3.6 and chain rule property (Proposition 3.9) can be immediately adapted to these definite integrals.

4 About the PDE operator L

Let σ, $b \in C^0(\mathbb{R})$ such that $\sigma > 0$. Without restriction of generality we will suppose $b(0) = 0$.

We consider formally a PDE operator of the following type:

$$Lg = \frac{\sigma^2}{2}g'' + b'g'. \tag{4.1}$$

If b is of class C^1, so that b' is continuous, we will say that L is a **classical** PDE operator.

For a given mollifier Φ we denote

$$\sigma_n^2 := (\sigma^2 \wedge n) * \Phi_n \quad b_n := (-n \wedge (b \vee n)) * \Phi_n.$$

We then consider

$$\begin{aligned} L_n g &= \frac{\sigma_n^2}{2} g'' + b_n' g', \text{for } g \in C^2(\mathbb{R}) \\ \mathcal{L}_n u &= \partial_t u + L_n u, \text{for } u \in C^{1,2}([0,T[\times\mathbb{R}) \end{aligned} \tag{4.2}$$

where L_n acts on x. A priori, σ_n^2, b_n and the operator L_n depend on the mollifier Φ.

Previous definitions are slightly different from those in papers [10, 11] but a great part of the analysis of L and the study of the martingale problem can be adapted. In those papers there was only regularization but no truncation; here truncation is used to study associated parabolic equations.

Definition 4.1 *A function $f \in C^1(\mathbb{R})$ is said to be a C^1-**generalized solution** to*

$$Lf = \dot{\ell}, \tag{4.3}$$

where $\dot{\ell} \in C^0$, if, for any mollifier Φ, there are sequences (f_n) in C^2, $(\dot{\ell}_n)$ in C^0 such that

$$L_n f_n = \dot{\ell}_n, \quad f_n \to f \text{ in } C^1, \quad \dot{\ell}_n \to \dot{\ell} \text{ in } C^0. \tag{4.4}$$

Proposition 4.2 *There is a solution $h \in C^1$ to $Lh = 0$ such that $h'(x) \neq 0$ for every $x \in \mathbb{R}$ if and only if*

$$\Sigma(x) := \lim_{n\to\infty} 2 \int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy$$

exists in C^0, independently from the mollifier. Moreover, in this case, any solution f to $Lf = 0$ fulfills

$$f'(x) = e^{-\Sigma(x)} f'(0). \tag{4.5}$$

Proof. This result follows in a very similar way the proof of Proposition 2.3 of [10]: first at the level of regularization and then passing to the limit.

■

From now on, throughout the whole paper, we will suppose the existence of this function Σ. We will consider $h \in C^1$ such that

$$h'(x) := \exp(-\Sigma(x)), h(0) = 0. \tag{4.6}$$

In particular, $h'(0) = 1$ holds. Even though in [10] we discuss the general case with related non-explosion conditions, here in order to ensure conservativeness we suppose that

$$\begin{aligned} \int_{-\infty}^{0} e^{-\Sigma(x)} dx &= \int_{0}^{\infty} e^{-\Sigma(x)} dx = +\infty \\ \int_{-\infty}^{0} \frac{e^{\Sigma(x)}}{\sigma^2} dx &= \int_{0}^{\infty} \frac{e^{\Sigma(x)}}{\sigma^2} dx = +\infty. \end{aligned} \tag{4.7}$$

Previous assumptions are of course verified if σ is lower bounded by a positive constant and b is constant outside a compact interval.

The condition (4.7) implies that the image set of h is $\mathbb{R}$.

Remark 4.3 *Proposition 4.2 implies uniqueness of the problem*

$$Lf = \dot{\ell}, \quad f \in C^1, \quad f(0) = x_0, \quad f'(0) = x_1 \tag{4.8}$$

for every $\dot{\ell} \in C^0$, $x_0, x_1 \in \mathbb{R}$.

Remark 4.4 *We present four important examples where Σ exists.*

a) If $b(x) = \alpha \left(\frac{\sigma^2(x)}{2} - \frac{\sigma^2(0)}{2} \right)$ for some $\alpha \in]0, 1]$ then

$$\Sigma(x) = \alpha \log \left(\frac{\sigma^2(x)}{\sigma^2(0)} \right)$$

and

$$h'(x) = \frac{\sigma^{2\alpha}(0)}{\sigma^{2\alpha}(x)}.$$

If $\alpha = 1$ the operator L can formally be expressed in divergence form as $Lf = (\frac{\sigma^2}{2} f')'$.

b) Suppose that b is locally of bounded variation. Then we get

$$\int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy = \int_0^x \frac{db_n(y)}{\sigma_n^2(y)} \to \int_0^x \frac{db}{\sigma^2},$$

since $db_n \to db$ in the weak-$*$ topology and $\frac{1}{\sigma^2}$ is continuous.

c) If σ has bounded variation then we have

$$\Sigma(x) = -2\int_0^x b\, d(\frac{1}{\sigma^2}) + \frac{2b}{\sigma^2}(x) - \frac{2b}{\sigma^2}(0).$$

In particular, this example contains the case where $\sigma = 1$ for any b.

d) Suppose that σ is locally Hölder continuous with parameter γ and b is locally Hölder continuous with parameter β so that $\beta + \gamma > 1$. Since σ is locally bounded, than σ^2 is also locally Hölder continuous with parameter γ. Proposition 2.3 implies that $\sigma_n^2 \to \sigma^2$ in $C^{\gamma'}$ and $b_n \to b$ in $C^{\beta'}$ for every $\gamma' < \gamma$ and $\beta' < \beta$. Since σ is strictly positive on each compact, $\frac{1}{\sigma_n^2} \to \frac{1}{\sigma^2}$ in $C^{\gamma'}$. By Remark 2.1, Σ is well defined and it is locally Hölder continuous with parameter β'.

Again the following lemma can be proved at the level of regularizations, see also Lemma 2.6 in [10].

Lemma 4.5 *The unique solution to problem (4.8) is given by*

$$\begin{aligned} f(0) &= x_0, \\ f'(x) &= h'(x)\left(2\int_0^x \frac{\dot{\ell}(y)}{(\sigma^2 h')(y)}\, dy + x_1\right). \end{aligned}$$

Remark 4.6 *If $b' \in C^0(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ is a classical solution to $Lf = \dot{\ell}$ then f is clearly also a C^1-generalized solution.*

Remark 4.7 *Given $\ell \in C^1$, we denote by $T\ell$ the unique C^1-generalized solution f to problem (4.8) with $\dot{\ell} = \ell', x_0 = 0, x_1 = 0$. The unique solution to the general problem (4.8) is given by*

$$f = x_0 + x_1 h + T\ell.$$

We denote $T^{x_1}\ell = T\ell + x_1 h$, i.e. the solution with $x_0 = 0$.

Remark 4.8 *Let $f \in C^1$. There is at most one $\dot{\ell} \in C^0$ such that $Lf = \dot{\ell}$. In fact, to see this, it is enough to suppose that $f = 0$. Lemma 4.5 implies that*

$$2\int_0^x \frac{\dot{\ell}}{\sigma^2 h'}(y)dy \equiv 0$$

consequently $\dot{\ell}$ is forced to be zero.

This consideration allows to define without ambiguity $L : \mathcal{D}_L \to C^0$, where $\mathcal{D}_L$ is the set of all $f \in C^1(\mathbb{R})$ which are C^1-generalized solution to $Lf = \dot{\ell}$, for some $\dot{\ell} \in C^0$. In particular $T\ell \in \mathcal{D}_L$.

A direct consequence of Lemma 4.5 is the following useful result.

Lemma 4.9 *$\mathcal{D}_L$ is the set of $f \in C^1$ such that there is $\psi \in C^1$ with $f' = e^{-\Sigma}\psi$.*

In particular it gives us the following density proposition.

Proposition 4.10 *$\mathcal{D}_L$ is dense in C^1.*

Proof. t is enough to show that every C^2-function is the C^1-limit of a sequence of functions in $\mathcal{D}_L$. Let (ψ_n) be a sequence in C^1 converging to $f'e^{\Sigma}$ in C^0. It follows that

$$f_n(x) := f(0) + \int_0^x e^{-\Sigma}(y)\psi_n(y)dy, \quad x \in \mathbb{R}$$

converges to $f \in C^1$ and $f_n \in \mathcal{D}_L$. ■

We need now to discuss technical aspects of the way L and its domain $\mathcal{D}_L$ are transformed by h. We recall that $Lh = 0$ and h' is strictly positive. Condition (4.7) implies that the image set of h is $\mathbb{R}$.

Let L^0 be the classical PDE operator

$$L^0\phi = \frac{\tilde{\sigma}_h^2}{2}\phi'', \quad \phi \in C^2, \tag{4.9}$$

where

$$\tilde{\sigma}_h(y) = (\tilde{\sigma}h')(h^{-1}(y)), y \in \mathbb{R}.$$

L^0 is a classical PDE map; however we can also consider it at the formal level and introduce $\mathcal{D}_{L^0}$.

Proposition 4.11 *a) $h^2 \in \mathcal{D}_L$, $Lh^2 = h'^2\sigma^2$,*

b) $\mathcal{D}_{L^0} = C^2$,

c) $\phi \in \mathcal{D}_{L^0}$ *holds if and only if* $\phi \circ h \in \mathcal{D}_L$. *Moreover, we have*

$$L(\phi \circ h) = (L^0\phi) \circ h \tag{4.10}$$

for every $\phi \in C^2$.

Proof. It follows similarly as for Proposition 2.13 of [10].

■

We will now discuss another operator related to L. Given a function f, we need to provide a suitable definition of $f \mapsto \int_0^x Lf(y)dy$, i.e. some primitive of Lf.

- One possibility is to define that map through previous expression on $\mathcal{D}_L$.
- Otherwise we try to define it as linear map on C^2. For this, suppose first that b' is continuous. Then, integrating by parts we obtain

$$\int_0^x Lf(y)dy = \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\, dy + (bf')(x) - (bf')(0). \tag{4.11}$$

  We remark that the right member of previous expression makes sense for any $f \in C^2$ and continuous b. We will so define $\hat{L} : C^2 \to C_0^0$ as

$$\hat{L}f := \int_0^x \left(\frac{\sigma^2}{2} - b\right) f''(y)\, dy + (bf')(x) - (bf')(0). \tag{4.12}$$

One may ask if, in the general case, the two definitions $f \to \int_0^x Lf(y)dy$ on $\mathcal{D}_L$ and $\hat{L}$ on C^2 are compatible. We will see later that, under Assumption $\mathcal{A}(\nu_0)$ this will be the case. However, in general $\mathcal{D}_L \bigcap C^2$ may be empty.

So far, we have learnt how to eliminate the first order term in a formal PDE operator through the transformation h introduced at (4.6); when L is classical, this was performed by Zvonkin, see [28]. Now we would like to introduce a transformation which puts the PDE operator in a divergence form.

Let L be a PDE operator which is formally of type (4.1)

$$Lg = \frac{\sigma^2}{2}g'' + b'g'.$$

We consider a function of class C^1, namely $k : \mathbb{R} \to \mathbb{R}$ such that

$$k(0) = 0 \quad \text{and} \quad k'(x) = \sigma^{-2}(x)\exp(\Sigma(x)). \tag{4.13}$$

According to assumptions (4.7) k is bijective on $\mathbb{R}$.

Remark 4.12 *If there is no drift term, i.e. $b = 0$, then we have $k'(x) = \sigma^{-2}(x)$.*

Lemma 4.13 *We consider the formal PDE operator given by*

$$L^1 g = \frac{\bar{\sigma}_k^2}{2}g'' + \left(\frac{\bar{\sigma}_k^2}{2}\right)' g' = (\frac{\bar{\sigma}_k^2}{2}g')' \tag{4.14}$$

where

$$\bar{\sigma}_k(z) = (\sigma k') \circ k^{-1}(z), \quad z \in \mathbb{R}$$

Then

(i) $g \in \mathcal{D}_{L^1}$ if and only if $g \circ k \in \mathcal{D}_L$,

(ii) for every $g \in \mathcal{D}_{L^1}$ we have $L^1 g = L(g \circ k) \circ k^{-1}$.

Proof. It is practically the same as in Lemma 2.16 of [10]. ■

We now give a lemma whose proof can be easily established by investigation. Suppose that L is a classical PDE operator. Then $\mathcal{L} = \partial_t + L$ is well defined for $C^{1,2}([0,T[\times\mathbb{R})$ functions where L acts on the second variable. Given a function $\varphi \in C([0,T] \times \mathbb{R})$ from now on we will set $\tilde{\varphi} : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ by $\tilde{\varphi}(t,y) = \varphi(t, h^{-1}(y))$.

Lemma 4.14 *Let us suppose $h \in C^2(\mathbb{R})$. We set $\sigma_h = \sigma h'$.*
We define the PDE operator $\mathcal{L}^0$ by $\mathcal{L}^0\varphi = \partial_t\varphi + L^0\varphi$ where L^0 is a classical operator acting on the space variable x and

$$L^0 f = \frac{\tilde{\sigma}_h^2}{2}f''.$$

If $f \in C^{1,2}([0,T[\times\mathbb{R})$ and $\mathcal{L}f = \gamma$ in the classical sense then $\mathcal{L}^0\tilde{f} = \tilde{\gamma}$.

We will now formulate a supplementary assumption which will be useful when we study singular stochastic differential equations in the proper sense and not only under the form of martingale problem.

Technical Assumption $\mathcal{A}(\nu_0)$.

Let ν_0 be a topological F-space which is a linear topological subspace of $C^0(\mathbb{R})$ (or eventually an inductive limit of sub F-spaces). The ν_0-convergence implies convergence in C^0 and therefore point-wise convergence.

We say that L fulfills Assumption $\mathcal{A}(\nu_0)$ if

(i) $C^1 \subset \nu_0$ which is dense.

(ii) For every $g \in C^1(\mathbb{R})$, the following multiplicative operator $\phi \to g\phi$ maps ν_0 into itself.

(iii) Let $T : C^1(\mathbb{R}) \subset \nu_0 \to C^1(\mathbb{R})$ as defined in lemma 4.5, i.e. $f = T\ell$ is such that

$$\begin{cases} f(0) &= 0, \\ f'(x) &= e^{-\Sigma}(x)\left(2\int_0^x \frac{e^{\Sigma}(y)\ell'(y)}{\sigma^2(y)}\,dy\right). \end{cases}$$

We recall that $f = T\ell$ solves problem $Lf = \ell'$ with $f(0) = f'(0) = 0$. We suppose that T admits a continuous extension to ν_0.

(iv) Let $x_1 \in \mathbb{R}$. For every $f \in C^2, f(0) = 0, f'(0) = x_1$, so that $\hat{L}f = \ell$ we have $\ell \in \nu_0$ and $T^{x_1}\ell = f$, where T^{x_1} denotes the continuous extension of T^{x_1} (see Remark 4.7) to ν_0 which exists by (iii).

(v) The set $\hat{L}C^2$ is dense in $\{\ell \in \nu_0 | \ell(0) = 0\}$.

Remark 4.15 *Let $x_1 \in \mathbb{R}$.*

(i) Remark 4.7 and point (iii) imply that $T^{x_1} : C^1(\mathbb{R}) \subset \nu_0 \to C^1(\mathbb{R})$ prolongates continuously to ν_0. Moreover

$$\{f \in C^2 | f(0) = 0, f'(0) = x_1\} \subset Im T^{x_1}.$$

(ii) Point (iv) of the previous Technical Assumption shows that $b \in \nu_0$ and $T^1 b = id$ where $id(x) = x$: in fact $id(0) = 0, id'(1) = 1$ and (4.12) implies that $\hat{L}id = b$.

(iii) Point (i) is verified for instance if the map T is closable, as a map from C^0 to C^1. In that case ν_0 may be defined as the domain of the closure of C^1 equipped with the graph topology related to $C^0 \times C^1$.

Below we give some sufficient conditions for the verification of points (iv) and (v) of the Technical Assumption.

We define by $C^1_{\nu_0}$ the vector space of functions $f \in C^1$ such that $f' \in \nu_0$; it will be an F-space if equipped with the following topology. A sequence (f_n) will be said to converge to f in $C^1_{\nu_0}$ if $f_n(0) \to f(0)$ and (f'_n) converges to f' in ν_0. In particular a sequence converging according to $C^1_{\nu_0}$, also converges with respect to C^1. On the other hand $C^2 \subset C^1_{\nu_0}$ and a sequence converging in C^2, also converges with respect to $C^1_{\nu_0}$. Moreover C^2 is dense in $C^1_{\nu_0}$ because C^1 is dense in ν_0.

Lemma 4.16 *Suppose that points (i) to (iii) of the previous Technical Assumption are fulfilled. We suppose moreover:*

a) $h \in C^1_{\nu_0}$.

b) For every $f \in C^2, f(0) = 0, f'(0) = 0, \quad \hat{L}f = \ell$ *we have* $\ell \in \nu_0$ *and* $T\ell = f$.

c) $\hat{L} : C^2 \to \nu_0$ *is well-defined and it has a continuous extension to* $C^1_{\nu_0}$, *still denoted by* $\hat{L}$. *Moreover* $\hat{L}h = 0$.

d) $ImT \subset C^1_{\nu_0}$.

e) $\hat{L}T$ *is the identity map on* $\{\ell \in \nu_0 | \ell(0) = 0\}$.

Then T, T^{x_1} *for every* $x_1 \in \mathbb{R}$ *are injective and points (iv), (v), of the Technical Assumption are verified.*

Proof. The injectivity of T follows from point e). The injectivity of T^{x_1} is a consequence of Remark 4.7.

We prove point (iv). Point c) says that $\hat{L}h = 0$. We set $\hat{f} = f - x_1 h$, $f \in C^2$, where $f(0) = 0$, $f'(0) = x_1$. Clearly $\hat{L}\hat{f} = \hat{L}f = \ell$ and $\hat{f}(0) = 0, \hat{f}'(0) = 0$.

Point b) implies that $T\ell = \hat{f}$. Hence $T^{x_1}\ell = T\ell + x_1 h = f$, and (iv) is satisfied.

Concerning point (v), let $\ell \in \nu_0$ with $\ell(0) = 0$ and set $f = T\ell$. Since f belongs to $C^1_{\nu_0}$ by c), f' belongs to ν_0. Point (i) of the Technical Assumption implies that there is a sequence (f'_n) of C^1 functions converging to f' in the ν_0 sense, and thus also in C^0. Let (f_n) be the sequence of primitives of (f'_n) (which are of class C^2) such that $f_n(0) = 0$. In particular we have that (f_n) converges to f in the $C^1_{\nu_0}$-sense. By c) there exists λ in ν_0 being limit of $\hat{L}f_n$ in the ν_0-sense. Observe that, because of b), $T(\hat{L}f_n) = f_n$. On the other hand $\lim_{n\to+\infty} f_n = f$ in C^1. Applying T and using (iii) of the Technical Assumption, we obtain

$$T\lambda = \lim_{n\to+\infty} T(\hat{L}f_n) = \lim_{n\to+\infty} f_n = f = T\ell.$$

The injectivity of T allows to conclude that $\ell = \lambda$.

■

Remark 4.17 *Under the assumptions of Lemma 4.16, we have*

- $\mathcal{D}_L \subset C^1_{\nu_0}$*:*
- $\hat{L}f = \int_0^x Lf(y)dy$*,* $f \in \mathcal{D}_L$*.*

In fact, let $f \in \mathcal{D}_L$. Without restriction of generality we can suppose $f(0) = 0$. Let $x_1 = f'(0)$ and we set $\hat{f} = f + x_1 h$ so that $\hat{f}(0) = \hat{f}'(0) = 0$. Setting $\dot{\ell} = L\hat{f}$, Lemma 4.5 implies that $\hat{f} = T\ell$ where $\ell = \int_0^x \dot{\ell}(y)dy$. So $\hat{f} \in ImT \subset C^1_{\nu_0}$. Since $h \in C^1_{\nu_0}$, $f \in C^1_{\nu_0}$, by additivity.

On the other hand,

$$\begin{aligned} Lf &= L\hat{f} + x_1 Lh = \hat{L}f = \dot{\ell}, \\ \hat{L}f &= \hat{L}\hat{f} + x_1 \hat{L}h = \hat{L}T\ell = \ell, \end{aligned}$$

because of point e) of Lemma 4.16.

Example 4.18 *We provide here a series of four significant examples when the Technical Assumption $\mathcal{A}(\nu_0)$ is verified. We only comment the points which are not easy to verify.*

(i) *First example is simple. It concerns the case when the drift b' is continuous. This problem to be studied later corresponds to an ordinary SDE where*

$$\nu_0 = C^1, C^1_{\nu_0} = C^2, \hat{L}f = \int_0^{\cdot} Lf(y)dy.$$

(ii) *L is* **close to divergence type**, *i.e. $b = \frac{\sigma^2 - \sigma^2(0)}{2} + \beta$ and β is bounded variation vanishing at zero. The operator is of divergence type plus a Radon measure term: we have $\Sigma = \ln \sigma^2 + 2\int_0^x \frac{d\beta}{\sigma^2}$. In this case we have $\nu_0 = C^0$. Points (i) and (ii) of the Technical Assumption are trivial.*

We have in fact

$$h'(x) = e^{-\Sigma} = \frac{1}{\sigma^2(x)} \exp\Big(-2\int_0^x \frac{d\beta}{\sigma^2}\Big).$$

T defined at point (iii) of the Technical Assumption is such that $T\ell = f$ where $f(0) = 0$ and

$$f'(x) = \frac{2\sigma^2(0)}{\sigma^2(x)} \exp\Big(-2\int_0^x \frac{d\beta}{\sigma^2}\Big) \int_0^x \ell'(y) \exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right) dy. \quad (4.15)$$

Consequently the prolongation of T to $\nu_0 = C^0$, always denoted by this letter, is given by $f = T\ell$ with $f(0) = 0$ and

$$\begin{aligned} f'(x) &= \frac{2}{\sigma^2(x)}\Big\{\ell(x) - 2\exp\Big(-2\int_0^x \frac{d\beta}{\sigma^2}\Big) \cdot \Big(\ell(0) \\ &+ \int_0^x \ell(y) \exp\left(2\int_0^y \frac{d\beta}{\sigma^2}\right) \frac{1}{\sigma^2(y)}\, d\beta(y)\Big)\Big\}. \end{aligned} \quad (4.16)$$

We verify points (iv) and (v) through lemma 4.16. We have $C^1_{\nu_0} = C^1$. Point a) is obvious since $h' \in C^0$ and so $h \in C^1_{\nu_0}$. Let $f \in C^2$. Using Lebesgue-Stieltjes calculus, we can easily show that

$$\ell(x) = \hat{L}f(x) = \frac{\sigma^2(x)}{2} f'(x) - \frac{\sigma^2(0)}{2} f'(0) + \int_0^x f' d\beta. \quad (4.17)$$

This shows that $\ell \in C^0 = \nu_0$ and therefore the first part of b). We remark that we can in fact consider $\hat{L} : C^2 \to \nu_0$ because

$$\hat{L}f = \hat{L}(f - x_1 h) + x_1 \hat{L}h = \hat{L}(f - x_1 h) \in \nu_0.$$

The expression of $\hat{L}f$ prolongates continuously to $f \in C^1$, which yields the first part of point c). Moreover inserting the expression of h' into f' in (4.17), one shows $\hat{L}h = 0$.

Suppose now in expression (4.17) $f \in C^2, f(0) = 0, f'(0) = 0$. A simple investigation shows that $T\ell = f$, so the second part of point b) is fulfilled; point d) is also clear because of (4.16). Finally point d) holds because one can prove by inspection that $\hat{L}T$ is the identity on C_0^0.

(iii) We recall the notation $D^\gamma(\mathbb{R})$ which indicates the topological vector space of locally Hölder continuous functions defined on $\mathbb{R}$ with parameter $\alpha > \gamma$. We recall that $D^\gamma(\mathbb{R})$ is a vector algebra.

Suppose $\sigma \in D^{\frac{1}{2}}$ and $b \in C^{\frac{1}{2}}$ (or $\sigma \in C^{\frac{1}{2}}$ and $b \in D^{\frac{1}{2}}$). Remark 4.4 d) implies that Σ also belongs to $D^{\frac{1}{2}}$. We set $\nu_0 = D^{\frac{1}{2}}$.

Technical Assumption $\mathcal{A}(\nu_0)$ is verified for the following reasons.

Since $\Sigma \in D^{\frac{1}{2}}$, $h' = e^{-\Sigma}$ belongs to the same space.

Point (i) follows because of Proposition 2.3, point (ii) because $D^{\frac{1}{2}}$ is an algebra. Corollary 2.4 yields that, for every $\ell \in D^{\frac{1}{2}}$, the function

$$f'(x) = e^{-\Sigma(x)} \int_0^x 2\frac{e^\Sigma}{\sigma^2}(y) d^{(y)}\ell(y), \tag{4.18}$$

is well-defined and it belongs to $D^{\frac{1}{2}}$. This shows that T can be continuously extended to ν_0 and point (iii) is established.

Concerning points (iv) and (v), we use again Lemma 4.16. We observe that

$$C^1_{\nu_0} = \{f \in C^1 | f' \in D^{\frac{1}{2}}\}.$$

Point a) is obvious since $h' = e^{-\Sigma} \in D^{\frac{1}{2}}$. Let $f \in C^2$. Observing b as a deterministic process, the definition of $\hat{L}$ as in (4.12), integration by parts in Remark 3.2 c) and Proposition 3.6 imply

$$\ell(x) = \int_0^x \frac{\sigma^2}{2} d^0 f' + \int_0^x f' d^\circ b \tag{4.19}$$

$$\ell(x) = \int_0^x \frac{\sigma^2}{2} d^{(y)} f' + \int_0^x f' d^{(y)} b. \tag{4.20}$$

First part of point b) follows because of Proposition 2.2. Of course, previous expression can be extended to $f \in C^1_{\nu_0}$ and this shows the first part of point c).

The second part of point c) of Lemma 4.16, consists in verifying $\hat{L}h = 0$. Plugging $h' = e^{-\Sigma}$ in previous expression, through Proposition 2.2, we obtain

$$\ell(x) = -\int_0^x \frac{\sigma^2}{2} e^{-\Sigma} d^{(y)}\Sigma + \int_0^x e^{-\Sigma} d^{(y)} b = 0.$$

Concerning the second part of point b), let $f \in C^2$ so that $f(0) = f'(0) = 0$. We want to show that $\varphi = T\ell$ coincides with f.

Since $\varphi(0) = 0$, it remains to check $\varphi' = f'$. We recall that

$$\varphi'(x) = e^{-\Sigma}(x) \left(2 \int_0^x \frac{e^{\Sigma}}{\sigma^2}(y) d^{(y)} \ell(y) \right).$$

Applying twice the chain rule of Proposition 2.2, (4.19), the fact that

$$e^{\Sigma}(x) = \int_0^x e^{\Sigma} \frac{2 d^{(y)} b}{\sigma^2} + 1,$$

and integration by parts we obtain

$$\begin{aligned}
\varphi'(x) &= e^{-\Sigma}(x) \left\{ \int_0^x e^{\Sigma} d^0 f' + \int_0^x 2 \frac{e^{\Sigma}}{\sigma^2} f' d^{(y)} b \right\} \\
&= e^{-\Sigma}(x) \left\{ \int_0^x e^{\Sigma} d^0 f' + \int_0^x f' d^{(y)} e^{\Sigma} \right\} \\
&= e^{-\Sigma}(x) \left\{ \int_0^x e^{\Sigma} d^0 f' + \int_0^x f' d^0 e^{\Sigma} \right\} \\
&= e^{-\Sigma}(x) \left\{ (f' e^{\Sigma})(x) - (f' e^{\Sigma})(0) \right\} \\
&= f'(x).
\end{aligned}$$

Point b) is therefore completely established.

Point d) follows because in (4.18), when $\ell \in \nu_0$, $f' \in \nu_0$ also.

Clearly, as for previous example, $ImT \subset C^1_{\nu_0}$. It remains to show that $\hat{L}T$ is the identity map $\{f \in D^{\frac{1}{2}} | f(0) = 0\}$.

For this we first remark that

$$\hat{L} f(x) = \int_0^x \frac{\sigma^2}{2} e^{-\Sigma} d^{(y)} \left(f' e^{\Sigma} \right) \tag{4.21}$$

In fact, by Proposition 3.6 and integration by parts Remark 3.2 c), we obtain

$$f'(x)e^{\Sigma(x)} = f'(0) + \int_0^x e^{\Sigma} d^{(y)} f' + \int_0^x f' d^{(y)} e^{\Sigma}.$$

By the chain rule of Proposition 2.2, we obtain the right member of (4.21).

At this point, by definition, if $f = T\ell$, we have

$$f'(x)e^{\Sigma(x)} = \int_0^x 2\frac{e^{\Sigma}}{\sigma^2} d^{(y)} \ell.$$

Therefore (4.21) and Proposition 2.2 allow to conclude

$$\hat{L}f(x) = \int_0^x \frac{\sigma^2}{2} e^{-\Sigma} 2\frac{e^{\Sigma}}{\sigma^2} d^{(y)} \ell = \ell(x) - \ell(0).$$

(iv) Suppose b locally with bounded variation. Then the Technical Assumption is verified for $\nu_0 = BV$ where BV is the space of continuous real functions, locally with bounded variation v, equipped with the following topology. A sequence (v_n) in BV converges to v if

$$\begin{cases} v_n(0) \to v(0) \\ dv_n \to dv \quad \text{in the weak}- * \text{topology.} \end{cases}$$

The arguments for proving that the Technical Assumption is verified are similar but easier than at previous point. Young type calculus is replaced by classical Lebesgue-Stieltjes calculus.

5 Martingale problem

In this section, we consider a PDE operator satisfying the same properties as in previous section, i.e.

$$Lg = \frac{\sigma^2}{2} g'' + b'g', \tag{5.1}$$

where $\sigma > 0$ and b are continuous. In particular, we assume that

$$\Sigma(x) = \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)\, dy \tag{5.2}$$

exists in C^0, independently from the chosen mollifier. Then h defined by $h'(x) := \exp(-\Sigma(x))$ and $h(0) = 0$, is a solution to $Lh = 0$ with $h' \neq 0$.

We aim here at introducing different notions of martingale problem trying, when possible, to clarify also the classical notion. For the next two definitions, we consider the following convention. Let $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ fulfilling the *usual conditions*.

Definition 5.1 *A process X is said to solve* **the martingale problem** *related to L (with respect to previous filtered probability space), with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if*

$$f(X_t) - f(x_0) - \int_0^t Lf(X_s)\, ds$$

is an $(\mathcal{F}_t)_{t\geq s}$-local martingale for $f \in \mathcal{D}_L$ and $X_0 = x_0$.

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq 0)$ solves the martingale problem related to L with initial value x at time s if for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\, dr, \quad t \geq s$$

is an $(\mathcal{F}_t)_{t\geq s}$-local martingale.

We remark that $X^{s,x}$ solves the martingale problem at time s if and only if $X_t := X_{t+s}^{s,x}$ solves the martingale problem at time 0.

Definition 5.2 *Let (W_t) be an $(\mathcal{F}_t)$-classical Wiener process. An $(\mathcal{F}_t)$-progressively measurable process $X = (X_t)$ is said to solve* **the sharp martingale problem** *related to L (on the given filtered probability space), with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}$, if*

$$f(X_t) - f(x_0) - \int_0^t Lf(X_r)\, dr = \int_0^t f'(X_r)\sigma(X_r) dW_r$$

for every $f \in \mathcal{D}_L$

More generally, for $s \geq 0$, $x \in \mathbb{R}$, we say that $(X_t^{s,x}, t \geq s)$ solves the sharp martingale problem related to L with initial value x at time s if for every $f \in \mathcal{D}_L$,

$$f(X_t^{s,x}) - f(x) - \int_s^t Lf(X_r^{s,x})\, dr = \int_s^t f'(X_r^{s,x})\sigma(X_r^{s,x}) dW_r, \quad t \geq s$$

Remark 5.3 *Let (W_t) be an $(\mathcal{F}_t)$-Wiener process. If b' is continuous then a process X solves the (corresponding) sharp martingale problem with respect to L if and only if it is a classical solution of the SDE*

$$X_t = x_0 + \int_0^t b'(X_r)\, dr + \int_0^t \sigma(X_r) dW_r.$$

For this, a simple application of the classical Itô formula gives the result.

Remark 5.4 *(i) In general, $f(x) = x$ does not belong to $\mathcal{D}_L$, otherwise a solution to the martingale problem with respect to L would be a semimartingale. According to Remark 5.18, this is generally not the case. In [11] we gave necessary and sufficient conditions on b so that X is a semimartingale.*

(ii) Given a solution X to the martingale problem related to L, we are interested in operators

$$\mathcal{A} : \mathcal{D}_L \to \mathcal{C},\ \textit{given by} \quad \mathcal{A}(f) = \int_0^{\cdot} Lf(X_s)\, ds$$

and

$$A : C^1 \to \mathcal{C},\ \textit{given by} \quad A(\ell) = \int_0^{\cdot} \ell'(X_s)\, ds.$$

where $\mathcal{C}$ is the vector algebra of continuous processes.

We may ask whether $\mathcal{A}$ and A are closable in C^1 and in C^0, respectively. We will see that $\mathcal{A}$ admits a continuous extension to C^1. However, A can be extended continuously to some topological vector subspace ν_0 of C^0, where ν_0 includes the drift, only when Assumption $\mathcal{A}(\nu_0)$ is verified.

Similarly to the case of classical stochastic differential equations it is possible to distinguish two types of existence and uniqueness for the martingale problem. Even, if we could treat initial conditions which are random $\mathcal{F}_0$-measurable solutions, here we will only discuss deterministic ones. We will denote by $MP(L, x_0)$ (resp. $SMP(L, x_0)$) the martingale problem (resp. sharp martingale problem) related to L with initial condition x_0. The notions will only be formulated with respect to the initial condition at time 0.

Definition 5.5 *(***Strong existence***) We will say that $SMP(L, x_0)$ admits* **strong existence** *if the following holds. Given any probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$, an $(\mathcal{F}_t)_{t\geq 0}$- Brownian motion $(W_t)_{t\geq 0}, x_0 \in \mathbb{R}$, there is a process $(X_t)_{t\geq 0}$ which is solution to the sharp martingale problem with respect to L and initial condition x_0.*

Definition 5.6 *(***Path-wise uniqueness***) We will say that $SMP(L, x_0)$ admits path-wise uniqueness if the following property is fulfilled.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space, a filtration $(\mathcal{F}_t)_{t\geq 0}$, an $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion $(W_t)_{t\geq 0}$. If two processes $X, \tilde{X}$ are two solutions of the sharp martingale problem with respect to L and x_0, such that $X_0 = \tilde{X}_0 = x_0$ a.s., then X and $\tilde{X}$ coincide.

Definition 5.7 *(***Existence in law** *or* **weak existence***) We will say that $MP(L; x_0)$ admits weak existence if there is a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$, and a process $(X_t)_{t\geq 0}$ which is a solution to the corresponding martingale problem.*

We say that $MP(L)$ admits weak existence if $MP(L; x_0)$ admits weak existence for every x_0.

Definition 5.8 *(***Uniqueness in law***) We say that $MP(L; x_0)$ has a* **unique solution in law** *if the following holds. We consider an arbitrary probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a solution X of the corresponding martingale problem. We also consider another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ equipped with another filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and a solution $\tilde{X}$. We suppose that $X_0 = x_0$, P- a.s. and $\tilde{X}_0 = x_0$, $\tilde{P}$- a.s. Then X and $\tilde{X}$ must have the same law as r.v. with values in $E = C(\mathbb{R}_+)$ (or $C[0, T]$).*

We say that $MP(L)$ has a unique solution in law if $MP(L, x_0)$ has a unique solution in law for every x_0.

Remark 5.9 *Let us suppose b' to be a continuous function, and we do not suppose σ to be strictly positive (only continuous).*

(i) Then the $SMP(L, x_0)$ admits strong existence and path-wise unique-

ness if the corresponding classical SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b'(X_s)ds$$

admits strong existence and path-wise uniqueness. In this case $\mathcal{D}_L = C^2$ *and to establish this, it is enough to use classical Itô formula.*

(ii) It is well-known, see [26, 14], that weak existence (resp. uniqueness in law) of the martingale problem is equivalent to weak existence (resp. uniqueness in law) of the corresponding SDE.

For the rest of the section, let $s \in [0, T]$, $x_0 \in \mathbb{R}$. Moreover let $(\Omega, (\mathcal{F}_t), P)$ be a fixed filtered probability space fulfilling the usual conditions.

The first result on solutions to the martingale problem related to L is the following.

Proposition 5.10 *Let* $y_0 = h(x_0)$.

(i) A process X *solves the martingale problem related to* L *with initial condition* x *at time* s *if and only if* $Y = h(X)$ *is a local martingale which solves on the same probability space*

$$Y_t = y_0 + \int_s^t \tilde{\sigma}_h(Y_s)\, dW_s, \tag{5.3}$$

where $\tilde{\sigma}_h(y) = (\sigma h')(h^{-1}(y))$ *and* (W_t) *is an* $(\mathcal{F}_t)$*-classical Brownian motion.*

(ii) Let (W_t) *be an* $(\mathcal{F}_t)$*-classical Brownian motion. If* Y *is a solution to equation (5.3), then* $X = h^{-1}(Y)$ *is a solution to the sharp martingale problem with respect to* L *with initial condition* x *at time* s.

Remark 5.11 *Let* X *be a solution to the martingale problem with respect to* L *and set* $Y = h(X)$ *as at point (i). Since* Y *is a local martingale, we know from Remark 3.5 (iv) that* $X = h^{-1}(Y)$ *is a* $(\mathcal{F}_t)$*-Dirichlet process with martingale part*

$$M_t^X = \int_0^t (h^{-1})'(Y_s)\, dY_s.$$

In particular, X is a finite quadratic variation process with

$$[X,X] = [M^X, M^X]_t = \int_0^t \sigma^2(X_s)\, ds.$$

Proof (of Proposition 5.10).

For simplicity we will set $s = 0$.

First, let X be a solution to the martingale problem related to L. Since $h \in \mathcal{D}_L$ and $Lh = 0$, we know that $Y = h(X)$ is an $(\mathcal{F}_t)$-local martingale. In order to calculate its bracket we recall that $h^2 \in \mathcal{D}_L$ and $Lh^2 = \sigma^2(h')^2$ hold by Proposition 4.11 a). Thus,

$$h^2(X_t) - \int_0^t (\sigma h')^2(X_s)\, ds$$

is an $(\mathcal{F}_t)$-local martingale. This implies

$$[Y,Y]_t = \int_0^t (\sigma h')^2(h^{-1}(Y_s))\, ds = \int_0^t \tilde{\sigma}_h^2(Y_s)\, ds.$$

Finally, Y is a solution to the SDE (5.3) with respect to the standard $\mathcal{F}_Y$-Brownian motion W given by

$$W_t = \int_0^t \frac{1}{\tilde{\sigma}_h(Y_s)}\, dY_s,$$

where $\mathcal{F}_Y$ is the canonical filtration generated by Y.

Now, let $Y = h(X)$ be a solution to (5.3) and $f \in \mathcal{D}_L$. Proposition 4.11 c) says that $\phi := f \circ h^{-1} \in \mathcal{D}_{L^0} \equiv C^2$, where

$$L^0\phi = \frac{\tilde{\sigma}_h^2}{2}\phi'' = (Lf) \circ h^{-1}. \tag{5.4}$$

So we can apply Itô formula to evaluate $\phi(Y)$ which coincides with $f(X)$. This gives

$$\phi(Y_t) = \phi(Y_0) + \int_0^t \phi'(Y_s)\, dY_s + \frac{1}{2}\int_0^t \phi''(Y_s)\, d[Y,Y]_s.$$

Using $d[Y,Y]_s = \tilde{\sigma}_h^2(Y_s)\, ds$ and taking into account (5.4), we conclude

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \int_0^t Lf(X_s)\, ds. \tag{5.5}$$

This establishes the proposition. ■

Remark 5.12 *From Proposition 5.10 in particular we have the following.*

Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space fulfilling the usual conditions. Let $x_0 \in \mathbb{R}$ and X be a solution to the martingale problem related to L with initial condition x_0. Then, there is a classical Brownian motion (W_t) so that X is a solution to the sharp martingale problem related to L with initial condition x_0.

Corollary 5.13 *Let X be a solution to the martingale problem related to L with initial condition x_0. Then, map $\mathcal{A}$ admits a continuous extension from $\mathcal{D}_L$ to C^1 with values in $\mathcal{C}$ which we will denote again by $\mathcal{A}$. Moreover, $\mathcal{A}(f)$ is a zero quadratic variation process for every $f \in C^1$.*

Proof. $\mathcal{A}$ has a continuous extension because of (5.5). $\mathcal{A}(f)$ is a zero quadratic variation process because X is a Dirichlet process with martingale part $\int_0^{\cdot} \sigma(X_s)\, dW_s$ and because of Remark 3.5. ■

Remark 5.14 *The extension of (5.5) to C^1 gives*

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)\, dW_s + \mathcal{A}(f). \tag{5.6}$$

Choosing $f = id$ in (5.6), we get

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \mathcal{A}(id).$$

We will see that, if there is a subspace ν_0 of C^0 such that Technical Assumption $\mathcal{A}(\nu_0)$ is verified then operator A will be extended to ν_0. If b is an element of that space, then it will be possible to write $\hat{L}id = b$ and $\mathcal{A}(id) = A(b)$. In that case we will be able to indicate that X is a solution of the generalized SDE with diffusion coefficient σ and distributional drift b'.

A similar result to Proposition 5.10 can be deduced for the case of a transformation through function k and the divergence type operator introduced at (4.13).

Proposition 5.15 *We consider the application k and the PDE operator L^1 introduced at (4.13) and in Lemma 4.13.*

A process X solves the martingale problem related to L with initial condition x_0 at time s if and only if $Z = k(X)$ solves the martingale problem related to L^1 with initial condition $k(x_0)$ at time s.

Proof. It is an easy consequence of Lemma 4.13. ■

Let $x_0 \in \mathbb{R}, y_0 = h(x_0)$. Let σ, b, Σ, h as in section 4.
We set $\tilde{\sigma}_h = (\sigma e^{-\Sigma}) \circ h^{-1}$.

From Proposition 5.10 the following yields.

Corollary 5.16 *(i) Strong existence (resp. path-wise uniqueness) holds for $MP(L, x_0)$ if and only if strong existence (resp. path-wise uniqueness) holds for the SDE*

$$dY_t = \tilde{\sigma}_h(Y_r)\, dW_r,$$

with initial condition $Y_0 = h(x_0)$.

(ii) An analogous equivalence holds for weak existence (resp. uniqueness in law).

From Proposition 5.10 we can deduce two other corollaries concerning well-posedness of our martingale problem.

Corollary 5.17 *Under the same assumptions as the previous corollary, $MP(L, x_0)$ admits weak existence and uniqueness in law.*

Proof. The statement follows from point (i) of Corollary 5.16 and from the fact that the SDE (5.3) admits weak existence and uniqueness in law because $\tilde{\sigma}_h > 0$, see Th. 5.7, ch. 5 of [14] or [7]. ■

Remark 5.18 • *By Corollary 5.11 of [11], it is immediate to see that the solution is a semimartingale for each initial condition if and only if Σ is locally of bounded variation.*

- *If L is in divergence form, see Remark 4.4 a) with $\alpha = 1$, then the solution corresponds to the process constructed and studied for instance by [25].*

Corollary 5.19 *Suppose that either $(\sigma, b) \in (D^{\frac{1}{2}}, C^{\frac{1}{2}})$ or $(b, \sigma) \in (D^{\frac{1}{2}}, C^{\frac{1}{2}})$ and moreover (4.7)is satisfied. Then $SMP(L, x_0)$ admits strong existence and path-wise uniqueness.*

Proof. In this case Σ is well defined, see Remark 4.4 d) and σ belongs to $D^{\frac{1}{2}}$. Since h^{-1} is of class C^1, $\tilde{\sigma}_h$ is Hölder continuous with parameter $\frac{1}{2}$. The SDE (5.3) admits path-wise uniqueness because of Theorem 3.5 ii) of [20] and weak existence again through Th. 5.7 of [14]. Yamada-Watanabe theorem, see [14] Corollary 3.23, ch. 5., implies also strong existence for (5.3). The result follows from point (i) of Corollary 5.16. ■

6 A significant stochastic differential equation with distributional drift.

In this section we will discuss the case where the martingale problem is equivalent to a stochastic differential equation to be specified. First of all one would need to give a precise sense to the generalized drift $\int_0^{\cdot} b'(X_s)ds$, b being a continuous function.

We will introduce a property related to a general process X. First we consider the linear map $A^X : \ell \to \int_0^{\cdot} \ell'(X_s)ds$ defined on $C^1(\mathbb{R})$ with values in $\mathcal{C}$.

Definition 6.1 *Let ν_1 be a topological F-space (or eventually an inductive limit of F-spaces) which is a topological linear subspace of $C^0(\mathbb{R})$, and $\nu_1 \supset C^1(\mathbb{R})$. We will say that X has the* **extended local time regularity with respect to** *ν_1 if*

- *A^X admits a continuous extension to ν_1, which will still be denoted by the same symbol;*
- *$\int_0^{\cdot} g(X)d^- A^X(\ell)$ exists for every $g \in C^2$ and every $\ell \in \nu_1$.*

Remark 6.2 *The terminology related to local time is natural in this context. To illustrate this, we consider a general continuous process X having a local time $(L_t(a), t \in [0,T], a \in \mathbb{R})$ with respect to Lebesgue measure, i.e. fulfilling the following density occupation identity:*

$$\int_0^t \varphi(X_s)ds = \int_{\mathbb{R}} \varphi(a)L_t(a)da, \quad t \in [0,T],$$

for every positive Borel function φ. X has trivially the extended local time regularity, at least with respect to $\nu_1 = C^1$.
Let $\ell \in C^1$. Suppose for a moment that $(L_t(a))$ is a semimartingale in a, for instance if X is a classical Brownian motion. In that case, one would have

$$\int_0^t \ell'(X_s)ds = \int_0^t \ell'(a)L_t(a)da = -\int_{\mathbb{R}} \ell(a)L_t(da).$$

Clearly, the right member can be extended continuously in probability to any $\ell \in C^0$, which implies that X has also the extended local time regularity related to $\nu_1 = C^0$. We remark that [4] gives general conditions on semimartingales X under which $L_t(da)$ is a good integrator, even if $(L_t(a))$ is not necessarily a semimartingale in a.

Definition 6.3 *Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space, (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion, Z be an $\mathcal{F}_0$-measurable random variable. A process X will be called ν_1-***solution*** of the SDE*

$$\left\{ \begin{array}{rcl} dX_t & = & b'(X_t)dt + \sigma(X_t)dW_t \\ X_0 & = & Z \end{array} \right.$$

if

- *X has the extended local time regularity with respect to ν_1,*
- *$X_t = Z + \int_0^t \sigma(X_s)dW_s + A^X(b)_t$,*
- *X is a finite quadratic variation process.*

Remark 6.4

- *Suppose that $b \in \nu_1$. If $\nu_1 \subset \nu_1'$, a ν_1'-solution is also a ν_1-solution.*

- *Previous definition is also new in the classical case, i.e. when b' is a continuous function. A ν_1-solution with $\nu_1 = C^1$ corresponds to a solution to the SDE in the classical sense. On the other hand a ν_1-solution with ν_1 strictly including C^1, is a solution whose local time has a certain additional regularity.*

Even in this generalized framework it is possible to introduce the notion of *strong ν_1-existence*, *weak ν_1-existence*, *path-wise ν_1-uniqueness* and *ν_1-uniqueness in law*. This can be done similarly as in Definition 5.8 according to the fact that the filtered probability space with the classical Brownian motion is fixed a priori or not.

Lemma 6.5 *We suppose that Technical Assumption $\mathcal{A}(\nu_0)$ is verified. If X is a solution to a martingale problem related to a PDE operator L then it fulfills the extended local time regularity with respect to $\nu_1 = \nu_0$.*

Proof. Let $\ell \in C^1$. Since X solves the martingale problem with respect to L, setting $f = T\ell$, it follows

$$\begin{aligned} A^X(\ell)_t &= \int_0^t \ell'(X_s)ds = \int_0^t Lf(X_s)ds \\ &= f(X_t) - f(X_0) - \int_0^t f'(X_s)\sigma(X_s)dW_s \end{aligned}$$

Continuity of T on ν_0 imply that A^X can be extended to ν_0.

Let now $\ell \in \nu_0$ and $f = T\ell \in C^1$. Since $f(X)$ equals a local martingale plus $A^X(\ell)$, it remains to show that

$$\int_0^{\cdot} g(X)d^-f(X) \tag{6.1}$$

exists for any $g \in C^2$. Integrating by parts, previous integral (6.1) equals

$$(gf)(X_\cdot) - (gf)(X_0) - \int_0^{\cdot} f(X)d^-g(X) - [f(X), g(X)].$$

Remark 3.3 b), f) tells that the right member is well-defined. ■

Lemma 6.6 *Let X be a process having the extended local time regularity with respect to some space F- space (or inductive limit) ν_1. Suppose that for fixed $g \in C^1$ the application $\ell \to g\ell$ is continuous from ν_1 to ν_1. Then, for every $g \in C^2$ and every $\ell \in \nu_1$, we have*

$$\int_0^{\cdot} g(X) d^- A^X(\ell) = A^X(\Phi(g, \ell)) \tag{6.2}$$

where

$$\Phi(g, \ell)(x) = (g\ell)(x) - (g\ell)(0) - \int_0^x (\ell g')(y)\, dy \tag{6.3}$$

Proof. The Banach-Steinhaus type Theorem 2.5 implies that, for every $g \in C^2$

$$\ell \mapsto \int_0^{\cdot} g(X)\, d^- A^X(\ell) \tag{6.4}$$

is continuous from ν_1 to $\mathcal{C}$. In fact, expression (6.4) is ucp limit of

$$\lim_{\varepsilon \to 0+} \int_0^{\cdot} g(X_s) \frac{A^X(\ell)_{s+\varepsilon} - A^X(\ell)_s}{\varepsilon}\, ds.$$

Note that Φ is a continuous bilinear map from $C^1 \times \nu_1$ to ν_1. Since $A^X : \nu_1 \to \mathcal{C}$ is continuous, the mapping $\ell \to A^X(\Phi(g, \ell))$ is also continuous from ν_1 to $\mathcal{C}$. In order to conclude the proof, we need to check identity (6.2) for $\ell \in C^1$. In that case, since

$$\Phi(g, \ell)(x) = \int_0^x (g\ell')(y)\, dy$$

both members of (6.2) equal

$$\int_0^{\cdot} (g\ell')(X_s) ds.$$

■

We are now going to explore the relation between the martingale problem associated with L and the stochastic differential equations with distributional drift.

Proposition 6.7 *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space, fulfilling the usual conditions and (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion.*

If X solves the sharp martingale problem with respect to L with initial condition x_0, then X is a ν_0-solution to the stochastic differential equation

$$\begin{cases} dX_t &= b'(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0 \end{cases} \tag{6.5}$$

Remark 6.8 *In particular, if L is close to divergence type as in Example 4.18 (ii) then X is a C^0-solution to previous equation with $b = \frac{\sigma^2}{2} + \beta - \frac{\sigma^2(0)}{2}$.*

Proof. Let X be a solution to the martingale problem related to L. We know by Lemma 6.5 that X fulfills the extended local time regularity with respect to ν_1. On the other hand, by Remark 5.11, X is a finite quadratic variation process. It remains to show

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + A^X(b)_t. \tag{6.6}$$

Let $\ell \in C^1$ and set $f = T^1\ell$. By definition of a sharp martingale problem we have

$$T^1\ell(X_t) = T^1\ell(X_0) + \int_0^t ((T^1\ell)'\sigma)(X_s)\, dW_s + A^X(\ell)_t \tag{6.7}$$

According to Remark 4.15 (i), which states the continuity of the map $T^1 : \nu_0 \to C^1$, previous expression can be extended to any $\ell \in \nu_0$.

By Remark 4.15 (ii) $\ell = b \in \nu_0$ and $f = T^1\ell = id$. Replacing this in (6.7) we obtain

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s + A^X(b).$$

Since $X_0 = Z$ proof is completed. ■

Corollary 6.9 *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. If $MP(L, x_0)$ (resp. $SMP(L, x_0)$) admits weak (resp. strong) existence then the SDE (6.5) also admits weak (resp. strong) existence.*

Proof. The statement about strong solutions is obvious. Concerning weak solutions, let us admit the existence of a filtered probability space, where there is a solution to the martingale problem with respect to L with initial condition x_0. Then, according to Remark 5.12, the mentioned solution is also a solution to a sharp martingale problem and the result follows. ■

If X is some ν_1-solution to (6.6), is it a solution to the (sharp) martingale problem related to some operator L? The answer is delicate. In the following proposition we only provide the converse of Proposition 6.7 as partial answer.

Proposition 6.10 *Suppose the PDE operator L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Let $(\Omega, (\mathcal{F}_t), P)$ be a filtered probability space, fulfilling the usual conditions and (W_t) be a classical $(\mathcal{F}_t)$-Brownian motion. Let X be a progressively measurable process.*

X solves the sharp martingale problem related to L with respect to some initial condition x_0 if and only if it is a ν_0-solution to the stochastic differential equation

$$\left\{ \begin{array}{rcl} dX_t & = & b'(X_t)dt + \sigma(X_t)dW_t \\ X_0 & = & x_0 \end{array} \right. \tag{6.8}$$

Corollary 6.11 *Let $x_0 \in \mathbb{R}$. Suppose that L fulfills Technical Assumption $\mathcal{A}(\nu_0)$. Then weak existence and uniqueness in law (resp. strong existence and path-wise uniqueness) holds for equation (6.8) if and only if the same holds for $MP(L, x_0)$ (resp. $SMP(L, x_0)$) .*

Proof (of the Proposition).

Suppose that X is a ν_0-solution to (6.8). Then it is a finite quadratic variation process. Let $f \in C^3$. Since X solves (6.6) and $\int_0^\cdot f'(X_s)\, d^-X_s$ always exists by the classical Itô formula (see Remark 3.3 e) of section 1), we know that $\int_0^\cdot f'(X)\, d^-A^X(b)$ also exists and is equal to $\int_0^\cdot f'(X)\, d^-X - \int_0^\cdot (f'\sigma)(X)\, dW$. Therefore, this Itô formula says that

$$\begin{array}{rcl} f(X_t) & = & f(X_0) + \displaystyle\int_0^t f'(X_s)\sigma(X_s)\, dW_s + \int_0^t f'(X)\, d^-A^X(b) \\ & & + \dfrac{1}{2}\displaystyle\int_0^t f''(X_s)\sigma^2(X_s)\, ds \end{array}$$

holds.

By Lemma 6.6, the linearity of mapping A^X and (4.12), we get

$$\begin{array}{rc} & \int_0^t f'(X) d^- A^X(b) + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)ds \\ = & A^X\left(\Phi(f',b)\right)_t + \frac{1}{2}\int_0^t (f''\sigma^2)(X_s)ds \\ = & \int_0^t (\frac{\sigma^2}{2} - b)(X_s) f''(X_s)ds + A^X(bf') = A^X(\hat{L}f) \end{array}$$

This shows

$$f(X_t) - f(X_0) - \int_0^t (f'\sigma)(X_s)dW_s = A^X(\hat{L}f), \tag{6.9}$$

for every $f \in C^3$. In reality it is possible to show the previous equality for any $f \in C^2$. In fact, the left member extends continuously to C^2, and even to C^1. The right member is also allowed to be prolongated to C^2 for the following reason. For $f \in C^2$, let (f_n) be a sequence of functions in C^3 converging to f when $n \to \infty$, according to the C^2 topology. In particular the convergence also holds in $C^1_{\nu_0}$. Since $\hat{L}$ is continuous with respect to the $C^1_{\nu_0}$ topology with values in ν_0, we have $\hat{L}f_n \to \hat{L}f$ in ν_0. Finally $A^X(\hat{L}f_n) \to A^X(\hat{L}f)$ ucp because of the extended local time regularity with respect to ν_0.

We will use in fact the validity of (6.9) for $f \in C^2$ with $f(0) = 0$. We set $x_1 = f'(0)$. We set $\ell = \hat{L}f$. According to Technical Assumption $\mathcal{A}(\nu_0)$ (iv), we have $f = T^{x_1}\ell$. Therefore (6.9) gives

$$T^{x_1}\ell(X_t) = T^{x_1}\ell(X_0) + \int_0^t ((T^{x_1}\ell)'\sigma)(X_s)dW_s + A^X(\ell).$$

Using again the extended local time regularity with respect to ν_0, and the continuity of T^{x_1} we can state the validity of previous expression to each $\ell \in \nu_0$ with $\ell(0) = 0$, in particular for $\ell \in C^1$ with $\ell(0) = 0$. But in this case, for any $f \in \mathcal{D}_L$ with $f(0) = 0$ and $\ell' = Lf$, we obtain

$$f(X_t) = f(X_0) + \int_0^t (f'\sigma)(X_s)dW_s + \int_0^t Lf(X_s)ds.$$

This shows the validity of the identity in Definition 5.2 for $f \in \mathcal{D}_L$ and $f(0) = x_0$ and $x_0 = 0$. If $x_0 \neq 0$, we replace f by $f - x_0$ in previous identity, we use that $L(f - x_0) = Lf$ for any $f \in \mathcal{D}_L$.

It follows that X fulfills a sharp martingale problem with respect to L.

This shows the reversed sense of the statement. The direct implication was proved in Proposition 6.7. ■

Corollary 6.12 *We suppose $\sigma \in D^{\frac{1}{2}}$ and $b \in C^{\frac{1}{2}}$ or $\sigma \in C^{\frac{1}{2}}$ and $b \in D^{\frac{1}{2}}$ with conditions (4.7). We set $\nu_0 = D^{\frac{1}{2}}$.*

Then equation (6.8) admits ν_0-strong existence and path-wise uniqueness.

Proof. The result follows from Corollaries 6.11 and 5.19. ■

7 About C_b^0- generalized solutions of parabolic equations

In this section we want to discuss the related parabolic Cauchy problem with final condition, associated with our stochastic differential equations with distributional drift.

We will make the same assumptions and conventions as in section 4. We consider the formal operator $\mathcal{L} = \partial_t + L$ where L will act, from now on, on the second variable.

Definition 7.1 *Let λ be an element of $C_b^0([0,T] \times \mathbb{R})$ and $u^0 \in C_b^0(\mathbb{R})$. A function $u \in C_b^0([0,T] \times \mathbb{R})$ will be said to be a C_b^0-**generalized solution** to*

$$\begin{cases} \mathcal{L}u &= \lambda \\ u(T,\cdot) &= u^0 \end{cases} \tag{7.1}$$

if the following is verified.

(i) For any sequence (λ_n) in $C_b^0([0,T] \times \mathbb{R})$, converging to λ in a bounded way,

(ii) for any sequence (u_n^0) in $C_b^0(\mathbb{R})$ converging in a bounded way to u^0,

(iii) such there are classical solutions (u_n) in $C_b^0([0,T]\times\mathbb{R})$ of class $C^{1,2}([0,T[\times\mathbb{R})$ to $\mathcal{L}_n u_n = \lambda_n$, $u_n(T,\cdot) = u_n^0$,

then (u_n) converges in a bounded way to u.

Remark 7.2 *a) u is said to solve $\mathcal{L}u = \lambda$ if there is $u^0 \in C_b^0(\mathbb{R})$ such that (7.1) holds.*

b) Previous definition depends in principle on the mollifier but it could be easily adapted so as not to depend on it.

c) The regularized problem admits a solution: if $u_n^0 \in C_b^3(\mathbb{R})$, $\lambda_n \in C_b^{0,1}([0,T]\times\mathbb{R})$ there is a classical solution u_n in $C^{1,2}([0,T]\times\mathbb{R})$ of

$$\begin{cases} \mathcal{L}_n v &= \lambda_n \\ v(T,\cdot) &= u_n^0 \end{cases}$$

For this it is enough to apply Theorem 5.19 of [15].

We now state a result concerning the case when operator L is classical. Even if next proposition could be stated when the drift b' is a continuous function, we will suppose it to be zero. In fact it will be later applied to $L = L^0$.

Proposition 7.3 *We suppose that $b = 0$. Let $\varphi, \varphi_n \in C_b^0(\mathbb{R})$, $g, g_n \in C_b^0([0,T]\times\mathbb{R}), n \in \mathbb{N}$, such that $\varphi_n \longrightarrow \varphi$, $g_n \longrightarrow g$ in a bounded way on $\mathbb{R}$ and $[0,T]\times\mathbb{R}$.*
Let σ be a strictly positive real continuous function.

Suppose there exist $u_n \in C^{1,2}([0,T[\times\mathbb{R}) \bigcap C_b^0([0,T]\times\mathbb{R})$ such that

$$\begin{cases} \mathcal{L}_n u_n &= g_n \\ u_n(T,\cdot) &= \varphi_n \end{cases}$$

Then (u_n) will converge to $u \in C_b^0([0,T]\times\mathbb{R})$ in a bounded way to the function u defined by

$$u(s,x) = \mathbb{E}\left(\varphi(Y_T^{s,x}) + \int_s^T g(r, Y_T^{r,x})dr\right) \tag{7.2}$$

where $Y = Y^{s,x}$ is the unique solution (in law) to

$$Y_t = x + \int_s^t \sigma(X_r)dW_r. \tag{7.3}$$

and (W_t) is a classical Brownian motion on some suitable filtered probability space.

Remark 7.4 • *Usual Itô calculus implies that*

$$u_n(s,x) = \mathbb{E}\left(\varphi_n(Y_T^{s,x}(n)) + \int_s^T g_n(r, Y_T^{r,x}(n))dr\right) \tag{7.4}$$

where $Y(n) = Y^{s,x}(n)$ is the unique solution in law to problem

$$Y_t(n) = x + \int_s^t \sigma_n(Y_r(n))dW_r. \tag{7.5}$$

- *Theorem 5.4 ch. 5 of [14], affirms it is possible to construct a solution (unique in law) $Y = Y^{s,x}$ to the SDE (7.3), resp. $Y(n) = Y^{s,x}(n)$ to (7.5).*

- *Suppose that L is a classical PDE operator. Let $u \in C^{1,2}([0,T[\times\mathbb{R})$ bounded and continuous on $[0,T]\times\mathbb{R}$. Again Itô calculus shows that u can be represented by (7.2) and (7.3). In particular a classical solution u to $\mathcal{L}u = g$ is also a C_b^0-generalized solution.*

Proof (of Proposition 7.3). We fix $s \in [0,T], x \in \mathbb{R}$. Using the Engelbert-Schmidt construction, see for instance the proof of Theorem 5.4 ch.5 and 5.7 of [14], it is possible to construct a solution $Y = Y^{s,x}$ of the SDE on some fixed probability space which solves (7.3) with respect to some classical Wiener process (W_t). We set $s = 0$ for simplicity. The procedure is based as follows. One fixes a standard Brownian motion (B_t) on some fixed probability space and one sets

$$R_t := \int_0^t \frac{du}{\sigma^2(x + B_u)}.$$

R is a.s. an homeomorphism on $\mathbb{R}_+$ and one defines A as the inverse of R. A solution Y will be then given by $Y_t = x + B_{A_t}$; in fact it is possible to show that the quadratic variation of the local martingale Y is

$$< Y, Y >_t = \int_0^t \sigma^2(Y_s)ds;$$

The Brownian motion W is constructed a posteriori and it is adapted to the natural filtration of Y by setting $W_t = \int_0^t \frac{dY_s}{\sigma(Y_s)}$.

So, on the same probability space we can set $Y_t(n) = x + B_{A_t(n)}$, $A(n)$ being the inverse of $R(n)$ where $R(n)_t; = \int_0^t \frac{du}{\sigma_n^2(x+B_u)}$.

Consequently, on the same probability space, we construct $Y_t(n) = x + B_{A_t(n)}$ where $A(n)$ is the inverse of $R(n)$ and $R(n)_t := \int_0^t \frac{du}{\sigma_n^2(x+B_u)}$. $Y(n)$ solves equation (7.5) with respect to a Brownian motion depending on n.

By construction, the family $Y_T^{s,x}(n)$ converges a.s. to $Y_T^{s,x}$, Using Lebesgue dominated convergence theorems and the bounded convergence of (φ_n) and (g_n), we can take the limit when $n \to \infty$ in expression (7.4) and obtain the desired result. ■

Remark 7.5 *In particular the corresponding laws of random variables* $(Y^{s,x}(n))$ *are tight.*

Again, we will make the same conventions as at section 4.

We set $\sigma_h = \sigma h'$ and L^0 is the classical operator defined at (4.9). Let us consider $\mathcal{L}^0 = \partial_t + L^0$ as formal operator.

Corollary 7.6 *Let* $g \in C_b^0([0,T]\times\mathbb{R}), \varphi \in C_b^0(\mathbb{R})$. *There is a* C_b^0*-generalized solution* u *to* $\mathcal{L}^0 u = g,\ u(T,\cdot) = \varphi$. *This solution is unique and it is given by (7.2).*

We go now back to the original PDE operator $\mathcal{L}$ with distributional drift. We denote again by h the same application defined at section 5 and discuss existence and uniqueness of C_b^0-generalized solutions of related parabolic Cauchy problems.

A useful consequence of Proposition 7.3 is the following.

Theorem 7.7 *For* $\varphi \in C^0([0,T]\times\mathbb{R})$ *or* $C^0(\mathbb{R})$ *we set again* $\tilde{\varphi} = \varphi \circ h^{-1}$ *according to the conventions of section 2. Let again* $\mathcal{L}^0 = \partial_t + L^0$ *as formal operator.*

Let $\lambda \in C_b^0([0,T]\times\mathbb{R}),\ u^0 \in C_b^0(\mathbb{R})$.
There is a unique solution $u \in C_b^0([0,T]\times\mathbb{R})$ *to*

$$\begin{cases} \mathcal{L}u &= \lambda \\ u(T,\cdot) &= u^0 \end{cases} \tag{7.6}$$

Moreover $\tilde{u}$ *solves*

$$\begin{cases} \mathcal{L}^0\tilde{u} &= \tilde{\lambda} \\ \tilde{u}(T,\cdot) &= \tilde{u}^0. \end{cases} \tag{7.7}$$

Proof. In accordance with section 4, let $(h_n)_{n\in\mathbb{N}}$ be an approximating sequence which is related to $Lh = 0$. Let us consider the PDE operators $\mathcal{L}_n$ defined at (4.2). Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $C_b^0([0,T)\times\mathbb{R})$, such that $\lambda_n \to \lambda$, $u_n^0 \to u^0$ in a bounded way, for which there are classical solutions u_n of

$$\begin{cases} \mathcal{L}_n u_n &= \lambda_n \\ u_n(T,\cdot) &= u_n^0 \end{cases}$$

We recall that those sequences always exist because of Remark 7.2 c).

We set

$$g_n = \lambda_n \circ h_n^{-1}, \quad \varphi_n = \varphi \circ h_n^{-1}, \quad v_n = u_n \circ h_n^{-1}.$$

By Lemma 4.14, we have

$$\begin{cases} \mathcal{L}_n^0 v_n &= g_n \\ v_n(T,\cdot) &= \varphi_n \end{cases}$$

where

$$\mathcal{L}_n^0\varphi(t,y) = \partial_t\varphi(t,y) + \sigma_{h_n}^2 \circ h_n^{-1}(t,y)\partial_{xx}^2\varphi(t,y)$$

By Proposition 7.3 and Corollary 7.6 $v_n \to \tilde{u}$ in a bounded way, where

$$\begin{cases} \mathcal{L}^0\tilde{u} &= \tilde{\lambda} \\ \tilde{u}(T,\cdot) &= \tilde{u}_0. \end{cases}$$

This concludes the proof of the proposition. ■

Now we discuss how C_b^0-generalized solutions are transformed under the action of function k introduced at (4.13). A similar result to Lemma 4.13 for the elliptic case, is the following.

Proposition 7.8 *For $\varphi \in C^0([0,T]\times\mathbb{R})$ or $C^0(\mathbb{R})$ we set $\bar{\varphi} = \varphi \circ k^{-1}$. We set $\sigma_k = \sigma k'$ and consider the formal operator*

$$\mathcal{L}^1 f = \partial_t f + \frac{1}{2}\bar{\sigma}_k^2\partial_{xx}^2 f + \frac{1}{2}(\bar{\sigma}_k^2)'\partial_x f.$$

Informally we can write

$$\mathcal{L}^1 f = \partial_t f + \frac{1}{2}\partial_x(\bar{\sigma}_k^2\partial_x f).$$

Let $\lambda \in C_b^0([0,T]\times\mathbb{R})$, $u^0 \in C_b^0(\mathbb{R})$.
Let u *be the unique* C_b^0*-generalized solution in* $C_b^0([0,T]\times\mathbb{R})$ *to*

$$\left\{\begin{array}{rcl} \mathcal{L}u & = & \lambda \\ u(T,\cdot) & = & u^0 \end{array}\right. \tag{7.8}$$

Then $\bar{u}$ *solves*

$$\left\{\begin{array}{rcl} \mathcal{L}^1\bar{u} & = & \bar{\lambda} \\ \bar{u}(T,\cdot) & = & \bar{u}^0. \end{array}\right.$$

Proof. Let v be the unique solution to

$$\left\{\begin{array}{rcl} \mathcal{L}^1 v & = & \bar{\lambda} \\ v(T,\cdot) & = & \bar{u}^0, \end{array}\right.$$

which exists because of Theorem 7.7 taking $\mathcal{L} = \mathcal{L}^1$.

We define $H : \mathbb{R} \to \mathbb{R}$ such that

$$H(0) = 0, \quad H'(z) = \frac{1}{\sigma_k^2}(z).$$

Again (4.7) implies that H is bijective on $\mathbb{R}$. This case corresponds to Example a) in Remark 4.4 with $\alpha = 1$.

We set $\tilde{v} = v \circ H^{-1}$. By Theorem 7.7 again, we have

$$\left\{\begin{array}{rcl} \mathcal{L}^{0,1}\tilde{v} & = & \bar{\lambda} \circ H^{-1} \\ \tilde{v}(T,\cdot) & = & u^0 \circ (k^{-1} \circ H^{-1}) \end{array}\right.$$

where $\mathcal{L}^{0,1} f = \frac{a^2}{2}\partial_{xx}^2 f$ and

$$a = (\sigma_k H') \circ H^{-1} = \frac{1}{\sigma_k} \circ H^{-1}.$$

Since

$$\sigma_k = (\sigma k') \circ k^{-1} = \frac{e^{\Sigma}}{\sigma} \circ k^{-1},$$

it yields

$$a = (\sigma e^{-\Sigma}) \circ (H \circ k)^{-1}.$$

On the other hand $H \circ k = h$ since

$$\begin{array}{rcl} H \circ k(0) & = & 0 = h(0) \\ (H \circ k(x))' & = & H'(k(x))k'(x) = \dfrac{1}{\sigma_k^2}k'(x) = \dfrac{1}{\sigma^2 k'} = e^{-\Sigma} \\ & = & h'. \end{array}$$

We can therefore conclude that $\mathcal{L}^{0,1} \equiv \mathcal{L}^0$. Since problem (7.7) has a unique solution then $\tilde{v} = \tilde{u}$ where u solves (7.6) and $\tilde{u} = u \circ h^{-1}$. Finally

$$\begin{aligned} v &= \tilde{v} \circ H = \tilde{u} \circ H = u \circ H \circ h^{-1} \\ &= u \circ k^{-1} = \bar{u}. \end{aligned}$$

■

Proposition 7.9 *The unique C_b^0- generalized solution to (7.6) admits a probabilistic representation in the sense that*

$$u(s,x) = \mathbb{E}\left(u^0(X_T^{s,x}) + \int_s^T \lambda(r, X_T^{r,x})dr\right) \tag{7.9}$$

where $X^{s,x}$ is the solution to the martingale problem related to L at time s and point x.

Proof. It follows from Theorem 7.7, Corollary 7.6 and Proposition 5.10 which implies the following: if X is a solution to the martingale problem related to L at point x at time s, then $Y = h(X)$ solves the stochastic differential equation (5.3) with initial condition $h(x)$ at time s. ■

8 Density of the associated semi-groups

We discuss now the existence of a law density for the solutions $X^{s,x}$ of the martingale problem related to L. First of all we suppose that L is an operator in divergence form with $Lf = (\frac{\sigma^2}{2}f')'$ and there are positive constants such that $c \leq \sigma^2 \leq C$. We will say in this case that L *has the Aronson form*. This denomination refers to the fundamental paper [1] about exponential estimates of fundamental solutions of non-degenerate parabolic equations. We start with some properties (partly classical) stated in [11]. We just observe that point (ix) is slightly modified with respect to [11] but this new asset can be immediately deduced from the proof in [11]. This preparatory work will be applied for the operator L^1 introduced in (4.14).

Lemma 8.1 *We suppose $0 < c \leq \sigma^2 \leq C$. Let σ_n, $n \in \mathbb{N}$, be smooth functions such that $0 < c \leq \sigma_n^2 \leq C$ and $\sigma_n^2 \to \sigma^2$ in C^0 as at the beginning*

of Section 4. We set $L_n g = (\frac{\sigma_n^2}{2} g')'$. *There exists a family of probability measures* $(\nu_t(dx,y), t \geq 0, y \in \mathbb{R})$, *resp.* $(\nu_t^n(dx,y), t \geq 0, y \in \mathbb{R})$, *enjoying the following properties:*

(i) $\nu_t(dx,y) = p_t(x,y)\,dx$, $\nu_t^n(dx,y) = p_t^n(x,y)\,dy$.

(ii) (Aronson estimates) There exists $M > 0$, *only depending on constants* c, C *with*

$$\frac{1}{M\sqrt{t}} \exp\left(-\frac{M|x-y|^2}{t}\right) \leq p_t(x,y) \leq \frac{M}{\sqrt{t}} \exp\left(-\frac{|x-y|^2}{Mt}\right).$$

(iii) We have

$$\partial_t \nu_t(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y \tag{8.1}$$

and

$$\partial_t \nu_t^n(\cdot, y) = L_n \nu_t^n(\cdot, y), \quad \nu_0^n(\cdot, y) = \delta_y.$$

ν *(resp.* ν^n*) is called the fundamental solution related to the previous parabolic linear equation.*

(iv) We have

$$\begin{aligned} \partial_t \nu_t(x, \cdot) &= L\nu(x, \cdot) \\ \partial_t \nu_t^n(x, \cdot) &= L_n \nu^n(x, \cdot) \end{aligned}$$

(v) The map $(t,x,y) \mapsto p_t(x,y)$ *is continuous from* $]0,\infty[\times\mathbb{R}^2$ *to* $\mathbb{R}$.

(vi) The p^n *are smooth on* $]0,\infty[\times\mathbb{R}^2$.

(vii) We have $\lim_{n\to\infty} p_t^n(x,y) = p_t(x,y)$ *uniformly on each compact subset of* $]0,\infty[\times\mathbb{R}^2$.

(viii) $p_t(x,y) = p_t(y,x)$ *holds for every* $t > 0$ *and every* $x, y \in \mathbb{R}$.

(ix) $\int_0^T \sup_y \Big(\int_{\mathbb{R}} |\partial_x p_t(x,y)|^2 dx \Big)^{\frac{1}{2}} dt < \infty$.

Previous lemma allows to establish the following.

Theorem 8.2 *Let $Z^{s,x}$ be the solution to the martingale problem related to L at time s and point x. Suppose L to be of divergence type having the Aronson form. Then, there is fundamental solution $\nu_t = r_t(x,y)$ of*

$$\partial_t \nu_t(\cdot, y) = L\nu_t(\cdot, y), \quad \nu_0(\cdot, y) = \delta_y.$$

with the following properties.

(i) Let $g \in C_b^0([0,T] \times \mathbb{R}), \varphi \in C_b(\mathbb{R})$. The C_b^0-generalized solution u to $\mathcal{L}u = g,\ u(T,\cdot) = \varphi$ is given by

$$u(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) r_{T-r}(x,y)dy. \quad (8.2)$$

(ii) The law of $Z_T^{s,x}$ has $r_{T-s}(x,\cdot)$ as density with respect to Lebesgue.

Proof. Let $(r_t^n(x,y))$ be the fundamental solution corresponding to the parabolic equation associated with the $L_n f(x) = (\frac{\sigma_n^2 f'}{2})'$ as introduced at section 4. We observe that (σ_n^2) converges in a bounded way to σ^2.

(i) We define

$$u_n(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}^n(x,y)dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) r_{T-r}^n(x,y)dy. \quad (8.3)$$

Point (vi) and (ii) of Lemma 8.1 imply that functions u_n belong to $C^{1,2}([0,T[\times\mathbb{R})$ so they are classical solutions to

$$\begin{cases} \mathcal{L}_n u_n &= g, \\ u_n(T,\cdot) &= u^0. \end{cases}$$

According to points (ii) and (vii) of the same lemma, one can prove that u_n converges in a bounded way to u defined by (8.2). In fact the coefficients σ_n^2 are lower and upper bounded with a common constant, related to c and C. Therefore this u is the C_b^0- generalized solution of the considered Cauchy problem, which is known to exist. By uniqueness, point (i) is established.

(ii) Setting $g = 0$, point (i) implies that $u(s,x) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)dy$ is the C_b^0-generalized solution to $\mathcal{L}u = 0$ with $u(T,x) = \varphi(x)$. By Proposition 7.9, in particular using the probabilistic representation, we get $\mathbb{E}\left(\varphi(Z_T^{s,x})\right) = \int_{\mathbb{R}} \varphi(y) r_{T-s}(x,y)dy$.

■

Remark 8.3 *If L is in the divergence form as before then $\mathcal{D}_L = \{f \in C^1$ such that there is $g \in C^1$ with $f' = \frac{g}{\sigma^2}\}$. This is a consequence of Lemma 4.9 and that $e^{-\Sigma} = \frac{1}{\sigma^2}$.*

From now on, we will consider a general PDE operator L with distributional drift, as in section 4, for which the Assumption (*Aronson*) below

$$(Aronson) \qquad c \leq \frac{e^{\Sigma}}{\sigma^2} \leq C.$$

We observe that the PDE operator in divergence form of the type $L^1 f = (\frac{\sigma_k^2 f'}{2})'$, where $\sigma_k = (\sigma k') \circ k^{-1}$ has the Aronson form and so previous Theorem can be applied.

Theorem 8.4 *Let $X^{s,x}$ be the solution to the martingale problem related to L at time s and point x. Suppose that L fulfills Assumption (Aronson). Then there is a kernel $p_t(x,y)$ such that*

(i) The law of $X_t^{s,x}$ has $p_{t-s}(x,\cdot)$ as density with respect to Lebesgue for each $t \in]s,T]$.

(ii) Let $g \in C_b^0([0,T] \times \mathbb{R}), \varphi \in C_b^0(\mathbb{R})$. The C_b^0-generalized solution u to $\mathcal{L}u = g,\ u(T,\cdot) = \varphi$ is given by

$$u(s,x) = \int_{\mathbb{R}} \varphi(y) p_{T-s}(x,y)dy + \int_s^T dr \int_{\mathbb{R}} g(r,y) p_{T-r}(x,y)dy. \quad (8.4)$$

Proof.

(i) Proposition 5.15 says that $Z^{s,x} = k(X^{s,x})$ solves the martingale problem with respect to L^1. Let $r_t(x,y)$ be the fundamental solution associated with the parabolic PDE $\mathcal{L}^1 = \partial_t + L^1$. The first point follows then from next observation.

Remark 8.5 *By a change of variable it is easy to see that the law density of $X_t^{s,x}$ equals*

$$p_t(x,x_1) = r_t(k(x),k(x_1))k'(x_1) = r_t(k(x),k(x_1))\frac{e^{\Sigma}}{\sigma^2}(x_1).$$

(ii) This is a consequence of point (i), Fubini theorem and Proposition 7.9.

■

At this point we need a lemma which extends to the kernel $p_t(x, x_1)$ the integrability property of the kernel $r_t(x, x_1)$ stated in (8.3) concerning the divergence case.

Lemma 8.6 *Let $p_t(x, x_1)$ be the kernel introduced in Theorem 8.4. Then*

(i) it is continuous in all variables $(t, x, x_1) \in]0, T[\times\mathbb{R}^2$;

(ii) it fulfills Aronson estimates;

(iii) $\int_0^T (\sup_{x_1} \int_{\mathbb{R}} \partial_x p_t(x, x_1)^2 dx)^{\frac{1}{2}} dt < \infty$.

Proof.

We recall by Remark 8.5 that

$$p_t(x, x_1) = r_t(k(x), k(x_1))k'(x_1)$$

where $r_t(z, z_1)$ is the fundamental solution associated with the operator $L^1 f = (\frac{\sigma_k^2}{2} f')'$, $\quad k' = \frac{e^{\Sigma}}{\sigma^2}$. This, and point (v) of Lemma 8.1 directly imply the validity of first point.

Taking in account Assumption (*Aronson*), Aronson estimates for $(r_t(z, z_1))$ and

$$|k(x) - k(x_1)| = \int_0^1 k'(\alpha x + (1-\alpha)x_1) d\alpha |x - x_1|,$$

result (ii) follows easily.

With the same conventions as before, we have

$$\partial_x p_t(x, x_1) = \partial_z r_t(k(x), k(x_1))k'(x)k'(x_1).$$

So, for $x \in \mathbb{R}$,

$$\begin{aligned} \left(\int_{\mathbb{R}} (\partial_x p_t(x, x_1))^2 dx\right)^{\frac{1}{2}} &= \left(k'(x_1) \int_{\mathbb{R}} (\partial_z r_t(z, k(x_1)))^2 dz\right)^{\frac{1}{2}} \\ &\leq \sqrt{C} \sup_{z_1} \left(\int_{\mathbb{R}} dz (\partial_z r_t(z, z_1))^2\right)^{\frac{1}{2}}. \end{aligned}$$

(iii) follows after integration with respect to t and because of Lemma 8.1 (ix). ■

Proposition 8.7 *Let $g \in C_b^0([0,T]\times\mathbb{R}) \bigcap L^1([0,T]\times\mathbb{R})$, $\varphi \in C_b^0(\mathbb{R}) \bigcap L^1(\mathbb{R})$ Let $u : [0,T]\times\mathbb{R} \to \mathbb{R}$ be the C_b^0-generalized solution to $\mathcal{L}u = g, u(T,\cdot) = \varphi$. Then*

a) $\int_0^T dt \int_{\mathbb{R}} u^2(t,x)dx < \infty$.

b) $x \mapsto u(t,x)$ is absolutely continuous and

$$\int_0^T dt \left(\int_{\mathbb{R}} (\partial_x u)^2(t,x)dx \right)^{\frac{1}{2}} < \infty.$$

In particular for a.e. $t \in [0,T]$, $\partial_x u(t,\cdot)$ is square integrable.

Remark 8.8 *Previous assumptions imply that g and φ are also square integrable.*

Proof. *We recall the expression given in Theorem 8.4:*

$$u(t,x) = \int_{\mathbb{R}} \varphi(x_1) p_{T-t}(x,x_1)dx_1 + \int_t^T dr \int_{\mathbb{R}} g(r,x_1) p_{T-r}(x,x_1)dx_1.$$

Using Lemma 8.6 and classical integration theorems, we have

$$\begin{aligned} \partial_x u(t,x) &= \int_{\mathbb{R}} \varphi(x_1)\partial_x p_{T-t}(x,x_1)dx_1 \\ &+ \int_t^T dr \int_{\mathbb{R}} ds g(s,x_1)\partial_x p_{T-s}(x,x_1)dx_1. \end{aligned} \tag{8.5}$$

Using Jensen inequality we have

$$|u(t,x)|^2 \leq \int_{\mathbb{R}} \varphi(x_1)^2 p_{T-t}(x,x_1)dx_1 + (T-t)\int_t^T ds \int_{\mathbb{R}} g^2(s,x_1)p_{T-s}(x,x_1)dx_1.$$

Therefore

$$\int_{\mathbb{R}} u^2(t,x)dx =$$

$$\int_{\mathbb{R}} dx_1 \varphi(x_1)^2 \int_{\mathbb{R}} dx p_{T-t}(x,x_1) + \int_t^T ds(T-t)\int_{\mathbb{R}} dx_1 \int g^2(s,x_1)\int_{\mathbb{R}} dx p_{T-s}(x,x_1).$$

Using Aronson estimates, this quantity is bounded by

$$\text{const}\left(\int_{\mathbb{R}} dx_1\varphi(x_1)^2\int_{\mathbb{R}} dx\frac{1}{\sqrt{T-t}}p\left(\frac{x-x_1}{\sqrt{T-t}}\right)\right.$$
$$\left.+\int_t^T ds\int_{\mathbb{R}} dx_1\int g^2(s,x_1)\int_{\mathbb{R}} dx\frac{1}{\sqrt{T-s}}p\left(\frac{x-x_1}{\sqrt{T-s}}\right)\right),$$

where p is the Gaussian $N(0,1)$ density. This is clearly equal to

$$\text{const}\left(\int_{\mathbb{R}} dx_1\varphi(x_1)^2+\int_0^T ds\int_{\mathbb{R}} dx_1 g^2(s,x_1)\right)$$

This establishes point a).

Concerning point b), in order not to over-charge the notations we will suppose $g=0$. Expression (8.4) implies

$$\partial_x u(t,x)=\int_{\mathbb{R}}\varphi(x_1)\partial_x p_{T-t}(x,x_1)dx_1.$$

Jensen inequality implies

$$\partial_x u(t,x)^2\leq\left(\int_{\mathbb{R}} dx_1|\varphi(x_1)||\partial_x p_{T-t}(x,x_1))|^2\right)\int_{\mathbb{R}} dx_1|\varphi(x_1)|.$$

Integrating with respect to x and taking the square root, we get

$$\begin{aligned}\sqrt{\int_{\mathbb{R}} dx\partial_x u(t,x)^2} &\leq \sqrt{\int_{\mathbb{R}}|\varphi(x_1)|dx_1}\sqrt{\int_{\mathbb{R}} dx_1|\varphi(x_1)|\int_{\mathbb{R}} dx|\partial_x p_{T-t}(x,x_1)|^2}\\ &\leq \int_{\mathbb{R}} dx_1|\varphi(x_1)|\sqrt{\sup_{x_1}\int_{\mathbb{R}}|\partial_x p_t(x,x_1)|^2dx}\end{aligned}$$

Integrating with respect to t, it gives

$$\int_0^T dt\|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})}\leq\int_{\mathbb{R}} dx_1|\varphi(x_1)|\int_0^T dt\sqrt{\sup_{x_1}\int_{\mathbb{R}}\partial_x p_t(x,x_1)^2dx}$$

This quantity is finite because of Lemma 8.6 (iii). ■

9 Relation with weak solutions of stochastic partial differential equations

As in previous section, we will make assumption (*Aronson*). At this point we wish to investigate the link between C_b^0-generalized solutions and the notion of SPDE's weak solutions for a corresponding Cauchy problem.

We will use the same conventions as in Section 4. In this section we will suppose that coefficients σ, b are realizations of stochastic processes indexed by $\mathbb{R}$. Let us consider the formal operator $\mathcal{L} = \partial_t + L$ where L acts on the second variable.

We consider the equation

$$\left\{ \begin{array}{rcl} \mathcal{L}u & = & \lambda \\ u(T,\cdot) & = & u^0. \end{array} \right. \tag{9.1}$$

The aim of this section is to show that a C_b^0-generalized solution to (9.1) provides, when $\sigma = 1$, a solution to the (stochastic) PDE of the type (1.1) as defined in the introduction, i.e. with the help of symmetric integral via regularization defined at section 3. We denote by $\mathcal{D}(\mathbb{R})$ the linear space of C^∞ real functions with compact support.

The link between the SPDE (1.1) and (1.2) is given in the following.

Proposition 9.1 *Let $u(t,x), v(t,x), t \in [0,T], x \in \mathbb{R}$ be two continuous random fields a.s. in $C^{0,1}(]0,T[\times\mathbb{R})$ such that $v(t,x) = u(T-t,x)$. v is a solution to the SPDE (1.1) if and only if v is a solution to the SPDE (1.2)*

Proof. We observe that $\partial_x v(t,x) = -\partial_x u(T-t,x)$. The proof is elementary. The only point to check is the following:

$$\int_{\mathbb{R}} d^\circ\eta(x)\alpha(x) \left(\int_t^T ds \partial_x u(s,x) \right) = - \int_{\mathbb{R}} d^\circ\eta(x)\alpha(x) \left(\int_0^t ds \partial_x v(s,x) \right).$$

This follows by the definition of symmetric integral and the following obvious identity

$$\begin{array}{rcl} \displaystyle\int_{\mathbb{R}} dx & \displaystyle\frac{\eta(x+\varepsilon)-\eta(x-\varepsilon)}{2\varepsilon}\alpha(x) \left(\int_t^T ds \partial_x u(s,x) \right) = \\ - & \displaystyle\int_{\mathbb{R}} dx \frac{\eta(x+\varepsilon)-\eta(x-\varepsilon)}{2\varepsilon}\alpha(x) \left(\int_0^t ds \partial_x v(s,x) \right), \end{array}$$

for every $\varepsilon > 0$. ■

We go on with a lemma, still supposing σ to be general.

Lemma 9.2 *Let λ (resp. u^0) be a random field with parameter $(t,x) \in [0,T]\times\mathbb{R}$ (resp. $x\in\mathbb{R}$) whose paths are bounded and continuous. Let σ, b be continuous stochastic processes such that Σ is defined a.s. and Assumption (Aronson) is verified. Let u be the random field being a.s. the C_b^0-generalized solution to (9.1). Then it holds*

$$\begin{aligned}\int_{\mathbb{R}} dx \quad & \alpha(x)\left(u(t,x)-u^0(x)+\int_t^T \lambda(s,x)ds\right)\\ = \quad & \int_{\mathbb{R}} e^{\Sigma(x)}\left(\int_t^T ds\partial_x u(s,x)\right) d^\circ\left(\alpha\frac{\sigma^2}{2}e^{-\Sigma}(x)\right),\end{aligned}$$

for every $\alpha\in\mathcal{D}(\mathbb{R})$.

Proof. We fix a realization ω. Theorem 8.4 says that the unique solution to equation (9.1) is given by

$$u(s,x)=\int_{\mathbb{R}} u^0(y)p_{T-s}(x,y)dy+\int_s^T dr\int_{\mathbb{R}}\lambda(r,y)p_{T-r}(x,y)dy. \tag{9.2}$$

where $(p_t(x,y))$ is the law density of the solution to the martingale problem related to L at point x at time s.

Proposition 8.7 b) implies that $\partial_x u$ exists and it is integrable on $]0,T[\times\mathbb{R}$.

According to Proposition 7.8 we know that

$$\bar{u}(t,z)=u(t,k^{-1}(z))$$

is a C_b^0-generalized solution to

$$\left\{\begin{array}{rcl}\mathcal{L}^1\bar{u} & = & \bar{\lambda}\\ \bar{u}(T,\cdot) & = & \bar{u}^0\end{array}\right. \tag{9.3}$$

where

$$\bar{\lambda}(t,z)=\lambda(t,k^{-1}(z)),\bar{u}^0(z)=u^0(k^{-1}(z)).$$

On the other hand $\bar{u}$ can be represented via (8.2) in Theorem 8.4 through fundamental solutions $(\nu_t)=(r_t(x,y))$ of

$$\partial_t\nu_t(\cdot,y)=L^1\nu_t(\cdot,y),\quad \nu_0(\cdot,y)=\delta_y.$$

Since previous equation holds in the Schwartz distribution sense, by inspection, it is not difficult to show that $\bar{u}$ is a solution (in the sense of distributions) to (9.3), which means the following:

$$\int_{\mathbb{R}} \alpha(z)(\bar{u}^0(z)-\bar{u}(t,z))dz-\frac{1}{2}\int_t^T ds \int_{\mathbb{R}} \alpha'(z)\partial_z \bar{u}(s,z)\sigma_k^2(z) = \int_t^T ds \int_{\mathbb{R}} \alpha(z)\bar{\lambda}(s,z). \tag{9.4}$$

for every test function $\alpha \in \mathcal{D}(\mathbb{R}), t \in [0,T]$. We recall in particular that $\partial_z \bar{u}$ is in $L^1(]0,T[\times\mathbb{R})$.

We set

$$D(t,z) = \int_t^T \partial_z \bar{u}(s,z)ds, \quad \mathcal{D}(t,z) = D(t,z)\frac{\sigma_k^2(z)}{2}.$$

Expression (9.4) shows that

$$\partial_z \mathcal{D}(t,\cdot) = -\bar{u}^0 + \bar{u}(t,\cdot) + \int_t^T \bar{\lambda}(s,\cdot)ds. \tag{9.5}$$

in the sense of distributions. So, for each $t \in [0,T]$, $\mathcal{D}$ is of class C^1.

For $t \in [0,T]$ and $x \in \mathbb{R}$, we set $A(t,x) = \int_t^T \partial_x u(s,x)ds, \mathcal{A}(t,x) = A(t,x)e^{\Sigma(x)}$. We remind that

$$u(s,x) = \bar{u}(s,k(x)), \quad \partial_x u(s,x) = \partial_x \bar{u}(s,k(x))k'(x).$$

Therefore

$$A(t,x) = D(t,k(x))k'(x)$$

so that

$$\begin{aligned} A(t,x) &= 2\mathcal{D}(t,k(x))\frac{k'(x)}{\sigma_k^2(k(x))} = \mathcal{D}(t,k(x))\frac{2}{\sigma^2(x)k'(x)} \\ &= 2\mathcal{D}(t,k(x))e^{-\Sigma(x)}. \end{aligned}$$

Therefore $\mathcal{A}(t,x) = 2\mathcal{D}(t,k(x))$ and so $\mathcal{A}$ is of class C^1.

Since $\partial_x \mathcal{A}(t,x) = 2\partial_z \mathcal{D}(t,k(x))k'(x)$, (9.5) gives

$$\partial_x \mathcal{A}(t,x) = \left(-u^0(x) + u(t,x) + \int_t^T \lambda(s,x)ds\right) 2\frac{e^{\Sigma}}{\sigma^2}(x). \tag{9.6}$$

Consequently

$$u(t,x) - u^0(x) + \int_t^T \lambda(s,x)ds = \partial_x \left(e^{\Sigma(x)} \int_t^T ds \partial_x u(s,x)\right) e^{-\Sigma(x)}\frac{\sigma^2(x)}{2}.$$

We integrate previous expression against a test function $\alpha \in \mathcal{D}(\mathbb{R})$ to get

$$\int_{\mathbb{R}} dx\alpha(x) \left(u(t,x) - u^0(x) + \int_t^T \lambda(s,x)ds \right) =$$

$$\int_{\mathbb{R}} dx\alpha(x) \left\{ \partial_x \left(e^{\Sigma(x)} \int_t^T ds\partial_x u(s,x) \right) e^{-\Sigma(x)} \frac{\sigma^2(x)}{2} \right\}$$

Remark 3.1 and integration by parts for the symmetric integral provided by Remark 3.2 c) allow to conclude the proof of the lemma. ■

Finally we are able to state the theorem concerning existence of weak solutions for the SPDE.

Theorem 9.3 *Let λ (resp. u^0) be a random field with parameter in $(t,x) \in [0,T] \times \mathbb{R}$ (resp. $x \in \mathbb{R}$) whose paths are bounded and continuous.*

We suppose that $\sigma = 1$ and η being a (two-sided) zero strong cubic variation process such that there are two finite and strictly positive random variables Z_1, Z_2 a.s. with $Z_1 \leq e^{\eta(x)} \leq Z_2$ a.s.

Let u be the random field which is ω a.s. C_b^0-generalized solution to (9.1), for $b = \eta(\omega)$. We set $v(t,x) = u(T-t,x)$. Then, v is a (weak) solution of the SPDE (1.1).

Proof. Proposition 9.1 says that it will be enough to verify

$$\begin{aligned} -\int_{\mathbb{R}} \alpha(x)u(t,x)dx &+ \int_{\mathbb{R}} \alpha(x)u^0(x)dx - \frac{1}{2}\int_{\mathbb{R}} \alpha'(x) \left(\int_t^T ds\partial_x u(s,x) \right) dx \\ &+ \int_{\mathbb{R}} \alpha(x) \left(\int_t^T ds\partial_x u(s,x) \right) d^\circ\eta(x) \\ &= \int_t^T ds \int_{\mathbb{R}} dx\alpha(x)\lambda(s,x). \end{aligned}$$

for every test function α and $t \in [0,T]$.

After identification $b = \eta(\omega)$, previous Lemma 9.2 says

$$\begin{aligned} \int_{\mathbb{R}} dx \quad & \alpha(x) \left(u(t,x) - u^0(x) + \int_t^T \lambda(s,x)ds \right) \\ = & \int_{\mathbb{R}} e^{2\eta(x)} \left(\int_t^T ds\partial_x u(s,x)) \right) d^\circ \left(\frac{\alpha e^{-2\eta}}{2}(x) \right) \end{aligned}$$

Since η is a zero strong cubic variation process, Proposition 3.8 implies that e^{η} is also a zero strong cubic variation process. Then Itô chain rule of Proposition 3.9 applied with $F(x,\eta(x)) = \alpha(x)e^{\eta(x)}$, and Remark 3.1 say that the right member of previous expression gives

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\left(\int_t^T ds\partial_x u(s,x)\right) d^0(\alpha e^{-2\eta(x)}) &= \frac{1}{2}\int_{\mathbb{R}}\left(\int_t^T ds\partial_x u(s,x)\right) e^{2\eta(x)} \\ &\quad \left(\alpha'(x)e^{-2\eta}(x)dx + \alpha(x)d^\circ e^{-2\eta(x)}\right) \\ &= \frac{1}{2}\int_{\mathbb{R}}\left(\int_t^T ds\partial_x u(s,x)\right)\alpha'(x)dx \\ &- \int_{\mathbb{R}}\left(\int_t^T ds\partial_x u(s,x)\right)\alpha(x)d^\circ\eta(x).\end{aligned}$$

This concludes the proof. ■

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