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Hermine Biermé

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ESTIMATION OF ANISOTROPIC GAUSSIAN FIELDS THROUGH RADON TRANSFORM

HERMINE BIERMÉ

ABSTRACT. We estimate the anisotropic index of an anisotropic fractional Brownian field. For all directions, we give a convergent estimator of the value of the anisotropic index in this direction, based on generalized quadratic variations. We also prove a central limit theorem. First we present a result of identification based on the asymptotic behavior of the spectral density of a process. Then, we define Radon transforms of the anisotropic fractional Brownian field and prove that these processes admit a spectral density satisfying the previous assumptions.

INTRODUCTION

The one dimensional fractional Brownian motion (fBm) was defined through a stochastic integral by Mandelbrot and Van Ness [18] in 1968 for the modeling of irregular data such as the level of water flows or economic series. Let us recall that this process is a Gaussian zero mean process with stationary increments characterized by its so-called Hurst index $H \in (0, 1)$ and denoted by $B_H = \{B_H(t); t \in \mathbb{R}\}$. A generalization of Bochner's Theorem allows to give a spectral representation of its covariance function, namely

$$(1) \quad \text{Cov}(B_H(t), B_H(s)) = \int_{\mathbb{R}} (e^{-it\xi} - 1)(e^{is\xi} - 1) \frac{1}{|\xi|^{2H+1}} d\xi.$$

The function $\frac{1}{|\xi|^{2H+1}}$ is called the spectral density of the fBm. Processes with that kind of spectral density are called "1/f-noises" in the terminology of signal theory. The Hurst parameter is the index of irregularity of the fBm. It corresponds to the order of self-similarity of the process and to the critical Hölder exponent of its paths. Many estimators for the Hurst parameter have been proposed based for example on time domain methods or spectral methods (see [9] and [3] and references therein). The quadratic variations give relevant estimators of the Hölder exponent of more general Gaussian processes with stationary increments [15].

For d-dimensional data, one can consider a natural extension of the 1-dimensional fBm, still called fBm characterized by its covariance function given by (1) where now

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$s, t \in \mathbb{R}^d$, the integral is over \mathbb{R}^d , the products are replaced by $t \cdot \xi$ and $s \cdot \xi$, the usual scalar products on \mathbb{R}^d , and the spectral density is given by $\frac{1}{|\xi|^{2H+d}}$, where $|\cdot|$ is the euclidean norm on \mathbb{R}^d . This yields to a zero mean Gaussian field with stationary increments that is isotropic.

Other generalizations have been proposed for anisotropic data modeling like the fractional Brownian sheet [17] or the multifractional Brownian motion introduced simultaneously in [5] and [21], where H is replaced by a Hurst parameter depending on the point. However such generalizations yield to models with non stationary increments. In order to keep this property and to get anisotropic fields the authors of [7] define anisotropic fractional Brownian fields as zero mean Gaussian random fields with stationary increments and spectral density of the form $\frac{1}{|\xi|^{2h(\xi)+d}}$, where the power $h(\xi) \in (0, 1)$ depends on the direction of ξ . However, these fields look like isotropic when the analysis is done line by line since the regularity along the line does not depend on the direction. Thus the authors of [7] give another method to recover anisotropy. Here we will use their method and give consistent estimators of the anisotropic index h of such fields with asymptotic normality.

The paper is organized as follows. In the first section we will recall the general setting for the study of fields with stationary increments. The second section is devoted to estimators of the Hölder exponent of Gaussian processes ($d = 1$) with spectral density, using generalized quadratic variations. We give estimators with asymptotic normality under assumptions that rely on the asymptotic behavior of the spectral density. In the third part we estimate the anisotropic index of an anisotropic fractional Brownian field, using the method developed in [7]. It is based on the study of Radon transforms of the field. These transformations yield to Gaussian processes with spectral densities for which we give an asymptotic expansion. Then we apply the results of the first part to this process. We illustrate this method in the fourth part using exact syntheses of fBm surfaces.

1. ANISOTROPIC GAUSSIAN MODELS

We consider a real d -parameter Gaussian field $X = \{X(t); t \in \mathbb{R}^d\}$ with zero mean and stationary increments. We assume that X is mean square continuous, that is

$$\mathbb{E}(|X(t) - X(t_0)|^2) \xrightarrow{t \rightarrow t_0} 0,$$

for any point $t_0 \in \mathbb{R}^d$. Actually X is mean square continuous when its covariance function $\Gamma(s, t) = \mathbb{E}(X(s)X(t))$ is continuous with respect to each variable. Then, according to Yaglom [27], there exists a single non-negative symmetric measure σ on \mathbb{R}^d , called spectral measure of X , with

$$\int_{\mathbb{R}^d} \frac{|\xi|^2}{1 + |\xi|^2} d\sigma(\xi) < +\infty,$$

and a single non-negative definite matrix $A = (a_{j,k})_{1 \leq j,k \leq d}$ such that

$$\mathbb{E}((X(s) - X(0))(X(t) - X(0))) = \int_{\mathbb{R}^d} (e^{-is \cdot \xi} - 1) \overline{(e^{-it \cdot \xi} - 1)} d\sigma(\xi) + s \cdot A t.$$

It follows that X can be represented as

$$(2) \quad X(t) - X(0) \stackrel{L^2(\Omega)}{=} \int_{\mathbb{R}^d \setminus \{0\}} (e^{-it \cdot \xi} - 1) dZ(\xi) + t \cdot \mathbf{Y}, \text{ for } t \in \mathbb{R}^d$$

where $\mathbf{Y} = (Y_1, \dots, Y_d)$ is a Gaussian zero mean random vector with covariance A , and Z is a zero mean, complex-valued, Gaussian random measure on the Borel sets of $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$, with

$$\mathbb{E}(Z(A) \overline{Z(B)}) = \sigma(A \cap B) \text{ and } Z(-A) = \overline{Z(A)},$$

for any $A, B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$.

From now on, we will consider that $\mathbf{Y} = 0$ since the corresponding term does not affect the regularity. We assume moreover that the spectral measure σ is absolutely continuous with respect to Lebesgue's measure. Then, its density function f is an even function of $L^1(\mathbb{R}^d, \min(1, |\xi|^2) d\xi)$ called the spectral density of X .

Let us remark that the finite dimensional distributions of $\{X(t) - X(0); t \in \mathbb{R}^d\}$ are completely given by the function $v : t \in \mathbb{R}^d \mapsto \mathbb{E}((X(t) - X(0))^2)$, called the variogramme of X . In our setting

$$v(t) = 4 \int_{\mathbb{R}^d} \sin^2\left(\frac{\xi \cdot t}{2}\right) f(\xi) d\xi,$$

and most properties of X can be characterized by its spectral density. For instance, the Gaussian field $X - X(0)$ is isotropic, i.e. the finite dimensional distributions of $X - X(0)$ are invariant under rotations of center 0, if and only if f is a radial map. In the same vein, $X - X(0)$ is self-similar of order H , i.e. a dilation of rate λ in the set of indices induces a dilation of rate λ^H in the state space, if and only if f is an homogeneous map of the shape $\frac{c(\xi)}{|\xi|^{2H+d}}$, where c is an even homogeneous function of degree 0, ie $c(\lambda\xi) = c(\xi)$ for all $\lambda \in \mathbb{R}^*$ and $\xi \in \mathbb{R}^d$.

We are mainly interested to identify the anisotropic index of an anisotropic fractional Brownian field as introduced by [7]. Such a field is defined to have a spectral density given by

$$f_h(\xi) = \frac{1}{|\xi|^{2h(\xi)+d}},$$

where h is an even homogeneous function of degree 0, with values in $(0, 1)$. When h is a constant function equal to H_0 , the field is just a fractional Brownian motion with Hurst parameter H_0 . Thus it is isotropic. An estimator of the function h is given in [2] through a directional analysis of the field, using projections. However, no speed convergence can be found under their weak conditions that the spectral density

behaves like f_h at high frequencies. We consider here the same estimator in broader context (h can take arbitrarily large values), but under stronger assumptions, which allow to give the speed of convergence.

Let us remark that the asymptotic homogeneity of the spectral density is linked to the Hölder regularity of the Gaussian field. It is proved (see Prop.1.3 of [7]) that if

$$f(\xi) \leq \frac{c}{|\xi|^{2H_0+d}},$$

for large $|\xi|$, with $H_0 \in (0, 1)$ and a positive constant c , then there exists a version \tilde{X} of X whose paths a.s. satisfy a uniform Hölder condition of any order less than H_0 . More precisely, with probability one, for all $\alpha < H_0$, for all $K > 0$, one can find a positive random variable $C > 0$ such that $\forall s, t \in [-K, +K]^d$,

$$|\tilde{X}(t) - \tilde{X}(s)| < C|t - s|^\alpha.$$

The main idea of the anisotropic index identification given in [2] and here is to use the former result, not for the field itself but for one dimensional associated processes.

2. IDENTIFICATION OF THE EXPONENT FOR A 1D-PROCESS

We will prove a first identification result in a general setting. It states that a consistent estimator of the critical Hölder exponent of a Gaussian process with stationary increments, when given by the asymptotic behavior of its spectral density, can be recovered using generalized quadratic variations. This is not a new idea (see [15] for instance) but the required assumptions rely directly on the spectral density and not on the variogramme. In fact, our assumptions are stronger than those of [2] where an estimator of the Hölder exponent is given using asymptotic behavior of the spectral density. They can be linked to those of [15], where an asymptotic expansion of the variogramme is required.

Let us consider a zero mean Gaussian process $X = \{X(t); t \in \mathbb{R}\}$, with stationary increments and spectral density $f \in L^1(\mathbb{R}, \min(1, |\xi|^2)d\xi)$. We want to estimate the asymptotic degree of homogeneity of f through discrete observations of X on $[0, 1]$. The *generalized quadratic variations* [15] of X are given by

$$(3) \quad V_K(N, X) = \sum_{p=0}^{N-K} \left(\sum_{k=0}^K (-1)^{K-k} \binom{K}{k} X\left(\frac{p+k}{N}\right) \right)^2$$

where

- K is an integer greater than 1 (the order of the increments of X),
- $\binom{K}{k}$ is the binomial coefficient $\frac{K!}{k!(K-k)!}$,
- N is an integer greater than K ($\frac{1}{N}$ is the step of discretization),

so that $\sum_{k=0}^K (-1)^{K-k} \binom{K}{k} X\left(\frac{p+k}{N}\right)$ is the increment of order K of X at point $\frac{p}{N}$ with step $\frac{1}{N}$.

Let us remark that one can also consider k -variations of the process. We will focus here on the quadratic variations ($k = 2$). It is motivated by a result of J. F. Coeurjolly [10] who proves that, in the fractional Brownian motion case, the asymptotic variance of the Hurst parameter estimator is the lowest for $k = 2$.

Recall that, up to a constant, the spectral density of a 1D fractional Brownian motion of Hurst parameter $H \in (0, 1)$ is given by the function

$$\frac{1}{|\xi|^{2H+1}}.$$

Remark that no processes with stationary increments can admit such spectral density whenever $H \geq 1$ since this function does not belong to $L^1(\mathbb{R}, \min(1, |\xi|^2)d\xi)$ anymore. However one can obtain such spectral densities by considering fields with higher order stationary increments (see [5] and [22] for instance). Moreover there is no restriction by considering processes with spectral density asymptotically equivalent to that kind of function for any $H > 0$. We can now state our first identification result.

Proposition 2.1. *Let $X = \{X(t); t \in \mathbb{R}\}$ be a zero mean Gaussian process, with stationary increments and spectral density f . We assume that f is differentiable on $\mathbb{R} \setminus [-r, r]$, for r large enough and that there exists $H > 0$, $c > 0$ and $s \geq 0$ such that,*

$$(4) \quad f(\xi) = \frac{c}{|\xi|^{2H+1}} + o_{|\xi| \rightarrow +\infty} \left(\frac{1}{|\xi|^{2H+1+s}} \right),$$

and

$$(5) \quad f'(\xi) = -(2H+1) \frac{c}{|\xi|^{2H+2}} + o_{|\xi| \rightarrow +\infty} \left(\frac{1}{|\xi|^{2H+2}} \right).$$

Let $K > H + \frac{1}{2}$ an integer. Then, almost surely,

$$\widehat{H}_{N,K} = \frac{1}{2} \left(\log_2 \left(\frac{V_K(N, X)}{V_K(2N, X)} \right) + 1 \right) \xrightarrow{N \rightarrow +\infty} H.$$

Moreover, when $s < \frac{1}{2}$,

$$N^s \left(\widehat{H}_{N,K} - H \right) \xrightarrow{L^2(\Omega)} 0,$$

and, when $s \geq \frac{1}{2}$, there exists $\gamma^2 > 0$ such that

$$N \mathbb{E} \left(\left(\widehat{H}_{N,K} - H \right)^2 \right) \xrightarrow{N \rightarrow +\infty} \gamma^2, \text{ with } \sqrt{N} \left(\widehat{H}_{N,K} - H \right) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \gamma^2),$$

where \xrightarrow{d} means the convergence in distribution and $\mathcal{N}(0, \gamma^2)$ is the centered normal law with variance γ^2 .

Proof. First, let us remark that it is enough to consider the case where $c = 1$ since $\widehat{H}_{N,K}$ is also equal to

$$\frac{1}{2} \left(\log_2 \left(\frac{V_K(N, X/\sqrt{c})}{V_K(2N, X/\sqrt{c})} \right) + 1 \right).$$

Let us denote the increment of order K of X with a step $1/N$ by

$$(6) \quad Z_{N,K}(t) = \sum_{k=0}^K (-1)^{K-k} \binom{K}{k} X \left(\frac{t+k}{N} \right).$$

Since by assumption X has stationary increments and spectral density f it follows from (2) that, when $K \geq 1$,

$$Z_{N,K}(t) \stackrel{L^2(\Omega)}{=} \int_{\mathbb{R} \setminus \{0\}} e^{-i\frac{t\xi}{N}} \left(e^{-i\frac{\xi}{N}} - 1 \right)^K dZ(\xi),$$

with Z a zero mean, complex-valued, Gaussian random measure on the Borel sets of $\mathcal{B}(\mathbb{R} \setminus \{0\})$, such that

$$\mathbb{E} \left(Z(A) \overline{Z(B)} \right) = \int_{A \cap B} f(\xi) d\xi \quad \text{and} \quad Z(-A) = \overline{Z(A)}.$$

Thus, when $K \geq 1$, the process $Z_{N,K}$ is Gaussian, stationary and with zero mean. Moreover,

$$(7) \quad \mathbb{E}(V_K(N, X)) = (N - K + 1) \text{Var}(Z_{N,K}(0)) ,$$

$$(8) \quad \text{Var}(V_K(N, X)) = 2(N - K + 1) \sum_{p=-(N-K)}^{N-K} \text{Cov}(Z_{N,K}(p), Z_{N,K}(0))^2 ,$$

where (8) is obtained by using on one hand the fact that $\text{Cov}(Z_{N,K}(p)^2, Z_{N,K}(p')^2) = 2\text{Cov}(Z_{N,K}(p), Z_{N,K}(p'))^2$ since $Z_{N,K}$ is Gaussian and, on the other hand the stationarity of $Z_{N,K}$.

First step. Asymptotics of $\mathbb{E}(V_K(N, X))$ and $\text{Var}(V_K(N, X))$.

Let us compute the covariance of $Z_{N,K}$ and give asymptotics using the asymptotic behavior of the spectral density. For $p \in \mathbb{Z}$, we write $\Gamma_{N,K}(p) = \text{Cov}(Z_{N,K}(p), Z_{N,K}(0))$, as in Lemma 3.2 of [2]. Then

$$\Gamma_{N,K}(p) = \int_{\mathbb{R}} h_K \left(p, \frac{\xi}{N} \right) f(\xi) d\xi, \quad \text{with} \quad h_K(p, \xi) = 4^K \cos(p\xi) \sin^{2K} \left(\frac{\xi}{2} \right).$$

We will prove the following result.

Lemma 2.2. For $K > H + s/2$,

$$(9) \quad \mathbb{E}(V_K(N, X)) = C_1 N^{-2H+1} + o(N^{-2H-s+1}),$$

and, when $K > H + 1/2$,

$$(10) \quad \text{Var}(V_K(N, X)) = C_2 N^{-4H+1} + o(N^{-4H+1}),$$

where

$$C_1 = \int_{\mathbb{R}} h_K(0, \xi) \frac{1}{|\xi|^{2H+1}} d\xi \quad \text{and} \quad C_2 = 2 \sum_{p \in \mathbb{Z}} \left(\int_{\mathbb{R}} h_K(p, \xi) \frac{1}{|\xi|^{2H+1}} d\xi \right)^2.$$

Proof. Let us remark that in the case of a standard fBm ($X = B_H$)

$$\Gamma_{N,K}^H(p) := \int_{\mathbb{R}} h_K\left(p, \frac{\xi}{N}\right) \frac{1}{|\xi|^{2H+1}} d\xi = N^{-2H} \left(\int_{\mathbb{R}} h_K(p, \xi) \frac{1}{|\xi|^{2H+1}} d\xi \right).$$

Using (7), it is straightforward to see that in this case, for $K > H$,

$$\mathbb{E}(V_K(N, B_H)) = C_1 N^{-2H+1} + o(N^{-2H}).$$

Moreover an integration by parts proves that, for $K > H + 1/2$, one can find $C > 0$ such that

$$(11) \quad \left| \int_{\mathbb{R}} h_K(p, \xi) \frac{1}{|\xi|^{2H+1}} d\xi \right| \leq C(1 + |p|)^{-1}.$$

Hence C_2 is well defined and

$$\text{Var}(V_K(N, B_H)) = C_2 N^{-4H+1} + o(N^{-4H+1}).$$

Let us choose $K \geq 1$ such that $K > H$. Then

$$\Gamma_{N,K}(p) = \Gamma_{N,K}^H(p) + \int_{\mathbb{R}} h_K\left(p, \frac{\xi}{N}\right) \Delta_H(\xi) d\xi,$$

where $\Delta_H(\xi)$ denotes the difference $f(\xi) - \frac{1}{|\xi|^{2H+1}}$. Using the asymptotic behavior of f and f' we will prove that, whenever $K > H + s/2$,

$$(12) \quad \Gamma_{N,K}(0) - \Gamma_{N,K}^H(0) = o_{N \rightarrow +\infty}(N^{-2H-s}).$$

And, for $K > H + 1/2$, for all $p \in \mathbb{Z}$, with $|p| \leq N$,

$$(13) \quad (1 + |p|) (\Gamma_{N,K}(p) - \Gamma_{N,K}^H(p)) = o_{N \rightarrow +\infty}(N^{-2H}).$$

Using the estimates for the fBm, this will conclude for the proof of Lemma 2.2.

Let $\epsilon > 0$ and let us choose r large enough such that for $|\xi| > r$,

$$(14) \quad |\Delta_H(\xi)| \leq \epsilon |\xi|^{-2H-s-1} \quad \text{and} \quad |\Delta_H'(\xi)| \leq \epsilon |\xi|^{-2H-2}.$$

We write

$$\int_{\mathbb{R}} h_K\left(p, \frac{\xi}{N}\right) \Delta_H(\xi) d\xi = \int_{|\xi| \leq r} + \int_{|\xi| > r}.$$

For the first integral, we remark that when $K > H$, since $\Delta_H \in L^1(\mathbb{R}, \min(1, |\xi|^{2K})d\xi)$,

$$\left| \int_{|\xi| \leq r} h_K \left(p, \frac{\xi}{N} \right) \Delta_H(\xi) d\xi \right| \leq C(r) N^{-2K}.$$

For the second integral, on one hand, for $p = 0$, as soon as $K > H + s/2$,

$$\left| \int_{|\xi| > r} h_K \left(0, \frac{\xi}{N} \right) \Delta_H(\xi) d\xi \right| \leq \left(4^K \int_{\mathbb{R}} \sin^{2K} \left(\frac{\xi}{2} \right) \frac{1}{|\xi|^{2H+1+s}} d\xi \right) \epsilon N^{-2H-s},$$

and we get (12). On the other hand, for $p \neq 0$, since Δ_H is differentiable over $\mathbb{R} \setminus [-r, r]$, we can integrate by parts. Then, by (14), for $K > H + 1/2$ and N sufficiently large, one can find $C > 0$ such that

$$\left| p \int_{|\xi| > r} h_K \left(p, \frac{\xi}{N} \right) \Delta_H(\xi) d\xi \right| \leq C \epsilon N^{-2H},$$

which yields to (13). \square

Second step. Proposition 3.1 of [2] states that, almost surely, when $K > H + 1/2$,

$$\frac{V_K(N, X)}{\mathbb{E}(V_K(N, X))} \xrightarrow{N \rightarrow +\infty} 1.$$

For sake of completeness, we recall the proof. Let $\eta > 0$ and let us denote

$$A_N = \left\{ \left| \frac{V_K(N, X)}{\mathbb{E}(V_K(N, X))} - 1 \right| > \eta \right\}.$$

By Markov Inequality,

$$\mathbb{P}(A_N) \leq \eta^{-4} |\mathbb{E}(V_K(N, X))|^{-4} \mathbb{E}(|V_K(N, X) - \mathbb{E}(V_K(N, X))|^4).$$

Since $Z_{N,K}$ is Gaussian, one can prove that there exists $c > 0$ such that

$$\mathbb{E}(|V_K(N, X) - \mathbb{E}(V_K(N, X))|^4) \leq c \text{Var}(V_K(N, X))^2.$$

By (9) and (10), one can find $C > 0$ such that, for N large enough,

$$\mathbb{P}(A_N) \leq C \eta^{-4} N^{-2}.$$

Then, the series $\sum_N \mathbb{P}(A_N)$ converges and by Borel-Cantelli Lemma, almost surely,

$$\left| \frac{V_K(N, X)}{\mathbb{E}(V_K(N, X))} - 1 \right| \leq \eta \text{ for } N \text{ sufficiently large.}$$

That concludes the proof of the second step.

Third step. We will sketch the proof of the following asymptotic normality. Whenever $K > H + 1/2$,

$$\sqrt{N} \left(\frac{V_K(N, X)}{\mathbb{E}(V_K(N, X))} - 1 \right) \xrightarrow{N \rightarrow +\infty} \mathcal{N} \left(0, \frac{C_2}{C_1^2} \right)$$

with

$$N\mathbb{E} \left(\left(\frac{V_K(N, X)}{\mathbb{E}(V_K(N, X))} - 1 \right)^2 \right) \xrightarrow{N \rightarrow +\infty} \frac{C_2}{C_1^2}.$$

The second assertion is clear from Lemma 2.2. To prove the asymptotic normality, let us recall the following version of Lindeberg's condition [11] used in [15].

Lemma 2.3. [15] *Consider the sequence of variables V_N defined by*

$$V_N = \sum_{p=0}^{N-K} \lambda_{p,N} (\epsilon_{p,N}^2 - 1),$$

where the $\epsilon_{p,N}$ are i.i.d zero mean normalized Gaussian variables and the $\lambda_{p,N}$ are positive. Let λ_N be the maximum of the $\lambda_{p,N}$. If $\lambda_N = o_{N \rightarrow +\infty} \left(\sqrt{\text{Var}(V_N)} \right)$, then $\frac{V_N}{\sqrt{\text{Var}(V_N)}}$ tends in distribution to a zero mean normalized Gaussian variable.

Let us denote by Γ_N the covariance matrix of the Gaussian vector $(Z_{N,K}(p))_{0 \leq p \leq N-K}$. Then

$$V_K(N, X) \stackrel{d}{=} \sum_{p=0}^{N-K} \lambda_{p,N} \epsilon_{p,N}^2,$$

where $(\lambda_{p,N})_{0 \leq p \leq N-K}$ are the positive eigenvalues of Γ_N . Since $\lambda_N \leq \sum_{p=-(N-K)}^{N-K} |\Gamma_{N,K}(p)|$, equations (11) and (13), compared to (10), show that $\lambda_N = o_{N \rightarrow +\infty} \left(\sqrt{\text{Var}(V_K(N, X))} \right)$.

By Lemma 2.3

$$\frac{V_K(N, X) - \mathbb{E}(V_K(N, X))}{\sqrt{\text{Var}(V_K(N, X))}} \xrightarrow{N \rightarrow +\infty} \mathcal{N}(0, 1).$$

Finally it is enough to recall that, for $K > H + 1/2$, from (9) and (10),

$$\frac{\mathbb{E}(V_K(N, X))}{\sqrt{\text{Var}(V_K(N, X))}} = \frac{C_1}{C_2^{1/2}} \sqrt{N} (1 + o(1)).$$

Fourth step. Conclusion.

To conclude for the proof we will use the following lemma about asymptotic normality of an estimator.

Lemma 2.4. *Let (T_n) a sequence of random vectors of \mathbb{R}^k that tends almost surely to $m \in \mathbb{R}^k$. Let g be a function defined on a neighborhood U of m with real values. We assume that g is twice differentiable on U with partial derivatives bounded on U .*

(1) *If $\sqrt{n}(T_n - m) \xrightarrow{n \rightarrow +\infty} \mathcal{N}(0, \Gamma)$, then*

$$\sqrt{n}(g(T_n) - g(m)) \xrightarrow{n \rightarrow +\infty} \mathcal{N}(0, \nabla g(m)^t \Gamma \nabla g(m)).$$

(2) If $n\mathbb{E}((T_n - m)(T_n - m)^t) \xrightarrow{n \rightarrow +\infty} \Gamma$ and $n\mathbb{E}\left(\left((T_n - m)^t(T_n - m)\right)^2\right) \xrightarrow{n \rightarrow +\infty} 0$, then

$$n\mathbb{E}\left(\left(g(T_n) - g(m)\right)^2\right) \xrightarrow{n \rightarrow +\infty} \nabla g(m)^T \Gamma \nabla g(m).$$

We do not give the details of the proof, which is a variant of the proof of Theorem 3.3.11 in [12]. The main tool is Taylor formula for g at m .

We will use Lemma 2.4 for the pair of estimators (T_N, T_{2N}) , with $T_N = \frac{V_K(N, X)}{\mathbb{E}(V_K(N, X))}$. By the second step, for $K > H + 1/2$, the pair (T_N, T_{2N}) tends to $(1, 1)$ almost surely. Moreover, by the third step, for $K > H + 1/2$,

$$\sqrt{N}(T_N - 1) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}\left(0, \frac{C_2}{C_1^2}\right) \quad \text{and} \quad \sqrt{N}(T_{2N} - 1) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}\left(0, \frac{C_2}{2C_1^2}\right),$$

with

$$N\mathbb{E}\left((T_N - 1)^2\right) \xrightarrow[N \rightarrow +\infty]{} \frac{C_2}{C_1^2} \quad \text{and} \quad N\mathbb{E}\left((T_{2N} - 1)^2\right) \xrightarrow[N \rightarrow +\infty]{} \frac{C_2}{2C_1^2}.$$

Equations (9) and (10) show that

$$N\mathbb{E}\left((T_N - 1)^4\right) \leq cN \frac{\text{Var}(V_K(N, X))^2}{\mathbb{E}(V_K(N, X))^4} \leq CN^{-1}.$$

We only sketch the proof of the convergence in distribution to a Gaussian vector of the pair $\sqrt{N}(T_N - 1, T_{2N} - 1)$, since it follows along the same line as in [15]. Similar arguments as in the third step allow us to prove the convergence in law to a Gaussian variable, for any linear combination

$$(15) \quad a\sqrt{N}(T_N - 1) + b\sqrt{N}(T_{2N} - 1), \quad \text{with } a, b > 0.$$

To prove that (15) holds for any $a, b \in \mathbb{R}$, let us remark that on one hand the law of the pair $\sqrt{N}(T_N - 1, T_{2N} - 1)$ is tight. On the other hand it admits a unique accumulation value, since the characteristic function of a Gaussian vector is an entire function, which is uniquely determined by its values on (a, b) , with $a, b > 0$. We obtain that

$$\sqrt{N}(T_N - 1, T_{2N} - 1) \xrightarrow[N \rightarrow +\infty]{} \mathcal{N}(0, \Gamma),$$

where $\Gamma = \begin{pmatrix} \frac{C_2}{C_1^2} & \frac{C_3}{2^{-2H+1}C_1^2} \\ \frac{C_3}{2^{-2H+1}C_1^2} & \frac{C_2}{2C_1^2} \end{pmatrix}$, with

$$C_3 = 2 \sum_{p \in \mathbb{Z}} \left(\int_{\mathbb{R}} e^{-i\frac{p\xi}{2}} (e^{-i\xi} - 1)^K (e^{i\xi/2} - 1)^K \frac{1}{|\xi|^{2H+1}} d\xi \right)^2,$$

which is well defined since $K > H + 1/2$.

We can now apply Lemma 2.4 with $g(x, y) = \log_2\left(\frac{x}{y}\right)$. It is straightforward to see

that

$$\sqrt{N} \log_2 \left(\frac{T_N}{T_{2N}} \right) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \nu^2), \text{ and } N \mathbb{E} \left(\left(\log_2 \left(\frac{T_N}{T_{2N}} \right) \right)^2 \right) \xrightarrow[N \rightarrow +\infty]{} \nu^2,$$

with

$$\nu^2 = \frac{1}{(\log 2)^2} \left((1 \ -1) \Gamma \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{1}{(\log 2C_1)^2} \left(C_2 \left(1 + \frac{1}{2^d} \right) - 2C_3 \frac{1}{2^{-2H+d}} \right).$$

But

$$\log_2 \left(\frac{T_N}{T_{2N}} \right) = \log_2 \left(\frac{V_K(N, X)}{V_K(2N, X)} \right) - \log_2 \left(\frac{\mathbb{E}(V_K(N, X))}{\mathbb{E}(V_K(2N, X))} \right),$$

where by (9), for $K > H + s/2$,

$$\log_2 \left(\frac{\mathbb{E}(V_K(N, X))}{\mathbb{E}(V_K(2N, X))} \right) = 2H - 1 + o(N^{-s}).$$

Thus, almost surely, for $K > H + 1/2$,

$$\widehat{H}_{N,K} = \frac{1}{2} \left(\log_2 \left(\frac{V_K(N, X)}{V_K(2N, X)} \right) + 1 \right) \xrightarrow[N \rightarrow +\infty]{} H.$$

Moreover, if $s < \frac{1}{2}$, whenever $K > H + 1/2$,

$$N^s \left(\widehat{H}_{N,K} - H \right) \xrightarrow[N \rightarrow +\infty]{L^2(\Omega)} 0,$$

and, for $\gamma^2 = \frac{\nu^2}{4}$, if $s \geq \frac{1}{2}$, whenever $K > H + 1/2$,

$$\sqrt{N} \left(\widehat{H}_{N,K} - H \right) \xrightarrow[N \rightarrow +\infty]{d} \mathcal{N}(0, \gamma^2),$$

with

$$N \mathbb{E} \left(\left(\widehat{H}_{N,K} - H \right)^2 \right) \xrightarrow[N \rightarrow +\infty]{} \gamma^2.$$

□

Let us add a few remarks. The proof uses similar arguments as in [15]. However the asymptotic expansion of the spectral density required in our assumption does not imply that the variogramme fulfills the asymptotic expansion needed in [15]. Our result is connected to the identification of filtered white noise introduced in [4]. In the simplest case of this paper, the authors consider a spectral density f given by

$$f(\xi) = |\xi|^{-2H-1} + R(\xi),$$

with $H \in \mathbb{R}^+ \setminus \mathbb{N}$ and $R \in \mathcal{C}^2(\mathbb{R})$, satisfying $|R^{(j)}(\xi)| \leq C|\xi|^{-2H-1-s-j}$, with $s > 0$, for $0 \leq j \leq 2$. These assumptions are close to ours but here we do not need such regularity on the reminder R and moreover H can be an integer. For more details we refer to [6] where a complete proof is given for a similar result in the more general case of Gaussian random fields, indexed by \mathbb{R}^d , with stationary increments of order $m \in \mathbb{N}$ and spectral density asymptotically equivalent to $|\xi|^{-2H-d}$.

3. IDENTIFICATION OF THE ANISOTROPIC INDEX OF ANISOTROPIC FRACTIONAL BROWNIAN FIELDS

In this section we consider an anisotropic fractional Brownian field $X = \{X(t); t \in \mathbb{R}^d\}$, as introduced in [7], that is a zero mean Gaussian random field, with stationary increments and spectral density given by

$$(16) \quad f_h(\xi) = \frac{1}{|\xi|^{2h(\xi)+d}},$$

where h is an even homogeneous function of degree 0 with values in $(0, 1)$, called anisotropic index. To determine anisotropy of such a field one could try to estimate its directional regularity by extracting lines of the field along various directions. However, for anisotropic fractional Brownian fields, this method fails. Actually, when θ is a fixed direction of the sphere S^{d-1} , one can prove that the process $\{X(t\theta); t \in \mathbb{R}\}$ is still a zero mean Gaussian process with stationary increments with a critical Hölder exponent almost surely equals to the essential infimum H of the function h (see [7] for definition and elements of proof). Then, the study of the generalized quadratic variations of such a process only allows us to recover H . To deal with that obstruction and in order to study processes rather than fields, the authors of [7] have introduced the Radon transform of such fields.

When a function f is integrable over \mathbb{R}^d , one can define its Radon transform on \mathbb{R} (see [25] for instance), in the direction θ , by

$$\mathcal{R}_\theta f(t) = \int_{\langle \theta \rangle^\perp} f(s + t\theta) ds, \text{ for all } t \in \mathbb{R},$$

where $\langle \theta \rangle^\perp$ stands for the hyperplane orthogonal to θ . For a function f that does not decay sufficiently at infinity, one can integrate it against a window. Let ρ be a smooth function defined on $\langle \theta \rangle^\perp$, that compensates the behavior at infinity of f . Then one can define the windowed Radon transform of f on \mathbb{R} , in the direction θ , by

$$\mathcal{R}_{\theta, \rho} f(t) = \int_{\langle \theta \rangle^\perp} f(s + t\theta) \rho(s) ds, \text{ for all } t \in \mathbb{R}.$$

For sake of simplicity we will deal with the direction θ to be $\theta_0 = (0, \dots, 0, 1)$ and identify the space $\mathbb{R}^{d-1} \times \{0\}$ to \mathbb{R}^{d-1} . Let us choose ρ a function of the Schwartz class $\mathcal{S}(\mathbb{R}^{d-1})$, with real values, ie ρ is a smooth function rapidly decreasing

$$(17) \quad \forall N \in \mathbb{N}, \forall x \in \mathbb{R}^{d-1}, |\rho(x)| \leq C_N (1 + |x|)^{-N}.$$

According to Proposition 4.1 of [7], one can define the Radon transform of X , with the window ρ , in the direction θ_0 , denoted by $R_\rho X = \{R_\rho X(t); t \in \mathbb{R}\}$. This process is obtained as the limit in $L^2(\Omega)$ of the finite dimensional distributions of the process

$$2^{-n(d-1)} \sum_{s \in 2^{-n}\mathbb{Z}^{d-1}} X((s, t)) \rho(s), \quad \forall t \in \mathbb{R},$$

when n tends to infinity. This corresponds to a discretization of the integral. The existence of such a process is proved in both [7] and [6] and relies on the slow increase of the covariance function of X due to its stationary increments and on its mean square continuity. Let us remark that one can define the Radon transform of X , with the window ρ , under less restrictive assumptions on ρ , as soon as the previous limit exists. By linearity of such a transformation, this new process is still a zero mean Gaussian process with stationary increments and it admits a spectral density given by the Radon transform of f_h against the window $|\widehat{\rho}|^2$,

$$(18) \quad \mathcal{R}_{|\widehat{\rho}|^2} f_h(p) = \int_{\mathbb{R}^{d-1}} f_h((\gamma, p)) |\widehat{\rho}(\gamma)|^2 d\gamma, \text{ for all } p \in \mathbb{R}.$$

Here, $\widehat{\rho}$ is the $(d-1)$ -dimensional Fourier transform of the window ρ and f_h is the spectral density of X given by (16). To estimate the Hölder regularity of this process, we will use its generalized quadratic variations as introduced in the section 2. Therefore, we have to study the asymptotic behavior of $\mathcal{R}_{|\widehat{\rho}|^2} f_h$ in order to apply Proposition 2.1. We will prove and use the following general result on the windowed Radon transform.

Proposition 3.1. *Let h and c be given functions on \mathbb{R}^d . Let $\alpha > 0$ and H_0, H_1 such that $0 < H_0 \leq H_1$. We assume that h and c are even homogeneous functions of degree 0, Lipschitz of order α on the sphere, with h taking its values in $[H_0, H_1]$. Let $\delta_0 > 0$. Let f be a function defined on \mathbb{R}^d such that, for all $\delta \in (0, \delta_0)$,*

$$f(\xi) = \frac{c(\xi)}{|\xi|^{h(\xi)}} + o\left(\frac{1}{|\xi|^{h(\xi)+\delta}}\right) \text{ when } |\xi| \rightarrow +\infty.$$

Choose $\rho \in \mathcal{S}(\mathbb{R}^{d-1})$ such that $\int_{\mathbb{R}^{d-1}} \rho(\gamma) d\gamma = 1$. Then, the Radon transform of f with the window ρ satisfies, for all $\delta \in (0, \delta_1)$,

$$\mathcal{R}_\rho f(p) = \frac{c(\theta_0)}{|p|^{h(\theta_0)}} + o\left(\frac{1}{|p|^{h(\theta_0)+\delta}}\right) \text{ when } p \in \mathbb{R} \text{ and } |p| \rightarrow +\infty,$$

with $\delta_1 = \min(\delta_0, \alpha)$.

Proof. Let ρ be a function of $\mathcal{S}(\mathbb{R}^{d-1})$ with $\int_{\mathbb{R}^{d-1}} \rho(\gamma) d\gamma = 1$. For $p \in \mathbb{R}$, with $|p|$ large enough, one can define the integral

$$\mathcal{R}_\rho f(p) = \int_{\mathbb{R}^{d-1}} f((\gamma, p)) \rho(\gamma) d\gamma.$$

We want to estimate its asymptotics when $|p| \rightarrow +\infty$. First, let us assume that there exists $A > 1$ such that, for $\xi \in \mathbb{R}^d$ and $|\xi| > A$,

$$f(\xi) = \frac{c(\xi)}{|\xi|^{h(\xi)}},$$

with h and c satisfying assumptions of Proposition 3.1. In this case, we will prove that for all $0 < \delta < \alpha$,

$$(19) \quad \mathcal{R}_\rho f(p) = f(p\theta_0) + o(|p|^{-h(\theta_0)-\delta}) \text{ when } p \in \mathbb{R} \text{ and } |p| \rightarrow +\infty.$$

For $|p| > A$, since $\int_{\mathbb{R}^{d-1}} \rho(\gamma) d\gamma = 1$, let us write

$$\mathcal{R}_\rho f(p) = f(p\theta_0) + \int_{\mathbb{R}^{d-1}} (f((\gamma, p)) - f(p\theta_0)) \rho(\gamma) d\gamma.$$

Then, it is enough to give an upper bound for

$$\int_{\mathbb{R}^{d-1}} (f((\gamma, p)) - f(p\theta_0)) \rho(\gamma) d\gamma.$$

Since ρ is rapidly decreasing, for all $s > 0$ and $N \in \mathbb{N}$,

$$\int_{|\gamma| > |p|^s} (f((\gamma, p)) - f(p\theta_0)) \rho(\gamma) d\gamma = \mathcal{O}_{|p| \rightarrow +\infty}(|p|^{-H_0 - Ns}),$$

which is negligible compared to $|p|^{-h(\theta_0)-\delta}$ as soon as $N > \frac{\delta + H_1 - H_0}{s}$.

Thus, it is sufficient to consider

$$\Delta_s(p) = \int_{|\gamma| \leq |p|^s} (f((\gamma, p)) - f(p\theta_0)) \rho(\gamma) d\gamma.$$

But,

$$\begin{aligned} |\Delta_s(p)| &\leq \int_{|\gamma| \leq |p|^s} |c((\gamma, p))| \left| \frac{1}{(|\gamma|^2 + p^2)^{h((\gamma, p))/2}} - \frac{1}{|p|^{h(\theta_0)}} \right| |\rho(\gamma)| d\gamma \\ &+ \frac{1}{|p|^{h(\theta_0)}} \int_{|\gamma| \leq |p|^s} |c((\gamma, p)) - c(\theta_0)| |\rho(\gamma)| d\gamma. \end{aligned}$$

Let us use the Lipschitz assumptions on h and c .

Lemma 3.2. *If g is an homogeneous function of degree 0, Lipschitz of order α on the sphere S^{d-1} , then there exists $C > 0$ such that for all $p \in \mathbb{R}^*$ and $\gamma \in \mathbb{R}^{d-1}$,*

$$|g((\gamma, p)) - g((0, p))| \leq C \min \left(\left(\frac{|\gamma|}{|p|} \right)^\alpha, 1 \right).$$

Proof. The function g is continuous on the sphere and thus it is bounded. Then, for $p \in \mathbb{R}^*$ and $\gamma \in \mathbb{R}^{d-1}$,

$$|g((\gamma, p)) - g((0, p))| \leq 2\|g\|_\infty.$$

Moreover, g is Lipschitz of order α on the sphere. Then, there exists $C > 0$ such that, for $p \neq 0$,

$$|g((\gamma, p)) - g((0, p))| \leq \left| \frac{(\gamma, p)}{(|\gamma|^2 + p^2)^{1/2}} - \frac{(0, p)}{|p|} \right|^\alpha.$$

But,

$$\left| \frac{(\gamma, p)}{(|\gamma|^2 + p^2)^{1/2}} - \frac{(0, p)}{|p|} \right|^2 = 2 \left(1 - \left(1 + \frac{|\gamma|^2}{p^2} \right)^{-1/2} \right) \leq \frac{|\gamma|^2}{p^2},$$

which concludes the proof of Lemma 3.2. \square

Let us recall that c is an even homogeneous function thus $c(\theta_0) = c((0, p))$, for all $p \neq 0$. Then, since c is Lipschitz of order α , one can find $C_1 > 0$ such that

$$\Delta_s^2(p) = \frac{1}{|p|^{h(\theta_0)}} \int_{|\gamma| \leq |p|^s} |c((\gamma, p)) - c((0, p))| |\rho(\gamma)| d\gamma \leq C_1 |p|^{-h(\theta_0) - \alpha(1-s)},$$

which is negligible compared to $|p|^{-h(\theta_0) - \delta}$ as soon as $\delta < \alpha(1-s)$.

It remains to consider

$$\begin{aligned} \Delta_s^1(p) &= \int_{|\gamma| \leq |p|^s} |c((\gamma, p))| \left| \frac{1}{(|\gamma|^2 + p^2)^{h((\gamma, p))/2}} - \frac{1}{|p|^{h(\theta_0)}} \right| |\rho(\gamma)| d\gamma \\ &= \frac{1}{|p|^{h(\theta_0)}} \int_{|\gamma| \leq |p|^s} |c((\gamma, p))| \left| \frac{|p|^{h(\theta_0)}}{(|\gamma|^2 + p^2)^{h((\gamma, p))/2}} - 1 \right| |\rho(\gamma)| d\gamma. \end{aligned}$$

Let us write

$$\frac{|p|^{h(\theta_0)}}{(|\gamma|^2 + p^2)^{h((\gamma, p))/2}} = e^{l(p)},$$

where, for $p \neq 0$,

$$\begin{aligned} l(p) &= h(\theta_0) \ln |p| - \frac{1}{2} h((\gamma, p)) \ln (|p|^2 + |\gamma|^2) \\ &= (h((0, p)) - h((\gamma, p))) \ln |p| - \frac{1}{2} h((\gamma, p)) \ln \left(1 + \frac{|\gamma|^2}{|p|^2} \right), \end{aligned}$$

writing $h(\theta_0) = h((0, p))$ since h is an even homogeneous function. Since h is Lipschitz of order α , by Lemma 3.2, for $s < 1$, there exists $C_2 > 0$ such that, for $|p| \geq A > e$ and $|\gamma| \leq |p|^s$,

$$\begin{aligned} |l(p)| &\leq C_2 \left(\left(\frac{|\gamma|}{|p|} \right)^\alpha \ln |p| + \frac{|\gamma|^2}{|p|^2} \right) \\ &\leq 2C_2 \left(\frac{|\gamma|}{|p|} \right)^\alpha \ln |p| \leq 2C_2 |p|^{-\alpha(1-s)} \ln |p|. \end{aligned}$$

The function $t \mapsto |t|^{-\alpha(1-s)} \ln |t|$ tends to 0 at infinity. Thus one can find $A_s > 0$ such that for $|p| > A_s$ we get $|l(p)| < 1$. Thus, for $|p| > A_s$

$$|e^{l(p)} - 1| \leq e |l(p)| \leq 2eC_2 |p|^{-\alpha(1-s)} \ln |p|,$$

and finally

$$\Delta_s^1(p) \leq 2eC_2 \|c\|_\infty |p|^{-h(\theta_0) - \alpha(1-s)} \ln |p|.$$

For $\delta < \alpha$, since $|\Delta_s(p)| \leq \Delta_s^1(p) + \Delta_s^2(p)$, taking $s \in (0, \frac{\alpha-\delta}{\alpha}) \subset (0, 1)$ we get

$$\Delta_s(p) = o_{|p| \rightarrow +\infty} \left(\frac{1}{|p|^{h(\theta_0)+\delta}} \right)$$

and (19) follows. In the general case, let us assume that, for all $\delta \in (0, \delta_0)$ and $\xi \in \mathbb{R}^d$,

$$f(\xi) = \frac{c(\xi)}{|\xi|^{h(\xi)}} + o \left(\frac{1}{|\xi|^{h(\xi)+\delta}} \right) \text{ when } |\xi| \rightarrow +\infty.$$

Replacing ρ by $|\rho|$ and h by $h + \delta$ in the special case above, we get the result for the remainder. \square

Let us remark that the result still holds for a window $\rho \in L^1(\mathbb{R}^{d-1})$ that satisfies

$$(20) \quad |\rho(\gamma)| = \mathcal{O} \left(\frac{1}{|\gamma|^{M+d-1}} \right) \text{ when } \gamma \in \mathbb{R}^{d-1} \text{ and } |\gamma| \rightarrow +\infty,$$

with $M > H_1 - H_0$. In this case $\delta_1 = \min \left(\delta_0, \alpha \frac{M+H_0-H_1}{M+\alpha} \right)$.

We can now state our main result concerning the identification of the anisotropic index of an anisotropic fractional Brownian field. We keep the notations of part 2 for the generalized quadratic variations of a 1D-process and recall that we fix the direction $\theta_0 = (0, \dots, 0, 1) \in S^{d-1}$.

Theorem 3.3. *Let $X = \{X(t); t \in \mathbb{R}^d\}$ be an anisotropic fractional Brownian field, with anisotropic index h given by an even homogeneous function of degree 0 with values in $[H_0, H_1] \subset (0, 1)$, which is assumed to be in $\mathcal{C}^1(S^{d-1})$.*

Let ρ be a window in $\mathcal{S}(\mathbb{R}^{d-1})$. Let $R_\rho X$ be the Radon transform of the field X . Let $K \in \mathbb{N}^$. If $K > h(\theta_0) + \frac{d}{2}$ then, almost surely,*

$$\widehat{h_{N,K}(\theta_0)} = \frac{1}{2} \left(\log_2 \left(\frac{V_K(N, R_\rho X)}{V_K(2N, R_\rho X)} \right) - (d-2) \right) \xrightarrow{N \rightarrow +\infty} h(\theta_0),$$

Moreover,

$$\sqrt{N} \left(\widehat{h_{N,K}(\theta_0)} - h(\theta_0) \right) \xrightarrow{N \rightarrow +\infty} \mathcal{N}(0, \gamma^2),$$

with

$$N \mathbb{E} \left(\left(\widehat{h_{N,K}(\theta_0)} - h(\theta_0) \right)^2 \right) \xrightarrow{N \rightarrow +\infty} \gamma^2.$$

Proof. It is sufficient to show that the spectral density of $R_\rho X$ satisfies the assumption of Proposition 2.1. From (18), this spectral density is given by the function

$$\mathcal{R}_{|\widehat{\rho}|^2} f_h(p) = \int_{\mathbb{R}^{d-1}} f_h((\gamma, p)) |\widehat{\rho}(\gamma)|^2 d\gamma, \text{ for all } p \in \mathbb{R},$$

with f_h given by (16). Since we can divide $R_\rho X$ by a constant, we can assume that $\int_{\mathbb{R}^{d-1}} |\widehat{\rho}(\gamma)|^2 d\gamma = 1$. Then, since $2h+d$ satisfies assumptions of Proposition 3.1 with $\alpha = 1$, by Proposition 3.1, for all $\delta < 1$,

$$\mathcal{R}_{|\widehat{\rho}|^2} f_h(p) = \frac{1}{|p|^{2h(\theta_0)+d}} + \underset{|p| \rightarrow +\infty}{o} \left(\frac{1}{|p|^{2h(\theta_0)+d+\delta}} \right).$$

We can also write this as

$$\mathcal{R}_{|\widehat{\rho}|^2} f_h(p) = \frac{1}{|p|^{2(h(\theta_0)+\frac{d-1}{2})+1}} + \underset{|p| \rightarrow +\infty}{o} \left(\frac{1}{|p|^{2(h(\theta_0)+\frac{d-1}{2})+1+\delta}} \right).$$

Moreover, since $h \in \mathcal{C}^1(S^{d-1})$, the function $g(\gamma) := h((\gamma, 1))$ is differentiable on \mathbb{R}^{d-1} and, for $\gamma, x \in \mathbb{R}^{d-1}$,

$$D_\gamma g(x) = \frac{1}{(|\gamma|^2 + 1)^{1/2}} D_{\frac{(\gamma, 1)}{|\gamma, 1|}} h \left((x, 0) - \frac{\gamma \cdot x}{|\gamma|^2 + 1} (\gamma, 1) \right),$$

such that

$$|D_\gamma g(x)| \leq 2 \|Dh\|_\infty |x|.$$

Thus, the spectral density f_h is differentiable on $\mathbb{R}^d \setminus \{0\}$. Let $(\gamma, p) \in \mathbb{R}^{d-1} \times \mathbb{R}^*$, since $h((\gamma, p)) = g(\gamma/p)$, we get

$$\frac{\partial}{\partial p} f_h((\gamma, p)) = f_h((\gamma, p)) \left(-D_{\frac{\gamma}{p}} g \left(-\frac{\gamma}{p^2} \right) \ln(|\gamma|^2 + p^2) - \frac{p(2h((\gamma, p)) + d)}{|\gamma|^2 + p^2} \right).$$

It follows that the spectral density $\mathcal{R}_{|\widehat{\rho}|^2} f_h$ is differentiable on \mathbb{R}^* and for $p \neq 0$ we have

$$\left(\mathcal{R}_{|\widehat{\rho}|^2} f_h \right)'(p) = \int_{\mathbb{R}^{d-1}} \frac{\partial}{\partial p} f_h((\gamma, p)) |\widehat{\rho}(\gamma)|^2 d\gamma.$$

Let us write

$$\frac{\partial}{\partial p} f_h((\gamma, p)) = \frac{|\gamma|}{p^2} F_1((\gamma, p)) - p F_2((\gamma, p)),$$

with

$$F_1((\gamma, p)) = D_{\frac{\gamma}{p}} g \left(-\frac{\gamma}{|\gamma|} \right) \ln(|\gamma|^2 + p^2) f_h((\gamma, p))$$

and

$$F_2((\gamma, p)) = (2h((\gamma, p)) + d) f_{h+1}((\gamma, p)).$$

Therefore,

$$\left(\mathcal{R}_{|\widehat{\rho}|^2} f_h \right)'(p) = \frac{1}{p^2} \mathcal{R}_{|\gamma||\widehat{\rho}|^2} F_1(p) - p \mathcal{R}_{|\widehat{\rho}|^2} F_2(p).$$

Then, by Proposition 3.1, whenever $\delta < 1$,

$$\mathcal{R}_{|\widehat{\rho}|^2} F_2(p) = \frac{2h(\theta_0) + d}{|p|^{2h(\theta_0)+d+2}} + \underset{|p| \rightarrow +\infty}{o} \left(\frac{1}{|p|^{2h(\theta_0)+d+2+\delta}} \right).$$

Moreover, for any $\epsilon > 0$ small enough

$$|F_1((\gamma, p))| \leq 2 \|Dh\|_\infty f_{h-\epsilon}((\gamma, p)).$$

Since $|\gamma| |\widehat{\rho}|^2$ is integrable over \mathbb{R}^{d-1} and rapidly decreasing, following the same lines as in the proof of Proposition 3.1, we get

$$\left| \mathcal{R}_{|\gamma| |\widehat{\rho}|^2} F_1(p) \right| \leq C |p|^{-2h(\theta_0) - d + 2\epsilon}.$$

This allows us to conclude that

$$\left(\mathcal{R}_{|\widehat{\rho}|^2} f_h \right)'(p) = -\frac{2h(\theta_0) + d}{|p|^{2h(\theta_0) + d + 1}} + \underset{|p| \rightarrow +\infty}{o} \left(\frac{1}{|p|^{2h(\theta_0) + d + 1}} \right).$$

Therefore $R_\rho X$ satisfies the assumption of Proposition 2.1 with $H = h(\theta_0) + \frac{d-1}{2}$ and $s < 1$, which concludes the proof. \square

Let us remark that if we choose a window such that $|\widehat{\rho}|^2$ only satisfies (20) for $M > 2(H_1 - H_0)$, the estimator $\widehat{h_{N,K}}(\theta_0)$ still tends almost surely to $h(\theta_0)$. However the speed of convergence will depend on M . Actually the result of Theorem 3.3 holds when $M \geq 1 + 4(H_1 - H_0)$ whereas for $M < 1 + 4(H_1 - H_0)$, Proposition 2.1 shows that, for all $s < \frac{M - 2(H_1 - H_0)}{M + 1}$,

$$N^s \left(\widehat{h_{N,K}}(\theta_0) - h(\theta_0) \right) \xrightarrow[N \rightarrow +\infty]{L^2(\Omega)} 0.$$

We would still obtain similar results by considering fields with spectral density asymptotically of the order of f_h and with partial derivatives of order 1 asymptotically of the order of the partial derivatives of f_h . Moreover, we have restrict h to have values in $(0, 1)$. This can be weakened by considering fields with higher order stationary increments or with spectral density asymptotically of the order of f_h (see [6] for example). This result allows us to estimate the anisotropic index h for all directions $\theta \in S^{d-1}$. Actually, it is sufficient to chose κ_θ a rotation of \mathbb{R}^d that maps $\theta_0 = (0, \dots, 0, 1)$ onto θ . Since $X \circ \kappa_\theta$ is still an anisotropic fractional Brownian field, with anisotropic index given by $h \circ \kappa_\theta$, which satisfies the same assumption as h , we just have to consider the Radon transform of this field.

For numerical applications, one has to approximate the Radon transform of X . In [2], the authors replace $R_\rho X(t)$ by

$$(21) \quad I_N(t) = N^{-\nu(d-1)} \sum_{s \in \mathbb{Z}^{d-1}} X((N^{-\nu} s, t)) \rho(N^{-\nu} s),$$

for ρ a smooth window with compact support. They give conditions on ν to get estimators of the anisotropic index using generalized quadratic variations of I_N with step N instead of $R_\rho X(t)$. We can do the same here. Let us denote

$$T_{N,K}(X) = \sum_{p=0}^{N-K} \left(\sum_{k=0}^K (-1)^{K-k} \binom{K}{k} I_N \left(\frac{p+k}{N} \right) \right)^2.$$

The key point of the proof is to estimate the error due to the approximation of $V_K(N, R_\rho X)$ by $T_{N,K}(X)$. Let us denote $H = h(\theta_0) + \frac{d-1}{2}$. Under the assumptions of Theorem 3.3, following the same lines as in [7], for $\alpha > 0$ with $\alpha < H_0 \leq \min_{S^{d-1}} h$, one can prove that there exists a positive finite random variable C such that, a.s.

$$\left| T_{N,K}(X)^{1/2} - V_K(N, R_\rho X)^{1/2} \right| \leq CN^{-\nu\alpha + \frac{1}{2}}.$$

Moreover, since for $K > H + 1/2$, a.s. $\frac{V_K(R_\rho X)}{\mathbb{E}(V_K(N, R_\rho X))} \xrightarrow{N \rightarrow +\infty} 1$, using (9) one can find a positive finite random variable C' such that, a.s.

$$V_K(N, R_\rho X)^{-1/2} \leq C' N^{H-1/2}.$$

Then, for $\nu > \frac{H}{H_0}$, $\alpha \in (H/\nu, H_0)$ and N large enough, writing

$$\log_2 \left(\frac{T_{N,K}(X)}{V_K(N, R_\rho X)} \right) = 2 \log_2 \left(1 + \frac{T_{N,K}(X)^{1/2} - V_K(N, R_\rho X)^{1/2}}{V_K(N, R_\rho X)^{1/2}} \right),$$

one can find a positive finite random variable C'' such that, a.s.

$$\left| \log_2 \left(\frac{T_{N,K}(X)}{V_K(N, R_\rho X)} \right) \right| \leq C'' N^{-\nu\alpha + H}.$$

We can state the following result.

Proposition 3.4. *We keep the assumptions of Theorem 3.3 and take ρ with compact support. Let $\nu > \frac{H}{H_0}$ with $H = h(\theta_0) + \frac{d-1}{2}$. Let $K \in \mathbb{N}^*$. If $K > H + 1/2$ then, almost surely,*

$$\widehat{H_{N,K}(\theta_0)} = \frac{1}{2} \left(\log_2 \left(\frac{T_{N,K}(X)}{T_{2N,K}(X)} \right) - (d-2) \right) \xrightarrow{N \rightarrow +\infty} h(\theta_0),$$

Moreover, when $\nu H_0 - H > 1/2$, there exists $\gamma^2 > 0$ such that

$$\sqrt{N} \left(\widehat{H_{N,K}(\theta_0)} - h(\theta_0) \right) \xrightarrow{N \rightarrow +\infty} \mathcal{N}(0, \gamma^2).$$

4. SIMULATION

In this section we present a preliminary simulation study to test estimators given by Theorem 3.3 on simulated paths. In order not to add bias due to approximate syntheses we would like to consider exact synthesis of anisotropic fractional Brownian fields. This is a hard numerical problem, not yet solved in the general case. Actually, lots of numerical methods have been proposed these last years to simulate 1-dimensional fractional Brownian motion (fBm). Most of them give rise to approximate syntheses, such as the midpoint displacement method (see [20], for instance), the wavelet based decomposition ([19], [1], [24], etc...), or more recently a method based on correlated random walks [14]. A few of them can be applied not only for 1-dimensional fBm but also to simulate 2-dimensional (anisotropic) fractional Brownian fields. However they still yield to approximate syntheses and their use

would add errors when we test our estimators. Of course there exist exact synthesis methods based on the Choleski decomposition of the covariance function. These give rise to numerical problems due to the size of the matrix. In order to have fast synthesis one can use the stationarity of the increments by applying the embedding circulant matrix method [13]. By this way, we easily obtain fast and exact synthesis of 1-dimensional fBm [23]. Some authors, as in [8] and [16], apply this method for higher dimension but this does not yield to exact synthesis. Finally, M. L. Stein proposed a fast and exact synthesis method for isotropic fBm surfaces in [26]. We use the simulated paths obtained by this method for this preliminary study. We have implemented, in collaboration with A. Fraysse and C. Lacaux, the matlab code corresponding, available at http://ciel.ccsd.cnrs.fr/index.php?paol_sid=47313ed821793db9d30ef328a7537dc8&view_this_doc=ciel-00000016&version=1. We refer to [6] for more details.

With this algorithm we generate realizations of points of fBm of Hurst parameter $H \in (0, 0.7)$ on an equispaced grid with mesh $N_0 = 2^{10}$, namely

$$\left\{ B_H \left(\frac{k}{N_0}, \frac{l}{N_0} \right); 0 \leq k, l \leq M_0 \right\},$$

where $M_0 = 764$ is the greatest integer less than $N_0/\sqrt{2}$.

In order to perform a Radon transform, we have thus chosen $\rho = \mathbf{1}_{[0, M_0/N_0]}$. Let us remark that the Fourier transform of this window is given, for $\gamma \in \mathbb{R}$, by

$$\widehat{\rho}(\gamma) = \frac{e^{-iM_0\gamma/N_0} - 1}{-i\gamma},$$

such that $|\widehat{\rho}|^2$ satisfies (20) for any $M < 1$.

With such data on a grid, we have to choose particular directions to compute the discretized Radon transform. Otherwise we do not have enough points or regular step. Here we chose the vertical and horizontal directions. Then, the Radon transforms of B_H in the vertical and horizontal directions, correspond to

$$R_\rho^1 B_H \left(\frac{l}{N_0} \right) = \frac{1}{N_0} \sum_{k=0}^{M_0} B_H \left(\frac{k}{N_0}, \frac{l}{N_0} \right),$$

and

$$R_\rho^2 B_H \left(\frac{k}{N_0} \right) = \frac{1}{N_0} \sum_{l=0}^{M_0} B_H \left(\frac{k}{N_0}, \frac{l}{N_0} \right),$$

for $0 \leq k, l \leq M_0$. Hence we obtain discretizations of the integrals as in (21) for $N \in \mathbb{N}$ and $\nu > 0$ such as $N^\nu = N_0 = 2^{10}$ is fixed.

Finally, we estimate $H + \frac{1}{2}$ through the estimator $\widehat{H}_{N,2}(\theta)$ given in Proposition 3.4, for $N = 2^n$ with $n \in \mathbb{N}^*$, $H \in \{0.2, 0.5, 0.7\}$, $\theta \in \{(0, 1), (1, 0)\}$ and $K = 2$ since $H + \frac{1}{2} + \frac{1}{2} \in (1, 2)$. Let us remark that in that case, the exponent ν is equal to $\frac{10}{n}$ and the condition $\nu > \frac{H+1/2}{H}$ of Proposition 3.4 becomes $n < \frac{10}{1+1/2H}$. Hence,

for $H = 0.2$ we should take $n < 3$, for $H = 0.5$, $n < 5$ and for $H = 0.7$, $n < 6$. However this makes no sense for the choice of the step $N = 2^n$ of the variations since the asymptotics of the estimators are only valid when N tends to infinity. As a compromise we have chosen $n \in \{7, 8, 9\}$.

We obtain the following results, averaged on 32 simulations of fBm surfaces. The bias corresponds to $H + \frac{1}{2} - \widehat{h_{N,2}(\theta)}$, where $\widehat{h_{N,2}(\theta)}$ is the mean value obtained over the 32 realizations, whereas $\widehat{\sigma}^2 = \frac{1}{32} \sum_{k=1}^{32} \left(\widehat{h_{N,2}^k(\theta)} - \widehat{h_{N,2}(\theta)} \right)^2$, where $\left(\widehat{h_{N,2}^k(\theta)} \right)_{1 \leq k \leq 32}$ denotes the values obtained over the 32 realizations.

		H=0.2		H=0.5		H=0.7	
		$R_\rho^1 B_H$	$R_\rho^2 B_H$	$R_\rho^1 B_H$	$R_\rho^2 B_H$	$R_\rho^1 B_H$	$R_\rho^2 B_H$
$N = 2^9$	Bias	0.234	0.231	0.077	0.088	0.069	0.047
	$\widehat{\sigma}$	0.064	0.075	0.051	0.048	0.038	0.042
$N = 2^8$	Bias	0.124	0.117	0.039	0.039	0.024	0.037
	$\widehat{\sigma}$	0.109	0.087	0.067	0.076	0.064	0.066
$N = 2^7$	Bias	0.098	0.076	0.091	0.014	0.035	0.02
	$\widehat{\sigma}$	0.130	0.124	0.100	0.084	0.090	0.086

All estimated values are under the theoretical values and $\widehat{\sigma}$ is high. This is not surprising since ν is not large enough. Besides let us remark that the bias decreases as the values of H increases. Moreover values are quite the same for both directions in all cases. That shows the isotropy of the fBm. We have to deal with the main problem that we can not generate a fBm with more than $2^{10} \times 2^{10}$ points. This is due to a numerical obstruction given by the method of simulation. One can expect that this method could give better estimates for the identification of the Hölder exponent for a larger step N_0 . However it might be relevant in order to estimate the anisotropy of a field.

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E-mail address: `hermine.bierme@math-info.univ-paris5.fr`

MAP5-UMR 8145, UNIVERSITÉ RENÉ DESCARTES 45, RUE DES SAINTS-PÈRES, 75270
PARIS CEDEX 06 FRANCE, www.univ-orleans.fr/sciences/mapmo/membres/bierme/index.html.
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