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## RANDOM SERIES OF FUNCTIONS AND APPLICATIONS

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ABSTRACT. In this article we study the continuity properties of trajectories for some random series of functions,  $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$  where  $(a_k)_{k \geq 0}$  is a complex sequence,  $(X_k)_{k \geq 0}$  is a sequence of real independent random variables,  $f$  is a real valued function with period one and summable Fourier coefficients. We obtain almost sure continuity results for these periodic or almost periodic series for a large class of functions  $f$ , where the "almost sure" does not depend on the function. The proof relies on gaussian randomization. We show optimality of the results in some cases.

## SERIES DE FONCTIONS ALEATOIRES ET APPLICATIONS

ABSTRACT. (RÉSUMÉ) Dans ce travail, nous étudions des propriétés de continuité de trajectoires de séries de fonctions aléatoires du type  $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$  où  $(a_k)_{k \geq 0}$  est une suite de nombres complexes,  $(X_k)_{k \geq 0}$  une suite de variables aléatoires réelles et indépendantes,  $f$  une fonction 1-périodique à coefficients de Fourier sommables. Nous montrons que, presque sûrement, ces séries de fonctions aléatoires (périodiques ou presque périodiques) sont à trajectoires continues pour une grande classe de fonctions  $f$ . Le "presque sûr" est indépendant de  $f$ . Les preuves s'appuient sur un procédé de randomisation gaussien. Dans certains cas, nous montrerons l'optimalité des résultats obtenus.

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**Keywords** : random fourier series, almost sure continuity of trajectories, gaussian randomization, almost periodic functions, random trigonometric polynomial

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## 1. INTRODUCTION. MAIN RESULTS

In [1], Berkes studies the almost sure convergence of series defined by :

$$\sum_{k \geq 1} a_k f(\alpha n_k)$$

where the sequence  $(n_k)$  is lacunary and the function  $f$  verifies :

$$f(x+1) = f(x) \quad \int_0^1 f(x) dx = 0 \quad \int_0^1 f^2(x) dx = 1$$

He shows that the important property of  $f$  to ensure the almost sure convergence is  $f \in \text{Lip}(\gamma)$  with  $\gamma > 1/2$  and  $\sum_{k \geq 1} |a_k|^2 < +\infty$ . In his case, the  $n_k$  are strictly lacunary, more precisely, they satisfy the Hadamard gap condition :

$$\frac{n_{k+1}}{n_k} \geq q > 1$$

We can naturally address the question whether the convergence still holds when  $n_k$  is polynomial, and for which class of functions. We are going to answer the question when the sequence  $(n_k)$  is randomly generated.

Let us mention that the result exists when  $(n_k)$  is a deterministic polynomial sequence and  $(a_k)$  is randomly distributed (see [5] and [6]).

We want to study the convergence properties of series of functions sampled by a random process. More precisely, consider the torus  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  and define  $A(\mathbb{T})$  as the set of complex valued functions whose Fourier coefficients are absolutely summable :

$$A(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{C}, f(\alpha) = \sum_{j \in \mathbb{Z}} \hat{f}(j) \exp(2i\pi\alpha j), \sum_{j \in \mathbb{Z}} |\hat{f}(j)| < +\infty\}$$

$(a_k)_{k \geq 0}$  will denote a sequence of real numbers and  $(X_k)_{k \geq 0}$  a sequence of independent real random variables defined on the probabilised space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Our aim is to study the convergence, when  $\omega \in \Omega$  is fixed, of the series of functions

$$\forall \alpha \in \mathbb{R}, \quad F(\alpha, \omega) = \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$$

Is it possible to give conditions on the sequence  $(a_k)_{k \geq 0}$  in order to find a  $\mathcal{A}$ -measurable set  $\Omega_0$  independent of the function  $f$ , such that  $\mathbb{P}(\Omega_0) = 1$ , on which the series uniformly converges?

Note that when  $X_k$  does not take integer values,  $F$  is not a periodical function of the torus. For us,  $\alpha$  will be real and we will deal with this "almost periodical" case. That is why we have to study the properties of  $F$  on a compact  $[-M, M]$  and not only  $[0, 1]$  (see for example [3])

For all  $f \in A(\mathbb{T})$ , define

$$\|f\| := \sum_{j \in \mathbb{Z}} |\hat{f}(j)| < +\infty.$$

$$|||f||| := \sum_{j \in \mathbb{Z}} |\hat{f}(j)| \sqrt{\log(|j| + 3)} < +\infty.$$

and

$$B(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{C}, f(\alpha) = \sum_{j \in \mathbb{Z}} \hat{f}(j) \exp(2i\pi\alpha j), |||f||| < +\infty\}$$

Remark that  $B(\mathbb{T}) \subset A(\mathbb{T})$ .

In the following, we will give conditions for  $\alpha \mapsto F(\alpha, \omega)$  to have continuous trajectories  $\mathbb{P}$ -almost surely.

We will denote by  $\varphi_X$  the characteristic function of the random variable  $X$

$$\forall t \in \mathbb{R}, \varphi_X(t) = \mathbb{E}(e^{2i\pi t X})$$

**Theorem 1.1.** *Let  $(X_k)_{k \geq 0}$  be a sequence of independent real valued random variables and let  $(a_k)_{k \geq 1}$  be a sequence of complex numbers such that, for any compact  $K$  which does not contain 0:*

$$(\mathcal{H}) \quad \forall \varepsilon > 0, \exists N > 0, \sup_{m > n \geq N} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \left| \sum_{k=n}^m a_k \varphi_{X_k}(j\alpha) \right| < \varepsilon.$$

Assume moreover that:

**case 1:(polynomial)** *there exists  $\beta > 0$  and  $d > 0$  with  $\mathbb{E}|X_k|^\beta = \mathcal{O}(k^d)$  and*

$$(1) \quad \sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n \sqrt{\log n}} < +\infty$$

**case 2:(subexponential)** *there exists  $\beta > 0$  and  $\gamma \in ]0, 1[$  with  $\mathbb{E}|X_k|^\beta = \mathcal{O}(2^{k^\gamma})$  and*

$$(2) \quad \sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n^{1-\frac{\gamma}{2}}} < +\infty$$

*then in both cases, there exists a measurable set  $\Omega_0$  with  $\mathbb{P}(\Omega_0) = 1$  such that for all  $\omega \in \Omega_0$ , for any  $f \in B(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$  : for  $\alpha \in \mathbb{R} - \{0\}$ ,  $F(\alpha, \omega)$  is well defined,  $\alpha \mapsto F(\alpha, \omega)$  is continuous and the series defining  $F$  converges uniformly on every compact which does not contain  $\{0\}$ .*

**Remark 1.1.**

- (1) *It is worth noticing that the set  $\Omega_0$  does not depend on the class of functions  $f$  ( $A(\mathbb{T})$  or  $B(\mathbb{T})$ ).*
- (2) *when  $(X_k)_{k \geq 0}$  takes integer values, condition  $|||f||| < \infty$  becomes  $||f|| < \infty$ .*
- (3) *we will give conditions on the law of the process  $(X_k)_{k \geq 0}$  to fulfill hypothesis  $(\mathcal{H})$ .*
- (4) *For example, when  $|a_k| = \mathcal{O}(k^{-\delta})$ , in case 1, if  $\delta > 1/2$ , then condition 1 holds and in case 2, if  $\delta > \frac{\gamma+1}{2}$ , then condition 2 holds.*

- (5) Concerning case 2, if  $\gamma \geq 1$  ( $\mathbb{E}|X_k|^\beta$  grows exponentially), one can prove using remark 2.1 that the series  $\sum a_k$  has to converge. The function  $F$  is then obviously well defined using only Cauchy Schwarz inequality.

In case condition  $(\mathcal{H})$  is hard to check, it is possible to split up the hypothesis on the sequence  $(a_k)$  and the characteristic function  $\varphi_{X_k}$  either using Abel's summation method or using Cauchy Schwarz inequality.

Define:

$$c_n = \begin{cases} 1 + \sqrt{\log n} & \text{in the polynomial case} \\ n^{\frac{\gamma}{2}} & \text{in the subexponential case} \end{cases}$$

**Corollary 1.2.** Let  $(X_k)_{k \geq 0}$  be a sequence of independent real valued random variables

Assume that, for any compact  $K$  which does not contain 0 :

$$(\mathcal{H}') \quad \sup_{N \geq 1} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \left| \sum_{k=0}^N \varphi_{X_k}(j\alpha) \right| < \infty.$$

Let  $(a_k)_{k \geq 1}$  be a sequence of complex numbers enjoying the following properties

- (1)  $\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{nc_n} < +\infty$
- (2)  $\sum_{k \geq 1} |a_k - a_{k+1}|$  converges

then there exists a measurable set  $\Omega_0$  with  $\mathbb{P}(\Omega_0) = 1$  such that for all  $\omega \in \Omega_0$ , for any  $f \in B(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$ : for  $\alpha \in \mathbb{R} - \{0\}$ ,  $F(\alpha, \omega)$  is well defined,  $\alpha \mapsto F(\alpha, \omega)$  is continuous and the series defining  $F$  converges uniformly on every compact which does not contain  $\{0\}$ .

**Corollary 1.3.** Let  $(X_k)_{k \geq 0}$  be a sequence of independent real valued random variables

Assume that, for any compact  $K$  which does not contain 0 :

$$(\mathcal{H}'') \quad \forall \varepsilon > 0, \exists N > 0, \sup_{m > n \geq N} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \left( \sum_{k=n}^m |\varphi_{X_k}(j\alpha)|^2 \right) < \varepsilon.$$

Let  $(a_k)_{k \geq 1}$  be a sequence of complex numbers enjoying

$$\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{nc_n} < +\infty$$

then there exists a measurable set  $\Omega_0$  with  $\mathbb{P}(\Omega_0) = 1$  such that for all  $\omega \in \Omega_0$ , for any  $f \in B(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$ : for  $\alpha \in \mathbb{R} - \{0\}$ ,  $F(\alpha, \omega)$  is well defined,  $\alpha \mapsto F(\alpha, \omega)$  is continuous and the series defining  $F$  converges uniformly on every compact which does not contain  $\{0\}$ .

**Remark 1.2.**

The previous corollaries will be useful for example when the law of  $X_k$  is obtained by convolution product (see corollary 4.2). As the condition

$\sum_{k \geq 1} |a_k - a_{k+1}|$  is often hard to check, corollary 1.3 is sometimes better to use.

The proof of theorem 1.1 will start by looking separately at  $F(\alpha, \omega) - \mathbb{E}(F(\alpha, \cdot))$  and  $\mathbb{E}(F(\alpha, \cdot))$ . It turns out that hypothesis  $(\mathcal{H})$  will be used only to deal with the expectation. That is why we think interesting to state the result for:

$$F(\alpha, \omega) - \mathbb{E}(F(\alpha, \cdot)) := \sum_k a_k [f(\alpha X_k(\omega)) - \mathbb{E}(f(\alpha X_k))]$$

**Theorem 1.4.** *Let  $(X_k)_{k \geq 0}$  be a sequence of independent real valued random variables such that there exists  $\beta > 0$  and  $d > 0$  with  $\mathbb{E}|X_k|^\beta = \mathcal{O}(k^d)$  or  $\gamma \in ]0, 1[$  with  $\mathbb{E}|X_k|^\beta = \mathcal{O}(2^{k^\gamma})$ . Let  $(a_k)_{k \geq 1}$  be a sequence of complex numbers enjoying the following property*

$$\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{nc_n} < +\infty$$

*then there exists a measurable set  $\Omega_0$  with  $\mathbb{P}(\Omega_0) = 1$  such that for all  $\omega \in \Omega_0$ , for any  $f \in B(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$  : for  $\alpha \in \mathbb{R}$ ,  $F(\alpha, \omega) - \mathbb{E}(F(\alpha, \cdot))$  is well defined,  $\alpha \mapsto F(\alpha, \omega) - \mathbb{E}(F(\alpha, \cdot))$  is continuous, the series defining  $F - \mathbb{E}(F)$  converges uniformly on every compact and there exists  $C_\omega > 0$  such that for all  $\alpha \in \mathbb{R}$ :*

$$|F(\alpha, \omega) - \mathbb{E}(F(\alpha, \cdot))| \leq C_\omega \|f\| \sqrt{\log(|\alpha| + 2)}$$

**Remark 1.3.**

- (1) *We also discuss the optimality of hypothesis on  $(a_k)$  of theorem 1.4 in section 2.*
- (2) *We also have:*

$$\mathbb{E} \sup_{T > 1} \frac{\sqrt{\int_0^T |F(t, \omega) - \mathbb{E}(F(t, \cdot))|^2 dt}}{\sqrt{T \log T}} < \infty$$

This result relies on uniform estimations of the size of some trigonometric polynomials, more precisely on the following :  
Recall that  $\log^+ = \max(\log, 0)$ .

**Theorem 1.5.** *Let  $\lambda$  and  $\Lambda$  be two integers with  $\lambda \leq \Lambda$ ,  $(X_k)_{k \geq 0}$  be a sequence of independent real valued random variables such that there exists  $\beta > 0$  such that,  $\forall N \geq 0$ ,  $\mathbb{E}|X_N^\beta| < \infty$ . Define*

$$\forall N \geq 0, \Phi_\beta(N) = 2 + \max(N, \mathbb{E}|X_N^\beta|)$$

*Let  $M \geq 1$  and  $I_M = [-M, M]$ . Let  $(a_k)_{k \geq 1}$  be a sequence of real or complex numbers.*

*Define*

$$A_{\lambda, \Lambda, M} = \sqrt{\log(M \Phi_\beta(\Lambda)) \sum_{k=\lambda}^{\Lambda} |a_k|^2},$$

then

$$\mathbb{E} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} a_k [\exp 2i\pi\alpha j X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha j X_k]}{\sqrt{A_{\lambda, \Lambda, M}^2 \log(|j| + 3)}} \right| < \infty$$

**Remark 1.4.** When  $(X_k)_{k \geq 0}$  takes integer values, the proof of theorem 1.5 is easier. Namely, using the fact that  $\alpha \mapsto j\alpha \pmod{1}$  is onto for  $j \neq 0$ , we get

$$\begin{aligned} & \sup_{j \in \mathbb{Z}^*} \sup_{\alpha \in \mathbb{T}} \left| \sum_{k=\lambda}^{\Lambda} a_k [\exp 2i\pi\alpha j X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha j X_k] \right| \\ &= \sup_{\alpha \in \mathbb{T}} \left| \sum_{k=\lambda}^{\Lambda} a_k [\exp 2i\pi\alpha X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha X_k] \right| \end{aligned}$$

the result of theorem 1.5 becomes then :

$$\mathbb{E} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} a_k [\exp 2i\pi\alpha X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha X_k]}{\sqrt{A_{\lambda, \Lambda, M}^2}} \right| < \infty$$

When  $(X_k)_{k \geq 1}$  takes real values, the proof is more tedious. It relies on a fine inequality about decoupling gaussian random functions (see section 3.). We can see here why, for integer-valued  $X_k$ , we can work with the functional space  $A(\mathbb{T})$ , whereas for real-valued  $X_k$ , we need to introduce the space  $B(\mathbb{T})$ .

## 2. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

First, we split  $F$  into two parts as follows :

$$\sum_k a_k f(\alpha X_k(\omega)) = \sum_k a_k [f(\alpha X_k(\omega) - \mathbb{E}(f(\alpha X_k))) + \sum_k a_k \mathbb{E}(f(\alpha X_k))]$$

**-Step 1 :** (first part of the sum)

Let  $(N_k)_{k \geq 1}$  be a strictly increasing sequence of integers and define

$$\forall k \geq 1, \quad P_k(\alpha) = \sum_{l=N_k+1}^{N_{k+1}} a_l [f(\alpha X_l(\omega)) - \mathbb{E} f(\alpha X_l)]$$

where  $f \in B(\mathbb{T})$ . We want to study the following series, for all  $M \geq 1$  :

$$\sum_k \sup_{\alpha \in [-M, M]} |P_k(\alpha)|$$

We have :

$$|P_k(\alpha)| \leq \sum_{j \in \mathbb{Z}} |\hat{f}(j)| \left| \sum_{l=N_k+1}^{N_{k+1}} a_l [\exp(2\pi j \alpha X_l(\omega)) - \mathbb{E} \exp(2\pi j \alpha X_l)] \right|$$

Hence, using theorem 1.5, there exists a positive integrable random variable  $\xi$  such that

$$(3) \quad \sup_{\alpha \in [-M, M]} |P_k(\alpha)| \leq \xi \|f\| \sqrt{\log(M\Phi_\beta(N_{k+1})) \sum_{j=N_k+1}^{N_{k+1}} |a_j|^2}$$

where

$$\xi = \sup_{j \in \mathbb{Z}} \sup_{k \geq 1} \sup_{\alpha \in I_M} \left| \frac{1}{\sqrt{A_{k,M}^2 \log(|j|+3)}} \sum_{l=N_k+1}^{N_{k+1}} a_l \left[ e^{2i\pi\alpha j X_l(\omega)} - \mathbb{E} e^{2i\pi\alpha j X_l} \right] \right|$$

with

$$A_{k,M}^2 = \log(M\Phi_\beta(N_{k+1})) \sum_{l=N_k+1}^{N_{k+1}} |a_l|^2$$

First, in the polynomial case, that is to say when there exists  $d > 0$  with

$$\Phi_\beta(N) = \mathcal{O}(N^d)$$

then we choose  $N_k = 2^{2^k}$  and we need to prove that

$$\sum_k 2^{k/2} \left( \sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} |a_l|^2 \right)^{1/2} < +\infty$$

now we use the following equivalent:

$$\sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} \frac{1}{l(\log(l))^{1/2}} \approx 2^{k/2}$$

which may be computed by comparing series and integral, hence :

$$\begin{aligned} 2^{k/2} \left( \sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} |a_l|^2 \right)^{1/2} &\leq C \sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} \frac{1}{l(\log(l))^{1/2}} \left( \sum_{l=2^{2^k+1}}^{\infty} |a_l|^2 \right)^{1/2} \\ &\leq C \sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} \frac{\left( \sum_{j=l}^{\infty} |a_j|^2 \right)^{1/2}}{l(\log(l))^{1/2}} \end{aligned}$$

and, using condition 1 :

$$\begin{aligned} \sum_k 2^{k/2} \left( \sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} |a_l|^2 \right)^{1/2} &\leq \sum_k C \sum_{l=2^{2^k+1}}^{2^{2^{k+1}}} \frac{\left( \sum_{j=l}^{\infty} |a_j|^2 \right)^{1/2}}{l(\log(l))^{1/2}} \\ &\leq \sum_{n \geq 2} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n \sqrt{\log n}} < +\infty \end{aligned}$$



this implies :

$$\sum_{k \geq 1} \sup_{\alpha \in [-M, M]} |P_k(\alpha)| < \infty$$

almost everywhere on the measurable set  $\Omega_o = \{\omega \in \Omega, \xi(\omega) < \infty\}$ . By construction, this set does not depend on the choice of  $f$ .

Secondly, in the subexponential case, that is when there exists  $\gamma \in ]0, 1[$  with

$$\Phi_\beta(N) = \mathcal{O}(2^{N^\gamma})$$

we choose  $N_k = 2^k$  and we need to prove that

$$\sum_k 2^{\gamma k/2} \left( \sum_{l=2^{k+1}}^{2^{k+1}} |a_l|^2 \right)^{1/2} < +\infty$$

Using the following equivalent:

$$\sum_{l=2^{k+1}}^{2^{k+1}} \frac{1}{l^{1-\frac{\gamma}{2}}} \approx 2^{\gamma k/2}$$

and doing the same kind of computation as before, using condition 2:

$$\sum_{n \geq 2} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n^{1-\frac{\gamma}{2}}} < +\infty$$

implies

$$\sum_{k \geq 1} \sup_{\alpha \in [-M, M]} |P_k(\alpha)| < \infty$$

We also get from (3), for all  $\alpha \in \mathbb{R}$ :

$$|P_k(\alpha)| \leq \xi \|f\| \sqrt{\log(|\alpha| + 1)} \sqrt{\log(\Phi_\beta(N_{k+1})) \sum_{j=N_k+1}^{N_{k+1}} |a_j|^2}$$

summing on  $k$ , we get the inequality:

$$|F(\alpha, \omega) - \mathbb{E}(F(\alpha, \cdot))| \leq C \xi(\omega) \|f\| \sqrt{\log(|\alpha| + 2)}$$

where  $C$  only depends on  $(a_k)$  and  $\mathbb{E}(\xi) < \infty$

This ends the proof of theorem 1.4 which is also the first step of the proof of theorem 1.1

**-Step 2 :** (second part of the sum)

Let  $K$  be a compact which does not contain zero and  $\alpha \in K$ . Let  $n < m$  be

two integers,

$$(4) \quad \left| \sum_{k=n}^m a_k \mathbb{E}(f(\alpha X_k)) \right| = \left| \sum_{j \in \mathbb{Z}^*} \sum_{k=n}^m a_k \hat{f}(j) \mathbb{E}(\exp(2i\pi j \alpha X_k)) \right|$$

$$(5) \quad = \left| \sum_{j \in \mathbb{Z}^*} \sum_{k=n}^m a_k \hat{f}(j) \varphi_{X_k}(j\alpha) \right|$$

$$(6) \quad = \left| \sum_{j \in \mathbb{Z}^*} \hat{f}(j) \left( \sum_{k=n}^m a_k \varphi_{X_k}(j\alpha) \right) \right|$$

$$(7) \quad \leq \left( \sum_{j \in \mathbb{Z}^*} |\hat{f}(j)| \right) \sup_{j \in \mathbb{Z}^*} \left| \sum_{k=n}^m a_k \varphi_{X_k}(j\alpha) \right|$$

At this point, to prove theorem 1.1, we can conclude directly by using hypothesis  $(\mathcal{H})$  to get

$$\sup_{n < m} \sup_{\alpha \in K} \left| \sum_{k=n}^m a_k \mathbb{E}(f(\alpha X_k)) \right| < \varepsilon$$

as long as  $m$  and  $n$  are large enough.

To prove corollary 1.2, we use Abel's summation. Let  $\phi_p = \sum_{k=0}^{p-1} \varphi_{X_k}$ , we have :

$$(8) \quad \sum_{k=n}^m a_k \varphi_{X_k}(j\alpha) = \sum_{k=n}^m a_k (\phi_{k+1}(j\alpha) - \phi_k(j\alpha))$$

$$(9) \quad = \sum_{k=n+1}^{m+1} a_{k-1} \phi_k(j\alpha) - \sum_{k=n}^m a_k \phi_k(j\alpha)$$

$$(10) \quad = -a_n \phi_n(j\alpha) + a_m \phi_{m+1}(j\alpha)$$

$$(11) \quad + \sum_{k=n+1}^m (a_{k-1} - a_k) \phi_k(j\alpha)$$

and :

$$(7) \leq \|f\| \sup_{N \geq 1} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z}^*} |\phi_N(j\alpha)| \left[ |a_m| + |a_n| + \sum_{k=n+1}^m |a_k - a_{k-1}| \right]$$

we now conclude using hypothesis  $\mathcal{H}'$  and hypothesis (1) on the sequence  $(a_n)$  in corollary 1.2 :

$$\begin{aligned} & \sup_{n < m} \sup_{\alpha \in K} \left| \sum_{k=n}^m a_k \mathbb{E}(f(\alpha X_k)) \right| \\ & \leq \|f\| \left[ |a_m| + |a_n| + \sum_{k=n+1}^m |a_k - a_{k-1}| \right] \sup_{N \geq 1} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z}^*} |\phi_N(j\alpha)| < \varepsilon \end{aligned}$$

as long as  $m$  and  $n$  are large enough.

To prove corollary 1.3, we use Cauchy Schwarz inequality in the following way:

$$\left| \sum_{k=n}^m a_k \varphi_{X_k}(j\alpha) \right| \leq \sqrt{\sum_{k=n}^m |a_k|^2} \sqrt{\sum_{k=n}^m |\varphi_{X_k}(j\alpha)|^2}$$

and we conclude using condition  $H''$ .

**Remark 2.1.**

- (1) *the most general hypothesis we can put on the sequence  $(a_k)_{k \geq 1}$  is the following : there exists a strictly increasing sequence  $(N_k)_{k \geq 1}$  such that*

$$\sum_{k=1}^{\infty} \sqrt{\log \Phi_{\beta}(N_{k+1}) \sum_{l=N_k+1}^{N_{k+1}} |a_l|^2} < \infty$$

- (2) *if the process  $(X_k)_{k \geq 1}$  takes integer values then  $f \in A(\mathbb{T})$  can be assumed without any other hypothesis.*
- (3) *If  $\sum |a_k|^2$  diverges, then we can construct a stochastic process  $(X_k)_{k \geq 1}$  verifying the hypothesis of theorem 1.4 and find  $f \in B(\mathbb{T})$  such that the convergence of the series is not uniform on any compact. In that sense, the conditions imposed to the sequence  $(a_k)_{k \geq 1}$  are optimal. Remark that in this case, condition 1 is not fulfilled.*

*Namely consider a sequence of independent random variables with disjoint supports. For all  $k \geq 1$  the support of  $X_k$  is the set of integers belonging to  $[k^2, (k+1)^2 - 1]$  and hence, the hypothesis on the moment is verified. We will come back to the law of  $X_k$  later. Now choose  $f$  in the following way : for all  $\alpha \in \mathbb{T}$ ,  $f(\alpha) = \exp(2i\pi\alpha)$ . Thus  $f \in A(\mathbb{T})$  et  $\|f\| < \infty$ . As a consequence, if the convergence of the series defining  $F$  was uniform in  $\alpha$  on  $\mathbb{T}$ , then we would have :*

$$\begin{aligned} & \sqrt{\int_0^1 \left| \sum_{k=1}^{\infty} a_k [\exp 2i\pi\alpha X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha X_k] \right|^2 d\alpha} \\ & \leq \sup_{\alpha \in [0,1]} \left| \sum_{k=1}^{\infty} a_k [\exp 2i\pi\alpha X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha X_k] \right| < \infty \end{aligned}$$

By construction,  $\mathbb{P}$ - almost surely :

$$\begin{aligned} & \int_0^1 \left| \sum_{k=1}^{\infty} a_k [\exp 2i\pi\alpha X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha X_k] \right|^2 d\alpha \\ &= \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |\exp(2i\pi\alpha X_k(\omega)) - \mathbb{E} \exp(2i\pi\alpha X_k)|^2 d\alpha \\ &\geq \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |1 - |\mathbb{E} \exp(2i\pi\alpha X_k)||^2 d\alpha \end{aligned}$$

Assume now that the law of  $X_k$  is uniform on the  $2k+1$  integers of  $[k^2, (k+1)^2 - 1]$ . for all  $k \geq 1$ .

$$|\mathbb{E} \exp(2i\pi\alpha X_k)| = \frac{1}{2k+1} \left| \frac{\sin \pi\alpha(2k+1)}{\sin \pi\alpha} \right|$$

Using Lebesgue convergence theorem, we get

$$\lim_{k \rightarrow +\infty} \int_0^1 |1 - |\mathbb{E} \exp(2i\pi\alpha X_k)||^2 d\alpha = 1$$

and we also get the divergence of the series with positive terms

$$\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |1 - |\mathbb{E} \exp(2i\pi\alpha X_k)||^2 d\alpha = \infty$$

A contradiction with uniform convergence of the centered part.

### 3. PROOF OF THEOREM 1.5

Let us begin by restating some inequalities obtained by Fernique ([4]) which will be useful in the proof of theorem 1.5. For more information on gaussian techniques in this framework, see [7] and [2].

**Inequality 3.1.** Let  $(G_k)_{k \geq 1}$  be a sequence of Banach space valued gaussian random variables  $(B, \|\cdot\|)$  defined on a probabilised space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Then :

$$\mathbb{E} \sup_{k \geq 1} \|G_k\| \leq K_1 \left\{ \sup_{k \geq 1} \mathbb{E} \|G_k\| + \mathbb{E} \sup_{k \geq 1} |\lambda_k \sigma_k| \right\}$$

where  $(\lambda_k)_{k \geq 1}$  is an isonormal sequence,  $K_1$  a universal constant and for all  $k \geq 1$ ,

$$\sigma_k = \sup_{f \in B', \|f\| \leq 1} \|\langle G_k, f \rangle_B\|_{2, \mathbb{P}}$$

**Inequality 3.2.** Let  $g$  be a real valued stationary gaussian random variable, separable and continuous in quadratic mean. Let  $m$  be its associated spectral measure on  $\mathbb{R}^+$  defined by

$$\mathbb{E}[|g(s) - g(t)|^2] = 2 \int_0^{\infty} [1 - \cos 2\pi u(s-t)] m(du)$$

We have

$$\mathbb{E} \sup_{\alpha \in [0,1]} g(\alpha) \leq K \left\{ \sqrt{\int_0^\infty \min(u^2, 1) m(du)} + \int_0^\infty \sqrt{m \left( \left[ \frac{e^{x^2}}{2M}, \infty \right] \right)} dx \right\}$$

where  $K$  is a universal constant.

**Inequality 3.3. (decoupling)** Let  $X = \{X(t) : t \in \mathbb{T}\}$  be a gaussian random function defined on a finite or countable set  $T$ . Let  $\{T_k, k \in [1, n]\}$  be a covering of  $T$ . Let  $S = T_1 \times \dots \times T_n$ . The following inequality holds :

$$\mathbb{E} \left\{ \sup_{k \in [1, n]} \left[ \sup_{t \in T_k} X(t) - \mathbb{E} \sup_{t \in T_k} X(t) \right] \right\} \leq \frac{\pi}{2} \cdot \sup_{s \in S} \left\{ \mathbb{E} \left[ \sup_{k \in [1, n]} X(s_k) \right] \right\}$$

The following estimation generalizes inequality 3.2 and will be useful in the almost periodic case because it gives estimations on arbitrarily large intervals.

**Inequality 3.4.** Let  $g$  a real valued stationary gaussian random function, separable and continuous in quadratic mean. Let  $m$  its associated spectral measure on  $\mathbb{R}^+$  defined as in inequality 3.2. There exists a universal constant  $K$  such that

$$\begin{aligned} & \mathbb{E} \sup_{\alpha \in [-M, M]} g(\alpha) \\ & \leq K \left\{ \sqrt{\int_0^\infty \min(2Mu^2, 1) m(du)} + \int_0^\infty \sqrt{m \left( \left[ \frac{e^{x^2}}{2M}, \infty \right] \right)} dx \right\} \end{aligned}$$

Let us now come to the proof

**-Step 1:** In this part, we replace our problem by a question of regularity of trajectories of random gaussian functions.

Let us consider an independent copy of  $X = (X_k)_{k \geq 1}$  denoted by  $X' = (X'_k)_{k \geq 1}$  defined on another probabilized space  $(\Omega', \mathcal{A}', \mathbb{P}')$ . We call  $\mathbb{E}_*$  the integration symbol whose index refers to the space of integration.

Using classical convexity properties, to prove (1), it is enough to show

$$(12) \quad \mathbb{E}_{X, X'} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \frac{\sum_{k=\lambda}^\Lambda a_k \left[ e^{2i\pi\alpha j X_k} - e^{2i\pi\alpha j X'_k} \right]}{\sqrt{A_{\Lambda, \lambda, M}^2 \log(|j| + 3)}} \right| < \infty$$

Let us now symmetrize the problem: consider the following separable family of random functions, with continuous trajectories

$$(f_k)_{k \geq 1} = \{f_k(\alpha, j) = a_k(\exp 2i\pi\alpha j X_k - \exp 2i\pi\alpha j X'_k), \alpha \in I_M, j \in \mathbb{Z}\}_{k \geq 1}$$

By construction  $f$  is a symmetric family of random functions, that is to say their law is sign-invariant. More precisely, call  $\{\varepsilon_k, k \geq 1\}$  a sequence of independent Rademacher random variables (taking the values  $+1$  and  $-1$  with probability  $1/2$ ), defined on a third space  $(\Omega'', \mathcal{A}'', \mathbb{P}'')$ , independent of  $X$  and  $X'$ .  $\{f_k, k \geq 1\}$  and  $\{\varepsilon_k f_k, k \geq 1\}$  have the same law. Thus for all

integers  $(\Lambda, \lambda)$  such that  $\Lambda \geq \lambda$ ,  $\sum_{k=\lambda}^{\Lambda} f_k$  and  $\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k$  also have the same law.

That is why (12) can be written on a larger space of integration in the following way:

$$\mathbb{E}_{X, X', \varepsilon} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k(\alpha, j)}{\sqrt{A_{\Lambda, \lambda, M}^2 \log(|j| + 3)}} \right|$$

We deduce a sufficient condition for (12) to be realised

$$(13) \quad \mathbb{E}_{X, \varepsilon} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k a_k \exp 2i\pi\alpha j X_k}{\sqrt{A_{\Lambda, \lambda, M}^2 \log(|j| + 3)}} \right|$$

We then use a precious tool in the theory of gaussian random functions: the contraction principle. This tool is built on a quite simple idea: replace the choice of signs by a sequence of gaussian random variables with mean zero and variance one. This idea can be explained by the following property: given  $g$  a gaussian random variable with mean zero and variance 1 and  $\varepsilon$  a Rademacher random variable, if  $g$  and  $\varepsilon$  are independent, then  $g$  and  $\varepsilon|g|$  have the same law.

As a consequence, in order to prove (13), we show

$$\mathbb{E}_{X, g, g'} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} a_k (g_k \cos 2\pi\alpha j X_k + g'_k \sin 2\pi\alpha j X_k)}{\sqrt{A_{\Lambda, \lambda, M}^2 \log(|j| + 3)}} \right| < +\infty$$

where  $\{g_k, k \geq 1\}$  et  $\{g'_k, k \geq 1\}$  are two sequences of independent identically distributed random variables with law  $\mathcal{N}(0, 1)$ , independent of  $X$  and  $\varepsilon$ , defined on two other probabilised spaces.

Conditionally to  $X$ , the problem is reduced to studying the regularity of the trajectories of stationary gaussian random variables. This concludes the first step of the proof.

**-Step 2 :** In this part, we use the gaussian tools introduced in the beginning.

Conditionally to  $X$ , call  $G(\lambda, \Lambda, j, \alpha)$  the following quantity

$$\frac{1}{\sqrt{A_{\Lambda, \lambda, M}^2 \log(|j| + 3)}} \sum_{k=\lambda}^{\Lambda} a_k \left[ g_k \cos(2\pi\alpha j X_k) + g'_k \sin(2\pi\alpha j X_k) \right]$$

If  $j$ ,  $\lambda$  and  $\Lambda$  are fixed,  $G(\alpha) := G(\lambda, \Lambda, j, \alpha)$  is a random function with almost surely continuous trajectories (up to a modification of trajectories). That is why it is enough to show that  $G$  is bounded on  $I_M \cap \mathbb{Q}$ . Moreover, we will assume that  $|j| \leq J$  where  $J$  is a large fixed integer.

Let us begin by finding an upper bound for

$$\mathbb{E}_{g, g'} \sup_{|j| \leq J} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M \cap \mathbb{Q}} |G(\lambda, \Lambda, j, \alpha)|$$

First remark that if  $Y_t$  ( $t \in T$ ) is a gaussian random function defined on  $T$ , then for all  $t_0 \in T$  we have (see [4] (480))

$$\mathbb{E} \sup_{t \in T} |Y_t| \leq \mathbb{E} |Y_{t_0}| + \mathbb{E} \sup_{t \in T} Y_t$$

In this way, we get rid of the absolute value. Apply this remark to  $G(\lambda, \Lambda, j, \alpha)$  with  $\alpha = 0$ ,  $\lambda = \Lambda = 1$  and  $j = 0$  and let us find an upper bound for

$$(14) \quad \mathbb{E}_{g, g'} \sup_{|j| \leq J} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(\lambda, \Lambda, j, \alpha)$$

In order to apply the decoupling inequality 3.3, define

$$T = \{-J, \dots, J\} \times H \times (I_M \cap \mathbb{Q})$$

where  $J$  is a large enough integer and  $H$  is the upper triangle of dimension 2 in  $\mathbb{N} \times \mathbb{N}$  (see figure 1 below) ( $\lambda \in \mathbb{N}$  and  $\Lambda \geq \lambda$ .) A point in  $t \in T$  will be written  $t = (j, \lambda, \Lambda, \alpha)$ .

This set  $T$  is at most countable and we will find an upper bound for  $\mathbb{E}_{g, g'} \sup_{t \in T} G(t)$  independently of  $J$  and then conclude by taking the supremum on  $j \in J$ . Define

$$T_j = \{j\} \times H \times (I_M \cap \mathbb{Q})$$

It is obvious that  $\{T_j\}_{j=-J, \dots, J}$  is a covering of  $T$ . Define

$$S = T_{-J} \times \dots \times T_J$$

Using inequality 3.3, we have

$$\mathbb{E}_{g, g'} \sup_{t \in T} G(t) \leq \frac{\pi}{2} \sup_{s \in S} \mathbb{E}_{g, g'} \sup_{-J \leq j \leq J} G(s_j) + \sup_{-J \leq j \leq J} \mathbb{E}_{g, g'} \sup_{t \in T_j} G(t)$$

where  $s_j$  is a point in  $T_j$ .

**-Step 3 :** We study now

$$\sup_{s \in S} \mathbb{E} \sup_{-J \leq j \leq J} G(s_j).$$

We can rewrite this in the following way :

$$(15) \quad \sup_{g, g'} \mathbb{E} \sup_{-J \leq j \leq J} \frac{\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g'_k \sin(2\pi j \alpha_j X_k) \right]}{\sqrt{A_{\lambda_j, \Lambda_j, M}^2 \log(|j| + 3)}}$$

where the first supremum is taken on

$$\{(\alpha_{-J}, \dots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}, ((\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J)) \in H^{2J+1}\}$$

Fix  $\{(\alpha_{-J}, \dots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}$  and  $((\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J)) \in H^{2J+1}\}$ . Define the gaussian process

$$G_j := \frac{\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g'_k \sin(2\pi j \alpha_j X_k) \right]}{\sqrt{A_{\lambda_j, \Lambda_j, M}^2 \log(|j| + 3)}}$$

In order to get an upper bound for 15, we remark that

$$(16) \leq \sup_{((\lambda_{-J}, \lambda_{-J}), \dots, (\lambda_J, \lambda_J)) \in H^{2J+1}} \mathbb{E}_{g, g'} \sup_{-J \leq j \leq J} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)|$$

We will get an upper bound for the right hand side of 16 independently of  $J$ , which will give us an upper bound for 15 by taking the supremum on  $j \in J$ .

Applying inequality 3.1 to the finite sequence of random gaussian functions

$$(G(j, \lambda_j, \Lambda_j, \alpha))_{-J \leq j \leq J}$$

we prove that  $\mathbb{E}_{g, g'} \sup_{-J \leq j \leq J} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)|$  is less than

$$(17) \quad C \left\{ \sup_{-J \leq j \leq J} \mathbb{E}_{g, g'} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)| + \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| \right\}$$

where  $C$  is a universal constant,  $(\xi_j)_{-J \leq j \leq J}$  is an isonormal sequence and

$$q_j \leq \sup_{\alpha \in I_M} \|G(j, \lambda_j, \Lambda_j, \alpha)\|_{2, g, g'}$$

This gives us the following upper bound

$$q_j \leq \frac{1}{\sqrt{\log(M\Phi_\beta(\Lambda_j)) \log(|j| + 3)}}$$

As for all  $j$  we have  $\Lambda_j \geq 1$  and  $M \geq 1$  we easily get

$$(18) \quad \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| \leq \mathbb{E}_\xi \sup_{-J \leq j \leq J} \left| \frac{1}{\sqrt{\log(|j| + 3)}} \xi_j \right|$$

$$(19) \quad \leq C \mathbb{E}_\xi \sup_{j \in \mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log(|j| + 3)}} \right|$$

The exponential integrability of gaussian vectors gives

$$\mathbb{E}_\xi \sup_{j \in \mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log |j| + 3}} \right| = C < \infty$$

and hence

$$\sup_{(\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J) \in H^{2J+1}} \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| = C < \infty$$

independently of  $J$  and of the sequence  $(\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J) \in H^{2J+1}$ .

We then get independently of  $J$  on the whole integration space

$$\mathbb{E}_X \sup_{(\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J) \in H^{2J+1}} \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| \leq C < \infty$$

where  $C$  is a constant.



Let us now try to find an upper bound for

$$(20) \quad \sup_{((\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J)) \in H^{2J+1}} \sup_{-J \leq j \leq J} \mathbb{E}_{g, g'} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)|$$

independently of  $J$ .

We will use inequality 3.4. Let us choose a finite sequence

$$((\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J))$$

and an integer  $|j| \leq J$ . The gaussian random function  $G_j(\alpha) := G(j, \lambda_j, \Lambda_j, \alpha)$  is stationary. Its associated spectral measure on  $\mathbb{R}^+$  is defined by

$$m_j = \frac{1}{\log(|j| + 3) \log(M\Phi_\beta(\Lambda_j)) \left( \sum_{k=\lambda_j}^{\Lambda_j} |a_k|^2 \right)} \sum_{k=\lambda_j}^{\Lambda_j} |a_k|^2 \delta_{|X_k|}$$

where  $\delta_u$  is the Dirac measure in the point  $u$ .

We get

$$\begin{aligned} & \mathbb{E}_{g, g'} \sup_{\alpha \in I_M} |G_j(\alpha)| \\ & \leq C \left\{ \sqrt{\int_0^\infty \min\left(\frac{u^2}{2M}, 1\right) m_j(du)} + \int_0^\infty \sqrt{m_j\left(\left] \frac{e^{x^2}}{2M}, \infty[ \right)} dx \right\} \end{aligned}$$

It is obvious that the first term is less than

$$\sqrt{m_j(\mathbb{R}^+)} \leq C \frac{1}{\sqrt{\log(2M) \log(|j| + 3)}}$$

where  $C$  is a universal constant because for all  $j$  we have  $\Phi_\beta(\Lambda_j) \geq 2$ .

For the second term, it can be rewritten in the following way

$$\begin{aligned} & \int_0^\infty \sqrt{m_j\left(\left] \frac{\exp x^2}{2M}, \infty[ \right)} dx \\ & = \left[ \frac{1}{\log(|j| + 3) \log(M\Phi_\beta(\Lambda_j)) \left( \sum_{k=\lambda_j}^{\Lambda_j} |a_k|^2 \right)} \right]^{\frac{1}{2}} \\ & \times \int_0^\infty \sqrt{\sum_{k=\lambda_j}^{\Lambda_j} |a_k|^2 \mathbf{1}_{\{2M|jX_k| > e^{x^2}\}}} dx \end{aligned}$$

Using

$$\forall k \geq 1, \quad \mathbf{1}_{\{2M|jX_k| > e^{x^2}\}} \leq \mathbf{1}_{\{2M|j| \sup_{l \leq k} |X_l| > e^{x^2}\}}$$

we cut  $\mathbb{R}^+$  in the integral according to the increasing subdivision

$$\{0\} \cup \left\{ \sqrt{\log^+(2M|j| \sup_{l \leq k} |X_l|)}, \lambda_j \leq k \leq \Lambda_j \right\}$$

We get thus an upper bound for the previous integral

$$\begin{aligned}
& \sum_{k=\lambda_j}^{\Lambda_j} \left[ \sqrt{\log^+ (2M|j| \sup_{l \leq k} |X_l|)} - \sqrt{\log^+ (2M|j| \sup_{l \leq k-1} |X_l|)} \right] \left( \sum_{l=\lambda_j}^{\Lambda_j} |a_l|^2 \right)^{\frac{1}{2}} \\
& \leq \left( \sum_{l=\lambda_j}^{\Lambda_j} |a_l|^2 \right)^{\frac{1}{2}} \sum_{k=\lambda_j}^{\Lambda_j} \left[ \sqrt{\log^+ (2M|j| \sup_{l \leq k} |X_l|)} \right. \\
& \quad \left. - \sqrt{\log^+ (2M|j| \sup_{l \leq k-1} |X_l|)} \right] \\
& \leq 2 \left( \sum_{k=\lambda_j}^{\Lambda_j} |a_k|^2 \right)^{\frac{1}{2}} \sqrt{\log^+ (2M|j| \sup_{l \leq \Lambda_j} |X_l|)}
\end{aligned}$$

Consequently,  $\forall |j| \leq J$ ,

$$\begin{aligned}
\int_0^\infty \sqrt{m_j \left( \frac{\exp x^2}{M}, \infty \right)} dx & \leq 2 \sqrt{\frac{\log^+ (2M|j| \sup_{l \leq \Lambda_j} |X_l|)}{\log(|j| + 3) \log(M\Phi_\beta(\Lambda_j))}} \\
& \leq 4 \sqrt{\frac{\log^+ (2M \sup_{1 \leq l \leq \Lambda_j} |X_l|)}{\log(M\Phi_\beta(\Lambda_j))}} \\
& \leq 4 \sup_{1 \leq l \leq \Lambda_j} \sqrt{\frac{\log^+ (2M |X_l|)}{\log(M\Phi_\beta(l))}} \\
& \leq 4 \sup_{N \geq 1} \sqrt{\frac{\log^+ (2M |X_N|)}{\log(M\Phi_\beta(N))}} \\
& \leq 4 \left( 1 + \sup_{N \geq 1} \sqrt{\frac{\log^+ (|X_N|)}{\log \Phi_\beta(N)}} \right)
\end{aligned}$$

Finally, on the whole integration space, we get the following upper bound :

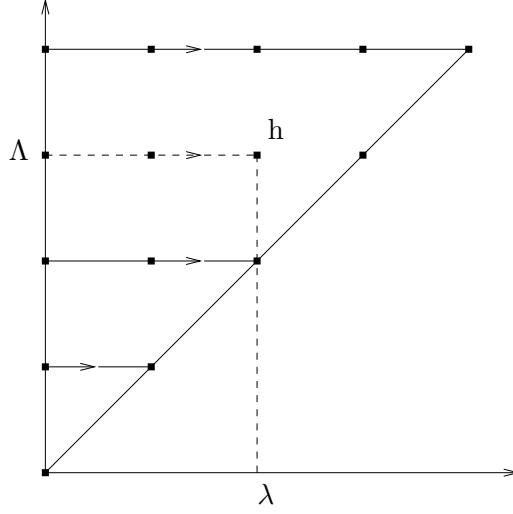
$$(21) \quad \mathbb{E}_X \sup_{(\lambda_{-J}, \Lambda_{-J}), \dots, (\lambda_J, \Lambda_J) \in H^{2J+1}} \sup_{-J \leq j \leq J} \mathbb{E}_{g, g'} \sup_{\alpha \in I_M} |G_j(\alpha)|$$

$$(22) \quad \leq C \left( 1 + \mathbb{E}_X \sup_{N \geq 1} \sqrt{\frac{\log^+ (|X_N|)}{\log \Phi_\beta(N)}} \right) < \infty$$

**-Step 4 :** To end the proof of theorem 1.5, it remains to deal with

$$\sup_{-J \leq j \leq J} \mathbb{E}_{g, g'} \sup_{(\lambda, \Lambda) \in H} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, \lambda, \Lambda, \alpha)$$

Let us fix  $j$ . In order to apply inequality 3.1, we need to replace the supremum on  $(\lambda, \Lambda) \in H$  by a supremum on only one variable, in other words,

FIGURE 1. renumbering the upper triangle in  $\mathbb{N}^2$ 

we need to renumber  $H$  using one variable  $h \in \mathbb{N}^2$  given by the formula :

$$h = \lambda + \frac{\Lambda(\Lambda - 1)}{2}$$

(see figure 1).

Let us fix  $j$ . Inequality 3.1 gives us :

$$\mathbb{E}_{g,g'} \sup_{h \geq 1} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, h, \alpha) \leq K_1 \left( \sup_{h \geq 1} \mathbb{E}_{g,g'} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, h, \alpha) + \mathbb{E}_{g,g'} \sup_{h \geq 1} |\lambda_h \sigma_h| \right)$$

where  $(\lambda_h)_{h \geq 1}$  is an isonormal sequence. The inequality :

$$\frac{\Lambda(\Lambda - 1)}{2} \leq h \leq \frac{\Lambda(\Lambda + 1)}{2}$$

gives a polynomial dependence between  $\Lambda$  and  $h$ , hence :

$$\sigma_h = \mathcal{O}\left(\frac{1}{\sqrt{\log h}}\right)$$

and the first term in the right hand side of inequality (23) is dealt with in the same way as before. Finally, we get :

$$\mathbb{E}_{g,g'} \sup_{(\lambda, \Lambda) \in H} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, \lambda, \Lambda, \alpha) \leq C \left( 1 + \sup_{N \geq 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}} \right)$$

That is to say, by integrating on the whole space,

(24)

$$\mathbb{E}_X \sup_{1 \leq j \leq n} \mathbb{E}_{g,g'} \sup_{(\lambda, \Lambda) \in H} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, \lambda, \Lambda, \alpha) \leq C \left( 1 + \mathbb{E}_X \sup_{N \geq 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}} \right)$$

Let us prove that

$$\mathbb{E}_X \sup_{N \geq 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}} < \infty$$

Using Jensen inequality, we can get rid of the square root. Let  $\delta > 0$ . For all  $N \geq 1$ , we have

$$\beta \log^+ |X_N| \leq \beta \log^+ \left[ \frac{|X_N|}{\Phi_\beta^\delta(N)} \right] + \beta \log^+ [\Phi_\beta^\delta(N)]$$

noticing that  $\Phi_\beta^\delta(N) \geq 2$  it is sufficient to show

$$\mathbb{E} \sup_{N \geq 1} \log^+ \left[ \frac{|X_N|^\beta}{\Phi_\beta^{\delta\beta}(N)} \right] < \infty$$

Using now the inequality  $\log^+(x) \leq x$  for any  $x \geq 0$ , it is sufficient to prove

$$\sum_{N \geq 1} \frac{\mathbb{E}|X_N|^\beta}{\Phi_\beta^{\delta\beta}(N)} < \infty$$

And as  $\mathbb{E}|X_N|^\beta \leq \Phi_\beta(N)$  and  $\Phi_\beta(N) \geq N$ , if we chose  $\delta = \frac{3}{\beta}$  we get the conclusion. Steps 3 (see 21), step 4 (see 24) lead us to the announced result of theorem 1.5.

#### 4. APPLICATIONS

Let us begin by giving an example where the  $(X_k)$  are uniformly distributed :

**Example 4.1.** Suppose that  $\mathcal{L}(X_k) = \mathcal{U}([\mu_k - \sigma_k/2, \mu_k + \sigma_k/2])$  with  $\sigma_k > 0$  et  $\mathbb{E}(X_k) = \mu_k$  with  $\mu_k = \mathcal{O}(k^d)$  for some  $d > 0$ . The characteristic function of  $X_k$  can easily be computed :

$$\varphi_{X_k}(t) = \frac{e^{2i\pi t\mu_k}}{\pi t\sigma_k} \sin(\pi t\sigma_k)$$

Using condition  $\mathcal{H}$  of theorem 1.1, the following condition

$$(25) \quad \sum_{n \geq 1} \frac{|a_n|}{\sigma_n} \text{ converges}$$

and

$$\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n\sqrt{\log n}} < +\infty$$

are sufficient to get the desired convergence.

Notice that using corollary 1.3, condition (25) is replaced by

$$\sum_{n \geq 1} \frac{1}{\sigma_n^2} < +\infty$$

The subexponential case could be dealt with in the same way.  
If we consider the border case  $a_k = \mathcal{O}(k^{-1/2-\varepsilon})$ , it is sufficient that :

$$\exists \eta > 0, \sigma_k \geq k^{\frac{1}{2}+\eta}$$

in this case :

$$\mathbb{P}\left\{\forall k, X_k \in \left[\mu_k - \frac{k^{\frac{1}{2}+\eta}}{2}, \mu_k + \frac{k^{\frac{1}{2}+\eta}}{2}\right]\right\} = 1$$

which gives an information on the possible dispersion of the variables  $X_k$ .

Here are other examples where the conditions of our theorems can be quite easily verified.

**Corollary 4.1.** *Let  $(X_k)_{k \geq 1}$  be a sequence of real independent random variables whose law can be written in the following way for all  $k \geq 1$  :  $\mathcal{L}(X_k) = \mathcal{L}(\sigma_k \cdot X + \mu_k)$  where  $X$  verifies  $\mathbb{E}|X|^\beta < \infty$  for some  $\beta > 0$ . Moreover, we assume that there exist  $d > 0$  and  $\delta > 0$  such that  $|\sigma_k| = \mathcal{O}(k^d)$ ,  $|\mu_k| = \mathcal{O}(k^d)$ , the application  $t \mapsto t^\delta \mathbb{E} \exp(2i\pi t X)$  is bounded on  $\mathbb{R}$ , and let  $(a_k)_{k \geq 1}$  be a sequence of real or complex numbers satisfying the following two conditions*

- (1)  $|a_k| = \mathcal{O}(k^{-\beta})$  with  $\beta > 1/2$
- (2)  $\sum_{k=1}^{\infty} \frac{1}{|\sigma_k|^{2\delta}} < \infty$

Then there exists a measurable set  $\Omega_o$  with full measure ( $\mathbb{P}(\Omega_o) = 1$ ) such that for any  $\omega \in \Omega_o$  for all  $f \in B(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$  : for any compact  $K$  which does not contain 0, the application  $t \in K \mapsto F(t) = \sum_{k \geq 1} a_k f(t X_k(\omega))$  is continuous and the series defining  $F$  converges uniformly on  $K$ .

The proof of corollary 4.1 relies on corollary 1.3.

**Example 4.2.** *The random variable  $X$  may have a gaussian law with mean zero and variance one, a Cauchy law, the first Laplace law, an exponential law with parameter  $\lambda > 0$ . Let us precise the gaussian case.*

Here  $\mathcal{L}(X) = \mathcal{N}(0, 1)$ , we have  $\mathbb{E} \exp 2i\pi t X = e^{-t^2/2}$ . Hence we can use the fact that  $t \mapsto e^{t^2/2} \mathbb{E} \exp 2i\pi t X$  is bounded on  $\mathbb{R}$ . In this case, the sufficient condition to obtain convergence is

$$\forall \varepsilon > 0, \exists N > 0, \sup_{m > n \geq N} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \left| \sum_{k=n}^m a_k e^{2i\pi \alpha \mu_k j} e^{-j^2 \alpha^2 \sigma_k^2 / 2} \right| < \varepsilon$$

Let  $d(0, K)$  be the distance between 0 and the compact  $K$ . Using the fact that  $|a_k| = \mathcal{O}(k^{-\beta})$  with  $\beta > 1/2$ , the previous condition will be satisfied as soon as :

$$\exists \varepsilon > 0, \sigma_k \geq \frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0, K)} \sqrt{\log k}$$

which, in terms of dispersion of the variables  $X_k$  means that infinitely often:

$$X_k \in \left[ \mu_k - 3 \frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0, K)} \sqrt{\log k}, \mu_k + 3 \frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0, K)} \sqrt{\log k} \right]$$

We discuss now the case when the laws of  $X_k$  are generated by a convolution product of a given law  $\mu$ . We distinguish two cases : on one hand when the support of  $\mu$  contains non integer values, on the other hand when the support of  $\mu$  is contained in  $\mathbb{Z}$ . The first case is discribed by the following corollary :

**Corollary 4.2.** *Let  $(X_k)_{k \geq 1}$  be an sequence of real valued independent random variables such that for all integer  $k \geq 1$ ,  $\mathcal{L}(X_k) = \mu^{*k}$  where  $\mu$  is a probability measure on  $\mathbb{R}$  with  $\mathbb{E}|X_1|^\delta < \infty$  for some  $\delta$ . Assume the following :*

- (a)  $\varphi_{X_1}(t) = 1 \iff t = 0$  ( $X_1$  aperiodic)
- (b)  $\exists \delta > 0$ ,  $\sup_{t \in \mathbb{R}} |t|^\delta \mathbb{E} \exp(2i\pi t X_1) = q < \infty$

Let  $(a_k)_{k \geq 1}$  be a sequence of real or complex numbers such that the sequence  $|a_k|$  is decreasing and fulfills the two following conditions :

- (1)  $|a_k| = O(k^{-\beta})$  avec  $\beta > 1/2$   
(2)  $\sum_{k=1}^{\infty} |a_k - a_{k+1}| < \infty$

Then there exists a measurable set  $\Omega_o$  with full measure ( $\mathbb{P}(\Omega_o) = 1$ ) such that for any  $\omega \in \Omega_o$ , for all  $f \in B(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$ , for any compact  $K$  which does not contain 0, the application  $t \in K \mapsto F(t, \omega) = \sum_{k \geq 1} a_k f(t X_k(\omega))$  is continuous and the series defining  $F$  converges uniformly on  $K$ .

**Remark 4.1.** *If  $X_1$  is strictly aperiodic ( $|\varphi_{X_1}(t)| = 1 \iff t = 0$ ), then the condition on the differences  $|a_k - a_{k+1}|$  may be removed, using corollary 1.3 and the same kind of computation as in the following proof.*

The random variable  $X_1$  being real valued, its characteristic function is not periodic. Take for example a gaussian law with mean zero and variance one.

**Proof :** Let  $K$  be a compact which does not contain 0. Using Abel 's summation, it is sufficient to prove

$$\sup_{t \in K} \sup_{|j| \geq 1} \left| \sum_{k=1}^N (\mathbb{E} \exp 2i\pi j t X_1)^k \right| < \infty$$

independently of  $N$ . Let us split the supremum on  $j$  respectively into the supremum on the indexes  $J(q)$  and  $\bar{J}(q)$  where  $J(q) = \{j \in \mathbb{Z}^* : |j|^\delta \leq \left\lceil \frac{2q}{\varepsilon^\delta} \right\rceil\}$  and  $2\varepsilon$  is the distance between 0 and the fixed compact  $K$ .

On one hand, using (a), it can be proved that :

$$\forall \varepsilon > 0 \quad \inf_{|t| > \varepsilon} |t|^\delta |1 - \mathbb{E} \exp(2i\pi t X_1)| > 0$$

this implies

$$\sup_{t \in K} \sup_{j \in J(q)} \left| \sum_{k=1}^N (\mathbb{E} \exp 2i\pi jt X_1)^k \right| \leq \sup_{t \in K} \sup_{j \in J(q)} C(\varepsilon) |jt|^\delta \leq C(K) \left[ \frac{2q}{\varepsilon^\delta} \right]$$

On the other hand, using (b),

$$\begin{aligned} \sup_{t \in K} \sup_{j \in J(q)} \left| \sum_{k=1}^N (\mathbb{E} \exp 2i\pi jt X_1)^k \right| &\leq \sup_{t \in K} \sup_{j \in J(q)} \sum_{k=1}^N \left( \frac{q}{|tj|^\delta} \right)^k \\ &\leq C \sum_{k=1}^N \frac{1}{2^k} \leq 2C \end{aligned}$$

where  $C$  is a universal constant.

□

As for the integer valued case, we have :

**Corollary 4.3.** *Let  $(X_k)_{k \geq 1}$  be an sequence of integer valued independent random variables such that for all integer  $k \geq 1$ ,  $\mathcal{L}(X_k) = \mu^{*k}$  where  $\mu$  is a probability measure on  $\mathbb{R}$  with  $\mathbb{E}|X_1|^\beta < \infty$  for some  $\beta > 0$ . Let  $(a_k)_{k \geq 1}$  be a sequence of complex numbers such that  $|a_k| = O(k^{-\beta})$  with  $\beta > 1/2$ .*

*Assume either :*

$$\varphi_{X_1}(t) = 1 \iff t = 0 \quad \text{and} \quad \sum_{k \geq 1} |a_k - a_{k+1}| \quad \text{converges}$$

*or :*

$$|\varphi_{X_1}(t)| = 1 \iff t = 0$$

*Then there exists a measurable set  $\Omega_o$  with full measure ( $\mathbb{P}(\Omega_o) = 1$ ) such that for any  $\omega \in \Omega_o$ , for all  $f \in A(\mathbb{T})$  such that  $\int_{\mathbb{T}} f(t) dt = 0$ , for any compact  $K$  of the torus which does not contain 0, the application  $t \in K \mapsto F(t, \omega) = \sum_{k \geq 1} a_k f(t X_k(\omega))$  is continuous and the series defining  $F$  converges uniformly on  $K$ .*

**Example 4.3.** *If the law of  $X_1$  is a Poisson law with parameter 1, we use :*

$$\forall t \in \mathbb{T}, |\varphi_{X_1}(t)| \leq e^{\cos(2\pi t) - 1}$$

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