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► **To cite this version:**

Vathana Ly Vath. Competitive market equilibrium under asymmetric information. 2006. hal-00019450v1

HAL Id: hal-00019450

<https://hal.science/hal-00019450v1>

Preprint submitted on 22 Feb 2006 (v1), last revised 9 Mar 2007 (v2)

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Competitive market equilibrium under asymmetric information

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February 20, 2006

Abstract

This paper studies the existence of competitive market equilibrium under asymmetric information. There are two agents involved in the trading of the risky asset: an “informed” trader and an “ordinary” trader. The market is competitive and the ordinary agent can infer the insider information from the risky asset’s price dynamics. The definition of market equilibrium is based on the law of supply-demand as described by a Rational Expectations Equilibrium of the Grossman and Stiglitz (1980) model. We show that equilibrium can be attained by linear dynamics of an admissible price process of the risky asset for a given linear supply dynamics.

Key words : insider trading, stochastic filtering theory, equilibrium, utility maximization.

JEL Classification : G10.

MSC Classification (2000) : 93E20, 60G35, 91B50.

1 Introduction

In recent years, financial mathematicians have been focusing on the model of asymmetric information. Asymmetric information arises when agents in the market do not have the same information filtration. They generally make an assumption regarding the extra information that is accessible uniquely by the “informed trader” or the “insider trader”. This extra information could be, for example, the future liquidation price of the risky asset. Using the results of enlargement of filtration first developed by Jeulin and then Jacod, many papers such as those of Pikovsky and Karatzas and Ghorud and Pontier focused on solving utility maximisation problems in a security market where two investors have different information levels. In these papers, the security prices are assumed to evolve according to an exogenous diffusion. In Hillairet [6], different types of asymmetric information, including “initial strong”, “progressive strong” and “weak” information are studied. However, the drawback of the above models is that “ordinary” or “uninformed” agents cannot infer the insider information.

On the other hand, in Kyle [7] and Back [1], the market is competitive and the ordinary agents can obtain feedbacks from the market regarding the insider information. There have also been several other studies, published in the economic literature, on the impact of asymmetric information on stock price. The first such paper is the seminal paper of Grossman-Stiglitz [5], followed by those of Glosten-Milgrom [3]. In [2], we may find a very insightful survey of the literature on these areas, including those cited above. In [5], the agents are competitive and market is Walrasian, ie. price equals supply and demand. The only exogenous part of this model may come from irrational traders, often called noise traders. In [2], the objective is to analyse the price formation in a dynamic version of Grossman and Stiglitz model where the stochastic control techniques can be used.

In the same framework, in our paper, we consider a financial market consisting of two traders, an “ordinary” agent and an “informed” agent and noise traders. While the ordinary agent can only observe the risky asset’s price dynamics, the “informed” agent has also access to the total supply of the risky asset. As in Back [1], based on the observation of risky asset’s price dynamics, “the ordinary” agent can infer the additional information of the “informed” agent. The purpose of the study is to see whether an equilibrium condition can be attained by linear dynamics of an admissible price process of the risky asset for a given linear supply dynamics. Like in the Grossman-Stiglitz model, the market is Walrasian, ie. the agents involved in the market are competitive agents.

Our studies show that the existence of linear competitive market equilibrium under asymmetric information is directly related to the existence of solution to some associated nonlinear equations. Indeed, the equilibrium condition can be explicitly expressed in the form of a system of nonlinear equations. However, we may not determine whether the associated system of nonlinear equations leads to a nonempty set of solution. We nevertheless find that in the particular case where the total supply is a Brownian motion, the equilibrium can be reached and we explicitly obtain the linear dynamics of an admissible price process.

The plan of the paper is organised as follows. We define the model and the equilibrium condition in the first section, while in the second section, we use stochastic control techniques and filtering theory to solve agents' CARA optimisation problem and then determine their optimal trading portfolio. In the third and fourth sections, we express the characterisation of a potential equilibrium price and explicitly calculate the linear dynamics of an admissible price process in the particular case where the total supply dynamics is a Brownian motion.

2 The model

We consider a financial market with a risky stock and a risk-free bond. The risk-free interest rate is assumed to be zero. We are given a standard Brownian motion, $W=(W_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions. T is a fixed time at which all transactions are liquidated.

2.1 Information and agents

There are two rational competitive traders:

- The first one is an “informed” trader (insider trader), agent I , whose information is described by the filtration, \mathbb{F} , as he can observe both the risky asset price $S = (S_t)$ and the total supply of the risky asset $Z = (Z_t)$. He has a constant absolute risk aversion (CARA) with coefficient $\eta_I > 0$, ie. his utility function is equal to $U_I(v) = -\exp(-\eta_I v)$.
- The second trader is an ordinary economic agent, agent O , whose information is only given by the price observation. We denote by \mathbb{F}^O the structure of his filtration. He also has a constant absolute risk-aversion (CARA) with coefficient $\eta_O > 0$, i.e. his utility function is in the form : $U_O(v) = -\exp(-\eta_O v)$.

We assume that the supply Z of the risky asset is a gaussian process, governed by the s.d.e:

$$dZ_t = (a(t)Z_t + b(t)) dt + \gamma(t)dW_t, \quad Z_0 = z_0 \in \mathbb{R} \quad (2.1)$$

where a , b , and γ are deterministic continuous functions from $[0, T]$ into \mathbb{R} .

2.2 Admissible price function

The purpose of this study is to find out whether an equilibrium condition can be attained by linear admissible price processes of the risky asset for a given linear supply dynamics as defined in (2.1). An admissible price process under is a process in the form of :

$$dS_t = S_t [(\alpha(t)Z_t + \beta(t)) dt + \sigma(t)dW_t], \quad 0 \leq t \leq T \quad (2.2)$$

where α and β are continuous functions from $[0, T]$ into \mathbb{R} , and σ a continuous function from $[0, T]$ into \mathbb{R}_+^* . We define \mathcal{S} as the set of admissible price processes of risky asset.

The purpose is therefore to determine all set of functions (α, β, σ) , ie. admissible price processes, satisfying an equilibrium condition.

2.3 Equilibrium

Given an admissible price process S , a trading strategy for the “informed” agent (resp. the ordinary agent) is a \mathbb{F} (resp. \mathbb{F}^O)-predictable process X integrable with respect to S . $X = (X_t)$ represents here the amount invested in the stocks at time t . We denote by $\mathcal{A}(\mathbb{F})$ (resp. $\mathcal{A}(\mathbb{F}^O)$) this set of trading strategies, $X = (X_t)_{0 \leq t \leq T}$, which satisfy the integrability criteria:

$$\int_0^T |X_t|^2 dt < \infty, \quad \mathbb{P} \text{ a.s.} \quad (2.3)$$

Each rational agent’s goal, with its own filtration, is to maximize his expected utility from terminal wealth. We now formulate the definition of market equilibrium based on the law of supply-demand as described by a Rational Expectations Equilibrium of the Grossman and Stiglitz model.

Definition 2.1 *A market equilibrium is a pair (\hat{X}^I, \hat{X}^O) and an element $\hat{S} \in \mathcal{S}$ such that :*

(i) \hat{X}^I is the solution of the insider agent’s optimization problem :

$$\max_{X \in \mathcal{A}(\mathbb{F})} \mathbb{E} \left[U_I \left(v_I + \int_0^T X_t \frac{d\hat{S}_t}{\hat{S}_t} \right) \right],$$

where $v_I \in \mathbb{R}$ is the initial capital of the insider.

(ii) \hat{X}^O is the solution of the ordinary agent’s optimization problem :

$$\max_{X \in \mathcal{A}(\mathbb{F}^O)} \mathbb{E} \left[U_O \left(v_O + \int_0^T X_t \frac{d\hat{S}_t}{\hat{S}_t} \right) \right],$$

where $v_O \in \mathbb{R}$ is the initial capital of the ordinary agent.

(iii) the market clearing conditions hold :

$$\hat{X}_t^I + \hat{X}_t^O = Z_t, \quad 0 \leq t \leq T.$$

If $(\hat{X}^I, \hat{X}^O, \hat{S})$ is a market equilibrium, then we say that \hat{S} is an equilibrium pricing rule.

3 CARA utility maximisation

In this section, we determine the optimal trading portfolio of the ordinary and insider agents.

3.1 “Informed” agent’s optimisation problem

Given an admissible price process S , the self-financed wealth process of the investor with a trading portfolio $X \in \mathcal{A}(\mathbb{F})$ has a dynamics given by :

$$\begin{aligned} dV_t &= X_t \frac{dS_t}{S_t} \\ &= X_t [\alpha(t)Z_t + \beta(t)] dt + X_t \sigma(t) dW_t. \end{aligned}$$

The investor with initial wealth v_I and constant risk aversion $\eta_I > 0$ has to solve the optimization problem :

$$\mathcal{J}_I(v_I) = \sup_{X \in \mathcal{A}(\mathbb{F})} \mathbb{E} [-\exp(-\eta_I V_T)]. \quad (3.1)$$

We consider the related dynamic optimization problem : for all $(t, v, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$J_I(t, v, z) = \sup_{X \in \mathcal{A}(\mathbb{F})} \mathbb{E} [-\exp(-\eta_I V_T) | V_t = v, Z_t = z], \quad (3.2)$$

so that

$$\mathcal{J}_I(v_I) = J_I(0, v_I, z_0).$$

The nonlinear dynamic programming equation associated to the stochastic control problem (3.2) is :

$$\frac{\partial J_I}{\partial t}(t, v, z) + \sup_{x \in \mathbb{R}} \mathcal{L}^x J_I(t, v, z) = 0, \quad (3.3)$$

together with the terminal condition $J_I(T, v, z) = -\exp(-\eta_I v)$. Here \mathcal{L}^x is the second order linear differential operator associated to the diffusion (V, Z) for the constant control $X = x$:

$$\mathcal{L}^x J_I = x[\alpha z + \beta] \frac{\partial J_I}{\partial v} + [az + b] \frac{\partial J_I}{\partial z} + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 J_I}{\partial v^2} + x \sigma \gamma \frac{\partial^2 J_I}{\partial v \partial z} + \frac{1}{2} \gamma^2 \frac{\partial^2 J_I}{\partial z^2}.$$

We make the logarithm transformation:

$$J_I(t, v, z) = -\exp[-\eta_I v - \phi(t, z)].$$

Then the Bellman equation (3.3) becomes:

$$\frac{\partial \phi}{\partial t} + \mathcal{L}_Z \phi + \sup_{x \in \mathbb{R}} \left[\eta_I x (\alpha z + \beta) - \frac{1}{2} \left| \eta_I x \sigma + \gamma \frac{\partial \phi}{\partial z} \right|^2 \right] = 0, \quad (3.4)$$

together with the terminal condition :

$$\phi(T, z) = 0, \quad (3.5)$$

Here \mathcal{L}_Z is the second order linear operator associated to the diffusion Z :

$$\mathcal{L}_Z \phi = (az + b) \frac{\partial \phi}{\partial z} + \frac{1}{2} \gamma^2 \frac{\partial^2 \phi}{\partial z^2}.$$

The maximum in (3.4) is attained for :

$$\hat{x}(t, z) = \frac{1}{\eta_I \sigma^2} \left[\alpha(t)z + \beta(t) - \sigma(t)\gamma(t) \frac{\partial \phi}{\partial z}(t, z) \right]. \quad (3.6)$$

Substituting into (3.4) gives :

$$\frac{\partial \phi}{\partial t} + \mathcal{L}_Z \phi + \frac{1}{2\sigma^2} \left(\alpha z + \beta - \sigma \gamma \frac{\partial \phi}{\partial z} \right)^2 - \frac{1}{2} (\gamma \frac{\partial \phi}{\partial z})^2 = 0. \quad (3.7)$$

This is a semi-linear equation for ϕ but with a quadratic term in $\frac{\partial \phi}{\partial z}$. We are therefore looking for a quadratic solution :

$$\phi(t, z) = \frac{1}{2}P_I(t)z^2 + Q_I(t)z + \chi_I(t)$$

where P_I , Q_I , and χ_I are deterministic functions valued in \mathbb{R} . By substituting and cancelling quadratic terms in z , we see that (3.7) holds iff P_I , Q_I and χ_I satisfy:

$$0 = \dot{P}_I + 2 \left[a - \frac{\gamma\alpha}{\sigma} \right] P_I + \frac{\alpha^2}{\sigma^2} \quad (3.8)$$

$$P_I(T) = 0,$$

$$0 = \dot{Q}_I + \left[a - \frac{\gamma\alpha}{\sigma} \right] Q_I + \frac{\beta}{\sigma^2} [\alpha - \gamma\sigma P_I] + bP_I \quad (3.9)$$

$$Q_I(T) = 0,$$

$$0 = \dot{\chi}_I + \frac{\beta^2}{2\sigma^2} + \left[b - \frac{\gamma\beta}{\sigma} \right] Q_I + \frac{1}{2}\gamma^2 P_I \quad (3.10)$$

$$\chi_I(T) = 0.$$

By solving these differential equations, we obtain:

$$P_I(t) = \exp \left[2 \int_t^T \left(a - \frac{\gamma\alpha}{\sigma} \right) (u) du \right] \quad (3.11)$$

$$\int_t^T \frac{\alpha^2}{\sigma^2}(s) \exp \left[2 \int_s^T \left(a - \frac{\gamma\alpha}{\sigma} \right) (u) du \right] ds,$$

$$Q_I(t) = \exp \left[\int_t^T \left(a - \frac{\gamma\alpha}{\sigma} \right) (u) du \right] \quad (3.12)$$

$$\int_t^T \left[\frac{\beta}{\sigma^2} (\alpha - P_I \gamma \sigma) + P_I b \right] (s) \exp \left[\int_s^T \left(a - \frac{\gamma\alpha}{\sigma} \right) (u) du \right] ds,$$

$$\chi_I(t) = \int_t^T \left[\frac{\beta^2}{2\sigma^2} + \left[b - \frac{a\beta}{\sigma} \right] Q_I + \frac{1}{2}\gamma^2 P_I \right] (u) du. \quad (3.13)$$

The main result of this section can then be stated as follows:

Theorem 3.1 *The value function for problem (3.2) is equal to:*

$$J_I(t, v, z) = - \exp \left(-\eta_I v - \frac{1}{2} z^2 P_I(t) - Q_I(t) z - \chi_I(t) \right),$$

where P_I , Q_I and χ_I are expressed in (3.11), (3.12), and (3.13). Moreover, the optimal trading portfolio for problem (3.2) is given by $\hat{X}_t^I = \hat{x}_I(t, Z_t)$, $0 \leq t \leq T$, where $\hat{x}_I(t, z)$ is

defined on $[0, T] \times \mathbb{R}$ by :

$$\hat{x}_I(t, z) = \Phi_I(t)z + H_I(t), \quad (3.14)$$

$$\Phi_I(t) = \frac{1}{\eta_I \sigma^2(t)} (\alpha(t) - \sigma(t)\gamma(t)P_I(t)), \quad (3.15)$$

$$H_I(t) = \frac{1}{\eta_I \sigma^2(t)} (\beta(t) - \sigma(t)\gamma(t)Q_I(t)). \quad (3.16)$$

Proof. See Appendix 1. □

3.2 Ordinary agent's optimisation problem

We now focus on the ordinary agent's optimisation problem. To do so, we need to decompose the price process $(S_t)_t$ in its own filtration $\mathbb{F}^O = (\mathcal{F}_t^O)_{t \in [0, T]}$, which is generated by the price process, $\mathcal{F}_t^O = \sigma(S_s, s \leq t)$.

Let us define:

$$\begin{aligned} \tilde{Z}_t &= \mathbb{E} \left(Z_t | \mathcal{F}_t^O \right), \\ \Gamma(t) &= \mathbb{E} \left[(Z_t - \tilde{Z}_t)^2 \right]. \end{aligned}$$

From Kalman Bucy filter results (see Theorem 10.3 in [8]), \tilde{Z}_t and Γ_t are solution of the systems of equations:

$$d\tilde{Z}_t = [a(t)\tilde{Z}_t + b(t)]dt + \frac{1}{\sigma(t)} [\sigma(t)\gamma(t) + \alpha(t)\Gamma(t)]dW_t^O, \quad (3.17)$$

$$\dot{\Gamma}(t) = 2a(t)\Gamma(t) - \frac{1}{\sigma^2(t)} [\gamma(t)\sigma(t) + \alpha(t)\Gamma(t)]^2 + \gamma^2(t), \quad (3.18)$$

where W^O a $(\mathbb{P}, \mathbb{F}^O)$ -Brownian motion, the so-called innovation process.

We may obtain the expression of $\Gamma(t)$ by solving the Riccati equation (3.18) (See [9], page 4-7):

$$\Gamma(t) = \Gamma(0) \frac{\exp \left(- \int_0^t \left[-2a(s) + 2\frac{\gamma(s)\alpha(s)}{\sigma(s)} \right] ds \right)}{1 + \Gamma(0) \int_0^t \frac{\alpha^2(s)}{\sigma^2(s)} \exp \left(- \int_0^s \left[-2a(u) + 2\frac{\gamma(u)\alpha(u)}{\sigma(u)} \right] du \right) ds}. \quad (3.19)$$

The dynamics of an admissible price process under $(\mathbb{P}, \mathbb{F}^O)$ is then given by :

$$dS_t = S_t \left[\left(\alpha(t)\tilde{Z}_t + \beta(t) \right) dt + \sigma(t)dW_t^O \right]. \quad (3.20)$$

The equivalent optimisation problem for the ordinary agent with an initial wealth v_o and constant risk aversion $\eta_o > 0$ is :

$$\mathcal{J}_o(v_o) = \sup_{X \in \mathcal{A}(\mathbb{F}^O)} \mathbb{E} [-\exp(-\eta_o V_T)]. \quad (3.21)$$

We consider the related dynamic optimization problem : for all $(t, v, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$J_O(t, v, z) = \sup_{X \in A(\mathbb{F}^O)} \mathbb{E}[-\exp(-\eta_O V_T) | V_t = v, Z_t = z], \quad (3.22)$$

so that $\mathcal{J}_O(v_O) = J_O(0, v_O, z_O)$.

Using the same arguments as in Theorem 3.1, we obtain the following results for ordinary agent:

Theorem 3.2 *The optimal trading portfolio for problem (3.21) is given by $\hat{X}_t^O = \hat{x}_O(t, Z_t)$, $0 \leq t \leq T$, where $\hat{x}_O(t, z)$ is defined on $[0, T] \times \mathbb{R}$ by :*

$$\hat{x}_O(t, z) = \Phi_O(t)z + H_O(t), \quad (3.23)$$

$$\Phi_O(t) = \frac{1}{\eta_O \sigma^2(t)} (\alpha(t) - \sigma(t)\bar{\gamma}(t)P_O(t)), \quad (3.24)$$

$$H_O(t) = \frac{1}{\eta_O \sigma^2(t)} (\beta(t) - \sigma(t)\bar{\gamma}(t)Q_O(t)), \quad (3.25)$$

and P_O and Q_O are expressed as:

$$P_O(t) = \exp \left[2 \int_t^T \left(a - \frac{\bar{\gamma}\alpha}{\sigma} \right) (s) ds \right] \int_t^T \frac{\alpha^2}{\sigma^2} \exp \left[2 \int_s^T \left(a - \frac{\bar{\gamma}\alpha}{\sigma} \right) (u) du \right] (s) ds, \quad (3.26)$$

$$Q_O(t) = \exp \left[\int_t^T \left(a - \frac{\bar{\gamma}\alpha}{\sigma} \right) (s) ds \right] \int_t^T \left[\frac{\beta}{\sigma^2} [\alpha - P_O \bar{\gamma}\sigma] + P_O b \right] \exp \left[\int_s^T \left(a - \frac{\bar{\gamma}\alpha}{\sigma} \right) (u) du \right] (s) ds, \quad (3.27)$$

with

$$\bar{\gamma}(t) = \frac{1}{\sigma(t)} [\sigma(t)\gamma(t) + \alpha(t)\Gamma(t)]. \quad (3.28)$$

4 Characterisation of the equilibrium price

In this section, we give a characterisation of a market equilibrium as defined in Definition 2.1. Using the optimal strategy of each agent determined in the previous section, we find that the equilibrium condition can be explicitly expressed as a nonlinear system.

Theorem 4.1 *The equilibrium condition is equivalent to the following nonlinear system of at most three equations with three unknown variables, α , β , and σ :*

$$\begin{cases} \frac{1}{\eta_O} [\beta(t) - \sigma(t)\bar{\gamma}(t)Q_O(t)] + \frac{1}{\eta_I} [\beta(t) - \sigma(t)\gamma(t)Q_I(t)] = 0, \\ \frac{1}{\eta_I \sigma^2(t)} [\alpha(t) - \sigma(t)\gamma(t)P_I(t)] - 1 = 0, \\ [\alpha(t) - \sigma(t)\bar{\gamma}(t)P_O(t)] \mathbf{Var}(\tilde{Z}_t) = 0. \end{cases} \quad (4.1)$$

Proof. The equilibrium pricing rule is given by

$$\hat{X}_I(t, Z_t) + \hat{X}_O(t, \tilde{Z}_t) = Z_t. \quad (4.2)$$

To simplify the calculations, we assume w.l.o.g. that the gaussian process $(Z_t - \tilde{Z}_t, \tilde{Z}_t)$ is centered. The equilibrium (4.2) is equivalent to

$$\begin{cases} \mathbb{E}[\hat{X}_I(t, Z_t) + \hat{X}_O(t, \tilde{Z}_t)] &= \mathbb{E}[Z_t], \\ \mathbf{Var}[\hat{X}_I(t, Z_t) + \hat{X}_O(t, \tilde{Z}_t)] &= \mathbf{Var}(Z_t). \end{cases} \quad (4.3)$$

Using (3.14) and (3.23), the equilibrium condition becomes:

$$\begin{cases} H_O(t) + H_I(t) &= 0, \\ (\Phi_I(t) - 1)\Gamma(t) &= 0, \\ (\Phi_I(t) + \Phi_O(t) - 1)\mathbf{Var}(\tilde{Z}_t) &= 0. \end{cases} \quad (4.4)$$

Since $\Gamma(t) > 0$, the above equilibrium condition is also written as:

$$\begin{cases} H_O(t) + H_I(t) &= 0, \\ (\Phi_I(t) - 1) &= 0, \\ \Phi_O(t)\mathbf{Var}(\tilde{Z}_t) &= 0. \end{cases} \quad (4.5)$$

and the required results are obtained by substituting the expression of H_O, H_I, Φ_O , and Φ_I . \square

Remark 4.1 *While the explicit expression of the equilibrium condition is in the form of a nonlinear system, we do not know whether this system leads to a nonempty set of solution. Recall that Q_O, Q_I, P_O, P_I , and $\bar{\gamma}$ are dependent on the unknown variables α, β , and σ , see (3.11), (3.12), (3.26), and (3.27).*

5 Equilibrium in the case: $Z_t = W_t$

We take the particular case of $Z_t = W_t$, i.e. $a(t) = b(t) = 0$ and $\gamma(t) = 1$.

Proposition 5.1 *In the case of $Z_t = W_t$, the equilibrium is reached and the linear dynamics of an admissible price process is given by*

$$dS_t = S_t [\alpha(t)Z_t dt + \sigma(t)dW_t]. \quad (5.6)$$

with

$$\sigma(t) = \frac{1}{\eta_I} \left[\mu(t) + \frac{1}{3} \frac{\mu^2(t)}{\mu_T} \left(1 - \frac{\mu^3(t)}{\mu_T^3} \right) \right], \quad (5.7)$$

$$\alpha(t) = \sigma(t)\mu(t), \quad (5.8)$$

where

$$\mu(t) = \frac{\mu_T}{1 + \mu_T(T - t)} \quad \text{and } \mu_T \text{ is any arbitrary positive constant.}$$

Remark 5.2 *The equilibrium condition does not depend on the CARA coefficient of the ordinary agent. In economic sense, this means that the “informed” agent defines his trading strategy in order to maximise his expected utility from terminal wealth and imposes his optimal trading strategy upon the ordinary trader.*

Proof of proposition 5.1. Let us set $\mu(t) = \frac{\alpha(t)}{\sigma(t)}$. From (3.28), (3.18), and (3.8), we obtain :

$$\begin{cases} \bar{\gamma}(t) &= & 1 + \mu(t)\Gamma(t), \\ \dot{\Gamma}(t) &= & 1 - [1 + \mu(t)\Gamma(t)]^2, \\ 0 &= & \dot{P}_O + \mu(t)^2 - 2\mu(t)(1 + \mu(t)\Gamma(t))P_O(t). \end{cases} \quad (5.9)$$

While (3.17) becomes:

$$d\tilde{Z}_t = [1 + \mu(t)\Gamma(t)]dW_t^O. \quad (5.10)$$

Thus

$$\mathbf{Var}(\tilde{Z}_t) = \int_0^t [1 + \mu(s)\Gamma(s)]^2 ds. \quad (5.11)$$

The equilibrium pricing rule becomes :

$$\frac{1}{\eta_O} [\beta(t) - \sigma(t)(1 + \mu(t)\Gamma(t))Q_O(t)] + \frac{1}{\eta_I} [\beta(t) - \sigma(t)Q_I(t)] = 0, \quad (5.12)$$

$$\frac{1}{\eta_I \sigma(t)} [\mu(t) - P_I(t)] - 1 = 0, \quad (5.13)$$

$$[\mu(t) - (1 + \mu(t)\Gamma(t))P_O(t)] \mathbf{Var}(\tilde{Z}_t) = 0. \quad (5.14)$$

The latter equation (5.14) allows us to obtain the expression of $\mu(t)$. Indeed, it is equivalent to:

$$\begin{cases} \mu(t) - (1 + \mu(t)\Gamma(t))P_O(t) = 0, \\ \text{or} \\ 1 + \mu(t)\Gamma(t) = 0. \end{cases} \quad (5.15)$$

By deriving any of the two above equations and using the relationship we had on $\dot{\Gamma}$, and \dot{P}_O , we obtain the same following equation for μ :

$$\frac{\dot{\mu}(t)}{\mu^2(t)} = 1, \quad t \in [0, T] \quad (5.16)$$

As such,

$$\mu(t) = \mu_T \frac{1}{1 + \mu_T(T - t)} \quad (5.17)$$

which raises no problem of definition in the case of $\mu_T > 0$.

From equation (5.12) and (5.13), we obtain the explicit expression of σ and β , and therefore α .

$$\sigma(t) = \frac{1}{\eta_I} \left[\mu(t) + \frac{1}{3} \frac{\mu^2(t)}{\mu_T} \left(1 - \frac{\mu^3(t)}{\mu_T^3} \right) \right] \quad (5.18)$$

$$\alpha(t) = \sigma(t)\mu(t) \quad (5.19)$$

$$\beta(t) = 0 \quad (5.20)$$

$$\text{where } \mu(t) = \frac{\mu_T}{1 + \mu_T(T-t)}. \quad (5.21)$$

We check that when $\mu_T > 0$, μ and σ are positive for $t \in [0, T]$. \square

Appendix 1

Proof of Theorem 3.1

We set:

$$\begin{aligned} \phi(t, z) &= \frac{1}{2} P_I(t) z^2 + Q_I(t) z + \chi_I(t) \\ g(t, v, z) &= -\eta_I v - \phi(t, z) \end{aligned}$$

Where P_I , Q_I , and χ_I are expressed as above [see(3.11), (3.12), (3.13)]

By differentiating, we obtain:

$$\begin{aligned} \frac{\partial g}{\partial t} &= -\frac{\partial \phi}{\partial t}, \quad \frac{\partial g}{\partial v} = -\eta_I, \quad \frac{\partial g}{\partial z} = -\frac{\partial \phi}{\partial z} \\ \mathcal{L}^x g &= -\eta_I(\alpha z + \beta)x - \mathcal{L}_z \phi \end{aligned}$$

By applying Itô's formula to $g(t, V_t, Z_t)$ for any $X \in \mathcal{A}(\mathbb{F})$ between t and T , we obtain :

$$\begin{aligned} g(T, V_T, Z_T) &= g(t, V_t, Z_t) + \int_t^T \left(\frac{\partial g}{\partial t} + \mathcal{L}^{X_u} g \right) (u, V_u, Z_u) du \\ &\quad + \int_t^T \left(\frac{\partial g}{\partial v} X \sigma + \left(\frac{\partial g}{\partial z} \right) \gamma \right) (u, V_u, Z_u) dB_u \\ &= g(t, V_t, Z_t) + \int_t^T \left(-\frac{\partial \phi}{\partial t} - \mathcal{L}_Y \phi - \eta_I X(\alpha Z + \beta) \right) (u, Z_u) du \\ &\quad + \int_t^T \left(-\eta_I X_u \sigma(u) - \left(\frac{\partial \phi}{\partial z} \right) (u, Z_u) \gamma(u) \right) dB_u \\ &= g(t, V_t, Z_t) - \int_t^T \left(\frac{\partial \phi}{\partial t} + \mathcal{L}_Y \phi + \eta_I X_u(\alpha Z + \beta) - \frac{1}{2} \left| \eta_I X \sigma + \gamma \frac{\partial \phi}{\partial z} \right|^2 \right) (u, Z_u) du \\ &\quad - \int_t^T \left(\eta_I X_u \sigma(u) + \left(\frac{\partial \phi}{\partial z} \right) (u, Z_u) \gamma(u) \right) dB_u \\ &\quad - \frac{1}{2} \int_t^T \left| \eta_I X \sigma + \gamma \frac{\partial \phi}{\partial z} \right|^2 (u, Z_u) du. \end{aligned} \quad (5.22)$$

We now consider the exponential local (\mathbb{P}, \mathbb{F}) -martingale for any $X \in \mathcal{A}(\mathbb{F})$:

$$\xi_t^X = \exp \left\{ - \int_t^T \left(\eta_I X_u \sigma(u) + \left(\frac{\partial \phi}{\partial z} \right)(u, Z_u) \gamma(u) \right) dB_u - \frac{1}{2} \int_t^T \left| \eta_I X \sigma + \gamma \frac{\partial \phi}{\partial z} \right|^2 (u, Z_u) du. \right\}.$$

From PDE (3.4) satisfied by ϕ , relation (5.22) yields for all $X \in \mathcal{A}(\mathbb{F})$:

$$\exp(g(T, V_t, Z_T)) \geq \exp(g(t, V_t, Z_t)) \cdot \frac{\xi_T^X}{\xi_t^X}. \quad (5.23)$$

Since $g(T, v, z) = -\eta_I v$ and ξ^X is a (\mathbb{P}, \mathbb{F}) -supermartingale, we obtain by taking conditional expectation in the previous inequality :

$$\mathbb{E}[-\exp(-\eta_I V_T) | V_t = v, Y_t = y] \leq -\exp(g(t, v, z)),$$

for all $X \in \mathcal{A}(\mathbb{F})$ and so :

$$J_{\mathbb{F}}(t, v, z) \leq -\exp(g(t, v, z)). \quad (5.24)$$

Consider now the control strategy $\hat{X}_t = \hat{x}(t, Z_t)$, $0 \leq t \leq T$, where \hat{x} is defined in (3.6) or more explicitly in (3.14). Then, we clearly have $\hat{X} \in \mathcal{A}(\mathbb{F})$, and we have now equality in (5.23) since \hat{x} attains the supremum in the PDE (3.4) :

$$\exp(g(T, V_t, Z_T)) = \exp(g(t, V_t, Z_t)) \cdot \frac{\xi_T^{\hat{X}}}{\xi_t^{\hat{X}}}. \quad (5.25)$$

Observe that :

$$\begin{aligned} \eta_I \hat{X}_u \sigma(u) + \left(\frac{\partial \phi}{\partial z} \right)(u, Z_u) \gamma(u) &= Z_u (\eta_I \Phi(u) \sigma(u) + P_I(u) \gamma(u)) + \\ &\quad \eta_I H(u) \sigma(u) + Q_I(u) \gamma(u). \end{aligned}$$

Since Z is a Gaussian process, it follows that for some $\delta > 0$, we have :

$$\mathbb{E} \left[\exp \left(\delta \left| \eta_I X_u \sigma + \gamma \frac{\partial \phi}{\partial z} \right|^2 (u, Z_u) \right) \right] < \infty.$$

Therefore by Liptser, Shiryaev (1977, p.220), $\xi^{\hat{X}}$ is a (\mathbb{P}, \mathbb{F}) -martingale and so by taking conditional expectation in (5.25), we have :

$$\mathbb{E}[-\exp(-\eta_I V_T) | V_t = v, Z_t = z] = -\exp(g(t, v, z)),$$

for the wealth process V controlled by the trading portfolio \hat{X} . This last equality combined with (5.24) ends the proof. \square

References

- [1] Back, K. (1992) : “Insider trading in continuous time”, *Review of Financial Studies*, 5, pp.387-409.
- [2] Biais, B, Rochet, J.C. (1996) : “Risk Sharing, adverse selection and market structure”, *Financial Mathematics, Lectures Notes on Mathematics*, Springer Verlag, 1997.
- [3] Glosten, L.R., Milgrom, P.R. (1985) : “Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders.”, *Journal of Financial Economics*, 14.
- [4] Grorud, A., Pontier, M. (1998) : “Insider trading in a continuous time market model”, *International Journal of Theoretical and Applied Finance* 1, pp. 331-347.
- [5] Grossman, S.J., Stiglitz, J.E. (1980) : “On the Impossibility of Informationally Efficient Markets.”, *American Economic Review*, 70, pp. 393-408.
- [6] Hillairet, C. (2004) : “Existence of an equilibrium on a financial market with different assymetric information”, preprint, LSP.
- [7] Kyle A.S. (1985) : “Continuous Auctions and Insider Trading.”, *Econometrica*, 53, pp. 1335-1355.
- [8] Lipster, R. Sh., Shiryaev, A. N. (2001) : “Statistics of Random Processes. I”, Springer-Verlag, New York.
- [9] Reid, W. (1972) : “Riccati differential equations”, volume 86 of *Mathematics in Science and Engineering*, Academic Press, New York and London.