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Contact resolutions of projectivised nilpotent orbit closures

Baohua Fu

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Abstract

The projectivised nilpotent orbit closure $\mathbb{P}(\overline{\mathcal{O}})$ carries a natural contact structure on its smooth part. A resolution $\pi : X \rightarrow \mathbb{P}(\overline{\mathcal{O}})$ is called *contact* if the contact structure on $\mathbb{P}(\mathcal{O})$ extends to a contact structure on X . It turns out that contact resolutions, crepant resolutions and minimal models of $\mathbb{P}(\overline{\mathcal{O}})$ are all the same. In this note, we determine when $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution, and in the case of existence, we study the birational geometry among different contact resolutions.

1 Introduction

A Riemannian manifold M is called *quaternion-Kähler* if its holonomy group is $Sp(n)Sp(1)$ ($n \geq 2$). Associated to M , there is a twistor space Z which is the \mathbb{P}^1 -bundle (over M) consisting of almost complex structures compatible with the quaternion-Kähler structure on M . It turns out ([Sa]) that Z is a complex manifold with a contact structure, i. e. there exists a maximally non-integrable holomorphic sub-bundle $F \subset TZ$ of codimension 1. In [Le1], LeBrun gave the inverse twistor construction, which allowed one to recover the quaternion-Kähler structure on M from the contact structure (and some extra structures) on Z . We will consider the special case of projectivised nilpotent orbits.

For a nilpotent orbit \mathcal{O} in a semi-simple complex Lie algebra \mathfrak{g} , it enjoys the following properties:

- (i) it is \mathbb{C}^* -invariant, where \mathbb{C}^* acts on \mathfrak{g} by linear scalars;

- (ii) it carries the Kirillov-Kostant-Souriau symplectic 2-form ω ;
- (iii) $\lambda^*\omega = \lambda\omega$ for any $\lambda \in \mathbb{C}^*$.

One deduces from (iii) that this symplectic structure on \mathcal{O} gives a contact structure on the projectivisation $\mathbb{P}(\mathcal{O})$. The procedure in [Le1] can be realized as following ([Sw], Theorem 4.1): there exists a free action of $\mathbb{H}^*/\pm 1$ on \mathcal{O} such that the quotient M is a quaternion-Kähler manifold with positive scalar curvature and the twistor space associated to M is isomorphic to $\mathbb{P}(\mathcal{O})$ preserving the contact structure.

However the Riemannian manifold M is in general not compact (thus the quaternion-Kähler metric is not complete). In fact, when \mathfrak{g} is simple, the variety $\mathbb{P}(\mathcal{O}) \subset \mathbb{P}(\mathfrak{g})$ is closed if and only if \mathcal{O} is the minimal nilpotent orbit \mathcal{O}_{min} (Prop. 2.6 [Be]). In this case, $\mathbb{P}(\mathcal{O}_{min})$ is a Fano contact manifold. It is generally believed that these are the only examples of such varieties ([Be], [Le2]). A positive answer to this would imply that every compact quaternion-Kähler manifold with positive scalar curvature is homothetic to a Wolf space (Theorem 3.2 [LeSa]).

If we take the closure $\overline{\mathbb{P}(\mathcal{O})} = \mathbb{P}(\overline{\mathcal{O}})$, then it is in general singular. A natural idea to construct smooth candidates of twistor spaces from W is by desingularizations. We say that a resolution $\pi : X \rightarrow \mathbb{P}(\overline{\mathcal{O}})$ is *contact* if the contact structure on $\pi^{-1}(\mathbb{P}(\mathcal{O}))$ extends to a contact structure on X . It follows that X is a projective contact manifold. Such varieties have drawn much attention recently. In particular, many partial classification results have been obtained (see for example [Pe] and the references therein).

The first aim of this note is to find all contact resolutions that $\mathbb{P}(\overline{\mathcal{O}})$ can have. More precisely we prove that (Theorem 5.3) if the normalization $\mathbb{P}(\tilde{\mathcal{O}})$ of $\mathbb{P}(\overline{\mathcal{O}})$ is not smooth, then X is isomorphic to $\mathbb{P}(T^*(G/P))$ for some parabolic sub-group P in the adjoint group G of \mathfrak{g} and π is the natural resolution. The proof relies on the main result in [KPSW] and that in [Fu]. A classification (Corollary 5.4) of \mathcal{O} such that $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution can be derived immediately, with the help of [Be].

Once we have settled the problem of existence of a contact resolution, we turn to study the birational geometry among different contact resolutions in the last section, where a description (Theorem 6.2) of the movable cone of a contact resolution is given, based on the main result in [Na]. This gives another way to prove the aforesaid result under the condition that $\overline{\mathcal{O}}$ admits a symplectic resolution, since minimal models, contact resolutions and crepant resolutions of $\mathbb{P}(\overline{\mathcal{O}})$ are the same objects (Proposition 3.3).

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2 Singularities in $\mathbb{P}(\tilde{\mathcal{O}})$

Let \mathfrak{g} be a simple complex Lie algebra and \mathcal{O} a nilpotent orbit in \mathfrak{g} . The normalization of the closure $\overline{\mathcal{O}}$ will be denoted by $\tilde{\mathcal{O}}$. The scalar \mathbb{C}^* -action on $\overline{\mathcal{O}}$ lifts to $\tilde{\mathcal{O}}$. There is only one \mathbb{C}^* -fixed point on $\tilde{\mathcal{O}}$, say o . We denote by $\mathbb{P}(\tilde{\mathcal{O}})$ the geometric quotient of $\tilde{\mathcal{O}} - \{o\}$ by the \mathbb{C}^* -action. Similarly we denote by $\mathbb{P}(\overline{\mathcal{O}})$ the geometric quotient $\overline{\mathcal{O}} - \{0\} // \mathbb{C}^*$. One sees easily that $\mathbb{P}(\tilde{\mathcal{O}})$ is nothing but the normalization of $\mathbb{P}(\overline{\mathcal{O}})$.

Recall that a *contact structure* on a smooth variety X is a corank 1 subbundle $F \subset TX$ such that the O'Neil tensor $F \times F \rightarrow L := TX/F$ is everywhere non-degenerate. In this case, $K_X \simeq L^{-(n+1)}$, where $n = (\dim X - 1)/2$. If we regard the natural map $TX \rightarrow L$ as a section $\theta \in H^0(X, \Omega_X^1(L))$ (called a *contact form*), then the non-degenerateness is equivalent to the condition that $\theta \wedge (d\theta)^n \neq 0$ considered as an element in $H^0(X, \Omega_X^{2n+1}(L^{n+1})) = H^0(X, \mathcal{O}_X)$.

For a point $v \in \mathcal{O}$, the tangent space $T_v\mathcal{O}$ is naturally isomorphic to $[v, \mathfrak{g}]$. The map $[v, x] \mapsto \kappa(v, x)$ defines a one-form θ' on \mathcal{O} , where κ is the Killing form of \mathfrak{g} . Then $\omega := d\theta'$ is the Kirillov-Kostant-Souriau symplectic form on \mathcal{O} . Notice that $\lambda^*\theta' = \lambda\theta'$ for every $\lambda \in \mathbb{C}^*$, so it defines an element $\theta \in H^0(\mathbb{P}(\mathcal{O}), \Omega_{\mathbb{P}(\mathcal{O})}^1(L))$, where L is the pull-back of $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ to $\mathbb{P}(\mathcal{O})$. This is in fact a contact form, i. e. $\theta \wedge (d\theta)^n$ is everywhere non-zero, where $n = (\dim \mathcal{O} - 2)/2$. Since the codimension of the complement of $\mathbb{P}(\mathcal{O})$ in $\mathbb{P}(\tilde{\mathcal{O}})$ is at least 2, θ extends to a contact form on the smooth part of $\mathbb{P}(\tilde{\mathcal{O}})$.

Remark 2.1. Let G be the adjoint group of \mathfrak{g} . Then the contact structure on $\mathbb{P}(\mathcal{O})$ is G -invariant, which is precisely the contact structure on $\mathbb{P}(\mathcal{O})$ viewed as a twistor space of a quaternion-Kähler manifold ([Sw]).

For an element $v \in \mathcal{O}$, there exists an $\mathfrak{sl}_2(\mathbb{C})$ triplet h, v, u , i. e. $[h, v] = 2v$, $[h, u] = -2u$, $[v, u] = h$. The Lie algebra \mathfrak{g} is decomposed into a direct sum

$\oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where \mathfrak{g}_i consists of elements x such that $[h, x] = ix$. Put $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$ and $\mathfrak{n} = \oplus_{i \geq 2} \mathfrak{g}_i$, then \mathfrak{p} is a parabolic subalgebra and \mathfrak{n} is a nilpotent ideal of \mathfrak{p} . Let P be the parabolic subgroup of G with Lie algebra \mathfrak{p} . Then the natural map $G \times^P \mathfrak{n} \rightarrow \tilde{\mathcal{O}}$ is a \mathbb{C}^* -equivariant resolution, where the \mathbb{C}^* -action on $G \times^P \mathfrak{n}$ is induced by the scalar \mathbb{C}^* -action on \mathfrak{n} .

By a result of Panyushev and Hinich (see Lemma 4.3 [Be]), the symplectic form ω extends to a G -invariant 2-form $\tilde{\omega}$ on $G \times^P \mathfrak{n}$. By contracting $\tilde{\omega}$ with the vector field generating the \mathbb{C}^* -action, one obtains a 1-form $\tilde{\theta}'$ on $G \times^P \mathfrak{n}$ satisfying $\lambda^* \tilde{\theta}' = \lambda \tilde{\theta}'$ for every $\lambda \in \mathbb{C}^*$. If we denote by Y the geometric quotient $(G \times^P (\mathfrak{n} - 0)) // \mathbb{C}^* \simeq G \times^P \mathbb{P}(\mathfrak{n})$, then $\tilde{\theta}'$ defines a 1-form $\tilde{\theta}$ on Y with values in $\mathcal{O}_p(1)$, where $p : G \times^P (\mathfrak{n} - 0) \rightarrow Y$ is the natural projection.

Notice that the induced map $f : Y \rightarrow \mathbb{P}(\tilde{\mathcal{O}})$ is a resolution of singularities and $f^* \theta = \tilde{\theta}$ on $f^{-1}(\mathbb{P}(\mathcal{O}))$. If we choose a local trivialization of $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ over U , then $\theta \wedge (d\theta)^n$ is a local volume form, which extends to a global $(2n+1)$ -form on the pre-image of U , thus the normalization of $U \cap \mathbb{P}(\tilde{\mathcal{O}})$ has rational Gorenstein singularities. Since $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ is very ample, such U covers the whole variety, which gives

Proposition 2.1. *The projective variety $\mathbb{P}(\tilde{\mathcal{O}})$ is projectively normal with only rational Gorenstein singularities.*

Remark 2.2. Another way to prove the above proposition goes as following: By abusing the notation, we denote also by L the pull-back of $\mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ to $\mathbb{P}(\tilde{\mathcal{O}})$. Notice that L is a line bundle on $\mathbb{P}(\tilde{\mathcal{O}})$, so $K := L^{-n-1}$ is locally free, which implies that $\mathbb{P}(\tilde{\mathcal{O}})$ is Gorenstein. Notice that $\tilde{\mathcal{O}} - \{o\}$ has rational singularities by results of Panyushev and Hinich, so is its quotient by \mathbb{C}^* . Thus $\mathbb{P}(\tilde{\mathcal{O}})$ has rational Gorenstein singularities.

The following proposition is easily deduced from Proposition 5.2 in [Be], which plays an important role to our classification result (Corollary 5.4).

Proposition 2.2. *Let \mathfrak{g} be a simple Lie algebra and $\mathcal{O} \subset \mathfrak{g}$ a non-zero nilpotent orbit. Then $\mathbb{P}(\tilde{\mathcal{O}})$ is smooth if and only if either \mathcal{O} is the minimal nilpotent orbit in \mathfrak{g} or \mathfrak{g} is of type G_2 and \mathcal{O} is the nilpotent orbit of a short root vector.*

Singularities in $\mathbb{P}(\tilde{\mathcal{O}})$ are examples of the so-called *contact singularities* in [CF]. Projectivised nilpotent orbits have already been studied, for example, in [Be] (for relation with Fano contact manifolds), [Ko] (for relation with

harmonic maps) and [Sw] (from the twistor aspect). Their closures have also been studied, for example in [Po] (for the self-duality), which give examples of non-smooth, self-dual projective varieties. This note is to study their crepant resolutions of its closure and relationships among their minimal models.

3 Minimal models

For a proper morphism between normal varieties $f : X \rightarrow W$, we denote by $N_1(f)$ the free abelian group (over \mathbb{R}) generated by reduced irreducible curves contained in fibers of f modulo numerical equivalence. Let $N^1(f)$ be the group $\text{Pic}(X) \otimes \mathbb{R}$ modulo numerical equivalence (w. r. t. $N_1(f)$), then we have a perfect pairing $N_1(f) \times N^1(f) \rightarrow \mathbb{R}$.

If f is a resolution, then X is called a *minimal model* of W if K_X is f -nef.

Proposition 3.1. *Let W be a projective normal variety with only canonical singularities and $f : X \rightarrow W$ a resolution. Then f is crepant if and only if X is a minimal model of W .*

Proof. If f is crepant, then $K_X = f^*K_W$, which gives $K_X \cdot [C] = 0$ for every f -exceptional curve C , so X is a minimal model of W .

Suppose K_X is f -nef, then so is $K_X - f^*K_W = \sum_i a_i E_i$, where E_i are exceptional divisors of f . By the negativity lemma (see Lemma 13-1-4 [Ma]), $a_i \leq 0$ for all i . On the other hand, W has only canonical singularities, so $a_i \geq 0$, which gives $a_i = 0$ for all i , thus f is crepant. \square

Corollary 3.2. *Let W be a projective normal variety with only terminal singularities and $f : X \rightarrow W$ a resolution. Then the following statements are equivalent:*

- (i) f is crepant;
- (ii) X is a minimal model of W ;
- (iii) f is small, i.e. $\text{codim}(\text{Exc}(f)) \geq 2$.

Remark 3.1. Recall that a morphism $g : X \rightarrow Y$ is *IH-small* if for every irreducible subvariety $F \subset X$, one has $2 \text{codim}(F) > \text{codim } g(F)$. It is shown in [To] (Proposition 8.3) that a projective IH-small resolution is a minimal model.

By the previous section, there is a contact structure on $\mathbb{P}(\mathcal{O})$. A *contact resolution* of $\mathbb{P}(\tilde{\mathcal{O}})$ is a resolution $\pi : X \rightarrow \mathbb{P}(\tilde{\mathcal{O}})$ such that the contact structure on $\mathbb{P}(\mathcal{O})$ extends to a contact structure on X .

Proposition 3.3. *Let $\pi : X \rightarrow \mathbb{P}(\tilde{\mathcal{O}})$ be a resolution, then the following statements are equivalent:*

- (i) π is crepant;
- (ii) K_X is π -nef;
- (iii) π is a contact resolution.

Proof. The equivalence between (i) and (ii) follows from Prop. 2.1 and Prop. 3.1. The implication (iii) to (i) is clear from the definitions. Now suppose that π is crepant, then $K_X \simeq \pi^*(L^{-(n+1)}) \simeq (\pi^*L)^{-(n+1)}$. Let \hat{X} be the fiber product $X \times_{\mathbb{P}(\tilde{\mathcal{O}})} (\tilde{\mathcal{O}} - \{o\})$ and $h : \hat{X} \rightarrow \tilde{\mathcal{O}} - \{o\}$ the natural projection. Then h is a resolution of singularities and $h^*\omega$ extends to a 2-form $\tilde{\omega}$ on \hat{X} , where ω is the symplectic form on the smooth part of $\tilde{\mathcal{O}}$. \hat{X} inherits a \mathbb{C}^* -action from that on $\tilde{\mathcal{O}}$. Contracting $\tilde{\omega}$ with the vector field generating the \mathbb{C}^* -action, one obtains an element $\tilde{\theta} \in H^0(\hat{X}, \Omega_{\hat{X}} \otimes \pi^*L)$. Now it is clear that $\tilde{\theta}$ gives the contact form on X extending θ . \square

4 Symplectic resolutions

Let $f : Z \rightarrow \mathbb{P}(\overline{\mathcal{O}})$ be a resolution and let \hat{Z} be the fiber product $Z \times_{\mathbb{P}(\overline{\mathcal{O}})} W_0$ and $\tilde{f} : \hat{Z} \rightarrow W_0$ the natural projection, where $W_0 = \overline{\mathcal{O}} - \{0\}$.

Lemma 4.1. *\hat{Z} is isomorphic to the complement of the zero section in the total space of the line bundle $(f^*L)^*$ and \tilde{f} is a resolution of singularities.*

Proof. This follows from the fact that W_0 is naturally isomorphic to the complement of the zero section in L^* and the fact that the fiber product $Z \times_{\mathbb{P}(\overline{\mathcal{O}})} L^*$ is isomorphic to $f^*(L^*) \simeq (f^*L)^*$. \square

Proposition 4.2. *The map f is a contact resolution if and only if \tilde{f} is a symplectic resolution.*

Proof. Let ω be the Kostant-Kirillov-Souriau symplectic form on \mathcal{O} , then $(\tilde{f})^*\omega$ extends to $\tilde{\omega} \in H^0(\hat{Z}, \Omega_{\hat{Z}}^2)$. \hat{Z} admits a \mathbb{C}^* -action induced from the one on W_0 and for this action, one has $\lambda^*\tilde{\omega} = \lambda\tilde{\omega}$ for all $\lambda \in \mathbb{C}^*$. By contracting $\tilde{\omega}$ with the vector field generating the \mathbb{C}^* -action, we obtain a 1-form θ' on \hat{Z} satisfying $\lambda^*\theta' = \lambda\theta'$, this gives an element θ in $H^0(Z, \Omega_Z(f^*L))$. Then θ is a contact form if and only if $\tilde{\omega}$ is a symplectic form. \square

Proposition 4.3. *Let \mathcal{O} be a nilpotent orbit with finite Picard group. If $\text{Sing}(\mathbb{P}(\tilde{\mathcal{O}}))$ is non-empty and not of pure codimension 2, then $\mathbb{P}(\tilde{\mathcal{O}})$ admits no contact resolution.*

Proof. By assumption, the Weil divisor class group $Cl(\tilde{\mathcal{O}} - \{o\})$ is finite, so $\tilde{\mathcal{O}} - \{o\}$ is normal and \mathbb{Q} -factorial. By Cor. 1.3 [Fu], it admits no symplectic resolution if its singular part is non-empty and not of pure codimension 2. By Prop. 4.2, $\mathbb{P}(\tilde{\mathcal{O}})$ admits no contact resolution. \square

There are some errors in the calculus of the Picard groups in [Fu], which is due to the fact that $SO_2 \simeq \mathbb{C}^*$ has non-trivial character group. For $\mathfrak{g} = \mathfrak{sp}_{2n}$, this has no effect, so every nilpotent orbit has finite Picard group. By the precedent proposition, for $\mathcal{O} = \mathcal{O}_{[2,2,1,1]}$ in \mathfrak{sp}_6 , the variety $\mathbb{P}(\tilde{\mathcal{O}})$ does not admit any contact resolution. For $\mathfrak{g} = \mathfrak{so}_m$, \mathcal{O} has finite Picard group if in the partition of m corresponding to \mathcal{O} , there is no odd part which appears exactly twice. For example, in \mathfrak{so}_8 , there is only one nilpotent orbit $\mathcal{O} = \mathcal{O}_{[3,2,2,1]}$ such that $\mathbb{P}(\tilde{\mathcal{O}})$ admits no contact resolution.

5 Contact resolutions

The aim of this section is to find all contact resolutions that the projectivisation of a nilpotent orbit closure can have.

Proposition 5.1. *Let \mathcal{O} be a nilpotent orbit such that $\mathbb{P}(\tilde{\mathcal{O}})$ is singular. Let $\bar{\pi} : X \rightarrow \mathbb{P}(\tilde{\mathcal{O}})$ be a contact resolution. Then X is isomorphic to the projectivised cotangent bundle of a projective manifold.*

Proof. Let us denote by \tilde{L} the contact line bundle on X , then $K_X \simeq \tilde{L}^{-n-1}$, where $n = (\dim \mathcal{O})/2 - 1$. Notice that for a curve C in X , we have $K_X \cdot C = -(n+1)L \cdot \bar{\pi}_*[C]$, thus K_X is not nef. By [KPSW], X is either a Fano contact manifold or the projectivised cotangent bundle of a projective manifold.

The map $\bar{\pi}$ factorizes through the normalization, so we obtain a birational map $\nu : X \rightarrow \mathbb{P}(\tilde{\mathcal{O}})$. By assumption, $\mathbb{P}(\tilde{\mathcal{O}})$ is singular. Zariski's main theorem then implies that there exists a curve C contained in a fiber of ν . Now $K_X \cdot C = 0$, thus $-K_X$ is not ample, which shows that X is not Fano. \square

So X is isomorphic to $\mathbb{P}(T^*Y)$ for some smooth projective variety Y . Let us denote by $\pi_0 : \tilde{X} \rightarrow W_0$ the symplectic resolution provided by Proposition 4.2.

Notice that the canonical bundle of $\mathbb{P}(T^*Y)$ is $\mathcal{O}_q(-n-1)$, where $q : \mathbb{P}(T^*Y) \rightarrow Y$ is the natural projection and $\dim(Y) = n+1$. In particular, we have $\bar{\pi}^*(L)$ is isomorphic to $\mathcal{O}_q(1)$. By lemma 4.1, \tilde{X} is isomorphic to $\tilde{X} = T^*Y - Y$.

Lemma 5.2. π_0 extends to a morphism $\pi : T^*Y \rightarrow \overline{\mathcal{O}}$.

Proof. Write $W = \overline{\mathcal{O}}$. The map π_0 induces a homomorphism $H^0(W_0, \mathcal{O}_{W_0}) \rightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$. Notice that $H^0(W_0, \mathcal{O}_{W_0}) = H^0(W, \mathcal{O}_W)$ and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^0(T^*Y, \mathcal{O}_{T^*Y})$. On the other hand, we have a natural morphism $T^*Y \rightarrow \text{Spec}(H^0(T^*Y, \mathcal{O}_{T^*Y}))$, which composed with the map $\text{Spec}(H^0(T^*Y, \mathcal{O}_{T^*Y})) \rightarrow W$ gives the morphism π . \square

Notice that π is a symplectic resolution of $\overline{\mathcal{O}}$, thus the main theorem in [Fu] implies that π is isomorphic to the moment map of the G -action on $T^*(G/P)$ for some parabolic subgroup P in G . So we obtain

Theorem 5.3. *Let \mathcal{O} be a nilpotent orbit in a semi-simple Lie algebra \mathfrak{g} such that $\mathbb{P}(\tilde{\mathcal{O}})$ is singular. Suppose that we have a contact resolution $\pi : Z \rightarrow \mathbb{P}(\overline{\mathcal{O}})$, then $Z \simeq \mathbb{P}(T^*(G/P))$ for some parabolic subgroup P in the adjoint group G of \mathfrak{g} and the morphism π is the natural one.*

Now Proposition 2.2 implies the following

Corollary 5.4. *Suppose \mathfrak{g} is simple. The projectivised nilpotent orbit closure $\mathbb{P}(\overline{\mathcal{O}})$ admits a contact resolution if and only if either*

- (i) \mathcal{O} is the minimal nilpotent orbit, or
- (ii) \mathfrak{g} is of type G_2 and \mathcal{O} is of dimension 8, or
- (iii) $\overline{\mathcal{O}}$ admits a symplectic resolution.

The classification of \mathcal{O} satisfying case (iii) with \mathfrak{g} simple but different to E_7 and E_8 has been done in [Fu]. For example, every projectivised nilpotent orbit closure in \mathfrak{sl}_n admits a contact resolution, which is given by the projectivisation of cotangent bundles of some flag varieties. Usually it admits several different contact resolutions.

6 Birational geometry

The result in the precedent section can tell us if a projectivised nilpotent orbit closure admits a contact resolution or not. However, in the case of

existence, it can admit several different ones. We now try to understand the birational geometry between them.

Let \mathfrak{g} be a simple complex Lie algebra and \mathcal{O} a non-zero nilpotent orbit in \mathfrak{g} . We can assume that \mathcal{O} is not the minimal nilpotent orbit, since $\mathbb{P}(\overline{\mathcal{O}}_{\min})$ is smooth.

Suppose that $\overline{\mathcal{O}}$ admits a symplectic resolution, then by [Fu], it is given by the natural map $\pi : X := T^*(G/P) \rightarrow W := \overline{\mathcal{O}}$ for some parabolic sub-group P in G . Let us denote by π_0 the restriction of π to $X_0 := T^*(G/P) - G/P$, then π_0 is a symplectic resolution of $W_0 := \overline{\mathcal{O}} - \{0\}$.

I'm indebted to M. Brion for the proof of the following proposition, which allows us to remove the restriction that \mathfrak{g} is of classical type in an earlier version of this note.

Proposition 6.1. *We have $N_1(\pi_0) = N_1(\pi)$ and $N^1(\pi_0) = N^1(\pi)$.*

Proof. Consider the natural projections: $X_0 \xrightarrow{p_0} G/P \xleftarrow{p} X$, then $\text{Pic}(X_0)$ is identified with $\text{Pic}(G/P) \otimes \mathbb{R} = N^1(G/P)$ via p_0^* . Notice that for a complete curve C on X_0 and a divisor $D \in \text{Pic}(G/P)$, we have $C \cdot p_0^*D = (p_0)_*(C) \cdot D$. Thus we need to show that images of complete curves in X_0 under $(p_0)_*$ generate $H_2(G/P, \mathbb{R}) = N_1(G/P)$.

Let I be the set of simple roots of G which are not roots of the Levi subgroup of P , i.e. I is the set of marked roots in the marked Dynkin diagram of $\mathfrak{p} = \text{Lie}(P)$. A basis of $H_2(G/P)$ is given by Schubert curves $C_\alpha := P_\alpha/B$, where $\alpha \in I$ and P_α is the corresponding minimal parabolic subgroup containing the Borel subgroup B . We need to lift every C_α to a curve in X_0 . There are two cases:

(i) I consists of a single simple root α , then $b_2(G/P) = 1$. Since \mathcal{O} is supposed to be non-minimal, and the 8-dimensional nilpotent orbit closure in G_2 has no symplectic resolution (Proposition 3.21 [Fu]), by Proposition 2.2, we can assume that $\tilde{\mathcal{O}} - \{o\}$ is not smooth. By Zariski's main theorem, there exists a fiber of π_0 which has positive dimension. Take an irreducible component C of this fiber, then $(p_0)_*C$ is non-zero in $H_2(G/P, \mathbb{R}) \simeq \mathbb{R}$, which generates (over \mathbb{R}) $N_1(G/P)$.

(ii) I contains at least two simple roots. To lift C_α , we take a simple root $\beta \in I$ different to α , then \mathfrak{g}_β generates a G_α -submodule M of \mathfrak{g} contained in \mathfrak{n} , where G_α is the simple subgroup of G associated with the simple root α and \mathfrak{n} is the nilradical of \mathfrak{p} . Then in $T^*(G/P) \simeq G \times^P \mathfrak{n}$, there is the closed subvariety $P_\alpha \times^B M \simeq G_\alpha \times^{B_\alpha} M$ which is mapped to $G_\alpha M = M$ with fibers $G_\alpha/B_\alpha \simeq P_\alpha/B$, where $B_\alpha = B \cap G_\alpha$. Now any fiber of this map lifts C_α .

□

Let $\bar{\pi} : \mathbb{P}(X) \rightarrow \mathbb{P}(W)$ be the induced map, which is a contact resolution. The contact structure on $\mathbb{P}(X)$ is given by the line bundle $\tilde{L} = \mathcal{O}_{\bar{p}}(1)$, where $\bar{p} : \mathbb{P}(X) \rightarrow G/P$ is the natural map. We have $\text{Pic}(\mathbb{P}(X)) \simeq \text{Pic}(G/P) \oplus \mathbb{Z}[\tilde{L}]$. Notice that $\tilde{L} = \bar{\pi}^*L$, so for any $\bar{\pi}$ -exceptional curve C , one has $C \cdot L = C \cdot \bar{\pi}^*L = 0$. This provides the identifications $N^1(\bar{\pi}) = N^1(\pi_0) = N^1(\pi)$ and $N_1(\bar{\pi}) = N_1(\pi_0) = N_1(\pi)$.

Recall that the cone $NE(\pi) = NE(G/P)$ is generated by Schubert curves in G/P over $\mathbb{R}^{\geq 0}$. As shown in the proof of Proposition 6.1, these Schubert curves are images of curves in the fibers of π_0 , thus $NE(\pi_0) = NE(\pi)$. Since $NE(\pi_0) = NE(\bar{\pi})$, we obtain $NE(\bar{\pi}) = NE(\pi)$. By Kleiman's criterion, $\overline{Amp}(\pi_0) = \overline{Amp}(\pi) = \overline{Amp}(\bar{\pi})$. By [Na] Theorem 4.1 (ii), this is a simplicial polyhedral cone.

Let $g : X_0 \rightarrow \mathbb{P}(X)$ and $h : W_0 \rightarrow \mathbb{P}(W)$ be the natural projections, then $\bar{\pi}g = h\pi_0$. Take a π_0 -movable divisor p_0^*D , then $(\pi_0)_*p_0^*D = h^*\bar{\pi}_*\bar{p}^*D \neq 0$, which gives that $\bar{\pi}_*\bar{p}^*D \neq 0$. Notice that $\pi_0^*(\pi_0)_*p_0^*D = g^*\bar{\pi}^*\bar{\pi}_*\bar{p}^*D$ and $p_0^*D = g^*\bar{p}^*D$, so the cokernel of $\bar{\pi}^*\bar{\pi}_*\bar{p}^*D \rightarrow \bar{p}^*D$ has support of codimension ≥ 2 . In conclusion \bar{p}^*D is $\bar{\pi}$ -movable and vice versa. So we obtain $\overline{Mov}(\pi_0) = \overline{Mov}(\pi) = \overline{Mov}(\bar{\pi})$.

To remember the parabolic subgroup P , from now on, we will write π_P instead of π (similarly for $\pi_0, \bar{\pi}$). For two parabolic subgroups Q, Q' in G , we write $Q \sim Q'$ (called *equivalent*) if $T^*(G/Q)$ and $T^*(G/Q')$ give both symplectic resolutions of a same nilpotent orbit closure. In [Na], Namikawa found a way to describe all parabolic subgroups which are equivalent to a given one. By *loc. cit.* Theorem 4.1, we obtain

Theorem 6.2. *Let \mathcal{O} be a non-minimal nilpotent orbit in a simple complex Lie algebra \mathfrak{g} whose closure admits a symplectic resolution, say $T^*(G/P)$, where G is the adjoint group of \mathfrak{g} . Then*

- (i) $\overline{Mov}(\bar{\pi}_P) = \cup_{Q \sim P} \overline{Amp}(\bar{\pi}_Q)$;
- (ii) *two crepant resolutions of $\mathbb{P}(\overline{\mathcal{O}})$ are connected by some simple flops.*

By Mori theory (see [Ma]), the assertion (i) implies that every minimal model of $\mathbb{P}(\overline{\mathcal{O}})$ is of the form $\mathbb{P}(T^*(G/P))$. Now by Proposition 3.3, this gives another proof of Theorem 5.3 in the case that $\overline{\mathcal{O}}$ admits a symplectic resolution.

Similarly, as a by-product of our argument, we obtain the description of the movable cone of a symplectic resolution of W_0 , which gives

Corollary 6.3. *Let \mathfrak{g} be a simple Lie algebra and $\mathcal{O} \subset \mathfrak{g}$ a nilpotent orbit. Suppose that $\overline{\mathcal{O}}$ admits a symplectic resolution, then every symplectic resolution of $\overline{\mathcal{O}} - \{0\}$ can be extended to a symplectic resolution of $\overline{\mathcal{O}}$.*

Remark 6.1. The condition that $\overline{\mathcal{O}}$ admits a symplectic resolution cannot be removed, due to the following two examples:

- (i). if \mathfrak{g} is not of type A , then $\overline{\mathcal{O}}_{\min}$ admits no symplectic resolution ([Fu]), however $\overline{\mathcal{O}}_{\min} - \{0\}$ is smooth, so trivially it admits a symplectic resolution;
- (ii). if \mathfrak{g} is of type G_2 and \mathcal{O} is the 8-dimensional nilpotent orbit, then $W_0 := \overline{\mathcal{O}} - \{0\}$ is not smooth, and its normalization map $\mu : \widetilde{W}_0 \rightarrow W_0$ is a symplectic resolution which does not extend to $\overline{\mathcal{O}}$, since \mathcal{O} is not a Richardson nilpotent orbit (Prop. 3.21 [Fu]). Here we used the result in [LeSm] and [Kr] which says that \widetilde{W}_0 is in fact the minimal nilpotent orbit in \mathfrak{so}_7 , thus it is smooth and symplectic.

Are these two examples the only exceptions?

Appendix: twistor spaces

We return to the question we have at the beginning: can any contact resolution of $\mathbb{P}(\overline{\mathcal{O}})$ be the twistor space of a quaternion-Kähler manifold?

Proposition 6.4. *Let G be a simple complex Lie group with Lie algebra \mathfrak{g} and P a parabolic sub-group of G . Then $\mathbb{P}(T^*(G/P))$ is a twistor space of a quaternion-Kähler manifold if and only if $G/P \simeq \mathbb{P}^n$ for some n .*

Proof. Recall that the image of the moment map $T^*(G/P) \rightarrow \mathfrak{g}$ is a nilpotent orbit closure $\overline{\mathcal{O}}$, which gives a generically finite morphism $\pi : \mathbb{P}(T^*(G/P)) \rightarrow \mathbb{P}(\overline{\mathcal{O}})$. There are two cases:

(i) there is a fiber of positive dimension, then as proved in Proposition 5.1, $\mathbb{P}(T^*(G/P))$ is not Fano.

(ii) every fiber of π is zero-dimensional, then π is a finite G -equivariant surjective morphism. If $\mathbb{P}(T^*(G/P))$ is Fano, then by Proposition 6.3 [Be], either $\pi = id$ and $\mathcal{O} = \mathcal{O}_{\min}$ or π is one of the G -covering in the list of Brylinski-Kostant (see table 6.2 [Be]). In both cases, one has that $\mathbb{P}(T^*(G/P))$ is isomorphic to $\mathbb{P}(\mathcal{O}'_{\min})$ for some minimal nilpotent orbit $\mathcal{O}'_{\min} \subset \mathfrak{g}'$, which is possible only if G/P is isomorphic to \mathbb{P}^n for some n .

Now suppose that $\mathbb{P}(T^*G/P)$ is a twistor space of a quaternion-Kähler manifold M . Then the scalar curvature of M should be positive, which

implies ([Sa]) that $\mathbb{P}(T^*G/P)$ is Fano, so G/P is isomorphic to \mathbb{P}^n for some n . \square

As pointed out by Prof. A. Swann, this proposition follows also from [LeSa], where it is shown that a contact Fano variety with $b_2 \geq 2$ is isomorphic to $\mathbb{P}(T^*\mathbb{P}^n)$ for some n .

By [Sw], there exists a S^2 -bundle structure on $\mathbb{P}(\overline{\mathcal{O}})$ with base space \overline{M} , and the latter is a singular Riemann manifold with a piece-wise quaternion-Kähler metric. By our previous discussions, if \mathcal{O} is non-minimal, then \overline{M} admits a smoothing by a quaternion-Kähler manifold if and only if \mathcal{O} is the 8-dimensional nilpotent orbit in G_2 . It would be interesting to study the singularities in \overline{M} .

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