# Random series of functions and applications

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# RANDOM SERIES OF FUNCTIONS AND APPLICATIONS

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Abstract. In this article we study the continuity properties of trajectories for some random series of functions, $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$ where $(a_k)_{k\geqslant 0}$ is a complex sequence, $(X_k)_{k\geqslant 0}$ is a sequence of real independent random variables, $f$ is a real valued function with period one and summable Fourier coefficients. We obtain almost sure continuity results for these periodic or almost periodic series for a large class of functions $f$, where the "almost sure" does not depend on the function. The proof relies on gaussian randomization. We show optimality of the results in some cases.

# SERIES DE FONCTIONS ALEATOIRES ET APPLICATIONS

Abstract. (Résumé) Dans ce travail, nous étudions des propriétés de continuité de trajectoires de séries de fonctions aléatoires du type $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$ où $(a_k)_{k\geqslant 0}$ est une suite de nombres complexes, $(X_k)_{k\geqslant 0}$ une suite de variables aléatoires réelles et indépendantes, $f$ une fonction 1-périodique à coefficients de Fourier sommables. Nous montrons que, presque sûrement, ces séries de fonctions aléatoires (périodiques ou presque périodiques) sont à trajectoires continues pour une grande classe de fonctions $f$. Le "presque sûr" est indépendant de $f$. Les preuves s'appuient sur un procédé de randomisation gaussien. Dans certains cas, nous montrerons l'optimalité des résultats obtenus.



**Keywords** : random fourier series, almost sure continuity of trajectories, gaussian randomization, almost periodic functions, random trigonometric polynomial

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## 1. Introduction. Main results

In [1], Berkes studies the almost sure convergence of series defined by :

$$\sum_{k\geqslant 1} a_k f(\alpha\, n_k)$$

where the sequence $(n_k)$ is lacunary and the function $f$ verifies :

$$f(x+1)=f(x) \qquad \int_0^1 f(x)dx=0 \qquad \int_0^1 f^2(x)dx=1$$

He shows that the important property of $f$ to ensure the almost sure convergence is $f\in \mathrm{Lip}(\gamma)$ with $\gamma>1/2$ and $\sum_{k\geqslant 1}|a_k|^2<+\infty$. In his case, the $n_k$ are strictly lacunary, more precisely, they satisfy the Hadamard gap condition :

$$\frac{n_{k+1}}{n_k}\geqslant q>1$$

We can naturally adress the question whether the convergence still holds when $n_k$ is polynomial, and for which class of functions. We are going to answer the question when the sequence $(n_k)$ is randomly generated.
Let us mention that the result exists when $(n_k)$ is a deterministic polynomial sequence and $(a_k)$ is randomly distributed (see [5] and [6]).
We want to study the convergence properties of series of functions sampled by a random process. More precisely, consider the torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ and define $A(\mathbb{T})$ as the set of complex valued functions whose Fourier coefficients are absolutely summable :

$$A(\mathbb{T})=\{f:\mathbb{T}\to\mathbb{C}, f(\alpha)=\sum_{j\in\mathbb{Z}}\hat{f}(j)\exp(2i\pi\alpha j), \sum_{j\in\mathbb{Z}}|\hat{f}(j)|<+\infty\}$$

$(a_k)_{k\geqslant 0}$ will denote a sequence of real numbers and $(X_k)_{k\geqslant 0}$ a sequence of independent real random variables defined on the probabilised space $(\Omega,\mathcal{A},\mathbb{P})$. Our aim is to study the convergence, when $\omega\in\Omega$ is fixed, of the series of functions

$$\forall\alpha\in\mathbb{R}, \qquad F(\alpha,\omega)=\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$$

Is it possible to give conditions on the sequence $(a_k)_{k\geqslant 0}$ in order to find a $\mathcal{A}-$measurable set $\Omega_0$ independent of the function $f$, such that $\mathbb{P}(\Omega_0)=1$, on which the series uniformly converges?
Note that when $X_k$ does not take integer values, $F$ is not a periodical function of the torus. For us, $\alpha$ will be real and we will deal with this "almost periodical" case. That is why we have to study the properties of $F$ on a compact $[-M,M]$ and not only $[0,1]$ (see for example [3])
For all $f\in A(\mathbb{T})$, define

$$||f||:=\sum_{j\in\mathbb{Z}}|\hat{f}(j)|<+\infty\,.$$

$$|||f||| := \sum_{j\in\mathbb{Z}} |\hat{f}(j)|\sqrt{\log(|j|+3)} < +\infty\,.$$

and

$$B(\mathbb{T}) = \{f : \mathbb{T}\to\mathbb{C}, f(\alpha) = \sum_{j\in\mathbb{Z}} \hat{f}(j)\exp(2i\pi\alpha j), |||f||| < +\infty\}$$

Remark that $B(\mathbb{T}) \subset A(\mathbb{T})$.
In the following, we will give conditions for $\alpha \mapsto F(\alpha,\omega)$ to have continuous trajectories $\mathbb{P}-$almost surely.
We will denote by $\varphi_X$ the characteristic function of the random variable $X$

$$\forall t\in\mathbb{R}, \varphi_X(t) = \mathbb{E}(e^{2i\pi tX})$$

**Theorem 1.1.** *Let $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables and let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers such that, for any compact $K$ which does not contain $0$:*

$$(\mathcal{H}) \qquad \forall\varepsilon>0, \exists N>0, \sup_{m>n\geqslant N}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}-\{0\}}\left|\sum_{k=n}^{m} a_k\varphi_{X_k}(j\alpha)\right| < \varepsilon\,.$$

*Assume moreover that:*
**case 1:(polynomial)** *there exists $\beta>0$ and $d>0$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(k^d)$ and*

$$\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n\sqrt{\log n}} < +\infty \tag{1}$$

**case 2:(subexponential)** *there exists $\beta>0$ and $\gamma\in]0,1[$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(2^{k^\gamma})$ and*

$$\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n^{1-\frac{\gamma}{2}}} < +\infty \tag{2}$$

*then in both cases, there exists a measurable set $\Omega_0$ with $\mathbb{P}(\Omega_0)=1$ such that for all $\omega\in\Omega_0$, for any $f\in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$ : for $\alpha\in\mathbb{R}-\{0\}$, $F(\alpha,\omega)$ is well defined, $\alpha\mapsto F(\alpha,\omega)$ is continuous and the series defining $F$ converges uniformly on every compact which does not contain $\{0\}$.*

**Remark 1.1.**

1. *It is worth noticing that the set $\Omega_0$ does not depend on the class of functions $f$ ($A(\mathbb{T})$ or $B(\mathbb{T})$).*
2. *when $(X_k)_{k\geqslant 0}$ takes integer values, condition $|||f||| < \infty$ becomes $||f|| < \infty$.*
3. *we will give conditions on the law of the process $(X_k)_{k\geqslant 0}$ to fulfill hypothesis $(\mathcal{H})$.*
4. *For example, when $|a_k| = \mathcal{O}(k^{-\delta})$, in case 1, if $\delta > 1/2$, then condition 1 holds and in case 2, if $\delta > \frac{\gamma+1}{2}$, then condition 2 holds.*

(5) *Concerning case 2, if $\gamma \geqslant 1$ ($\mathbb{E}|X_k|^\beta$ growths exponentially), one can prove using remark 2.1 that the series $\sum a_k$ has to converge. The function $F$ is then obviously well defined using only Cauchy Schwarz inequality.*

In case condition $(\mathcal{H})$ is hard to check, it is possible to split up the hypothesis on the sequence $(a_k)$ and the characteristic function $\varphi_{X_k}$ either using Abel's summation method or using Cauchy Schwarz inequality.
Define:

$$c_n = \begin{cases} 1+\sqrt{\log n} & \text{in the polynomial case} \\ n^{\frac{\gamma}{2}} & \text{in the subexponential case} \end{cases}$$

**Corollary 1.2.** *Let $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables*

*Assume that, for any compact $K$ which does not contain $0$ :*

$$(\mathcal{H}') \qquad \sup_{N\geqslant 1}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}-\{0\}}\left|\sum_{k=0}^{N}\varphi_{X_k}(j\alpha)\right| < \infty\,.$$

*Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers enjoying the following properties*

(1) $\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{nc_n} < +\infty$
(2) $\sum_{k\geqslant 1}|a_k - a_{k+1}|$ *converges*

*then there exists a measurable set $\Omega_0$ with $\mathbb{P}(\Omega_0)=1$ such that for all $\omega\in\Omega_0$, for any $f\in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$: for $\alpha\in\mathbb{R}-\{0\}$, $F(\alpha,\omega)$ is well defined, $\alpha\mapsto F(\alpha,\omega)$ is continuous and the series defining $F$ converges uniformly on every compact which does not contain $\{0\}$.*

**Corollary 1.3.** *Let $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables*

*Assume that, for any compact $K$ which does not contain $0$ :*

$$(\mathcal{H}'') \qquad \forall\varepsilon>0, \exists N>0, \sup_{m>n\geqslant N}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}-\{0\}}\left(\sum_{k=n}^{m}|\varphi_{X_k}(j\alpha)|^2\right) < \varepsilon\,.$$

*Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers enjoying*

$$\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{nc_n} < +\infty$$

*then there exists a measurable set $\Omega_0$ with $\mathbb{P}(\Omega_0)=1$ such that for all $\omega\in\Omega_0$, for any $f\in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$: for $\alpha\in\mathbb{R}-\{0\}$, $F(\alpha,\omega)$ is well defined, $\alpha\mapsto F(\alpha,\omega)$ is continuous and the series defining $F$ converges uniformly on every compact which does not contain $\{0\}$.*

**Remark 1.2.**
*The previous corollaries will be useful for example when the law of $X_k$ is obtained by convolution product (see corollary 4.2). As the condition*

$\sum_{k\geqslant 1}|a_k-a_{k+1}|$ *is often hard to check, corollary 1.3 is sometimes better to use.*

The proof of theorem 1.1 will start by looking separately at $F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ and $\mathbb{E}(F(\alpha,.))$. It turns out that hypothesis $(\mathcal{H})$ will be used only to deal with the expectation. That is why we think interesting to state the result for:

$$F(\alpha,\omega)-\mathbb{E}(F(\alpha,.)):=\sum_k a_k[f(\alpha X_k(\omega)-\mathbb{E}(f(\alpha X_k))$$

**Theorem 1.4.** *Let $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables such that there exists $\beta>0$ and $d>0$ with $\mathbb{E}|X_k|^\beta=\mathcal{O}(k^d)$ or $\gamma\in]0,1[$ with $\mathbb{E}|X_k|^\beta=\mathcal{O}(2^{k^\gamma})$. Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers enjoying the following property*

$$\sum_{n\geqslant 1}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{nc_n}<+\infty$$

*then there exists a measurable set $\Omega_0$ with $\mathbb{P}(\Omega_0)=1$ such that for all $\omega\in\Omega_0$, for any $f\in B(\mathbb{T})$ such that $\int_{\mathbb{T}}f(t)dt=0$ : for $\alpha\in\mathbb{R}$, $F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ is well defined, $\alpha\mapsto F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ is continuous, the series defining $F-\mathbb{E}(F)$ converges uniformly on every compact and there exists $C_\omega>0$ such that for all $\alpha\in\mathbb{R}$:*

$$|F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))|\leqslant C_\omega|||f|||\sqrt{\log(|\alpha|+2)}$$

**Remark 1.3.**

(1) *We also discuss the optimality of hypothesis on $(a_k)$ of theorem 1.4 in section 2.*
(2) *We also have:*

$$\mathbb{E}\sup_{T>1}\frac{\sqrt{\int_0^T|F(t,\omega)-\mathbb{E}(F(t,.))|^2dt}}{\sqrt{T\log T}}<\infty$$

This result relies on uniform estimations of the size of some trigonometric polynomials, more precisely on the following :
Recall that $\log^+=\max(\log,0)$.

**Theorem 1.5.** *Let $\lambda$ and $\Lambda$ be two integers with $\lambda\leqslant\Lambda$, $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables such that there exists $\beta>0$ such that, $\forall N\geqslant 0$, $\mathbb{E}|X_N^\beta|<\infty$. Define*

$$\forall N\geqslant 0,\Phi_\beta(N)=2+\max(N,\mathbb{E}|X_N^\beta|)$$

*Let $M\geqslant 1$ and $I_M=[-M,M]$. Let $(a_k)_{k\geqslant 1}$ be a sequence of real or complex numbers.*

*Define*

$$A_{\lambda,\Lambda,M}=\sqrt{\log\left(M\Phi_\beta(\Lambda)\right)\sum_{k=\lambda}^{\Lambda}|a_k|^2},$$

*then*

$$\mathbb{E}\sup_{j\in\mathbb{Z}}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha jX_k(\omega)-\mathbb{E}\exp 2i\pi\alpha jX_k\right]}{\sqrt{A^2_{\lambda,\Lambda,M}\log\left(|j|+3\right)}}\right|<\infty$$

**Remark 1.4.** *When$(X_k)_{k\geqslant 0}$ takes integer values, the proof of theorem 1.5 is easier. Namely, using the fact that $\alpha\mapsto j\alpha$ (mod 1) is onto for $j\neq 0$, we get*

$$\sup_{j\in\mathbb{Z}^*}\sup_{\alpha\in\mathbb{T}}\left|\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha jX_k(\omega)-\mathbb{E}\exp 2i\pi\alpha jX_k\right]\right|$$

$$=\sup_{\alpha\in\mathbb{T}}\left|\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha X_k(\omega)-\mathbb{E}\exp 2i\pi\alpha X_k\right]\right|$$

*the result of theorem 1.5 becomes then :*

$$\mathbb{E}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda}a_k\left[\exp 2i\pi\alpha X_k(\omega)-\mathbb{E}\exp 2i\pi\alpha X_k\right]}{\sqrt{A^2_{\lambda,\Lambda,M}}}\right|<\infty$$

*When $(X_k)_{k\geqslant 1}$ takes real values, the proof is more tedious. It relies on a fine inequality about decoupling gaussian random functions (see section 3.). We can see here why, for integer-valued $X_k$, we can work with the functional space $A(\mathbb{T})$, whereas for real-valued $X_k$, we need to introduce the space $B(\mathbb{T})$.*

## 2. Proof of theorem 1.1 and corollary 1.2

First, we split $F$ into two parts as follows :

$$\sum_k a_k f(\alpha X_k(\omega))=\sum_k a_k[f(\alpha X_k(\omega)-\mathbb{E}(f(\alpha X_k))+\sum_k a_k\mathbb{E}(f(\alpha X_k))$$

**-Step 1 :** (first part of the sum)
Let $(N_k)_{k\geqslant 1}$ be a strictly increasing sequence of integers and define

$$\forall k\geqslant 1,\qquad P_k(\alpha)=\sum_{l=N_k+1}^{N_{k+1}}a_l\left[f(\alpha X_l(\omega))-\mathbb{E}f(\alpha X_l)\right]$$

where $f\in B(\mathbb{T})$. We want to study the following series, for all $M\geqslant 1$ :

$$\sum_k\sup_{\alpha\in[-M,M]}|P_k(\alpha)|$$

We have :

$$|P_k(\alpha)|\leq\sum_{j\in\mathbb{Z}}|\hat{f}(j)|\left|\sum_{l=N_k+1}^{N_{k+1}}a_l[\exp\left(2\pi j\alpha X_l(\omega)\right)-\mathbb{E}\exp\left(2\pi j\alpha X_l\right)]\right|$$

Hence, using theorem 1.5, there exists a positive integrable random variable $\xi$ such that

$$\sup_{\alpha\in[-M,M]} |P_k(\alpha)| \leq \xi |||f||| \sqrt{\log\left(M\Phi_\beta(N_{k+1})\right) \sum_{j=N_k+1}^{N_{k+1}} |a_j|^2} \tag{3}$$

where

$$\xi = \sup_{j\in\mathbb{Z}} \sup_{k\geqslant 1} \sup_{\alpha\in I_M} \left| \frac{1}{\sqrt{A_{k,M}^2 \log\left(|j|+3\right)}} \sum_{l=N_k+1}^{N_{k+1}} a_l \left[ e^{2i\pi\alpha j X_l(\omega)} - \mathbb{E}e^{2i\pi\alpha j X_l} \right] \right|$$

with

$$A_{k,M}^2 = \log\left(M\Phi_\beta(N_{k+1})\right) \sum_{l=N_k+1}^{N_{k+1}} |a_l|^2$$

First, in the polynomial case, that is to say when there exists $d>0$ with

$$\Phi_\beta(N) = \mathcal{O}(N^d)$$

then we choose $N_k = 2^{2^k}$ and we need to prove that

$$\sum_k 2^{k/2} \left( \sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} |a_l|^2 \right)^{1/2} < +\infty$$

now we use the following equivalent:

$$\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} \frac{1}{l(\log(l))^{1/2}} \approx 2^{k/2}$$

which may be computed by comparing series and integral, hence :

$$\begin{aligned} 2^{k/2} \left( \sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} |a_l|^2 \right)^{1/2} &\leqslant C \sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} \frac{1}{l(\log(l))^{1/2}} \left( \sum_{l=2^{2^k}+1}^{\infty} |a_l|^2 \right)^{1/2} \\ &\leqslant C \sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} \frac{\left(\sum_{j=l}^{\infty} |a_j|^2\right)^{1/2}}{l(\log(l))^{1/2}} \end{aligned}$$

and, using condition 1 :

$$\begin{aligned} \sum_k 2^{k/2} \left( \sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} |a_l|^2 \right)^{1/2} &\leqslant \sum_k C \sum_{l=2^{2^k}+1}^{2^{2^{k+1}}} \frac{\left(\sum_{j=l}^{\infty} |a_j|^2\right)^{1/2}}{l(\log(l))^{1/2}} \\ &\leqslant \sum_{n\geqslant 2} \frac{\sqrt{\sum_{k\geqslant n} |a_k|^2}}{n\sqrt{\log n}} < +\infty \end{aligned}$$

this implies :

$$\sum_{k\geqslant 1}\sup_{\alpha\in[-M,M]}|P_k(\alpha)|<\infty$$

almost everywhere on the measurable set $\Omega_o=\{\omega\in\Omega,\xi(\omega)<\infty\}$. By construction, this set does not depend on the choice of $f$.
Secondly, in the subexponential case, that is when there exists $\gamma\in]0,1[$ with

$$\Phi_\beta(N)=\mathcal{O}(2^{N^\gamma})$$

we choose $N_k=2^k$ and we need to prove that

$$\sum_k 2^{\gamma k/2}\left(\sum_{l=2^k+1}^{2^{k+1}}|a_l|^2\right)^{1/2}<+\infty$$

Using the following equivalent:

$$\sum_{l=2^k+1}^{2^{k+1}}\frac{1}{l^{1-\frac{\gamma}{2}}}\approx 2^{\gamma k/2}$$

and doing the same kind of computation as before, using condition 2:

$$\sum_{n\geqslant 2}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n^{1-\frac{\gamma}{2}}}<+\infty$$

implies

$$\sum_{k\geqslant 1}\sup_{\alpha\in[-M,M]}|P_k(\alpha)|<\infty$$

We also get from (3), for all $\alpha\in\mathbb{R}$:

$$|P_k(\alpha)|\leqslant\xi|||f|||\sqrt{\log(|\alpha|+1)}\sqrt{\log(\Phi_\beta(N_{k+1}))\sum_{j=N_k+1}^{N_{k+1}}|a_j|^2}$$

summing on $k$, we get the inequality:

$$|F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))|\leqslant C\xi(\omega)|||f|||\sqrt{\log(|\alpha|+2)}$$

where $C$ only depends on $(a_k)$ and $\mathbb{E}(\xi)<\infty$
This ends the proof of theorem 1.4 which is also the first step of the proof of theorem 1.1

**-Step 2 :** (second part of the sum)
Let $K$ be a compact which does not contain zero and $\alpha\in K$. Let $n<m$ be

two integers,

$$|\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))| = |\sum_{j\in\mathbb{Z}^*}\sum_{k=n}^{m} a_k \hat{f}(j)\mathbb{E}(\exp(2i\pi j\alpha X_k))| \tag{4}$$

$$= |\sum_{j\in\mathbb{Z}^*}\sum_{k=n}^{m} a_k \hat{f}(j)\varphi_{X_k}(j\alpha)| \tag{5}$$

$$= |\sum_{j\in\mathbb{Z}^*} \hat{f}(j)(\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha))| \tag{6}$$

$$\leqslant (\sum_{j\in\mathbb{Z}^*} |\hat{f}(j)|) \sup_{j\in\mathbb{Z}^*} |\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha)| \tag{7}$$

At this point, to prove theorem 1.1, we can conclude directly by using hypothesis $(\mathcal{H})$ to get

$$\sup_{n<m}\sup_{\alpha\in K} |\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))| < \varepsilon$$

as long as $m$ and $n$ are large enough.
To prove corollary 1.2, we use Abel's summation. Let $\phi_p = \sum_{k=0}^{p-1} \varphi_{X_k}$, we have :

$$\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) = \sum_{k=n}^{m} a_k(\phi_{k+1}(j\alpha) - \phi_k(j\alpha)) \tag{8}$$

$$= \sum_{k=n+1}^{m+1} a_{k-1}\phi_k(j\alpha) - \sum_{k=n}^{m} a_k\phi_k(j\alpha) \tag{9}$$

$$= -a_n\phi_n(j\alpha) + a_m\phi_{m+1}(j\alpha) \tag{10}$$

$$+ \sum_{k=n+1}^{m} (a_{k-1} - a_k)\phi_k(j\alpha) \tag{11}$$

and :

$$(7) \leqslant |||f||| \sup_{N\geqslant 1}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}^*} |\phi_N(j\alpha)| \left[|a_m| + |a_n| + \sum_{k=n+1}^{m} |a_k - a_{k-1}|\right]$$

we now conclude using hypothesis $\mathcal{H}'$ and hypothesis (1) on the sequence $(a_n)$ in corollary 1.2 :

$$\sup_{n<m}\sup_{\alpha\in K} |\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))|$$

$$\leqslant |||f||| \left[|a_m| + |a_n| + \sum_{k=n+1}^{m} |a_k - a_{k-1}|\right] \sup_{N\geqslant 1}\sup_{\alpha\in K}\sup_{j\in\mathbb{Z}^*} |\phi_N(j\alpha)| < \varepsilon$$

as long as $m$ and $n$ are large enough.
To prove corollary 1.3, we use Cauchy Schwarz inequality in the following way:

$$\left|\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha)\right| \leqslant \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} |\varphi_{X_k}(j\alpha)|^2}$$

and we conclude using condition $H''$.

**Remark 2.1.**

(1) *the most general hypothesis we can put on the sequence* $(a_k)_{k\geqslant 1}$ *is the following : there exists a strictly increasing sequence* $(N_k)_{k\geqslant 1}$ *such that*

$$\sum_{k=1}^{\infty} \sqrt{\log \Phi_\beta(N_{k+1}) \sum_{l=N_k+1}^{N_{k+1}} |a_l|^2} < \infty$$

(2) *if the process* $(X_k)_{k\geqslant 1}$ *takes integer values then* $f \in A(\mathbb{T})$ *can be assumed without any other hypothesis.*

(3) *If* $\sum |a_k|^2$ *diverges, then we can construct a stochastic process* $(X_k)_{k\geqslant 1}$ *verifying the hypothesis of theorem 1.4 and find* $f \in B(\mathbb{T})$ *such that the convergence of the series is not uniform on any compact. In that sense, the conditions imposed to the sequence* $(a_k)_{k\geqslant 1}$ *are optimal. Remark that in this case, condition 1 is not fulfilled.*

*Namely consider a sequence of independent random variables with disjoint supports. For all* $k \geqslant 1$ *the support of* $X_k$ *is the set of integers belonging to* $[k^2, (k+1)^2-1]$ *and hence, the hypothesis on the moment is verified. We will come back to the law of* $X_k$ *later. Now choose* $f$ *in the following way : for all* $\alpha \in \mathbb{T}$, $f(\alpha) = \exp(2i\pi\alpha)$. *Thus* $f \in A(\mathbb{T})$ *et* $|||f||| < \infty$. *As a consequence, if the convergence of the series defining* $F$ *was uniform in* $\alpha$ *on* $\mathbb{T}$, *then we would have :*

$$\begin{aligned} & \sqrt{\int_0^1 \left|\sum_{k=1}^{\infty} a_k \left[\exp 2i\pi\alpha X_k(\omega) - \mathbb{E}\exp 2i\pi\alpha X_k\right]\right|^2 d\alpha} \\ \leq \quad & \sup_{\alpha\in[0,1]} \left|\sum_{k=1}^{\infty} a_k \left[\exp 2i\pi\alpha X_k(\omega) - \mathbb{E}\exp 2i\pi\alpha X_k\right]\right| < \infty \end{aligned}$$

*By construction, $\mathbb{P}-$ almost surely :*

$$\begin{aligned}
&\int_0^1 \left| \sum_{k=1}^{\infty} a_k \left[ \exp 2i\pi\alpha X_k(\omega) - \mathbb{E} \exp 2i\pi\alpha X_k \right] \right|^2 d\alpha \\
= &\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |\exp\left(2i\pi\alpha X_k(\omega)\right) - \mathbb{E} \exp\left(2i\pi\alpha X_k\right)|^2 d\alpha \\
\geqslant &\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |1 - |\mathbb{E} \exp\left(2i\pi\alpha X_k\right)||^2 \, d\alpha
\end{aligned}$$

*Assume now that the law of $X_k$ is uniform on the $2k+1$ integers of $[k^2, (k+1)^2 - 1]$. for all $k \geqslant 1$.*

$$|\mathbb{E} \exp\left(2i\pi\alpha X_k\right)| = \frac{1}{2k+1} \left| \frac{\sin \pi\alpha(2k+1)}{\sin \pi\alpha} \right|$$

*Using Lebesgue convergence theorem, we get*

$$\lim_{k \to +\infty} \int_0^1 |1 - |\mathbb{E} \exp\left(2i\pi\alpha X_k\right)||^2 \, d\alpha = 1$$

*and we also get the divergence of the series with positive terms*

$$\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |1 - |\mathbb{E} \exp\left(2i\pi\alpha X_k\right)||^2 \, d\alpha = \infty$$

*A contradiction with uniform convergence of the centered part.*

## 3. Proof of theorem 1.5

Let us begin by restating some inequalities obtained by Fernique ([4]) which will be useful in the proof of theorem 1.5. For more information on gaussian techniques in this framework, see [7] and [2].

**Inequality 3.1.** *Let $(G_k)_{k \geqslant 1}$ be a sequence of Banach space valued gaussian random variables $(B, \| \cdot \|)$ defined on a probabilised space $(\Omega, \mathcal{A}, \mathbb{P})$. Then :*

$$\mathbb{E} \sup_{k \geqslant 1} \| G_k \| \leq K_1 \left\{ \sup_{k \geqslant 1} \mathbb{E} \| G_k \| + \mathbb{E} \sup_{k \geqslant 1} | \lambda_k \sigma_k | \right\}$$

*where $(\lambda_k)_{k \geqslant 1}$ is an isonormal sequence, $K_1$ a universal constant and for all $k \geqslant 1$,*

$$\sigma_k = \sup_{f \in B', \|f\| \leq 1} \| < G_k, f >_B \|_{2,\mathbb{P}}$$

**Inequality 3.2.** *Let $g$ be a real valued stationary gaussian random variable, separable and continuous in quadratic mean. Let $m$ be its associated spectral measure on $\mathbb{R}^+$ defined by*

$$\mathbb{E}[| g(s) - g(t) |^2] = 2 \int_0^{\infty} [1 - \cos 2\pi u(s-t)] m(du)$$

*We have*

$$\mathbb{E}\sup_{\alpha\in[0,1]} g(\alpha) \leq K\left\{\sqrt{\int_0^\infty \min(u^2,1)m(du)} + \int_0^\infty \sqrt{m\left(\left]e^{x^2},\infty\right[\right)}dx\right\}$$

*where $K$ is a universal constant.*

**Inequality 3.3.** *(**decoupling**) Let $X = \{X(t) : t \in \mathbb{T}\}$ be a gaussian random function defined on a finite or countable set $T$. Let $\{T_k, k \in [1,n]\}$ be a covering of $T$. Let $S = T_1 \times \cdots \times T_n$. The following inequality holds :*

$$\mathbb{E}\left\{\sup_{k\in[1,n]}\left[\sup_{t\in T_k} X(t) - \mathbb{E}\sup_{t\in T_k} X(t)\right]\right\} \leq \frac{\pi}{2}\cdot\sup_{s\in S}\left\{\mathbb{E}\left[\sup_{k\in[1,n]} X(s_k)\right]\right\}$$

The following estimation generalizes inequality 3.2 and will be useful in the almost periodic case because it gives estimations on arbitrarily large intervals.

**Inequality 3.4.** *Let $g$ a real valued stationnary gaussian random function, separable and continuous in quadratic mean. Let $m$ its associated spectral measure on $\mathbb{R}^+$ defined as in inequality 3.2. There exists a universal constant $K$ such that*

$$\mathbb{E}\sup_{\alpha\in[-M,M]} g(\alpha)$$
$$\leq K\left\{\sqrt{\int_0^\infty \min(2Mu^2,1)\ m(du)} + \int_0^\infty \sqrt{m\left(\left]\frac{e^{x^2}}{2M},\infty\right[\right)}dx\right\}$$

Let us now come to the proof

**-Step 1:** In this part, we replace our problem by a question of regularity of trajectories of random gaussian functions.

Let us consider an independent copy of $X = (X_k)_{k\geqslant 1}$ denoted by $X' = (X'_k)_{k\geqslant 1}$ defined on another probabilized space $(\Omega', \mathcal{A}', \mathbb{P}')$. We call $\mathbb{E}_*$ the integration symbol whose index refers to the space of integration.

Using classical convexity properties, to prove (1), it is enough to show

$$(12)\ \ \mathbb{E}_{X,X'}\sup_{j\in\mathbb{Z}}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda} a_k\left[e^{2i\pi\alpha jX_k} - e^{2i\pi\alpha jX'_k}\right]}{\sqrt{A^2_{\Lambda,\lambda,M}\log(|j|+3)}}\right| < \infty$$

Let us now symmetrize the problem: consider the following separable family of random functions, with continuous trajectories

$$(f_k)_{k\geqslant 1} = \{f_k(\alpha,j) = a_k(\exp 2i\pi\alpha jX_k - \exp 2i\pi\alpha jX'_k), \alpha\in I_M, j\in\mathbb{Z}\}_{k\geqslant 1}$$

By construction $f$ is a symmetric family of random functions, that is to say their law is sign-invariant. More precisely, call $\{\varepsilon_k, k \geqslant 1\}$ a sequence of independent Rademacher random variables (taking the values $+1$ and $-1$ with probability $1/2$), defined on a third space $(\Omega'', \mathcal{A}'', \mathbb{P}'')$, independent of $X$ and $X'$. $\{f_k, k \geqslant 1\}$ and $\{\varepsilon_k f_k, k \geqslant 1\}$ have the same law. Thus for all

integers $(\Lambda, \lambda)$ such that $\Lambda \geqslant \lambda$, $\sum_{k=\lambda}^{\Lambda} f_k$ and $\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k$ also have the same law.

That is why (12) can be written on a larger space of integration in the following way:

$$\mathbb{E}_{X,X',\varepsilon} \sup_{j\in\mathbb{Z}} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k(\alpha, j)}{\sqrt{A^2_{\Lambda,\lambda,M} \log(|j|+3)}} \right|$$

We deduce a sufficient condition for (12) to be realised

$$\mathbb{E}_{X,\varepsilon} \sup_{j\in\mathbb{Z}} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k a_k \exp 2i\pi\alpha j X_k}{\sqrt{A^2_{\Lambda,\lambda,M} \log(|j|+3)}} \right| \tag{13}$$

We then use a precious tool in the theory of gaussian random functions: the contraction principle. This tool is built on a quite simple idea: replace the choice of signs by a sequence of gaussian random variables with mean zero and variance one. This idea can be explained by the following property: given $g$ a gaussian random variable with mean zero and variance 1 and $\varepsilon$ a Rademacher random variable, if $g$ and $\varepsilon$ are independent, then $g$ and $\varepsilon|g|$ have the same law.

As a consequence, in order to prove (13), we show

$$\mathbb{E}_{X,g,g'} \sup_{j\in\mathbb{Z}} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M} \left| \frac{\sum_{k=\lambda}^{\Lambda} a_k (g_k \cos 2\pi\alpha j X_k + g'_k \sin 2\pi\alpha j X_k)}{\sqrt{A^2_{\lambda,\Lambda,M} \log(|j|+3)}} \right| < +\infty$$

where $\{g_k, k \geqslant 1\}$ et $\{g'_k, k \geqslant 1\}$ are two sequences of independent identically distributed random variables with law $\mathcal{N}(0,1)$, independent of $X$ and $\varepsilon$, defined on two other probabilised spaces.

Conditionally to $X$, the problem is reduced to studying the regularity of the trajectories of stationary gaussian random variables. This concludes the first step of the proof.

**-Step 2 :** In this part, we use the gaussian tools introduced in the beginning.

Conditionally to $X$, call $G(\lambda, \Lambda, j, \alpha)$ the following quantity

$$\frac{1}{\sqrt{A^2_{\lambda,\Lambda,M} \log(|j|+3)}} \sum_{k=\lambda}^{\Lambda} a_k \left[ g_k \cos(2\pi\alpha j X_k) + g'_k \sin(2\pi\alpha j X_k) \right]$$

If $j$, $\lambda$ and $\Lambda$ are fixed, $G(\alpha) := G(\lambda, \Lambda, j, \alpha)$ is a random function with almost surely continuous trajectories (up to a modification of trajectories). That is why it is enough to show that $G$ is bounded on $I_M \cap \mathbb{Q}$. Moreover, we will assume that $|j| \leq J$ where $J$ is a large fixed integer.

Let us begin by finding an upper bound for

$$\mathbb{E}_{g,g'} \sup_{|j|\leq J} \sup_{\lambda\geqslant 1} \sup_{\Lambda\geqslant\lambda} \sup_{\alpha\in I_M\cap\mathbb{Q}} |G(\lambda, \Lambda, j, \alpha)|$$

First remark that if $Y_t$ $(t \in T)$ is a gaussian random function defined on $T$, then for all $t_0 \in T$ we have (see [4] (480))

$$\mathbb{E} \sup_{t\in T} |Y_t| \leq \mathbb{E}|Y_{t_0}| + \mathbb{E} \sup_{t\in T} Y_t$$

In this way, we get rid of the absolute value. Apply this remark to $G(\lambda, \Lambda, j, \alpha)$ with $\alpha = 0$, $\lambda = \Lambda = 1$ and $j = 0$ and let us find an upper bound for

$$\mathbb{E}_{g,g'} \sup_{|j|\leq J} \sup_{\lambda \geqslant 1} \sup_{\Lambda \geqslant \lambda} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(\lambda, \Lambda, j, \alpha) \tag{14}$$

In order to apply the decoupling inequality 3.3, define

$$T = \{-J, \cdots, J\} \times H \times (I_M \cap \mathbb{Q})$$

where $J$ is a large enough integer and $H$ is the upper triangle of dimension 2 in $\mathbb{N} \times \mathbb{N}$ (see figure 1 below) ($\lambda \in \mathbb{N}$ and $\Lambda \geqslant \lambda$.) A point in $t \in T$ will be written $t = (j, \lambda, \Lambda, \alpha)$.

This set $T$ is at most countable and we will find an upper bound for $\mathbb{E}_{g,g'} \sup_{t\in T} G(t)$ independently of $J$ and then conclude by taking the supremum on $j \in J$. Define

$$T_j = \{j\} \times H \times (I_M \cap \mathbb{Q})$$

It is obvious that $\{T_j\}_{j=-J,\cdots,J}$ is a covering of $T$. Define

$$S = T_{-J} \times \cdots \times T_J$$

Using inequality 3.3, we have

$$\mathbb{E}_{g,g'} \sup_{t\in T} G(t) \leq \frac{\pi}{2} \sup_{s\in S} \mathbb{E}_{g,g'} \sup_{-J\leqslant j\leqslant J} G(s_j) + \sup_{-J\leqslant j\leqslant J} \mathbb{E}_{g,g'} \sup_{t\in T_j} G(t)$$

where $s_j$ is a point in $T_j$.

**-Step 3 :** We study now

$$\sup_{s\in S} \mathbb{E} \sup_{-J\leqslant j\leqslant J} G(s_j).$$

We can rewrite this in the following way :

$$\sup \mathbb{E}_{g,g'} \sup_{-J\leqslant j\leqslant J} \frac{\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g_k' \sin(2\pi j \alpha_j X_k) \right]}{\sqrt{A^2_{\lambda_j,\Lambda_j,M} \log(|j|+3)}} \tag{15}$$

where the first supremum is taken on

$$\{(\alpha_{-J}, \cdots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}, ((\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J)) \in H^{2J+1}\}$$

Fix $\{(\alpha_{-J}, \cdots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}$ and $((\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J)) \in H^{2J+1}\}$. Define the gaussian process

$$G_j := \frac{\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g_k' \sin(2\pi j \alpha_j X_k) \right]}{\sqrt{A^2_{\lambda_j,\Lambda_j,M} \log(|j|+3)}}$$

In order to get an upper bound for 15, we remark that

$$\sup \mathbb{E}_{g,g'} \sup_{-J\leqslant j\leqslant J} G_j$$

$$(16) \leq \sup_{((\lambda_{-J},\lambda_{-J}),\cdots,(\lambda_J,\lambda_J))\in H^{2J+1}} \mathbb{E}_{g,g'} \sup_{-J\leqslant j\leqslant J} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)|$$

We will get an upper bound for the right hand side of 16 independently of $J$, which will give us an upper bound for 15 by taking the supremum on $j \in J$.

Applying inequality 3.1 to the finite sequence of random gaussian functions

$$(G(j,\lambda_j,\Lambda_j,\alpha))_{-J\leq j\leq J}$$

we prove that $\mathbb{E}_{g,g'} \sup_{-J\leq j\leq J} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)|$ is less than

$$(17) \quad C\left\{ \sup_{-J\leq j\leq J} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)| + \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| \right\}$$

where $C$ is a universal constant, $(\xi_j)_{-J\leq j\leq J}$ is an isonormal sequence and

$$q_j \leq \sup_{\alpha\in I_M} ||G(j,\lambda_j\Lambda_j,\alpha)||_{2,g,g'}$$

This gives us the following upper bound

$$q_j \leq \frac{1}{\sqrt{\log(M\Phi_\beta(\Lambda_j))\log(|j|+3)}}$$

As for all $j$ we have $\Lambda_j \geqslant 1$ and $M \geqslant 1$ we easily get

$$(18) \qquad \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| \leq \mathbb{E}_\xi \sup_{-J\leq j\leq J} \left| \frac{1}{\sqrt{\log(|j|+3)}} \xi_j \right|$$

$$(19) \qquad \leq C\mathbb{E}_\xi \sup_{j\in\mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log(|j|+3)}} \right|$$

The exponential integrability of gaussian vectors gives

$$\mathbb{E}_\xi \sup_{j\in\mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log|j|+3}} \right| = C < \infty$$

and hence

$$\sup_{(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1}} \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| = C < \infty$$

independently of $J$ and of the sequence $(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J)) \in H^{2J+1})$.

We then get independently of $J$ on the whole integration space

$$\mathbb{E}_X \sup_{(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1}} \mathbb{E}_\xi \sup_{-J\leq j\leq J} |\xi_j q_j| \leq C < \infty$$

where $C$ is a constant.

Let us now try to find an upper bound for

$$\sup_{((\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))\in H^{2J+1}} \sup_{-J\leq j\leq J} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M} |G(j,\lambda_j,\Lambda_j,\alpha)| \tag{20}$$

independently of $J$.

We will use inequality 3.4. Let us choose a finite sequence

$$((\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J))$$

and an integer $|j|\leq J$. The gaussian random function $G_j(\alpha):=G(j,\lambda_j,\Lambda_j,\alpha)$ is stationary. Its associated spectral measure on $\mathbb{R}^+$ is defined by

$$m_j=\frac{1}{\log(|j|+3)\log\left(M\Phi_\beta(\Lambda_j)\right)\left(\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\right)}\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\delta_{j|X_k|}$$

where $\delta_u$ is the Dirac measure in the point $u$.

We get

$$\mathbb{E}_{g,g'} \sup_{\alpha\in I_M} |G_j(\alpha)|$$

$$\leq C\left\{\sqrt{\int_0^\infty \min(\frac{u^2}{2M},1)m_j(du)}+\int_0^\infty\sqrt{m_j\left(]\frac{e^{x^2}}{2M},\infty[\right)}dx\right\}$$

It is obvious that the first term is less than

$$\sqrt{m_j(\mathbb{R}^+)}\leq C\frac{1}{\sqrt{\log(2M)\log(|j|+3)}}$$

where $C$ is a universal constant because for all $j$ we have $\Phi_\beta(\Lambda_j)\geqslant 2$.

For the second term, it can be rewritten in the following way

$$\begin{aligned}&\int_0^\infty\sqrt{m_j\left(]\frac{\exp x^2}{2M},\infty[\right)}dx\\ =&\left[\frac{1}{\log(|j|+3)\log\left(M\Phi_\beta(\Lambda_j)\right)\left(\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\right)}\right]^{\frac{1}{2}}\\ \times&\int_0^\infty\sqrt{\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\mathbf{1}_{\{2M|jX_k|>e^{x^2}\}}}dx\end{aligned}$$

Using

$$\forall k\geqslant 1,\qquad \mathbf{1}_{\{2M|jX_k|>e^{x^2}\}}\leq\mathbf{1}_{\{2M|j|\sup_{l\leq k}|X_l|>e^{x^2}\}}$$

we cut $\mathbb{R}^+$ in the integral according to the increasing subdivision

$$\{0\}\bigcup\{\sqrt{\log^+(2M|j|\sup_{l\leq k}|X_l|)},\lambda_j\leq k\leq\Lambda_j\}$$

We get thus an upper bound for the previous integral

$$\begin{aligned}
\sum_{k=\lambda_j}^{\Lambda_j} & \left[\sqrt{\log^+\left(2M|j|\sup_{l\leq k}\mid X_l\mid\right)}-\sqrt{\log^+\left(2M|j|\sup_{l\leq k-1}\mid X_l\mid\right)}\right]\left(\sum_{l=\lambda_j}^{\Lambda_j}|a_l|^2\right)^{\frac{1}{2}}\\
&\leq \left(\sum_{l=\lambda_j}^{\Lambda_j}|a_l|^2\right)^{\frac{1}{2}}\sum_{k=\lambda_j}^{\Lambda_j}\left[\sqrt{\log^+\left(2M|j|\sup_{l\leq k}\mid X_l\mid\right)}\right.\\
&\left.- \sqrt{\log^+\left(2M|j|\sup_{l\leq k-1}\mid X_l\mid\right)}\right]\\
&\leq 2\left(\sum_{k=\lambda_j}^{\Lambda_j}|a_k|^2\right)^{\frac{1}{2}}\sqrt{\log^+\left(2M|j|\sup_{l\leq\Lambda_j}\mid X_l\mid\right)}
\end{aligned}$$

Consequently, $\forall |j|\leq J$,

$$\begin{aligned}
\int_0^\infty\sqrt{m_j\left(]\frac{\exp x^2}{M},\infty[\right)}dx &\leq 2\sqrt{\frac{\log^+\left(2M|j|\sup_{l\leq\Lambda_j}\mid X_l\mid\right)}{\log\left(|j|+3\right)\log(M\Phi_\beta(\Lambda_j))}}\\
&\leq 4\sqrt{\frac{\log^+\left(2M\sup_{1\leq l\leq\Lambda_j}\mid X_l\mid\right)}{\log(M\Phi_\beta(\Lambda_j))}}\\
&\leq 4\sup_{1\leq l\leq\Lambda_j}\sqrt{\frac{\log^+\left(2M\mid X_l\mid\right)}{\log(M\Phi_\beta(l))}}\\
&\leq 4\sup_{N\geqslant 1}\sqrt{\frac{\log^+\left(2M\mid X_N\mid\right)}{\log\left(M\Phi_\beta(N)\right)}}\\
&\leq 4\left(1+\sup_{N\geqslant 1}\sqrt{\frac{\log^+\left(\mid X_N\mid\right)}{\log\Phi_\beta(N)}}\right)
\end{aligned}$$

Finally, on the whole integration space, we get the following upper bound :

$$\mathbb{E}_X\sup_{(\lambda_{-J},\Lambda_{-J}),\cdots,(\lambda_J,\Lambda_J)\in H^{2J+1}}\sup_{-J\leq j\leq J}\mathbb{E}_{g,g'}\sup_{\alpha\in I_M}|G_j(\alpha)| \tag{21}$$

$$\leq C\left(1+\mathbb{E}_X\sup_{N\geqslant 1}\sqrt{\frac{\log^+\left(\mid X_N\mid\right)}{\log\Phi_\beta(N)}}\right)<\infty \tag{22}$$

**-Step 4 :** To end the proof of theorem 1.5, it remains to deal with

$$\sup_{-J\leq j\leq J}\mathbb{E}_{g,g'}\sup_{(\lambda,\Lambda)\in H}\sup_{\alpha\in I_M\cap\mathbb{Q}}G(j,\lambda,\Lambda,\alpha)$$

Let us fix $j$. In order to apply inequality 3.1, we need to replace the supremum on $(\lambda,\Lambda)\in H$ by a supremum on only one variable, in other words,

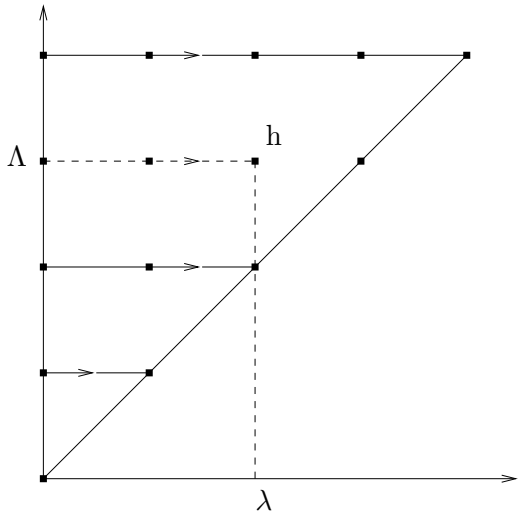


FIGURE 1. renumbering the upper triangle in $\mathbb{N}^2$

we need to renumber $H$ using one variable $h \in \mathbb{N}^2$ given by the formula :

$$h = \lambda + \frac{\Lambda(\Lambda-1)}{2}$$

(see figure 1).

Let us fix $j$. Inequality 3.1 gives us :

$$\mathbb{E}_{g,g'} \sup_{h\geqslant 1} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,h,\alpha) \leqslant K_1(\sup_{h\geqslant 1} \mathbb{E}_{g,g'} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,h,\alpha) + \mathbb{E}_{g,g'} \sup_{h\geqslant 1} |\lambda_h\sigma_h|) \tag{23}$$

where $(\lambda_h)_{h\geqslant 1}$ is an isonormal sequence. The inequality :

$$\frac{\Lambda(\Lambda-1)}{2} \leqslant h \leqslant \frac{\Lambda(\Lambda+1)}{2}$$

gives a polynomial dependence between $\Lambda$ and $h$, hence :

$$\sigma_h = \mathcal{O}(\frac{1}{\sqrt{\log h}})$$

and the first term in the right hand side of inequality (23) is dealt with in the same way as before. Finally, we get :

$$\mathbb{E}_{g,g'} \sup_{(\lambda,\Lambda)\in H} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,\lambda,\Lambda,\alpha) \leq C\left(1+\sup_{N\geqslant 1}\sqrt{\frac{\log^+(|X_N|)}{\log\Phi_\beta(N)}}\right)$$

That is to say, by integrating on the whole space,

$$\mathbb{E}_X \sup_{1\leq j\leq n} \mathbb{E}_{g,g'} \sup_{(\lambda,\Lambda)\in H} \sup_{\alpha\in I_M\cap\mathbb{Q}} G(j,\lambda,\Lambda,\alpha) \leq C\left(1+\mathbb{E}_X \sup_{N\geqslant 1}\sqrt{\frac{\log^+(|X_N|)}{\log\Phi_\beta(N)}}\right) \tag{24}$$

Let us prove that

$$\mathbb{E}_X \sup_{N\geqslant 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_\beta(N)}} < \infty$$

Using Jensen inequality, we can get rid of the square root. Let $\delta > 0$. For all $N \geqslant 1$, we have

$$\beta \log^+ |X_N| \leq \beta \log^+ \left[\frac{|X_N|}{\Phi_\beta^\delta(N)}\right] + \beta \log^+[\Phi_\beta^\delta(N)]$$

noticing that $\Phi_\beta^\delta(N)) \geqslant 2$ it is sufficient to show

$$\mathbb{E} \sup_{N\geqslant 1} \log^+ \left[\frac{|X_N|^\beta}{\Phi_\beta^{\delta\beta}(N)}\right] < \infty$$

Using now the inequality $\log^+(x) \leq x$ for any $x \geqslant 0$, it is sufficient to prove

$$\sum_{N\geqslant 1} \frac{\mathbb{E}|X_N|^\beta}{\Phi_\beta^{\delta\beta}(N)} < \infty$$

And as $\mathbb{E}|X_N|^\beta \leqslant \Phi_\beta(N)$ and $\Phi_\beta(N) \geqslant N$, if we chose $\delta = \frac{3}{\beta}$ we get the conclusion. Steps 3 (see 21), step 4 (see 24) lead us to the announced result of theorem 1.5.

## 4. Applications

Let us begin by giving an example where the $(X_k)$ are uniformly distributed :

**Example 4.1.** *Suppose that $\mathcal{L}(X_k) = \mathcal{U}([\mu_k - \sigma_k/2, \mu_k + \sigma_k/2])$ with $\sigma_k > 0$ et $\mathbb{E}(X_k) = \mu_k$ with $\mu_k = \mathcal{O}(k^d)$ for some $d > 0$. The characteristic function of $X_k$ can easily be computed :*

$$\varphi_{X_k}(t) = \frac{e^{2i\pi t\mu_k}}{\pi t\sigma_k} \sin(\pi t\sigma_k)$$

*Using condition $\mathcal{H}$ of theorem 1.1, the following condition*

$$\sum_{n\geqslant 1} \frac{|a_n|}{\sigma_n} \text{ converges} \tag{25}$$

*and*

$$\sum_{n\geqslant 1} \frac{\sqrt{\sum_{k\geqslant n} |a_k|^2}}{n\sqrt{\log n}} < +\infty$$

*are sufficient to get the desired convergence.*
*Notice that using corollary 1.3, condition (25) is replaced by*

$$\sum_{n\geqslant 1} \frac{1}{\sigma_n^2} < +\infty$$

*The subexponential case could be dealt with in the same way.*
*If we consider the border case $a_k = \mathcal{O}(k^{-1/2-\varepsilon})$, it is sufficient that :*

$$\exists \eta > 0, \sigma_k \geqslant k^{\frac{1}{2}+\eta}$$

*in this case :*

$$\mathbb{P}\{\forall k, X_k \in [\mu_k - \frac{k^{\frac{1}{2}+\eta}}{2}, \mu_k + \frac{k^{\frac{1}{2}+\eta}}{2}]\} = 1$$

*which gives an information on the possible dispersion of the variables $X_k$.*

Here are other examples where the conditions of our theorems can be quite easily verified.

**Corollary 4.1.** *Let $(X_k)_{k\geqslant 1}$ be a sequence of real independent random variables whose law can be written in the following way for all $k \geqslant 1$ : $\mathcal{L}(X_k) = \mathcal{L}(\sigma_k \cdot X + \mu_k)$ where $X$ verifies $\mathbb{E}|X|^\beta < \infty$ for some $\beta > 0$. Moreover, we assume that there exist $d > 0$ and $\delta > 0$ such that $|\sigma_k| = \mathcal{O}(k^d)$, $|\mu_k| = \mathcal{O}(k^d)$, the application $t \mapsto t^\delta \mathbb{E}\exp(2i\pi tX)$ is bounded on $\mathbb{R}$, and let $(a_k)_{k\geqslant 1}$ be a sequence of real or complex numbers satisfying the following two conditions*

(1) $|a_k| = O(k^{-\beta})$ *with* $\beta > 1/2$
(2) $\sum_{k=1}^{\infty} \frac{1}{|\sigma_k|^{2\delta}} < \infty$

*Then there exists a measurable set $\Omega_o$ with full measure ($\mathbb{P}(\Omega_o) = 1$) such that for any $\omega \in \Omega_o$ for all $f \in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$ : for any compact $K$ which does not contain 0, the application $t \in K \mapsto F(t) = \sum_{k\geqslant 1} a_k f(tX_k(\omega))$ is continuous and the series defining $F$ converges uniformly on $K$.*

The proof of corollary 4.1 relies on corollary 1.3.

**Example 4.2.** *The random variable $X$ may have a gaussian law with mean zero and variance one, a Cauchy law, the first Laplace law, an exponential law with parameter $\lambda > 0$. Let us precise the gaussian case.*
*Here $\mathcal{L}(X) = \mathcal{N}(0,1)$, we have $\mathbb{E}\exp 2i\pi tX = e^{-t^2/2}$. Hence we can use the fact that $t \mapsto e^{t^2/2}\mathbb{E}\exp 2i\pi tX$ is bounded on $\mathbb{R}$. In this case, the sufficient condition to obtain convergence is*

$$\forall \varepsilon > 0, \exists N > 0, \sup_{m>n\geqslant N} \sup_{\alpha\in K} \sup_{j\in\mathbb{Z}-\{0\}} |\sum_{k=n}^{m} a_k e^{2i\pi\alpha\mu_k j} e^{-j^2\alpha^2\sigma_k^2/2}| < \varepsilon$$

*Let $d(0,K)$ be the distance between 0 and the compact $K$. Using the fact that $|a_k| = O(k^{-\beta})$ with $\beta > 1/2$, the previous condition will be satisfied as soon as :*

$$\exists \varepsilon > 0, \sigma_k \geqslant \frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0,K)}\sqrt{\log k}$$

*which, in terms of dispersion of the variables $X_k$ means that infinitely often:*

$$X_k \in [\mu_k - 3\frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0,K)}\sqrt{\log k}, \mu_k + 3\frac{\sqrt{2(1-\beta+\varepsilon)}}{d(0,K)}\sqrt{\log k}]$$

We discuss now the case when the laws of $X_k$ are generated by a convolution product of a given law $\mu$. We distinguish two cases : on one hand when the support of $\mu$ contains non integer values, on the other hand when the support of $\mu$ is contained in $\mathbb{Z}$. The first case is discribed by the following corollary :

**Corollary 4.2.** *Let $(X_k)_{k\geqslant 1}$ be an sequence of real valued independent random variables such that for all integer $k \geqslant 1$, $\mathcal{L}(X_k) = \mu^{*k}$ where $\mu$ is a probability measure on $\mathbb{R}$ with $\mathbb{E}|X_1|^\delta < \infty$ for some $\delta$. Assume the following :*

$(a)$ $\qquad \varphi_{X_1}(t) = 1 \Longleftrightarrow t = 0 \qquad$ *($X_1$ aperiodic)*

$(b)$ $\quad \exists \delta > 0, \qquad \sup_{t\in\mathbb{R}} |t^\delta \mathbb{E}\exp(2i\pi t X_1)| = q < \infty$

*Let $(a_k)_{k\geqslant 1}$ be a sequence of real or complex numbers such that the sequence $|a_k|$ is decreasing and fulfills the two following conditions :*

(1) $|a_k| = O(k^{-\beta})$ *avec* $\beta > 1/2$
(2) $\sum_{k=1}^{\infty} |a_k - a_{k+1}| < \infty$

*Then there exists a measurable set $\Omega_o$ with full measure ($\mathbb{P}(\Omega_o) = 1$) such that for any $\omega \in \Omega_o$, for all $f \in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$, for any compact $K$ which does not contain 0, the application $t \in K \mapsto F(t,\omega) = \sum_{k\geqslant 1} a_k f(t\, X_k(\omega))$ is continuous and the series defining $F$ converges uniformly on $K$.*

**Remark 4.1.** *If $X_1$ is striclty aperiodic ($|\varphi_{X_1}(t)| = 1 \Longleftrightarrow t = 0$), then the condition on the differences $|a_k - a_{k+1}|$ may be removed, using corollary 1.3 and the same kind of conputation as in the following proof.*

The random variable $X_1$ being real valued, its characteristic function is not periodic. Take for example a gaussian law with mean zero and variance one.

**Proof** : Let $K$ be a compact which does not contain 0. Using Abel 's summation, it is sufficient to prove

$$\sup_{t\in K}\sup_{|j|\geqslant 1} \left| \sum_{k=1}^{N} (\mathbb{E}\exp 2i\pi j t X_1)^k \right| < \infty$$

independently of $N$. Let us split the supremum on $j$ respectively into the supremum on the indexes $J(q)$ and $\bar{J}(q)$ where $J(q) = \{j \in \mathbb{Z}^* : |j|^\delta \leq \left[\frac{2q}{\varepsilon^\delta}\right]\}$ and $2\varepsilon$ is the distance between 0 and the fixed compact $K$.
On one hand, using $(a)$, it can be proved that :

$$\forall \varepsilon > 0 \qquad \inf_{|t|>\varepsilon} |t|^\delta |1 - \mathbb{E}\exp(2i\pi t X_1)| > 0$$

this implies

$$\sup_{t\in K}\sup_{j\in J(q)}\left|\sum_{k=1}^{N}(\mathbb{E}\exp 2i\pi jtX_1)^k\right|\leq \sup_{t\in K}\sup_{j\in J(q)}C(\varepsilon)|jt|^\delta\leq C(K)\left[\frac{2q}{\varepsilon^\delta}\right]$$

On the other hand, using $(b)$,

$$\begin{aligned}\sup_{t\in K}\sup_{j\in \bar{J}(q)}\left|\sum_{k=1}^{N}(\mathbb{E}\exp 2i\pi jtX_1)^k\right| &\leq \sup_{t\in K}\sup_{j\in \bar{J}(q)}\sum_{k=1}^{N}\left(\frac{q}{|tj|^\delta}\right)^k\\ &\leq C\sum_{k=1}^{N}\frac{1}{2^k}\leq 2C\end{aligned}$$

where $C$ is a universal constant.
□

As for the integer valued case, we have :

**Corollary 4.3.** *Let $(X_k)_{k\geqslant 1}$ be an sequence of integer valued independent random variables such that for all integer $k\geqslant 1$, $\mathcal{L}(X_k)=\mu^{*k}$ where $\mu$ is a probability measure on $\mathbb{R}$ with $\mathbb{E}|X_1|^\beta<\infty$ for some $\beta>0$. Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers such that $|a_k|=O(k^{-\beta})$ with $\beta>1/2$. Assume either :*

$$\varphi_{X_1}(t)=1\Longleftrightarrow t=0\quad \textit{and}\quad \sum_{k\geqslant 1}|a_k-a_{k+1}|\quad \textit{converges}$$

*or :*

$$|\varphi_{X_1}(t)|=1\Longleftrightarrow t=0$$

*Then there exists a measurable set $\Omega_o$ with full measure ($\mathbb{P}(\Omega_o)=1$) such that for any $\omega\in\Omega_o$, for all $f\in A(\mathbb{T})$ such that $\int_{\mathbb{T}}f(t)dt=0$, for any compact $K$ of the torus which does not contain $0$, the application $t\in K\mapsto F(t,\omega)=\sum_{k\geqslant 1}a_kf(t\,X_k(\omega))$ is continuous and the series defining $F$ converges uniformly on $K$.*

**Example 4.3.** *If the law of $X_1$ is a Poisson law with parameter $1$, we use :*

$$\forall t\in\mathbb{T},|\varphi_{X_1}(t)|\leqslant e^{cos(2\pi t)-1}$$

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