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The Eulerian Distribution on Involutions is Indeed Unimodal

Victor J. W. Guo¹ and Jiang Zeng²

Institut Camille Jordan, Université Claude Bernard (Lyon I), F-69622, Villeurbanne Cedex, France

Abstract. Let $I_{n,k}$ (respectively, $J_{n,k}$) be the number of involutions (respectively, fixed-point free involutions) of $\{1, \dots, n\}$ with k descents. Motivated by Brenti's conjecture which states that the sequence $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$ is log-concave, we prove that the two sequences $I_{n,k}$ and $J_{2n,k}$ are unimodal in k , for all n . Furthermore, we conjecture that there are nonnegative integers $a_{n,k}$ such that

$$\sum_{k=0}^{n-1} I_{n,k} t^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}.$$

This statement is stronger than the unimodality of $I_{n,k}$ but is also interesting in its own right.

Keywords: Involutions, Descent number, Unimodality, Eulerian polynomial, Zeilberger's algorithm

AMS Subject Classifications (2000): Primary 05A15; Secondary 05A20

1 Introduction

A sequence a_0, a_1, \dots, a_n of real numbers is said to be *unimodal* if for some $0 \leq j \leq n$ we have $a_0 \leq a_1 \leq \dots \leq a_j \geq a_{j+1} \geq \dots \geq a_n$, and is said to be *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$. Clearly a log-concave sequence of *positive* terms is unimodal. The reader is referred to Stanley's survey [10] for the surprisingly rich variety of methods to show that a sequence is log-concave or unimodal. As noticed by Brenti [2], even though log-concave and unimodality have one-line definitions, to prove the unimodality or log-concavity of a sequence can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools.

Let \mathfrak{S}_n be the set of all permutations of $[n] := \{1, \dots, n\}$. We say that a permutation $\pi = a_1 a_2 \dots a_n \in \mathfrak{S}_n$ has a *descent* at i ($1 \leq i \leq n-1$) if $a_i > a_{i+1}$. The number of descents of π is called its descent number and is denoted by $d(\pi)$. A statistic on \mathfrak{S}_n is said to be *Eulerian*, if it is equidistributed with the descent number statistic. Recall that the polynomial

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{1+d(\pi)} = \sum_{k=1}^n A(n, k) t^k$$

is called an *Eulerian polynomial*. It is well-known that the *Eulerian numbers* $A(n, k)$ ($1 \leq k \leq n$) form a unimodal sequence, of which several proofs have been published: such

¹guo@math.univ-lyon1.fr, <http://math.univ-lyon1.fr/~guo>

²zeng@math.univ-lyon1.fr, <http://math.univ-lyon1.fr/~zeng>

as the analytical one by showing that the polynomial $A_n(t)$ has only real zeros [3, p. 294], by induction based on the recurrence relation of $A(n, k)$ (see [9]), or by combinatorial techniques (see [7, 11]).

Let \mathcal{I}_n be the set of all involutions in \mathfrak{S}_n and \mathcal{J}_n the set of all fixed-point free involutions in \mathfrak{S}_n . Define

$$I_n(t) = \sum_{\pi \in \mathcal{I}_n} t^{d(\pi)} = \sum_{k=0}^{n-1} I_{n,k} t^k,$$

$$J_n(t) = \sum_{\pi \in \mathcal{J}_n} t^{d(\pi)} = \sum_{k=0}^{n-1} J_{n,k} t^k.$$

The first values of these polynomials are given in Table 1.

Table 1: The polynomials $I_n(t)$ and $J_n(t)$ for $n \leq 6$.

n	$I_n(t)$	$J_n(t)$
1	1	0
2	$1 + t$	t
3	$1 + 2t + t^2$	0
4	$1 + 4t + 4t^2 + t^3$	$t + t^2 + t^3$
5	$1 + 6t + 12t^2 + 6t^3 + t^4$	0
6	$1 + 9t + 28t^2 + 28t^3 + 9t^4 + t^5$	$t + 3t^2 + 7t^3 + 3t^4 + t^5$

As one may notice from Table 1 that the coefficients of $I_n(t)$ and $J_n(t)$ are *symmetric* and *unimodal* for $1 \leq n \leq 6$. Actually, the symmetries had been conjectured by Dumont and were first proved by Strehl [12]. Recently, Brenti (see [5]) conjectured that the coefficients of the polynomial $I_n(t)$ are *log-concave* and Dukes [5] has obtained some partial results on the unimodality of the coefficients of $I_n(t)$ and $J_{2n}(t)$. Note that, in contrast to Eulerian polynomials $A_n(t)$, the polynomials $I_n(t)$ and $J_{2n}(t)$ may have *non-real zeros*.

In this paper we will prove that for $n \geq 1$, the two sequences $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$ and $J_{2n,1}, J_{2n,2}, \dots, J_{2n,2n-1}$ are unimodal. Our starting point is the known generating functions of polynomials $I_n(t)$ and $J_n(t)$:

$$\sum_{n=0}^{\infty} I_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u)^{r+1} (1-u^2)^{r(r+1)/2}}, \quad (1.1)$$

$$\sum_{n=0}^{\infty} J_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u^2)^{r(r+1)/2}}, \quad (1.2)$$

which have been obtained by Désarménien and Foata [4] and Gessel and Reutenauer [8] using different methods. We first derive linear recurrence formulas for $I_{n,k}$ and $J_{2n,k}$ in the next section and then prove the unimodality by induction in Section 3. We end this paper with further conjectures beyond the unimodality of the two sequences $I_{n,k}$ and $J_{2n,k}$.

2 Linear recurrence formulas for $I_{n,k}$ and $J_{2n,k}$

Since the recurrence formula for the numbers $I_{n,k}$ is a little more complicated than $J_{2n,k}$, we shall first prove it for the latter.

Theorem 2.1. *For $n \geq 2$ and $k \geq 0$, the numbers $J_{2n,k}$ satisfy the following recurrence formula:*

$$2nJ_{2n,k} = [k(k+1) + 2n - 2]J_{2n-2,k} + 2[(k-1)(2n-k-1) + 1]J_{2n-2,k-1} \\ + [(2n-k)(2n-k+1) + 2n - 2]J_{2n-2,k-2}. \quad (2.1)$$

Here and in what follows $J_{2n,k} = 0$ if $k < 0$.

Proof. Equating the coefficients of u^{2n} in (1.2), we obtain

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \sum_{r=0}^{\infty} \binom{r(r+1)/2 + n - 1}{n} t^r. \quad (2.2)$$

Since

$$\binom{r(r+1)/2 + n - 1}{n} = \frac{r(r-1)/2 + r + n - 1}{n} \binom{r(r+1)/2 + n - 2}{n-1},$$

it follows from (2.2) that

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \frac{t^2}{2n} \left(\frac{J_{2n-2}(t)}{(1-t)^{2n-1}} \right)'' + \frac{t}{n} \left(\frac{J_{2n-2}(t)}{(1-t)^{2n-1}} \right)' + \frac{n-1}{n} \frac{J_{2n-2}(t)}{(1-t)^{2n-1}},$$

or

$$J_{2n}(t) = \frac{t^2(1-t)^2}{2n} J_{2n-2}''(t) + \left[\frac{(2n-1)t^2(1-t)}{n} + \frac{t(1-t)^2}{n} \right] J_{2n-2}'(t) \\ + \left[(2n-1)t^2 + \frac{(2n-1)(1-t)t}{n} + \frac{(n-1)(1-t)^2}{n} \right] J_{2n-2}(t) \\ = \frac{t^4 - 2t^3 + t^2}{2n} J_{2n-2}''(t) + \left[\frac{(2-2n)t^3}{n} + \frac{(2n-3)t^2}{n} + \frac{t}{n} \right] J_{2n-2}'(t) \\ + \left[(2n-2)t^2 + \frac{t}{n} + \frac{n-1}{n} \right] J_{2n-2}(t). \quad (2.3)$$

Equating the coefficients of t^n in (2.3) yields

$$J_{2n,k} = \frac{(k-2)(k-3)}{2n} J_{2n-2,k-2} - \frac{(k-1)(k-2)}{n} J_{2n-2,k-1} + \frac{k(k-1)}{2n} J_{2n-2,k} \\ + \frac{(2-2n)(k-2)}{n} J_{2n-2,k-2} + \frac{(2n-3)(k-1)}{n} J_{2n-2,k-1} + \frac{k}{n} J_{2n-2,k} \\ + (2n-2)J_{2n-2,k-2} + \frac{1}{n} J_{2n-2,k-1} + \frac{n-1}{n} J_{2n-2,k}.$$

After simplification, we obtain (2.1). ■

Theorem 2.2. For $n \geq 3$ and $k \geq 0$, the numbers $I_{n,k}$ satisfy the following recurrence formula:

$$\begin{aligned} nI_{n,k} &= (k+1)I_{n-1,k} + (n-k)I_{n-1,k-1} + [(k+1)^2 + n-2]I_{n-2,k} \\ &\quad + [2k(n-k-1) - n+3]I_{n-2,k-1} + [(n-k)^2 + n-2]I_{n-2,k-2}. \end{aligned} \quad (2.4)$$

Here and in what follows $I_{n,k} = 0$ if $k < 0$.

Proof. Extracting the coefficients of u^{2n} in (1.1), we obtain

$$\frac{I_n(t)}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} t^r \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{r(r+1)/2 + k - 1}{k} \binom{r+n-2k}{n-2k}. \quad (2.5)$$

Let

$$T(n, k) := \binom{x+k-1}{k} \binom{y+n-2k}{n-2k},$$

and

$$s(n) := \sum_{k=0}^{\lfloor n/2 \rfloor} T(n, k).$$

Applying Zeilberger's algorithm, the Maple package `ZeilbergerRecurrence(T,n,k,s,0..n)` gives

$$(2x+y+n+1)s(n) + (y+1)s(n+1) - (n+2)s(n+2) = 0, \quad (2.6)$$

i.e.,

$$s(n) = \frac{y+1}{n}s(n-1) + \frac{2x+y+n-1}{n}s(n-2).$$

When $x = r(r+1)/2$ and $y = r$, we get

$$s(n) = \frac{r+1}{n}s(n-1) + \frac{r(r-1) + 3r + n - 1}{n}s(n-2). \quad (2.7)$$

Now, from (2.5) and (2.7) follows the identity

$$\begin{aligned} \frac{nI_n(t)}{(1-t)^{n+1}} &= t \left(\frac{I_{n-1}(t)}{(1-t)^n} \right)' + \frac{I_{n-1}(t)}{(1-t)^n} + t^2 \left(\frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)'' + 3t \left(\frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)' \\ &\quad + (n-1) \frac{I_{n-2}(t)}{(1-t)^{n-1}}, \end{aligned}$$

or

$$\begin{aligned} nI_n(t) &= (t-t^2)I'_{n-1}(t) + [1+(n-1)t]I_{n-1}(t) + t^2(1-t)^2I''_{n-2}(t) \\ &\quad + t(1-t)[3+(2n-5)t]I'_{n-2}(t) + (n-1)[1+t+(n-2)t^2]I_{n-2}(t). \end{aligned} \quad (2.8)$$

Comparing the coefficients of t^k in both sides of (2.8), we obtain

$$\begin{aligned} nI_{n,k} &= kI_{n-1,k} - (k-1)I_{n-1,k-1} + I_{n-1,k} + (n-1)I_{n-1,k-1} \\ &\quad + k(k-1)I_{n-2,k} - 2(k-1)(k-2)I_{n-2,k-1} + (k-2)(k-3)I_{n-2,k-2} \\ &\quad + 3kI_{n-2,k} + (2n-8)(k-1)I_{n-2,k-1} - (2n-5)(k-2)I_{n-2,k-2} \\ &\quad + (n-1)I_{n-2,k} + (n-1)I_{n-2,k-1} + (n-1)(n-2)I_{n-2,k-2}, \end{aligned}$$

which, after simplification, equals the right-hand side of (2.4). ■

Remark. The recurrence formula (2.6) can also be proved by hand as follows. It is easy to see that the generating function of $s(n)$ is

$$\sum_{n=0}^{\infty} s(n)u^n = (1-u^2)^{-x}(1-u)^{-y-1}. \quad (2.9)$$

Differentiating (2.9) with respect to u implies that

$$\sum_{n=0}^{\infty} ns(n)u^{n-1} = \left(\frac{2ux}{1-u^2} + \frac{y+1}{1-u} \right) (1-u^2)^{-x}(1-u)^{-y-1},$$

consequently,

$$\begin{aligned} (1-u^2) \sum_{n=0}^{\infty} ns(n)u^{n-1} &= [(2x+y+1)u + y+1](1-u^2)^{-x}(1-u)^{-y-1} \\ &= [(2x+y+1)u + y+1] \sum_{n=0}^{\infty} s(n)u^n. \end{aligned} \quad (2.10)$$

Comparing the coefficients of u^{n+1} in both sides of (2.10), we obtain

$$(n+2)s(n+2) - ns(n) = (2x+y+1)s(n) + (y+1)s(n+1),$$

which is equivalent to (2.6).

Note that the right-hand side of (2.1) (respectively, (2.4)) is invariant under the substitution $k \rightarrow 2n-k$ (respectively, $k \rightarrow n-1-k$), provided that the sequence $I_{n-1,k}$ (respectively, $J_{2n-2,k}$) is symmetric. Thus, by induction we derive immediately the symmetry properties of $J_{2n,k}$ and $I_{n,k}$ (see [4, 8, 12]).

Corollary 2.3. *For $n, k \in \mathbb{N}$, we have*

$$I_{n,k} = I_{n,n-1-k}, \quad J_{2n,k} = J_{2n,2n-k}.$$

It would be interesting to find a combinatorial proof of the recurrence formulas (2.1) and (2.4), since such a proof could hopefully lead to a combinatorial proof of the unimodality of these two sequences.

3 Unimodality of the sequences $I_{n,k}$ and $J_{2n,k}$

The following observation is crucial in our inductive proof of the unimodality of the sequences $I_{n,k}$ ($0 \leq k \leq n-1$) and $J_{2n,k}$ ($1 \leq k \leq 2n-1$).

Lemma 3.1. *Let x_0, x_1, \dots, x_n and a_0, a_1, \dots, a_n be real numbers such that $x_0 \geq x_1 \geq \dots \geq x_n \geq 0$ and $a_0 + a_1 + \dots + a_k \geq 0$ for all $k = 0, 1, \dots, n$. Then*

$$\sum_{i=0}^n a_i x_i \geq 0.$$

Indeed, the above inequality follows from the identity:

$$\sum_{i=0}^n a_i x_i = \sum_{k=0}^n (x_k - x_{k+1})(a_0 + a_1 + \dots + a_k),$$

where $x_{n+1} = 0$.

Theorem 3.2. *The sequence $J_{2n,1}, J_{2n,2}, \dots, J_{2n,2n-1}$ is unimodal.*

Proof. By the symmetry of $J_{2n,k}$, it is enough to show that $J_{2n,k} \geq J_{2n,k-1}$ for all $2 \leq k \leq n$. We proceed by induction on n . The $n = 2$ case is clear from Table 1. Suppose the sequence $J_{2n-2,k}$ is unimodal in k . By Theorem 2.1, one has

$$2n(J_{2n,k} - J_{2n,k-1}) = A_0 J_{2n-2,k} + A_1 J_{2n-2,k-1} + A_2 J_{2n-2,k-2} + A_3 J_{2n-2,k-3}, \quad (3.1)$$

where

$$\begin{aligned} A_0 &= k^2 + k + 2n - 2, & A_1 &= 4nk - 3k^2 - 6n + k + 6, \\ A_2 &= 3k^2 + 4n^2 - 8nk - 5k + 12n - 4, & A_3 &= 3k - k^2 + 4nk - 4n^2 - 8n. \end{aligned}$$

We have the following two cases:

- If $2 \leq k \leq n-1$, then

$$J_{2n-2,k} \geq J_{2n-2,k-1} \geq J_{2n-2,k-2} \geq J_{2n-2,k-3}$$

by the induction hypothesis, and clearly

$$\begin{aligned} A_0 &\geq 0, & A_0 + A_1 &= 2(k-1)(2n-k) + 4 \geq 0, \\ A_0 + A_1 + A_2 &= (2n-k)^2 - 3k + 8n \geq 0, & A_0 + A_1 + A_2 + A_3 &= 0. \end{aligned}$$

Therefore, by Lemma 3.1, we have

$$J_{2n,k} - J_{2n,k-1} \geq 0.$$

- If $k = n$, then

$$J_{2n-2,n-1} \geq J_{2n-2,n} = J_{2n-2,n-2} \geq J_{2n-2,n-3}$$

by symmetry and the induction hypothesis. In this case, we have $A_1 = (n-2)(n-3) \geq 0$ and thus the corresponding condition of Lemma 3.1 is satisfied. Therefore, we have

$$J_{2n,n} - J_{2n,n-1} \geq 0.$$

This completes the proof. ■

Theorem 3.3. *The sequence $I_{n,0}, I_{n,1}, \dots, I_{n,n-1}$ is unimodal.*

Proof. By the symmetry of $I_{n,k}$, it suffices to show that $I_{n,k} \geq I_{n,k-1}$ for all $1 \leq k \leq (n-1)/2$. From Table 1, it is clear that the sequences $I_{n,k}$ are unimodal in k for $1 \leq n \leq 6$.

Now suppose $n \geq 7$ and the sequences $I_{n-1,k}$ and $I_{n-2,k}$ are unimodal in k . Replacing k by $k-1$ in (2.4), we obtain

$$\begin{aligned} nI_{n,k-1} &= kI_{n-1,k-1} + (n-k+1)I_{n-1,k-2} + (k^2 + n-2)I_{n-2,k-1} \\ &\quad + [2(k-1)(n-k) - n + 3]I_{n-2,k-2} + [(n-k+1)^2 + n-2]I_{n-2,k-3}. \end{aligned} \quad (3.2)$$

Combining (2.4) and (3.2) yields

$$\begin{aligned} n(I_{n,k} - I_{n,k-1}) &= B_0I_{n-1,k} + B_1I_{n-1,k-1} + B_2I_{n-1,k-2} \\ &\quad + C_0I_{n-2,k} + C_1I_{n-2,k-1} + C_2I_{n-2,k-2} + C_3I_{n-2,k-3}, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} B_0 &= k+1, \quad B_1 = n-2k, \quad B_2 = -(n-k+1), \\ C_0 &= (k+1)^2 + n-2, \quad C_1 = 2nk - 3k^2 - 2k - 2n + 5, \\ C_2 &= n^2 - 4nk + 3k^2 + 4n - 2k - 5, \quad C_3 = -(n-k+1)^2 - n + 2. \end{aligned}$$

Notice that $I_{n-1,k} \geq I_{n-1,k-1} \geq I_{n-1,k-2}$ for $1 \leq k \leq (n-1)/2$. By Lemma 3.1, we have

$$B_0I_{n-1,k} + B_1I_{n-1,k-1} + B_2I_{n-1,k-2} \geq 0. \quad (3.4)$$

It remains to show that

$$C_0I_{n-2,k} + C_1I_{n-2,k-1} + C_2I_{n-2,k-2} + C_3I_{n-2,k-3} \geq 0, \quad \forall 1 \leq k \leq (n-1)/2. \quad (3.5)$$

We need to consider the following two cases:

- If $1 \leq k \leq (n-2)/2$, then

$$I_{n-2,k} \geq I_{n-2,k-1} \geq I_{n-2,k-2} \geq I_{n-2,k-3}$$

by the induction hypothesis, and

$$\begin{aligned} C_0 &= (k+1)^2 + n-2 \geq 0, \quad C_0 + C_1 = (2k-1)(n-k-1) + k+3 \geq 0, \\ C_0 + C_1 + C_2 &= (n-k+1)^2 + n-2 \geq 0, \quad C_0 + C_1 + C_2 + C_3 = 0. \end{aligned}$$

- If $k = (n - 1)/2$, then by symmetry and the induction hypothesis,

$$I_{n-2,k-1} \geq I_{n-2,k} = I_{n-2,k-2} \geq I_{n-2,k-3}.$$

In this case, we have $C_1 = (n - 3)(n - 7)/4 \geq 0$ for $n \geq 7$.

Therefore, by Lemma 3.1 the inequality (3.5) holds. It follows from (3.3)–(3.5) that

$$I_{n,k} - I_{n,k-1} \geq 0, \quad \forall 1 \leq k \leq (n - 1)/2.$$

This completes the proof. ■

4 Further remarks and open problems

Since $I_{n,k} = I_{n,n-1-k}$, we can rewrite $I_n(t)$ as follows:

$$I_n(t) = \sum_{k=0}^{n-1} I_{n,k} t^k = \begin{cases} \sum_{k=0}^{n/2-1} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is even,} \\ I_{n,(n-1)/2} t^{(n-1)/2} + \sum_{k=0}^{(n-3)/2} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is odd.} \end{cases}$$

Applying the well-known formula

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j},$$

we obtain

$$I_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}, \quad (4.1)$$

where

$$a_{n,k} = \begin{cases} \sum_{j=0}^k (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k+1 < n, \\ I_{n,k} + \sum_{j=0}^{k-1} (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k+1 = n. \end{cases}$$

The first values of $a_{n,k}$ are given in Table 2, which seems to suggest the following conjecture.

Conjecture 4.1. *For $n \geq 1$ and $k \geq 0$, the coefficients $a_{n,k}$ are nonnegative integers.*

Table 2: Values of $a_{n,k}$ for $n \leq 16$ and $0 \leq k \leq \lfloor (n-1)/2 \rfloor$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1			0	1	2	4	6	9	12	16	20	25	30	36	42	49
2					2	6	18	39	79	141	239	379	579	849	1211	1680
3							0	18	78	272	722	1716	3626	7160	13206	23263
4									20	124	668	2560	8360	23536	59824	139457
5											32	700	4800	24160	95680	325572
6													440	5480	44632	257964
7															2176	44376

Since the coefficients of $t^k(1+t)^{n-2k-1}$ are symmetric and unimodal with center of symmetry at $(n-1)/2$, Conjecture 4.1 is stronger than the fact that the coefficients of $I_n(t)$ are symmetric and unimodal. A more interesting question is to give a combinatorial interpretation of $a_{n,k}$. Note that the Eulerian polynomials can be written as

$$A_n(t) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} c_{n,k} t^k (1+t)^{n-2k+1},$$

where $c_{n,k}$ is the number of increasing binary trees on $[n]$ with k leaves and no vertices having left children only (see [1, 6, 7]).

We now proceed to derive a recurrence relation for $a_{n,k}$. Set $x = x(t) = t/(1+t)^2$ and

$$P_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} x^k.$$

Then we can rewrite (4.1) as

$$I_n(t) = (1+t)^{n-1} P_n(x). \quad (4.2)$$

Differentiating (4.2) with respect to t we get

$$I'_n(t) = (n-1)(1+t)^{n-2} P_n(x) + (1+t)^{n-1} P'_n(x) x'(t), \quad (4.3)$$

$$\begin{aligned} I''_n(t) &= (n-1)(n-2)(1+t)^{n-3} P_n(x) + 2(n-1)(1+t)^{n-2} P'_n(x) x'(t) \\ &\quad + (1+t)^{n-1} P''_n(x) (x'(t))^2 + (1+t)^{n-1} P'_n(x) x''(t), \end{aligned} \quad (4.4)$$

$$x'(t) = \frac{1-t}{(1+t)^3}, \quad x''(t) = \frac{2t-4}{(1+t)^4}. \quad (4.5)$$

Substituting (4.2)–(4.5) into (2.8), we obtain

$$\begin{aligned}
& n(1+t)^{n-1}P_n(x) \\
&= [1 + (2n-2)t + t^2](1+t)^{n-3}P_{n-1}(x) + t(1-t)^2(1+t)^{n-5}P'_{n-1}(x) \\
&\quad + [-(t^2 + 14t + 1)(1-t)^2 + (1 + 6t - 18t^2 + 6t^3 + t^4)n + 4t^2n^2](1+t)^{n-5}P_{n-2}(x) \\
&\quad + [3t(t^2 - 4t + 1)(1-t)^2 + 4t^2(1-t)^2n](1+t)^{n-7}P'_{n-2}(x) \\
&\quad + t^2(1-t)^4(1+t)^{n-9}P''_{n-2}(x). \tag{4.6}
\end{aligned}$$

Dividing the two sides of (4.6) by $(1+t)^{n-1}$ and noticing that $t/(1+t)^2 = x$, after a little manipulation we get

$$\begin{aligned}
nP_n(x) &= [1 + (2n-4)x]P_{n-1}(x) + (x-4x^2)P'_{n-1}(x) \\
&\quad + [(n-1) + (2n-8)x + 4(n-3)(n-4)x^2]P_{n-2}(x) \\
&\quad + [3x + (4n-30)x^2 + (72-16n)x^3]P'_{n-2}(x) + (x^2-8x^3+16x^4)P''_{n-2}(x).
\end{aligned}$$

Extracting the coefficients of x^k yields

$$\begin{aligned}
na_{n,k} &= a_{n-1,k} + (2n-4)a_{n-1,k-1} + ka_{n-1,k} - 4(k-1)a_{n-1,k-1} \\
&\quad + (n-1)a_{n-2,k} + (2n-8)a_{n-2,k-1} + 4(n-3)(n-4)a_{n-2,k-2} \\
&\quad + 3ka_{n-2,k} + (4n-30)(k-1)a_{n-2,k-1} + (72-16n)(k-2)a_{n-2,k-2} \\
&\quad + k(k-1)a_{n-2,k} - 8(k-1)(k-2)a_{n-2,k-1} + 16(k-2)(k-3)a_{n-2,k-2}.
\end{aligned}$$

After simplification, we obtain the following recurrence formula for $a_{n,k}$.

Theorem 4.2. *For $n \geq 3$ and $k \geq 0$, there holds*

$$\begin{aligned}
na_{n,k} &= (k+1)a_{n-1,k} + (2n-4k)a_{n-1,k-1} + [k(k+2) + n-1]a_{n-2,k} \\
&\quad + [(k-1)(4n-8k-14) + 2n-8]a_{n-2,k-1} + 4(n-2k)(n-2k+1)a_{n-2,k-2}, \tag{4.7}
\end{aligned}$$

where $a_{n,k} = 0$ if $k < 0$ or $k > (n-1)/2$.

Note that, if $n \geq 2k+3$, then

$$(k-1)(4n-8k-14) + 2n-8 > 0 \quad \text{for any } k \geq 1,$$

and so are the other coefficients in (4.7). Therefore, Conjecture 4.1 would be proved if one can show that $a_{2n+1,n} \geq 0$ and $a_{2n+2,n} \geq 0$.

Finally, from (4.1) it is easy to see that

$$a_{2n+1,n} = (-1)^n I_{2n+1}(-1) = \sum_{k=0}^{2n} (-1)^{n-k} I_{2n+1,k},$$

$$a_{2n+2,n} = (-1)^n I'_{2n+2}(-1) = \sum_{k=1}^{2n+1} (-1)^{n+1-k} k I_{2n+2,k}.$$

Thus, Conjecture 4.1 is equivalent to the *nonnegativity* of the above two alternating sums.

Since $J_{2n,k} = J_{2n,2n-k}$, in the same manner as $I_n(t)$ we obtain

$$J_{2n}(t) = \sum_{k=1}^n b_{2n,k} t^k (1+t)^{2n-2k},$$

where

$$b_{2n,k} = \begin{cases} \sum_{j=1}^k (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k < n, \\ J_{2n,k} + \sum_{j=1}^{k-1} (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k = n. \end{cases}$$

Now, it follows from (2.2) that

$$J_{2n,k} = \sum_{i=0}^k (-1)^{k-i} \binom{2n+1}{k-i} \binom{i(i+1)/2 + n - 1}{i(i+1)/2 - 1}$$

is a polynomial in n of degree $d := k(k+1)/2 - 1$ with leading coefficient $1/d!$, and so is $b_{2n,k}$. Thus, we have $\lim_{n \rightarrow +\infty} b_{2n,k} = +\infty$ for any fixed $k > 1$.

The first values of $b_{2n,k}$ are given in Table 3, which seems to suggest

Conjecture 4.3. *For $n \geq 9$ and $k \geq 1$, the coefficients $b_{2n,k}$ are nonnegative integers.*

Similarly to the proof of Theorem 4.2, we can prove the following result.

Theorem 4.4. *For $n \geq 2$ and $k \geq 1$, there holds*

$$2nb_{2n,k} = [k(k+1) + 2n - 2]b_{2n-2,k} + [2 + 2(k-1)(4n-4k-3)]b_{2n-2,k-1} \\ + 8(n-k+1)(2n-2k+1)b_{2n-2,k-2}.$$

where $b_{2n,k} = 0$ if $k < 1$ or $k > n$.

Theorem 4.4 allows us to reduce the verification of Conjecture 4.3 to the boundary case $b_{2n,n} \geq 0$ for $n \geq 9$.

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Table 3: Values of $b_{2n,k}$ for $2n \leq 24$ and $1 \leq k \leq n$.

$k \setminus 2n$	2	4	6	8	10	12	14	16	18	20	22	24
1	1	1	1	1	1	1	1	1	1	1	1	1
2		-1	-1	0	2	5	9	14	20	27	35	44
3			3	12	36	91	201	399	728	1242	2007	3102
4				-7	-10	91	652	2593	7902	20401	46852	98494
5					25	219	1710	10532	50165	194139	639968	1861215
6						-65	249	11319	122571	841038	4377636	18747924
7							283	6586	135545	1737505	15219292	101116704
8								-583	33188	1372734	24412940	277963127
9									4417	379029	16488999	367507439
10										1791	3350211	203698690
11											133107	36903128
12												761785

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