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# The covariance of GPS coordinates and frames

Marc Lachièze-Rey  
 CNRS APC, UMR 7164  
 Service d'Astrophysique, CE Saclay,  
 91191 Gif sur Yvette Cedex, France  
 e-mail: marclr@cea.fr

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## Abstract

We explore, in the general relativistic context, the properties of the recently introduced GPS coordinates, as well as those of the associated frames and coframes that they define. We show that they are covariant, and completely independent of any observer. We show that standard spectroscopic and astrometric observations allow any observer to measure, from (i) the values of the GPS coordinates at his position, (ii) the components of his [four-]velocity and (iii) the components of the metric in the GPS frame. This provides to this system an unique value both for conceptual discussion (no frame dependence) and for practical use (involved quantities are directly measurable): localisation, motion monitoring, astrometry, cosmography, tests of gravitation theories.

We show explicitly, in the general relativistic context, how an observer may estimate its position and motion, and reconstruct the components of the metric. This arises from two main results: the extension of the velocity fields of the probes to the whole (curved) spacetime; and the identification of the components of the observer's velocity in the GPS frame with the (inversed) observed redshifts of the probes. Specific cases (non relativistic velocities; Minkowski and Friedmann-Lemaître spacetimes; geodesic motions) are studied in details.

## 1 Introduction and notations

In general relativity, “ real ” quantities are represented by covariant objects, i.e., defined independently of any frame or system of coordinates. A moving object (hereafter, a *probe*) is represented by its worldline, and the associated [four-]velocity, [four-]acceleration vectors... All these entities are covariant. An observer, as a particular case of moving object,

is also represented by its (covariant) world line, velocity and acceleration. An observation requires an observer and an observed system, both described by covariant quantities. Any observable quantity is a scalar covariant combination (usually a tensorial contraction) of covariant quantities associated to the observer and the observed system.

Most often, calculations in general relativity cannot be performed without the auxiliary introduction of specific frames (or coordinates). This perfectly justified procedure introduces however a risk of misinterpretation of the intermediary frame-dependent (thus, non covariant) quantities introduced. In addition, one has to distinguish between observer-dependent and observer-independent results. Thus, one has usually the alternative between covariant and observable quantities on the one hand, and frame-dependent quantities on the other hand, which can be more easy to calculate but require special care to be converted into really observable quantities.

This motivates the introduction of the following GPS coordinates which are covariant and observable. Let us emphasize that the quasi totality of coordinate systems used in general relativity calculations do not offer these advantages. In addition, this system of coordinates is completely observer independent. This allows easy comparison of measurements made by different observers, or by the same observer at different locations. This makes possible to know what another observer would observe in the same situation. Finally, even for a well defined observer, there is no global frame canonically defined (see, e.g., [3]), so that any observer-dependent frame involves an undesirable arbitrariness, which is removed with the GPS coordinates.

GPS coordinates have been recently introduced by Rovelli [5]. They are covariant; they represent directly measurable quantities, and they are independent of the observer. They show some similarities with the *optical* coordinates introduced by [6], which however are observer dependent. The denomination “ GPS coordinates ” comes from the fact that their definition assumes the existence of four reference probes, a situation which corresponds to the well known Global Positioning System (GPS, see for instance [4]). However, their use is much more general, since they can be defined from any system of “ probes ”, that we define here as objects emitting a signal. They can be part of system of artificial satellites (like the GPS), but also stars, pulsars, galaxies... The only requirement is that an observer is able to measure the direction and the arrival time of the emitted signal (in his proper time) and its redshift. It is shown below how this allow to construct the GPS coordinates and different additional quantities. This makes GPS coordinates very convenient for localization, astrometry, tests of gravitation theories, cosmography. Note that the situation can be improved in especially designed experiments, where the probes send additional information, like for instance the instant (in any time coordinate) of

the signal emission, but we will not consider explicitly such cases in this paper.

In [5], Rovelli has introduced several fundamental properties of the GPS coordinates system, and derived their explicit calculation in Minkowski spacetime. Here we extend these calculations, and introduce the *frames* and *coframes* defined by the GPS coordinates. We show that they are also covariant, in the general relativistic meaning; that they do not depend on any observer, and that their components can be measured, in a manner that we specify explicitly. Our calculations apply to an arbitrary spacetime with curvature. They are valid in the context of any metric theory (including general relativity) and thus allow potential applications to the checks of gravitation theories.

#### Observer's frames

On the other hand, it may be convenient for an observer to chose a preferred frame for making his calculation, expressing and interpreting his observational results. The timelike direction defined by his four velocity allows a local (strictly speaking, infinitesimal ) space + time splitting. We will indicate precisely the passage between an arbitrary frame (for instance one linked to an observer) and the GPS frame. We show the nice property that the (inverse) velocity components of the observer, in the GPS frame, identify with the set of the redshifts he receives from the probes.

After recalling some geometrical results of differential geometry (1.1), section (2) first considers the question of the observation of a probe by an observer, in a metric theory. Then, in (3), we study the coordinates, frames and coframes defined by a set of four such probes. In (4), we present potential applications, and finally (5), we apply our results to the simplified case of Minkowski spacetime, and recover some results already found in the literature.

## 1.1 Geometrical preliminaries

At any point  $m$  of a differential manifold  $\mathcal{M}$  (here, spacetime), the tangent space  $T_m\mathcal{M}$  is dual to the cotangent space  $T_m^*\mathcal{M}$ . The duality is however not canonical, in the sense that there is no natural association of a one-form to a vector. There is however a natural way to define the dual of a frame  $(e_\mu)$  as the coframe  $(e^\mu)$  defined through

$$\langle e^\mu, e_\nu \rangle = \delta_\nu^\mu.$$

Each  $e_\mu$  is a vector; each  $e^\mu$  is a one form, and the brackets represent the action of one-forms on vectors. Note that this frame duality does not depend on any metric.

We assume now a (Lorentzian) metric  $g$ . A *null frame*  $(e_\mu)$  is such that

$$g_{\mu\mu} \equiv g(e_\mu, e_\mu) \equiv e_\mu \cdot e_\mu = 0, \quad \forall \mu$$

(no sum over indices). A *null coframe* is such that

$$g^{\mu\mu} \equiv g(e^\mu, e^\mu) \equiv e^\mu \cdot e^\mu = 0, \quad \forall \mu.$$

Note that the dual of a null coframe is not a null frame, and reciprocally.

To any vector  $V$ , the *musical isomorphism* (or “canonical isomorphism”) associates the one-form  $\flat V$  defined through

$$g(V, W) \equiv V \cdot W = \langle \flat V, W \rangle, \quad \forall W.$$

Similarly, to a one-form  $\theta$ , one associates the vector  $\sharp\theta$  such that  $\flat(\sharp\theta) = \theta$ .

To a given frame  $(e_\mu)$ , the musical isomorphism associates the coframe  $(\flat e_\mu)$ . In general, it does not coincide with the dual coframe  $(e^\mu)$ . Only when the frame is orthogonal (as it is common in standard calculations), the two coframes coincide, up to constants. This is not the case for null frames or coframes.

In the following, we will consider the holonomic coframe  $(\omega^\alpha = ds^\alpha)$  associated to the GPS coordinates  $(s^\alpha)$ , and its dual frame  $(\omega_\alpha = \frac{\partial}{\partial s^\alpha})$ ; but also the musically transformed frame  $(\Omega_\alpha \equiv \sharp\omega^\alpha)$ , and its dual coframe  $(\Omega^\alpha \equiv \flat\omega_\alpha)$ .

Let us also recall that a frame is holonomic, when it is linked to (local) coordinates. In this case, one can write (locally):

$$\omega^\alpha = ds^\alpha, \quad \omega_\alpha = \frac{\partial}{\partial s^\alpha} = \partial_\alpha.$$

## 2 Observing a probe

### 2.1 Probes

Very generally, we call a *probe* any object emitting an observable signal, which can be used as a reference. The only requirement is that an observer can measure the arrival time of a signal (in his proper time) and its redshift. As we will see below, this gives direct access to the proper time of the signal emission by the probe, which is at the basis of the formalism. Typically, the probe can be a *GPS satellite*, hence the denomination. It can also be a *star* (in particular a pulsar) sending its radiation, which may constitute an efficient astrometric tool. This can be also a distant *galaxy*, or a quasar, in cosmology. In some circumstances, the probe can be a member of an especially designed system of satellites (e.g., the GPS system or the LISA experiment). In this case, these satellites can send some additional information, offering additional possibilities, including the tests of metric theories. In this paper, we do not explore such opportunities, and only consider the

basic case where we can measure the arrival time and direction, as well as the redshift of the signals, from which results the (probe's) proper time of emission,  $s$ .

The motion of the probe (assumed to be given) is defined by its time line  $s \mapsto P(s)$ , with  $s$  its proper time. We do not assume geodesic motion. Although, in general, the observer has no direct access to  $s$ , he can in fact monitor its value by integration of the redshift, as shown below. Thus we consider  $s$  as a measurable quantity. The (normalized) velocity of the probe,  $u \equiv \frac{d}{ds} |_P$ , defines a vector field along its world line. It verifies  $u \cdot u = 1$ . We will show below how to extend this velocity field to the whole spacetime.

The proper time of the probe,  $s$ , is defined along the world line  $P$  only. We extend it as a function  $x \mapsto s(x)$  defined "everywhere" (hereafter, "everywhere" means the region of spacetime where these coordinates are well defined; see the discussion by [5]) : at any event  $x$  in spacetime,  $s(x)$  is defined as the proper time of the probe, when it emitted the light ray reaching  $x$  (see the discussion in [5] for the case where many light-rays reach the observer). In other words, the hypersurface  $\Sigma_{s_0}$ , of equation  $s = s_0$  is the future light cone of the probe at its proper time  $s_0$  (figure 1). This is a null hypersurface (for the properties of null hypersurfaces, see the paper by [2]). It is a fundamental fact that  $s$  is an observable quantity.

Each  $\Sigma_s$  admits a one dimensional vector space of vectors orthogonal to it, at any point. These vectors are all proportional to  $\sharp ds$ . Since the surface is null, these vectors are also tangent to it, and they are null vectors ( $\sharp ds \cdot \sharp ds = 0$ ). Their integral lines are null geodesics which constitute the null generators of  $\Sigma_s$ . They are the future directed light rays emanating from the event  $P(s)$ . In a null surface, the null vectors cannot be normalized like for a spatial surface. However, one may select here an unique orthogonal vector, from the form  $\omega \equiv ds$ , which is well defined over  $\Sigma_s$  (except on the line  $P$  itself, where  $\Sigma_s$  becomes singular). The canonical isomorphism (generated by the metric  $g$  of spacetime) transforms the one form  $\omega$  to the vector  $\Omega \equiv \sharp \omega$  such that :

$$\langle \omega, \Omega \rangle \equiv g(\Omega, \Omega) \equiv \Omega \cdot \Omega \equiv g(\omega, \omega) = 0$$

(see the detailed proofs in [2]). This implies

$$\Omega \cdot \nabla \Omega \equiv \nabla_{\Omega} \Omega = \Omega \cdot \nabla \omega = 0,$$

which means that  $\Omega$  and  $\omega$  are parallelly transported by  $\Omega$ , that the null lines generated by  $\Omega$  are geodesic, with  $\Omega$  an affine vector along them. The vector  $\Omega$  is the *frequency vector* [1] of the light ray (here normalized so that the emitted frequency is unity).

Note that  $\Omega$  and  $\omega$  are not defined on the world line  $P$  itself. However, for each light-ray, it is possible to extend them by continuity, so

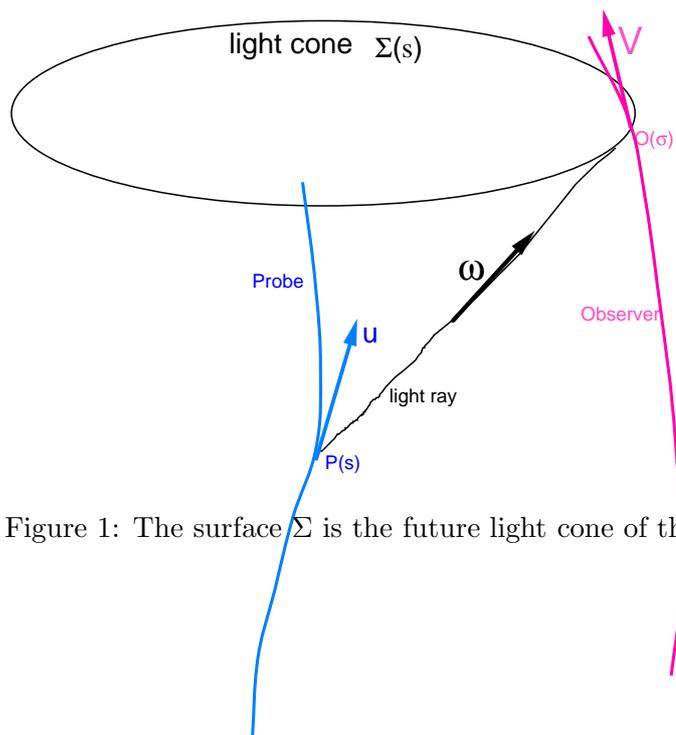


Figure 1: The surface  $\Sigma$  is the future light cone of the probe.

that they are parallelly transported : this allows us to consider parallel transport along the null geodesics, including their intersections with  $P$ .

Note that the function  $s(x)$  obeys the equation

$$\mathcal{W}[s(x), x] = 0, \tag{1}$$

where  $\mathcal{W}(x, y)$  is the *world function* [6], defined as half the geodesic distance between the two points of spacetime  $x$  and  $y$ .

## 2.2 Extending the velocity of the probe

Now let us define “ everywhere ” the time like vector field  $u$  obeying the following requirements:

- it is normalized:  $u \cdot u = 1$ ;
- it is parallelly transported along the null generators:

$$\Omega \cdot \nabla u = \nabla_{\Omega} u = 0.$$

Note that we have extended above the parallel transport up to the line  $P(s)$  itself, and we require that  $u$  is parallelly transported in this way.

- it coincides with the velocity of the probe  $P$  along its worldline.

Since  $\Omega$  is also parallelly transported by itself, this implies that  $\langle u, \Omega \rangle = u \cdot \Omega$  is constant along the null generators (the light rays from  $P$ ).

To apply the third condition, we consider the integral 2-surface  $\mathcal{H}$  generated by the two vector fields  $u$  and  $\Omega$  (see figure 2). These two vector fields constitute a moving frame of  $\mathcal{H}$ , including the world line  $P$ . On  $P$ , the scalar product  $\Omega \cdot u = 1$  because there the velocity  $u = \frac{d}{ds}$ . Since this product is preserved by the null generators, as shown above, this implies that it keeps the value 1 on  $\Sigma_s$ :

$$\Omega \cdot u = 1 \text{ everywhere,}$$

one main result of this paper. This allows the decomposition

$$\Omega = u + \nu,$$

where the (spacelike) vector  $\nu$  verifies  $\nu \cdot \nu = -1$ ,  $\nu \cdot u = 0$ .

The vector field  $u$  may be seen as the velocity of the probe, transported everywhere by the null generators.

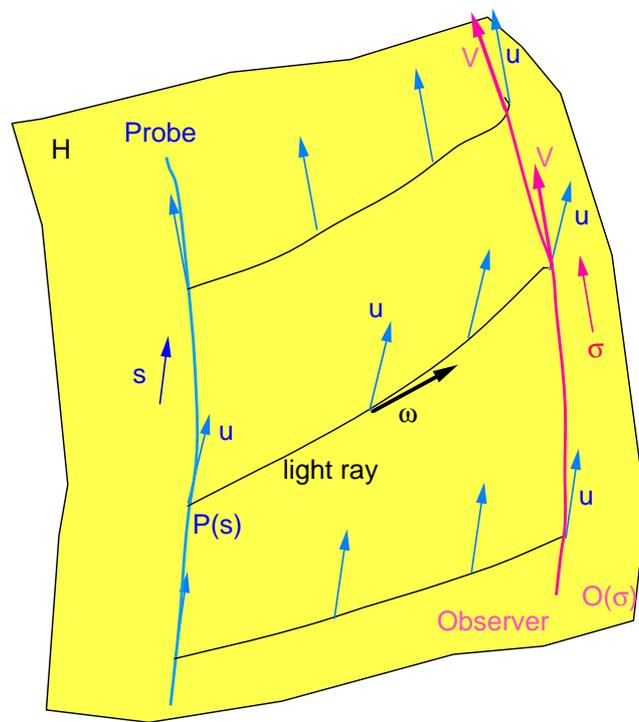


Figure 2: The surface  $\mathcal{H}$  contains the world line of the probe and of the observers, as well as the light rays from the former to the latter.

### 2.3 Observers and redshifts

An observer is defined by his worldline  $\sigma \mapsto O(\sigma)$ , along which flows his proper time  $\sigma$ . His normalized velocity  $V \equiv \frac{d}{d\sigma}$ , with  $V \cdot V = 1$ . By definition, he sees the probe with a redshift

$$1 + z = \frac{d\sigma}{ds},$$

along his world line.

Since he can monitor this redshift as a function of his proper time,  $z(\sigma)$ , this gives him access (up to an additional constant) to the proper time of the probe at the moment of emission:

$$s(\sigma) = \int d\sigma \frac{ds}{d\sigma} = \int d\sigma \frac{1}{1 + z(\sigma)} + C^{te}.$$

This makes  $s(\sigma)$  an observable quantity, as announced.

On the other hand, it is easy to show that

$$1 + z = \frac{u(P) \cdot \Omega(P)}{V \cdot \Omega(O)} = \frac{1}{V \cdot \Omega(O)} = \frac{1}{\langle \omega, V \rangle (O)}, \quad (2)$$

where  $(O)$  and  $(P)$  mean that the quantities are evaluated at observer's and probe's position respectively. Note that  $z(\sigma)$  is a perfectly measurable, and covariant quantity. But it is observer dependent (it depends on his velocity). On the other hand,  $s(\sigma)$ , also covariant and measurable, is observer-independent.

It is a fundamental fact that the observer is able to estimate the value of  $s$  by monitoring the redshift as a function of his proper time,  $z(s)$ , and by integrating this relation. (In some specifically designed experiments, like in a future version of the GPS system, the probe may emit explicitly the value of  $s$ , which would allow additional possibilities.)

By projecting the velocity  $V$  of the observer onto the vector  $\Omega$ , one obtains

$$\Omega = \frac{1}{1 + z} (V - n), \quad n \cdot n = -1, \quad n \cdot V = 0, \quad V \cdot V = 1.$$

The vector  $n$  represents the spatial (unit) direction in which the observer sees the probe. For the observer, this vector is purely spatial in the sense that it is orthogonal to his velocity  $V$ , which defines the time direction for him. Note that the observer is able to monitor this direction, which is also a covariant (but observer dependent) quantity.

Taking into account  $u \cdot \Omega = 1$ , this relation implies

$$1 + z = u \cdot V - u \cdot n. \quad (3)$$

We emphasize that these relations hold in any spacetime, for any observer, and for any metric theory.

### The surface $\mathcal{H}$

We have called  $\mathcal{H}$  the 2-dimensional surface generated by the probe world line, and all the light-rays reaching the observer, at successive moments (see figure 2). It also contains the observer's world line. The vectors  $u$  and  $\Omega$  are defined everywhere in this surface, thanks to the mentioned extension, and provide a (moving) frame for  $\mathcal{H}$ .

In  $\mathcal{H}$ , any tensor may be decomposed on this frame  $(u, \Omega)$ . The induced metric on  $\mathcal{H}$  reads

$$g_{\mathcal{H}} = ds \otimes bu + bu \otimes ds + -ds \otimes ds.$$

To the frame  $\omega, u$  corresponds the dual coframe  $(\underline{\Omega} = bu - \omega, \underline{u} = \omega)$ . The same metric also reads

$$g_{\mathcal{H}} = \underline{u} \otimes \underline{u} + \underline{u} \otimes \underline{\Omega} + \underline{\Omega} \otimes \underline{u}.$$

Note that  $u - V$  represents the relative [four-]velocity of the probe with respect to the observer.

## 2.4 Non relativistic motion

All formulae above hold for arbitrary velocities. If we work in the Solar System, or in the Galaxy (i.e., for instance, with pulsars), the velocities involved are most often non relativistic, which corresponds to  $z \ll 1$ . All scalar quantities can be developed in this small parameter. We have for instance

$$\begin{aligned} \Omega \cdot V &= \frac{1}{1+z} \simeq 1 - z, \\ u \cdot V &= 1 + \mathcal{O}(z^2). \end{aligned}$$

The first relation implies

$$\Omega \cdot (u - V) \approx z.$$

The expansion of (3) gives

$$u \cdot n = -z + \mathcal{O}(z).$$

This relation was obtained in Minkowski spacetime by [7], and is at the basis of their analysis of pulsar timing. We have shown here that it remains true in an arbitrary spacetime.

When the probe (e.g., a pulsar) has an intrinsic period  $T$ , the observer monitors a period

$$T_{obs} = T (1 + z) = 1 - u \cdot n + \mathcal{O}(z^2).$$

To continue, we may assume that the pulsar has an intrinsic variation  $\dot{T}$  of its period. Then, by writing (3) at two successive instants, one obtains the observed period variation

$$\dot{T}_{obs} \equiv \frac{dT_{obs}}{d\sigma} = \dot{T} (1 + z) + T \frac{dz}{d\sigma}. \quad (4)$$

This is an exact formula. However, the expression of  $\frac{dz}{d\sigma}$  is quite complicated. [7] have given an approximation of it, which is valid for small redshifts, and in Minkowski spacetime.

## 2.5 Geodesic motion

Other simplifications occur when both the probe and the observer have geodesic motion:

$$u \cdot \nabla u = 0, \quad V \cdot \nabla V = 0.$$

Let us work in the surface  $\mathcal{H}$ : since  $u$  is parallelly transported by the two basis vectors, it verifies  $\nabla u = 0$  in  $\mathcal{H}$ . This implies  $V \cdot \nabla u = 0$  and, thus

$$V \cdot \nabla(u \cdot V) \equiv \frac{d(u \cdot V)}{d\sigma} = 0.$$

Using the above decomposition  $\Omega = u + \nu$ , this implies, after derivation w.r.t. to the observer's proper time,

$$\frac{dz}{d\sigma} = -\frac{d(u \cdot n)}{d\sigma}.$$

This relation may be used by the observer to check that he is in inertial motion with respect to the probe (see [7], who derived an explicit expression for this quantity, at first order, in Minkowski spacetime).

## 2.6 Application to Minkowski spacetime

The case of Minkowski spacetime is particularly simple. Without loss of generality, we can place the observer at the center of spatial coordinates:  $O = (t = \sigma, 0, 0, 0)$ , so that his velocity has components  $V = (1, 0, 0, 0)$ . A radial light-ray arriving to the observer has a (future directed) affine tangent vector  $\Omega$ , with components  $(K, -K, 0, 0)$ , with  $K > 0$ . And the probe has a velocity  $(u^t, u^r, u^\theta, u^\varphi)$ , with  $u^r > 0$  for a probe moving away from the source. Then it is trivial to form the scalar products  $\Omega \cdot V = K$  and  $\Omega \cdot u = K (u^t + u^r)$ , from which it results

$$1 + z = u^t + u^r \approx 1 + u^r,$$

where the latter approximation holds for non relativistic motion.

## 2.7 Application to cosmology

In a Friedmann-Lemaître universe, with metric in the Robertson-Walker form

$$g = dt^2 - a(t)^2 [dr^2 + f(r)^2 (d\theta^2 + \sin^2 \theta d\varphi^2)],$$

the observer is assumed at the center of spatial coordinates, so that his velocity has components  $V = (1, 0, 0, 0)$ . The null radial vector  $\Omega$  has components  $(\Omega^t, -\Omega^t/a(t), 0, 0)$ , with metric dual  $\omega = (\Omega^t, +a(t)\Omega^t, 0, 0)$ . The requirement to be an affine vector implies  $\Omega^t = K/a(t)$ , with  $K$  a constant along the light ray. The value of the latter is fixed by the condition

$$1 = u \cdot \Omega = \frac{K}{a(P)} [u^t(P) + a(P) u^r(P)],$$

where  $(P)$  means evaluated at the position of the probe. This gives the extension of the velocity field of the probe everywhere, as

$$u = (u^t = \frac{a(t)}{a(P)} u^t(P), u^r = u^r(P), u^\theta = u^\theta(P), u^\varphi = u^\varphi(P)).$$

Using  $\Omega \cdot V = \Omega^t = K/a(t)$ , we obtain the redshift

$$1 + z = \frac{a(t)}{K} = \frac{a(t)}{a(P)} [u^t(P) + a(P) u^r(P)],$$

which reduces to the usual formula  $1 + z = \frac{a(t)}{a(P)}$  for a comoving galaxy, defined by  $u^r(P) = 0$ ,  $u^t(P) = 1$ .

## 3 GPS coordinates and frames

### 3.1 Coframes and frames

Now we assume four probes  $P^\alpha$  ( $\alpha = 1, 2, 3, 4$ ) and adopt, like in [5], the four corresponding functions  $s^\alpha$  as coordinates (see [5] for discussions about the range of validity of these coordinates, to which we refer as “everywhere”). As emphasized above, these coordinates are measurable. They define the coframe ( $\omega^\alpha \equiv ds^\alpha$ ). By definition, the contravariant components of the metric tensor are given by

$$g^{\alpha\beta} = \omega^\alpha \cdot \omega^\beta.$$

As we have seen in (2), each  $s^\alpha$  is a null function, so that

$$g^{\alpha\alpha} = \omega^\alpha \cdot \omega^\alpha = 0.$$

This means that  $(\omega^\alpha)$  is a null coframe.

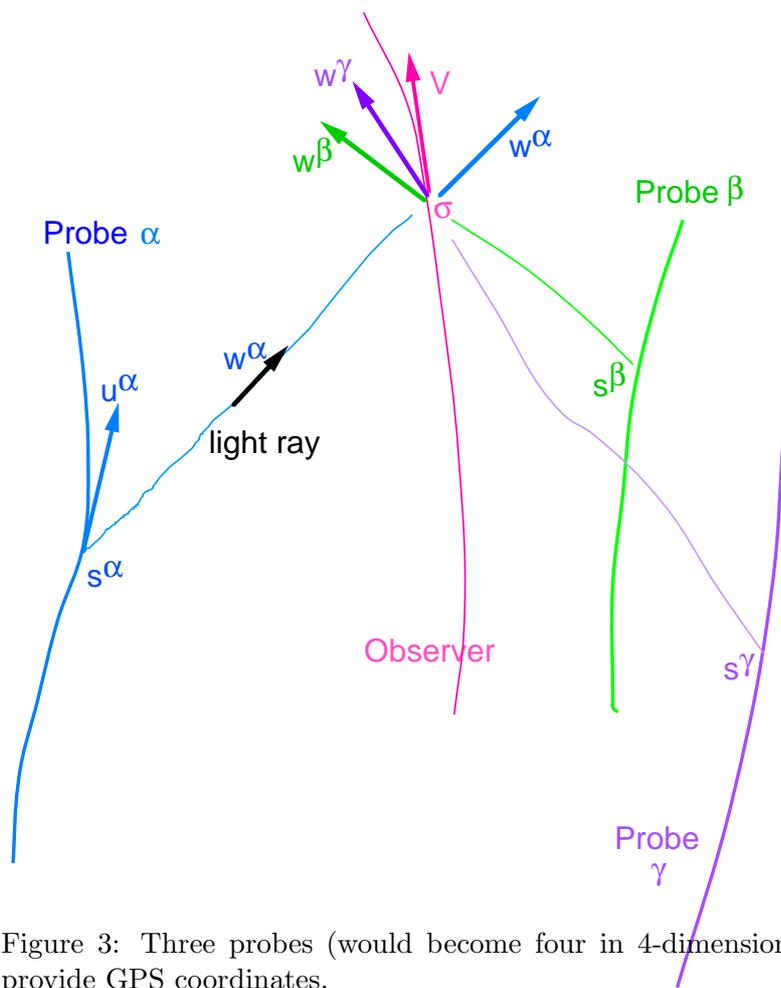


Figure 3: Three probes (would become four in 4-dimensional spacetime) provide GPS coordinates.

$\omega^\alpha = ds^\alpha$	$\omega_\alpha = \partial_\alpha$	$\Omega^\alpha = \flat\omega_\alpha = \flat\partial_\alpha$	$\Omega_\alpha = \sharp\omega^\alpha$
null holonomic coframe	holonomic frame	coframe	null frame

Table 1: Frames and coframes linked to the GPS coordinates

This coframe is holonomic. Its dual frame, given by  $(\omega_\alpha \equiv \partial_\alpha \equiv \frac{\partial}{\partial s^\alpha})$ , obeys the duality relations  $\langle \omega^\alpha, \partial_\beta \rangle = \delta_\beta^\alpha$ . We have by definition,

$$g_{\alpha\beta} = \partial_\alpha \cdot \partial_\beta.$$

The covariant components  $g_{\alpha\beta}$  of the metric are defined, as usual, from the relations  $g_{\alpha\beta} = g^{\gamma\delta} g_{\alpha\gamma} g_{\beta\delta}$ . Note that, in general,  $g_{\alpha\alpha} \neq 0$ , so that  $(\partial_\alpha)$  is not a null frame

To the system of coordinates  $(s^\alpha)$ , are naturally associated the (holonomic) frame and coframes  $\partial_\alpha$  and  $\omega^\alpha$ . These frames involve only covariant quantities. They are defined “everywhere”. They are completely independent of any observer.

The musical isomorphism (see 1) defines the vectors  $\sharp\omega^\alpha \equiv \Omega_\alpha$  (despite the upper index, the  $\sharp\omega^\alpha$  are vectors) through

$$\langle \omega^\alpha, \Omega_\beta \rangle = \omega^\alpha \cdot \omega^\beta = \Omega_\alpha \cdot \Omega_\beta = g^{\alpha\beta}.$$

Since  $g^{\alpha\alpha} = 0$ ,  $(\Omega_\alpha)$  form a null frame.

As vectors, the  $\Omega_\alpha$  's can be expanded in the  $(\partial_\alpha)$  basis:

$$\Omega_\alpha = g^{\alpha\beta} \partial_\beta \Rightarrow (\Omega_\alpha)^\beta = (\Omega_\beta)^\alpha = g^{\alpha\beta}.$$

See table 1 and figure 3 for illustrations of these frames and coframes.

### 3.2 Measurements

Measurements are performed by an observer, that we assume given by his world line  $\sigma \mapsto O(\sigma)$ ,  $\sigma$  being his proper time, that he is able to read on his clock. His (normalized) velocity vector,  $V \equiv \frac{d}{d\sigma}$ , defines a natural time direction for him, at each point of his world line; and, by orthogonality, a set of spacelike directions. In the following, we will assume that the observer used this natural time+space splitting to define (locally) an ON frame  $(e_I) \equiv (e_0 \equiv V, e_i)$ , with  $I = 0, 1, 2, 3$ ;  $i = 1, 2, 3$ . We do not care about the way the spatial part of this frame (the  $e_i$ 's) is defined.

By spectroscopic observations, the observer is able to estimate the redshifts corresponding to the signals emitted by the four probes,  $z^\alpha$ . As we will see, these observed quantities play a fundamental role. As

any vector, the observer's velocity can be expanded in the basis as  $V = V^\alpha \partial_\alpha$ . From (2), it results the very important relation

$$V \cdot \Omega_\alpha = \langle \omega^\alpha, V \rangle = V^\alpha \equiv \frac{1}{1 + z^\alpha}.$$

This implies

$$V = \frac{1}{1 + z^\alpha} \partial_\alpha :$$

the inverse components of the observer's velocity reduce to the redshifts with respect to the probes. It results that, from the observed redshifts, the observer can derive the components of his velocity in the GPS frame, and thus monitor his motion. Note that the four redshifts are linked by the relation

$$g_{\alpha\beta} \frac{1}{1 + z^\alpha} \frac{1}{1 + z^\beta} = 1.$$

The frames and coframes defined by these coordinates are also covariant and independent of the observers. However, observations do not give direct access to the components of vectors, forms or tensors. Those are generally evaluated in a given reference frame, chosen for convenience (although the physics should be independent of such a choice).

### 3.3 The reconstruction of the metric

All the components of the metric tensor (in the GPS frame) identify with covariant quantities (in general, a tensor is a covariant quantity, but not its components). In addition, they are (as they must) completely observer independent. Now we show that these components are directly measurable.

The observer is able to project any quantity (like a vector or a tensor), at his position, parallelly and orthogonally to his velocity vector  $V$ . He will interpret the projected quantities as temporal and spatial components. Note that this decomposition, although of course observer dependent, remains perfectly covariant (does not require the introduction of any frame). In particular,

$$\Omega_\alpha = V^\alpha (V - n_\alpha); \quad V^\alpha \equiv \frac{1}{1 + z^\alpha},$$

and its metric dual (musical) version

$$\omega^\alpha = V^\alpha (bV - bn_\alpha),$$

where

$$n_\alpha \cdot n_\alpha = bn_\alpha \cdot bn_\alpha = -1, \quad V \cdot n_\alpha = 0, \quad V \cdot V = 1.$$

The spatial normalized vectors  $n_\alpha$ 's represent the (spatial) directions of arrival of the signals of the four probes. The  $\flat n_\alpha$ 's are the spatial normalized one-forms, which represent the corresponding wavefronts.

The observer is able to monitor these directions (in any frame), and thus to estimate the scalar products  $K_{\alpha\beta} \equiv n_\alpha \cdot n_\beta$  (obeying  $K_{\alpha\alpha} = -1$ ). These measurable quantities are frame independent.

Some easy algebra shows

$$n_\alpha = -V^\beta K_{\alpha\beta} \partial_\beta \text{ (sum on } \beta\text{)}.$$

Also, from the relations above,

$$g^{\alpha\beta} = \Omega_\alpha \cdot \Omega_\beta = \frac{1}{1+z^\alpha} \frac{1}{1+z^\beta} (1 + K_{\alpha\beta})$$

(no sum on indices). It results that all the metric coefficients  $g^{\alpha\beta}$  are measurable by the observer. By matrix inversion, he may also reconstitute the matrix  $g_{\alpha\beta}$ .

#### Observer dependent frames

The observer may select three of these vectors,  $\Omega_i$ , that we label now with latin indices  $i, j = 1, 2, 3$ . With his velocity  $V$ , they constitute a perfectly valid frame, although observer-dependent. It was introduced by Synge [6] under the qualification of “ optical ”. Also, the three spatial vectors  $n_i$ , with  $V$ , constitute a frame of a more usual form, with one timelike, and three spacelike vectors. In this case, we will label  $V$  by the index 0, and it is easy to establish that, in this frame,

$$g_{00} = 1, \quad g_{0i} = 0, \quad g_{ij} = K^{ij}.$$

Thus, the observer is also perfectly able to reconstruct the metric coefficients in this frame.

## 4 Potential applications

Via the measured redshifts, the observer can monitor the  $s^\alpha$  as a function of his proper time. It is also able to measure the scalar products  $K^{\alpha\beta}$ , and thus to reconstruct completely the metric as indicated above. Since the GPS frame is defined everywhere, it is possible to perform the same measurements from other places : directly, following the observer's motion moving in spacetime; or indirectly, using measurements from other probes, which play the role of additional observers. The procedure above provides the metric coefficients in the different corresponding points of spacetime. By derivations, this would give access to the components of the connection and of the curvature.

In any case, the observer can monitor the redshifts along his world line, i.e., with respect to his proper time  $\sigma$ . Then,

$$\frac{dz^\alpha}{d\sigma} = A \cdot \Omega_\alpha + V \cdot \frac{d\omega_\alpha}{d\sigma}.$$

This relation allows him to calculate his own acceleration  $A$ . This could be at the basis of experimental tests of gravitation theories.

If the probes send an additional information, under the form of a variation of the period of a signal emitted by the probe (this is the case from a pulsar), the relation (4) brings then more constraints to the parameters of the system. They can be used for instance to determine the acceleration of the earth, or of the Solar System (see [7]).

#### 4.1 Localization

The observer may wish to use such coordinates for his localization. In some sense, this is a triviality since the  $s^\alpha$  are already *bona fide* coordinates, their measurements provide formally a perfect localization. In general, however, the observer is not interested by localization with respect to the probes, but rather with respect to a more usual system of coordinates  $x^\mu$ , like for instance a terrestrial one. In such a case, he would renounce to covariance, which present no interest for his purpose.

This requires no more than a simple conversion of variables ( $s^\alpha \rightsquigarrow x^\mu$ ). This possibility is guaranteed by the fact that the  $s^\alpha$  are true coordinates. Explicitly, it requires the explicit knowledge of the ephemeris  $P_{(\alpha)}^\mu(s^\alpha)$  of the four probes, in his favorite coordinate system. For a system of artificial probes, they are perfectly known, and the conversion is simply a matter of calculations, although there is no general analytic expression for them in non flat spacetime. In practice, the ephemeris has to be inserted in the world equation.

This provides the formulae for the change of coordinates,  $s^\alpha(x^\mu)$ , and its inverse  $x^\mu(s^\alpha)$ . From this, the Jacobian provides the transformation matrices for the change of frames:

$$E_\mu^\alpha \equiv \frac{\partial s^\alpha}{\partial x^\mu}, \quad E_\alpha^\mu \equiv \frac{\partial x^\mu}{\partial s^\alpha}.$$

This allows us to express the vectors [one-forms] of this frame in the other arbitrary frame [coframe].

$$\omega^\alpha = ds^\alpha = E_\mu^\alpha e^\mu, \quad \partial_\alpha = E_\alpha^\mu e_\mu,$$

$$e^\mu = E_\alpha^\mu \omega^\alpha, \quad e_\mu = E_\mu^\alpha \partial_\alpha.$$

Note that the last formulas apply even when the new frame is not holonomic (i.e., does not correspond to a system of coordinates).

These formulae would be useful when an observer wishes to evaluate tensorial components in his personal frame (at the price of losing covariance and observer's independence). Note that the knowledge of the explicit correspondence requires to know the ephemeris of the probes.

## 5 Application to Minkowski spacetime

In flat Minkowski spacetime, the properties of parallel transport imply that the four velocities  $u^\alpha$  are constant vectors. The positions  $X \equiv O(\sigma)$  of the observer, and  $P^\alpha$  of the probe may be considered as vectors, and we define the separation vectors (between the probes and the observer)

$$D^\alpha \equiv P^\alpha - X.$$

Note that  $D^\alpha \cdot D^\alpha = 0$ .

It is easy to establish the relations

$$\begin{aligned} \#ds^\alpha &\equiv \Omega_\alpha = \frac{D^\alpha}{D^\alpha \cdot u^\alpha}, \\ V^\alpha &= 1 + z^\alpha = \frac{D^\alpha \cdot V}{D^\alpha \cdot u^\alpha}, \\ \#n^\alpha &= V - \frac{D^\alpha}{D^\alpha \cdot V}. \end{aligned}$$

It results

$$g^{\alpha\beta} = \frac{D^\alpha \cdot D^\beta}{(D^\alpha \cdot u^\alpha)(D^\beta \cdot u^\beta)}.$$

If the motions of the four probes are geodesic, we have  $P^\alpha = P_0^\alpha + u^\alpha s^\alpha$ , where  $P_0^\alpha$  represents the origin of the geodesic motion of the probe  $\alpha$ . Defining the four vectors  $\xi^\alpha = X - P_0^\alpha$ , we obtain an explicit expression for the GPS coordinates,

$$s^\alpha = \xi^\alpha \cdot u^\alpha - \sqrt{(\xi^\alpha \cdot u^\alpha)^2 - \xi^\alpha \cdot \xi^\alpha}.$$

This expression has been found by [5], in the case where the four probes start at the same origin ( $P_0^\alpha = 0$ ).

### 5.1 Localization in Minkowski spacetime

The observer monitors the GPS coordinates ( $s^\alpha$ ), and wishes to know the coordinates  $x^\mu$  in his favorite frame, like the terrestrial one. The conversion requires to know the ephemeris of the four probes,  $P_{(\alpha)}^\mu(t)$ , where  $t$  is an arbitrary parameter, that we take to be the time function in the terrestrial frame. Each ephemeris may be easily explicitated, along the worldline of the probe, as a function of its proper time, giving after conversion:  $P_{(\alpha)}^\mu(t) = \Pi_{(\alpha)}^\mu(s^\alpha)$ . Then, we have simply to solve the system of four equations

$$[X^0 - \Pi_{(\alpha)}^0(s^\alpha)]^2 = [X^1 - \Pi_{(\alpha)}^1(s^\alpha)]^2 + [X^2 - \Pi_{(\alpha)}^2(s^\alpha)]^2 + [X^3 - \Pi_{(\alpha)}^3(s^\alpha)]^2,$$

for the four values of  $\alpha$ . Here, the  $\Pi_{(\alpha)}^\mu(s^\alpha)$  are the functions derived from the ephemeris, and the unknown are the four  $X^\mu$  which represent the coordinates of the observer in the desired frame.

Of course, the system becomes more complicated when there is curvature, but without problem to be solved numerically.

## 6 Conclusion and discussion

The GPS coordinates and corresponding frames offer a series of advantages. First, they are covariant quantities. It is not common in general relativity calculations to dispose of coordinates or frames with this property. This provide to them an absolute and intrinsic character, which is however balanced by the fact that they depend on the choice of a set of probes.

Second, they are completely independent of the observer. This allows different observers, or an observer at different locations in space-time, to compare different observational results and to interpret them, without involving different coordinates systems and frames.

We have shown how their use, with that of the associated frames and coframes, allow the complete reconstruction of the metric and, in some circumstances, of the curvature and connection.

In the previous analysis, we have considered four probes and an observer. But the treatment of the observer is perfectly identical to those of the probes (the velocity  $V$  of the observer is analog to that,  $u$ , of a probe; his proper time  $\sigma$  is analog to that,  $s$ , of the probe). Thus the observer can perfectly be a probe of the same type, i.e., a fifth satellite. Conversely, one of the probe can be chosen to be the terrestrial observer. In this case, one loses of course the independence w.r.t. the observer, but this case allows the easy reconstruction of a frame with more usual properties, involving timelike (for instance  $V$ ) and spacelike vectors.

The possible applications are numerous. This allows not only the (relative) localization of the terrestrial observer, but also the monitoring of its motion (velocity and acceleration), as well as the reconstruction of the metric, curvature and connection components. The latter possibilities open the opportunity to combine the different relations established above (which remain valid in any metric theory, not only in general relativity) to provide checks of the gravitation theories.

As we have remarked, the properties of a system of pulsars show many similarities with those of a GPS system. This allows to use such a system for precise astrometry, in the general relativistic context (see for instance the discussions in [7]).

Finally, the situation also applies to cosmography. Although the metric of a Friedmann-Lemaître universe model is generally put under the Robertson-Walker form, it is perfectly possible to express it by using GPS coordinates defined from a set of four distant galaxies, which may be assumed (or not) in free fall. This is feasible in different

manners, which correspond to different choices of the probes.

More generally, *any metric* can be expressed in a GPS frame. This can be done directly, by searching for the coefficients expressing the change of frames. More easily, this could result from the choice of a set of convenient probes, from which the coordinates and frames are defined as above.

The exploration of such possibilities, for cosmology, and for the Schwarzschild metric, are in progress.

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