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# Computing the average parallelism in trace monoids\*

Daniel Kroh<sup>†</sup>      Jean Mairesse<sup>†</sup>      Ioannis Michos<sup>†</sup>

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## Abstract

The *height* of a trace is the height of the corresponding heap of pieces in Viennot's representation, or equivalently the number of factors in its Cartier-Foata decomposition. Let  $h(t)$  and  $|t|$  stand respectively for the height and the length of a trace  $t$ . We prove that the bivariate commutative series  $\sum_t x^{h(t)} y^{|t|}$  is rational, and we give a finite representation of it. We use the rationality to obtain precise information on the asymptotics of the number of traces of a given height or length. Then, we study the average height of a trace for various probability distributions on traces. For the uniform probability distribution on traces of the same length (resp. of the same height), the asymptotic average height (resp. length) exists and is an algebraic number. To illustrate our results and methods, we consider a couple of examples: the free commutative monoid and the trace monoid whose independence graph is the ladder graph.

**Keywords:** Automata and formal languages, trace monoids, Cartier-Foata normal form, height function, generating series, speedup, performance evaluation.

## 1 Introduction

Traces are used to model the occurrence of events in concurrent systems [12]. Roughly speaking, a letter corresponds to an event and two letters commute when the corresponding events can occur simultaneously. In this context, the two basic performance measures associated with a trace  $t$  are its *length*  $|t|$  (the 'sequential' execution time) and its *height*  $h(t)$  (the 'parallel' execution

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time). The ratio  $|t|/h(t)$  captures in some sense the *amount of parallelism* (the *speedup* in [9]). Let  $\mathbb{M}$  be a trace monoid. Define the generating series

$$F = \sum_{t \in \mathbb{M}} x^{h(t)} y^{|t|}, \quad L = \sum_{t \in \mathbb{M}} y^{|t|}, \quad H = \sum_{t \in \mathbb{M}} x^{h(t)}.$$

It is well known that  $L$  is a rational series [8]. We prove that  $F$  and  $H$  are also rational and we provide finite representations for the series. Exploiting the symmetries of the trace monoid enables to obtain representations of reduced dimensions. We use the rationality to obtain precise information on the asymptotics of the number of traces of a given height or length.

Then, given a trace monoid and a measure on the traces, we study the *average parallelism* in the trace monoid. One notion of average parallelism is obtained by considering the measure over traces induced by the uniform distribution over words of the same length in the free monoid. In other terms, the probability of a trace is proportional to the number of its representatives in the free monoid. This quantity was introduced in [27] and later studied in [2, 5, 6, 14, 28]. Here we define alternative notions of average parallelism by considering successively the uniform distribution over traces of the same length, the uniform distribution over traces of the same height, and the uniform distribution over Cartier-Foata normal forms. We prove in particular that there exists  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}$  in  $\mathbb{R}_+^*$  such that

$$\frac{\sum_{t \in \mathbb{M}, |t|=n} h(t)}{n \cdot \#\{t \in \mathbb{M}, |t|=n\}} \xrightarrow{n \rightarrow \infty} \lambda_{\mathbb{M}}, \quad \frac{\sum_{t \in \mathbb{M}, h(t)=n} |t|}{n \cdot \#\{t \in \mathbb{M}, h(t)=n\}} \xrightarrow{n \rightarrow \infty} \gamma_{\mathbb{M}}.$$

Furthermore, the numbers  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}$  are algebraic. Explicit formulas involving the series  $L$  and  $H$  are given for  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}$ .

The present paper is an extended version with proofs of [24].

## 2 The Trace Monoid

We start by introducing all the necessary notions from the theory of trace monoids. The reader may refer to [11, 12] for further information.

In the sequel, a *graph* is a couple  $(N, A)$  where  $N$  is a finite non-empty set and  $A \subset N \times N$ . Hence we consider directed graphs, allowing for self-loops but not multi-arcs. Such a graph is *non-directed* if  $A$  is symmetric. We use without recalling it the basic terminology of graph theory. Given a graph and two nodes  $u$  and  $v$ , we write  $u \rightarrow v$  if there is a path from  $u$  to  $v$ .

Fix a finite alphabet  $\Sigma$ . Let  $D$  be a reflexive and symmetric relation on  $\Sigma$ , called the *dependence* relation, and let  $I$  be its complement in  $\Sigma \times \Sigma$ , known as the *independence* or *commutation* relation.

The *trace monoid*, or *free partially commutative monoid*,  $\mathbb{M} = \mathbb{M}(\Sigma, D)$  is defined as the quotient of the free monoid  $\Sigma^*$  by the least congruence

containing the relations  $ab \sim ba$  for every  $(a, b) \in I$ . The elements of  $\mathbb{M}$  are called *traces*. Two words are representatives of the same trace if they can be obtained one from the other by repeatedly commuting independent adjacent letters.

The *length* of the trace  $t$  is the length of any of its representatives and is denoted by  $|t|$ . Note that we also use the notation  $|S| = \#S$  for the cardinal of a set  $S$ . The set of letters appearing in (any representative of) the trace  $t$  is denoted by  $\text{alph}(t)$ . The graphs  $(\Sigma, D)$  and  $(\Sigma, I)$  are called respectively the *dependence* and the *independence graph* of  $\mathbb{M}$ . Let finally  $\psi$  denote the canonical projection from  $\Sigma^*$  into the trace monoid  $\mathbb{M}$ . In the sequel, we most often simplify the notations by denoting a trace by any of its representatives, that is by identifying  $w$  and  $\psi(w)$ .

**Example 2.1.** Let  $\Sigma = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$  (the set of subsets of cardinal two of  $\{1, 2, 3, 4\}$ ). Define the independence relation  $I = \{(u, v) : u \cap v = \emptyset\}$ . The dependence graph  $(\Sigma, D)$  is the line graph of the complete graph  $K_4$ , also called the triangular graph  $T_4$ . For notational

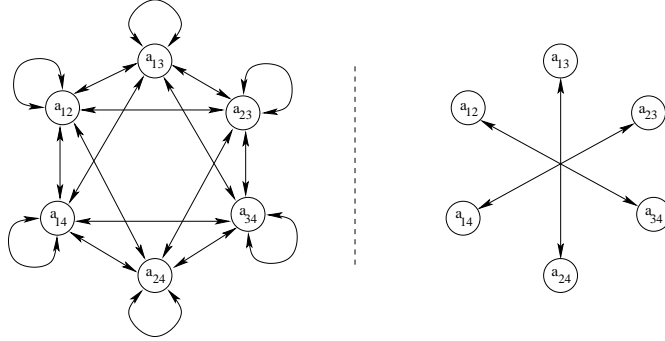


Figure 1: The dependence graph  $T_4$  (left) and its independence graph (right).

simplicity, set  $a_{ij} = \{i, j\}$ . The dependence graph is represented on the left of Figure 1 and the independence graph on the right. In the trace monoid  $\mathbb{M}(\Sigma, D)$ , we have  $\tau = a_{12}a_{34}a_{23}^2a_{14} = a_{34}a_{12}a_{23}a_{14}a_{23}$ .

A *clique* is a non-empty trace whose letters are mutually independent. Cliques are in one-to-one correspondence with the complete subgraphs (also called cliques in a graph theoretical context) of  $(\Sigma, I)$ . We denote the set of cliques of  $\mathbb{M}$  by  $\mathfrak{C}$ .

An element  $(u, v) \in \mathfrak{C} \times \mathfrak{C}$  is called *Cartier-Foata (CF-) admissible* if for every  $b \in \text{alph}(v)$ , there exists  $a \in \text{alph}(u)$  such that  $(a, b) \in D$ . Remark that the CF-admissibility of  $(u, v)$  does not imply the one of  $(v, u)$ . The *Cartier-Foata (CF) decomposition* of a trace  $t$  is the uniquely defined (see [8, Chap. I]) sequence of cliques  $(c_1, c_2, \dots, c_m)$  such that  $t = c_1c_2 \cdots c_m$ ,

and the couple  $(c_j, c_{j+1})$  is CF-admissible for all  $j$  in  $\{1, \dots, m-1\}$ . The positive integer  $m$  is called the *height* of  $t$  and is denoted by  $h(t)$ . In the visualization of traces using *heaps of pieces*, introduced by Viennot in [32], the height corresponds precisely to the height of the heap.

**Example 2.2.** Consider the trace monoid defined in Example 2.1. The set of cliques is  $\mathfrak{C} = \{a, a \in \Sigma\} \cup \{a_{12}a_{34}, a_{13}a_{24}, a_{14}a_{23}\}$ . The CF decomposition

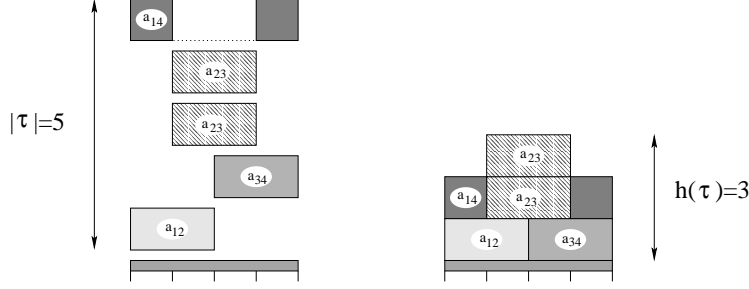


Figure 2: Heap of pieces.

of  $\tau$  is  $(a_{12}a_{34}, a_{14}a_{23}, a_{23})$ . We have  $|\tau| = 5$  and  $h(\tau) = 3$ . We represented the heap of pieces associated with  $\tau$  on Figure 2.

### 3 The Graph of Cliques

We define the *graph of cliques*  $\Gamma$  as the directed graph with  $\mathfrak{C}$  as its set of nodes and the set of all CF-admissible couples as its set of arcs. Note that  $\Gamma$  contains as a subgraph the dependence graph  $(\Sigma, D)$ . The graph  $\Gamma$  is in general complicated and looks like a maze.

**Example 3.1.** We continue with the model of Examples 2.1 and 2.2. For

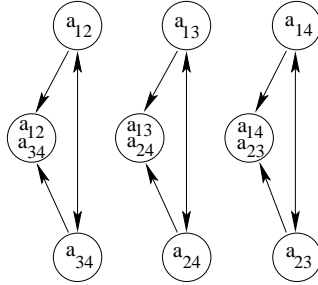


Figure 3: The complement of the graph of cliques of  $T_4$ .

simplicity, the graph represented Figure 3 is the complement of the corresponding graph of cliques (the complement of the graph  $(N, A)$  is the graph  $(N, (N \times N) - A)$ ).

**Lemma 3.2.** *If the dependence graph is connected, then the corresponding graph of cliques is strongly connected.*

*Proof.* Let  $(\Sigma, D)$  be the dependence graph,  $\mathfrak{C}$  the set of cliques, and  $\Gamma$  the graph of cliques. Given  $u, v \in \mathfrak{C}$ , we want to prove that there is a path from  $u$  to  $v$  in  $\Gamma$ . We argue by induction on the value of  $|u| + |v|$ . If  $|u| + |v| = 2$ , the result follows by the connectivity of the dependence graph  $(\Sigma, D)$ .

Now consider the case  $|u| + |v| > 2$ . Assume first that  $|u| > 1$ . Let  $a$  belong to  $\text{alph}(u)$ . Clearly  $(u, a)$  is CF-admissible. By induction, we have  $a \rightarrow v$  and we deduce that  $u \rightarrow v$ .

Assume now that  $|u| = 1$ . Then we have  $|v| > 1$  and let  $v = v'ab, a, b \in \Sigma$ . By induction, we have  $u \rightarrow v'a$ . Let us prove that  $v'a \rightarrow v'ab$ . By connectivity, there exists in  $(\Sigma, D)$  a path  $(c_0 = a, \dots, c_k = b)$ . For  $j \in \{0, \dots, k\}$ , set  $v_j = v'ac_j$  if  $v'ac_j \in \mathfrak{C}$  and otherwise set  $v_j = wc_j$  where  $w$  is the longest trace such that  $\text{alph}(w) \subset \text{alph}(v'a)$  and  $wc_j \in \mathfrak{C}$ . By construction, we obtain that  $(v_0 = v'a, \dots, v_k = v'ab)$  is a path in  $\Gamma$ . It completes the proof.  $\square$

The above lemma can be restated as follows: given two cliques  $u$  and  $v$  there exists at least one trace in  $\mathbb{M}$  such that the first factor in its CF-decomposition is  $u$  and the last one is  $v$ .

We now use a standard reduction technique for multi-graphs (see [10, Chap. 4] or [17, Chap. 5]). We partition the nodes of  $\Gamma$  based on their set of direct successors. An *equitable partition* of  $\mathfrak{C}$  is a partition  $\pi = \{\mathfrak{C}_1, \dots, \mathfrak{C}_s\}$  with the property that for all  $i$  and  $j$  the number  $a_{ij}$  of direct successors that a node in  $\mathfrak{C}_i$  has in  $\mathfrak{C}_j$  is independent of the choice of the node in  $\mathfrak{C}_i$ . Set  $A_\pi = (a_{ij})_{i,j}$ . The matrix  $A_\pi$  is called the *coloration matrix* corresponding to  $\pi$ . In the case of the partition  $\{\{c\}, c \in \mathfrak{C}\}$ , the coloration matrix is the *adjacency matrix* of  $\Gamma$ .

**Example 3.3.** We keep studying the model of Examples 2.1, 2.2 and 3.1. Consider the partition  $\pi$  of  $\mathfrak{C}$  defined by

$$\mathfrak{C}_1 = \{a_{12}, a_{13}, a_{14}\}, \quad \mathfrak{C}_2 = \{a_{23}, a_{24}, a_{34}\}, \quad \mathfrak{C}_3 = \{a_{12}a_{34}, a_{13}a_{24}, a_{14}a_{23}\}.$$

It is easily checked that the partition is equitable. The corresponding coloration matrix is

$$A_\pi = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

A natural family of equitable partitions is the one induced by the non-trivial subgroups of the full automorphism group of  $\Gamma$ . Given such a group  $G$ , the cells of the corresponding partition  $\pi_G$  are the orbits into which  $\mathfrak{C}$  is partitioned by  $G$ . The corresponding coloration matrix is denoted by  $A_G$ .

An automorphism of  $(\Sigma, D)$  induces an automorphism of  $\Gamma$ . Indeed, consider an automorphism  $\phi$  of  $(\Sigma, D)$ . The map  $\phi : \Sigma \rightarrow \Sigma$  can be extended into a map  $\phi' : \mathfrak{C} \rightarrow \mathfrak{C}$  as follows. Given  $c = u_1 \cdots u_k \in \mathfrak{C}$  with  $|u_i| = 1$  for all  $i$ , set  $\phi'(c) = \phi(u_1) \cdots \phi(u_k)$ . Note that the definition is unambiguous since the letters  $\phi(u_i)$  commute. It is immediate that  $\phi'$  is an automorphism of  $\Gamma$ .

Due to the complex structure of  $\Gamma$ , finding its automorphisms is in general difficult. Finding the automorphisms of  $(\Sigma, D)$  is often an easier task. This simple observation allows us to focus on the automorphism groups of  $(\Sigma, D)$  and to consider their action on the nodes of  $\Gamma$ . When  $(\Sigma, D)$  has a great amount of symmetries, the corresponding reduction can be very important (see Section 6.2).

Below we need to consider equitable partitions such that all the cliques in the same cell have a common length. This requirement is always satisfied for the equitable partitions associated with automorphism groups.

**Example 3.4.** The model is the one of Examples 2.1, 2.2, 3.1, and 3.3. The symmetric group  $\mathfrak{S}_4$  of degree 4 is a non-trivial group of automorphisms of  $(\Sigma, D)$ . It is of index 2 in the full automorphism group  $G$  of  $(\Sigma, D)$ . The partition of  $\mathfrak{C}$  induced by  $\mathfrak{S}_4$  (or by  $G$ ) is  $\mathfrak{C}_1 = \{a, a \in \Sigma\}$  and  $\mathfrak{C}_2 = \{a_{12}a_{34}, a_{13}a_{24}, a_{14}a_{23}\}$ . The coloration matrix is given by

$$A_{\mathfrak{S}_4} = \begin{pmatrix} 5 & 2 \\ 6 & 3 \end{pmatrix}.$$

## 4 Height and Length Generating Function

Let  $F \in \mathbb{N}[[x, y]]$  be the *height and length generating function* defined by

$$F(x, y) = \sum_{t \in \mathbb{M}} x^{h(t)} y^{|t|} = \sum_{k, l \in \mathbb{N}} f_{k, l} x^k y^l,$$

where  $x$  and  $y$  are commuting indeterminate and  $f_{k, l}$  is the number of traces of height  $k$  and length  $l$ . Set  $H(x) = F(x, 1)$  and  $L(y) = F(1, y)$ . Then  $H(x)$  and  $L(y)$  are respectively the generating functions of the height and of the length. The *Möbius polynomial*  $\mu(\Sigma, I)$  of the graph  $(\Sigma, I)$  is defined by

$$\mu(\Sigma, I) = 1 + \sum_{u \in \mathfrak{C}} (-1)^{|u|} y^{|u|}. \quad (1)$$

It is well known [8, Chap. II] that  $L(y)$  is equal to the inverse of the Möbius polynomial, i.e.  $L(y) = \mu(\Sigma, I)^{-1}$ . In particular, it is a rational series.

**Proposition 4.1.** *Let  $\mathbb{M} = \mathbb{M}(\Sigma, D)$  be a trace monoid and let  $\mathfrak{C}$  be the set of cliques of  $(\Sigma, I)$ . Define the matrix  $A(x, y) \in \mathbb{N}[x, y]^{\mathfrak{C} \times \mathfrak{C}}$  by setting*

$A(x, y)_{i,j} = xy^{|i|}$  if  $(i, j)$  is CF-admissible and 0 otherwise. Define also  $u = (1, \dots, 1) \in \mathbb{N}[x, y]^{1 \times \mathfrak{C}}$  and  $v(x, y) = (xy^{|i|})_i \in \mathbb{N}[x, y]^{\mathfrak{C} \times 1}$ . The height and length generating function is then given by

$$F - 1 = \sum_{n \in \mathbb{N}} u A(x, y)^n v(x, y) = u (I - A(x, y))^{-1} v(x, y), \quad (2)$$

where 1 is the identity of  $\mathbb{N}[[x, y]]$  and  $I$  is the  $\mathfrak{C} \times \mathfrak{C}$  identity matrix.

Proposition 4.1 states that  $F(x, y)$  is a rational series of  $\mathbb{N}[[x, y]]$  and that  $(u, A(x, y), v(x, y))$  is a finite representation of it.

**Corollary 4.2.** *The series  $L$  and  $H$  are rational, and we have  $L = 1 + u (I - A(1, y))^{-1} v(1, y)$  and  $H = 1 + u (I - xA(1, 1))^{-1} xv(1, 1)$ .*

Proposition 4.1 and Corollary 4.2, although easy to prove, do not seem to appear in the literature. In the case of the length generating series, the rationality is not new but Corollary 4.2 provides a new formula for  $L$ .

There exist related results in the context of directed animals. Indeed there is a bijection between directed animal of width  $k$  on a 2d triangular lattice and traces in the monoid  $\mathbb{M}(\Sigma, D)$  with  $\Sigma = \{a_1, \dots, a_k\}$  and  $D = \{(a_i, a_j), |i - j| \leq 1\}$ . The precise asymptotics for such directed animals are derived in [21, 25] with the same method as in the proof of Proposition 4.1. More generally, the method of proof of Proposition 4.1 can be viewed as an instance of the transfer matrix method [30, Chap. 4.7].

In the context of trace monoids, the idea of working with the alphabet of cliques  $\mathfrak{C}$  to study the height function appeared in [9] and was later used in [15].

Let  $\pi = \{\mathfrak{C}_1, \dots, \mathfrak{C}_s\}$  be an equitable partition of  $\mathfrak{C}$  such that all the cliques in  $\mathfrak{C}_i$  have a common length  $l_i$ . Let  $A_\pi = (a_{ij})_{ij} \in \mathbb{N}^{s \times s}$  be the coloration matrix. Define the matrix  $A_\pi(x, y) \in \mathbb{N}[x, y]^{s \times s}$  by  $A_\pi(x, y) = (a_{ij}xy^{l_i})_{i,j}$ . Define  $u_\pi = (|\mathfrak{C}_i|)_i \in \mathbb{N}[x, y]^{1 \times s}$  and  $v_\pi(x, y) = (xy^{l_i})_i \in \mathbb{N}[x, y]^{s \times 1}$ . Then formula (2) holds when replacing  $u, A(x, y)$ , and  $v(x, y)$ , by  $u_\pi, A_\pi(x, y)$ , and  $v_\pi(x, y)$ . The proof is similar to the one below.

*Proof of Proposition 4.1.* As recalled above, with each trace is associated its unique CF decomposition. We associate with a path  $p$  in  $\Gamma$  the sequence of its nodes  $(c_1, \dots, c_k)$ . By construction, the CF decomposition of the trace  $t = c_1 \cdots c_k$  is precisely  $(c_1, \dots, c_k)$ . In other words, the CF decompositions of traces are in one-to-one correspondence with the paths in  $\Gamma$ . The contribution of the trace  $t$  to the series  $F$  is  $x^{h(t)}y^{|t|}$ . The weight of the path  $p$  in the weighted automaton  $(u, A(x, y), v(x, y))$  is

$$u_{c_1} \left( \prod_{i=1}^{k-1} A(x, y)_{c_i c_{i+1}} \right) v(x, y)_{c_k} = \left( \prod_{i=1}^{k-1} xy^{|c_i|} \right) xy^{|c_k|} = x^{h(t)}y^{|t|}.$$

This completes the proof of the result.  $\square$

It is easily checked that the series  $F(x, y)$  is not recognizable in general. We recall that  $F = \sum_{k,l} f_{k,l} x^k y^l$  is a *recognizable* series of  $\mathbb{N}[[x, y]]$  if there exists  $K \in \mathbb{N}^*$ ,  $\alpha \in \mathbb{N}^{1 \times K}$ ,  $\mu(x) \in \mathbb{N}^{K \times K}$ ,  $\mu(y) \in \mathbb{N}^{K \times K}$ , and  $\beta \in \mathbb{N}^{K \times 1}$ , such that  $f_{k,l} = \alpha \mu(x)^k \mu(y)^l \beta$  for all  $k$  and  $l$ .

**Example 4.3.** We persevere with the model of Examples 2.1, 2.2, 3.1, 3.3, and 3.4. The height and length generating function is given by

$$\begin{aligned} F &= 1 + \begin{pmatrix} 6 & 3 \end{pmatrix} \begin{pmatrix} 1 - 5xy & -2xy \\ -6xy^2 & 1 - 3xy^2 \end{pmatrix}^{-1} \begin{pmatrix} xy \\ xy^2 \end{pmatrix} \\ &= \frac{1 + xy}{1 - 5xy - 3xy^2 + 3x^2y^3}. \end{aligned} \quad (3)$$

Setting  $x = 1$ , we check that the length generating function is the inverse of the Möbius polynomial, i.e.  $L = (1 - 6y + 3y^2)^{-1}$ . Setting  $y = 1$ , we obtain the height generating function  $H = (1 + x)(1 - 8x + 3x^2)^{-1}$ . The Taylor expansion of the series  $F$  around 0 is

$$\begin{aligned} F &= 1 + 6xy + 3xy^2 + 30x^2y^2 + 30x^2y^3 + 150x^3y^3 + 9x^2y^4 + 222x^3y^4 + \\ &\quad 750x^4y^4 + 126x^3y^5 + 1470x^4y^5 + \dots + 71910x^6y^8 + \dots \end{aligned}$$

For instance, there are 126 traces of length 5 and height 3, or 71910 traces of length 8 and height 6.

We now use Proposition 4.1 to provide some precise results on the asymptotics of the number of traces of a given length or height.

Given a complex function analytic at the origin, a *singularity* is a point where the function ceases to be complex-differentiable. A *dominant* singularity is a singularity of minimal modulus. Throughout the paper, given a series  $S \in \mathbb{N}[[x]]$ , we set  $S = \sum_n (S|n)x^n$ . When applicable, we denote the modulus of the dominant singularities of  $S$  (viewed as a function) by  $\rho_S$ . Classically, see [1, 13, 33], the asymptotic growth rate of  $(S|n)$  is linked to the values of the dominant singularities.

**Lemma 4.4.** *We have  $\rho_L = 1$  or  $\rho_H = 1$  if and only if  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid over  $\Sigma$ .*

*Proof.* We have  $\limsup_n (L|n)^{1/n} = 1/\rho_L$ , and  $\limsup_n (H|n)^{1/n} = 1/\rho_H$  (the ‘exponential growth formula’). It implies that  $\rho_L \leq 1$  and  $\rho_H \leq 1$ .

Assume there exists  $(a, b) \in D$  with  $a \neq b$ . Then all the traces  $t_1 \dots t_n$  with  $t_i \in \{a, b\}$  are of length  $n$  and height  $n$ . It implies that  $(L|n) \geq 2^n$  and that  $(H|n) \geq 2^n$ . It implies in turn that  $\rho_L \leq 1/2$  and  $\rho_H \leq 1/2$ .

Assume now that  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid. By direct computation or using the results from Section 6.1, we get  $(L|n) \sim n^{|\Sigma|-1}$  and  $(H|n) \sim n^{|\Sigma|-1}$ . It implies that  $\rho_L = 1$  and  $\rho_H = 1$ .  $\square$

**Proposition 4.5.** *Let  $(\Sigma, D)$  be a connected dependence graph. Then  $L$  and  $H$  have a unique dominant singularity which is positive real and of order 1.*

It follows (see [1, 13, 33]) that when  $(\Sigma, D)$  is connected, we have  $(L|n) \sim \alpha_L \rho_L^{-n}$  and  $(H|n) \sim \alpha_H \rho_H^{-n}$ , with  $\alpha_L = \rho_L^{-1} \cdot [L(y)(\rho_L - y)]|_{y=\rho_L}$  and  $\alpha_H = \rho_H^{-1} \cdot [H(x)(\rho_H - x)]|_{x=\rho_H}$ .

The proof of Proposition 4.5 is based on the representation given in Proposition 4.1. For convenience reasons, the proof is included in the proof of Proposition 5.1 and given in Appendix.

**Proposition 4.6.** *Let  $(\Sigma, D)$  be a non-connected dependence graph. Let  $(\Sigma_s, D_s)_{s \in S}$  be its partition into maximal connected subgraphs. Denote by  $L_s, H_s$ , the corresponding length and height generating functions. Then one has:*

- 1) *the series  $L$  has a unique dominant singularity equal to  $\rho_L = \min_s \rho_{L_s}$ , and whose order is  $\#\{s, \rho_{L_s} = \rho_L\}$ ;*
- 2) *the series  $H$  has a unique dominant singularity equal to  $\rho_H = \prod_s \rho_{H_s}$ . Its order is  $|\Sigma|$  if  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid, and  $1 + \#\{s, |\Sigma_s| = 1\}$  otherwise.*

Let  $k_L$  and  $k_H$  denote the respective orders of  $\rho_L$  in  $L$  and  $\rho_H$  in  $H$ . It follows from the above Proposition (see [1, 13, 33]) that we have  $(L|n) \sim \alpha_L n^{k_L-1} \rho_L^{-n}$ , and  $(H|n) \sim \alpha_H n^{k_H-1} \rho_H^{-n}$  with  $\alpha_L = (\rho_L^{-k_L} / (k_L - 1)!) \cdot [L(y)(\rho_L - y)^{k_L}]|_{y=\rho_L}$  and  $\alpha_H = (\rho_H^{-k_H} / (k_H - 1)!) \cdot [H(x)(\rho_H - x)^{k_H}]|_{x=\rho_H}$ .

*Proof.* We have  $L = \prod_s L_s = \prod_s \mu(\Sigma_s, I_s)^{-1}$  where  $\mu(\cdot)$  is defined in (1). It implies directly the result on  $\rho_L$ .

Consider now the height generating function. We prove the result by induction on  $|S|$ . Assume first that  $\#S = 2$  and set  $S = \{1, 2\}$ . We have  $H = \sum_{i,j} (H_1|i)(H_2|j)x^{\max(i,j)}$ . It implies that

$$(H|n) = (H_1|n) \sum_{i=0}^n (H_2|i) + (H_2|n) \sum_{i=0}^n (H_1|i) - (H_1|n)(H_2|n). \quad (4)$$

Applying Proposition 4.5, we obtain  $(H_1|n) = a_n \rho_{H_1}^{-n}$ , with  $\lim_n a_n = a \in \mathbb{R}_+^*$ , and  $(H_2|n) = b_n \rho_{H_2}^{-n}$ , with  $\lim_n b_n = b \in \mathbb{R}_+^*$ .

Consider first the case  $\rho_{H_1} < 1$  and  $\rho_{H_2} < 1$ . We have

$$\begin{aligned} (H_1|n) \sum_{i=0}^n (H_2|i) &= a_n \rho_{H_1}^{-n} \left( \sum_{i=0}^n b_i \rho_{H_2}^{-i} \right) \\ &= a_n b \rho_{H_1}^{-n} \rho_{H_2}^{-n} \left( \sum_{i=0}^n (b_{n-i}/b) \rho_{H_2}^i \right) \\ &\sim ab(1 - \rho_{H_2})^{-1} (\rho_{H_1} \rho_{H_2})^{-n}. \end{aligned}$$

The same type of identity also holds for the second term in (4). Going back to (4), we then obtain

$$(H|n) \sim ab((1 - \rho_{H_1})^{-1} + (1 - \rho_{H_2})^{-1} - 1)(\rho_{H_1}\rho_{H_2})^{-n}.$$

Hence we have  $\rho_H = \rho_{H_1}\rho_{H_2}$  and the order of  $\rho_H$  in  $H$  is 1.

We consider now the case  $\rho_{H_1} = 1$  and  $\rho_{H_2} = 1$ . By Lemma 4.4, we get that  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid over two letters. Applying (4), we get that  $(H|n) = (2n + 1)$ . Hence we have  $\rho_H = 1$  and the order of  $\rho_H$  in  $H$  is 2.

By symmetry, the last case to consider is  $\rho_{H_1} < 1$  and  $\rho_{H_2} = 1$ . By Proposition 4.5, we have  $(H_1|n) \sim a\rho_{H_1}^{-n}$ . We also have  $(H_2|n) = 1$ . Simplifying (4), we obtain that  $(H|n) \sim an\rho_{H_1}^{-n}$ . It implies that  $\rho_H = \rho_{H_1}$  and that the order of  $\rho_H$  in  $H$  is 2.

Consider now the case  $\#S > 2$ . Let  $(\Sigma_1, D_1)$  and  $(\Sigma_2, D_2)$  be a partition of  $(\Sigma, D)$  in two subgraphs such that  $(\Sigma_1, D_1)$  is connected. The induction hypothesis applies to  $(\Sigma_2, D_2)$  and the proof follows exactly the same steps as above.  $\square$

The results on  $L$  in Proposition 4.5 and Proposition 4.6 can be restated as results on the smallest root of the Möbius polynomial of a non-directed graph. They improve on a recent result by Goldwurm and Santini [19] stating that the Möbius polynomial has a unique and positive real root of smallest modulus. Our proof of Proposition 5.1 follows several of the steps of [19]. One central difference is that we work with Cartier-Foata representatives instead of minimal lexicographic representatives. Proving the strengthened statements while working with the latter does not appear to be easy.

A *matching* in a (non-directed) graph is a subset of arcs with no common nodes. The *matching polynomial* of a graph is equal to  $\sum_k (-1)^k m_k y^k$ , where  $m_k$  is the number of matchings of  $k$  arcs. Hence, the matching polynomial of a graph  $G$  is equal to the Möbius polynomial of the complement of the line graph of  $G$ . Matching polynomials have been studied quite extensively. It is known for instance that all the roots of a matching polynomial are real [16, 18]. It implies that the same is true for the Möbius polynomial of a graph which is the complement of a line graph. For a general graph, the result is not true and one has to settle for the weaker results in Proposition 4.5 and Proposition 4.6. Consider for instance the graph with nodes  $\{a, b, c, d\}$  and arcs  $\{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}$ . It is the smallest graph which is not the complement of a line graph. Its Möbius polynomial is  $\mu = 1 - 4y + 3y^2 - y^3$ , which has two non-real roots.

## 5 Asymptotic Average Height

We want to address questions such as: what is the amount of ‘parallelism’ in a trace monoid? Given several dependence graphs over the same alphabet, which one is the ‘most parallel’? To give a precise meaning to these questions, we define the following performance measures. Let  $\mathbb{M}_n$  denote the set of traces of length  $n$  of the trace monoid  $\mathbb{M}$ . We equip  $\mathbb{M}_n$  with a probability distribution  $P_n$  and we compute the corresponding average height

$$E_n[h] = \sum_{t \in \mathbb{M}_n} P_n\{t\} h(t) .$$

Assuming the limit exists, we call  $\lim_n E_n[h]/n$  the (*asymptotic*) *average height*. Obviously this quantity belongs to  $[C^{-1}, 1]$ , where  $C$  is the maximal length of a clique. Clearly the relevance of the average height as a measure of the parallelism in the trace monoid depends on the relevance of the chosen family of probability measures. This may vary depending on the application context. A very common choice is to consider uniform probabilities. It is the natural solution in the absence of precise information on the structure of the traces to be dealt with. Let us consider different instances of uniform probabilities over traces.

### 5.1 Uniform probability on words

Let  $\mu_n$  be the uniform probability distribution over  $\Sigma^n$  which is defined by setting  $\mu_n\{u\} = 1/|\Sigma|^n$ , for every  $u \in \Sigma^n$ . We set  $P_n = \mu_n \circ \psi^{-1}$ , i.e.  $P_n\{t\} = \mu_n\{w : \psi(w) = t\}$ . The limit below exists:

$$\lambda_* = \lambda_*(\Sigma, D) = \lim_n \frac{E_n[h]}{n} = \lim_n \frac{\sum_{w \in \Sigma^n} h(\psi(w))}{n|\Sigma|^n} . \quad (5)$$

This is proved using Markovian arguments in [27]. The existence of  $\lambda_*$  can also be proved using sub-additive arguments. More precisely, it is shown in [14] that  $h(\psi(.))$  is recognized by an automaton with multiplicities over the  $(\max, +)$  semiring, which provides a different proof of the existence of  $\lambda_*$ . In fact a stronger result holds. Let  $(x_n)_{n \in \mathbb{N}^*}$  be a sequence of independent random variables valued in  $\Sigma$  and uniformly distributed:  $P\{x_n = u\} = 1/|\Sigma|, u \in \Sigma$ . The probability distribution of  $(x_1 \cdots x_n)$  is then the uniform distribution over  $\Sigma^n$ . It is proved in [27, 14] that

$$P\left\{ \lim_n \frac{h(\psi(x_1 \cdots x_n))}{n} = \lambda_* \right\} = 1 . \quad (6)$$

Except for small trace monoids,  $\lambda_*$  is neither rational, nor algebraic. The problem of approximating  $\lambda_*$  is NP-hard [3]. Non-elementary bounds are proposed in [6]. Exact computations for simple trace monoids are proposed in [5, 28]. A software package named ERS [22] enables to simulate and compute bounds for  $\lambda_*$ .

## 5.2 Uniform probability on traces

A natural counterpart of the above case consists in considering the uniform probability distribution over  $\mathbb{M}_n$ , i.e.  $Q_n\{t\} = 1/|\mathbb{M}_n|$  for every  $t \in \mathbb{M}_n$ . Assuming existence, we define the limit

$$\lambda_{\mathbb{M}} = \lambda_{\mathbb{M}}(\Sigma, D) = \lim_n \frac{E_n[h]}{n} = \lim_n \frac{\sum_{t \in \mathbb{M}_n} h(t)}{n|\mathbb{M}_n|} = \lim_n \frac{\sum_{m \in \mathbb{N}} m f_{m,n}}{\sum_{m \in \mathbb{N}} n f_{m,n}}. \quad (7)$$

Dually, let  ${}_m\mathbb{M}$  be the set of traces of height  $m$ , and let  $\tilde{Q}_m$  be the uniform probability measure on  ${}_m\mathbb{M}$ , i.e.  $\tilde{Q}_m\{t\} = 1/|{}_m\mathbb{M}|$  for every  $t \in {}_m\mathbb{M}$ . The average length of a trace in  ${}_m\mathbb{M}$  is equal to  $E_m[l] = \sum_{t \in {}_m\mathbb{M}} \tilde{Q}_m\{t\} |t|$ . Assuming existence, we define the limit

$$\gamma_{\mathbb{M}} = \gamma_{\mathbb{M}}(\Sigma, D) = \lim_m \frac{E_m[l]}{m} = \lim_m \frac{\sum_{t \in {}_m\mathbb{M}} |t|}{m|{}_m\mathbb{M}|} = \lim_m \frac{\sum_{n \in \mathbb{N}} n f_{m,n}}{\sum_{n \in \mathbb{N}} m f_{m,n}}. \quad (8)$$

The quantity  $\gamma_{\mathbb{M}}$  is an (*asymptotic*) *average length*. The analog of  $\lambda_*$  and  $\lambda_{\mathbb{M}}$  is then the quantity  $\gamma_{\mathbb{M}}^{-1}$ .

**Proposition 5.1.** *The limits  $\lambda_{\mathbb{M}}$  in (7) and  $\gamma_{\mathbb{M}}$  in (8) exist. Furthermore,  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}$  are algebraic numbers.*

The proof, based on Proposition 4.1, is rather long and we postponed it to the Appendix. In fact, the proof of Proposition 5.1 provides a formula for  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}$ . Define  $G = (\partial F / \partial x)(1, y)$  and  $\tilde{G} = (\partial F / \partial y)(x, 1)$ . Then, with the notations of Section 4, we have

$$\lambda_{\mathbb{M}} = \frac{[G(y)(\rho_L - y)^{k_L+1}]_{|y=\rho_L}}{k_L \rho_L [L(y)(\rho_L - y)^{k_L}]_{|y=\rho_L}}, \quad \gamma_{\mathbb{M}} = \frac{[\tilde{G}(x)(\rho_H - x)^{k_H+1}]_{|x=\rho_H}}{k_H \rho_H [H(x)(\rho_H - x)^{k_H}]_{|x=\rho_H}}. \quad (9)$$

## 5.3 Uniform probability on CF decompositions

In this section, we use some basic results on Markov chains, for details see for instance [4, 26, 29]. Let  $A \in \{0, 1\}^{\mathfrak{C} \times \mathfrak{C}}$  be the adjacency matrix of  $\Gamma$ . We associate with  $A = (a_{ij})_{i,j}$ , the Markovian matrix

$$\hat{A} = (\hat{a}_{ij})_{i,j}, \quad \hat{a}_{ij} = a_{ij} \left( \sum_k a_{ik} \right)^{-1}. \quad (10)$$

We define the vector  $\vec{1} \in \mathbb{R}^{1 \times \mathfrak{C}}$  by  $\vec{1}_i = 1/|\mathfrak{C}|$  for all  $i$ . We define the probability measure  $R_m$  on  ${}_m\mathbb{M}$  as follows: for a trace  $t \in {}_m\mathbb{M}$  with Cartier-Foata decomposition  $(c_1, \dots, c_m)$ , we set  $R_m\{t\} = \vec{1}_{c_1} \hat{a}_{c_1 c_2} \cdots \hat{a}_{c_{m-1} c_m}$ .

An interpretation for the family  $(R_m)_m$  is as follows. Consider a Markov chain  $(X_n)_n$  on the state space  $\mathfrak{C}$  with transition matrix  $\hat{A}$  and with initial

distribution  $\vec{1}$ . Then  $R_m\{t\} = P\{X_1 \cdots X_m = t\}$ . Equivalently, given a trace  $t$  of height  $m$ , we get a trace  $t'$  of height  $m+1$  by picking at random and uniformly an admissible clique  $c$  and by setting  $t' = tc$ . This can be loosely described as a ‘uniform probability on CF decompositions’.

The average length of a trace in  ${}_m\mathbb{M}$  is equal to  $E_m[l] = \sum_{t \in {}_m\mathbb{M}} R_m\{t\}|t|$ . Assuming existence, the analog of  $\lambda_*$ ,  $\lambda_{\mathbb{M}}$  or  $\gamma_{\mathbb{M}}^{-1}$  is then the *(asymptotic) average height*

$$\lambda_{\text{cf}} = \lambda_{\text{cf}}(\Sigma, D) = \lim_m \frac{m}{E_m[l]}. \quad (11)$$

Let  $p = (p(c))_{c \in \mathfrak{C}}$  be defined by

$$p = \lim_n \vec{1}(I + \hat{A} + \cdots + \hat{A}^{n-1})/n. \quad (12)$$

The vector  $p$  can be interpreted as the limit distribution of the Markov chain  $(X_n)_n$ . According to the ergodic theorem for Markov chains, the limit exists in (11) and we have

$$\lambda_{\text{cf}} = \left( \sum_{c \in \mathfrak{C}} p(c)|c| \right)^{-1}. \quad (13)$$

When  $(\Sigma, D)$  is connected, it follows from Lemma 3.2 that  $\hat{A}$  is irreducible. Then  $p$  is entirely determined by  $p\hat{A} = p$  and  $\sum_i p_i = 1$  (Perron-Frobenius Theorem). It implies that  $\lambda_{\text{cf}}$  is explicitly computable and rational. When  $(\Sigma, D)$  is non-connected,  $\lambda_{\text{cf}}$  is still explicitly computable and rational according to Proposition 5.4.

Consider an equitable partition  $\pi = \{\mathfrak{C}_1, \dots, \mathfrak{C}_s\}$  such that all the cliques in  $\mathfrak{C}_i$  have a common length  $l_i$ . There exists an analog of (13) corresponding to this partition. Let  $\hat{A}_\pi$  be the Markovian matrix associated with the coloration matrix  $A_\pi$ . Let  $p_\pi$  be defined by  $p_\pi = \lim_n \vec{1}(I + \hat{A}_\pi + \cdots + \hat{A}_\pi^{n-1})/n$ . Then, we have  $\lambda_{\text{cf}} = (\sum_i p_\pi(i)l_i)^{-1}$ .

## 5.4 Non-connected dependence graphs

Assume that  $(\Sigma, D)$  is *non-connected* and let  $(\Sigma_s, D_s)_{s \in S}$  be the maximal connected subgraphs of  $(\Sigma, D)$ . We now propose formulas to express the average height of  $(\Sigma, D)$  as a function of the ones of  $(\Sigma_s, D_s)$ .

First, it is simple to prove using (6) and the Strong Law of Large Numbers (see also Theorem 5.7 in [27]) that we have

$$\lambda_*(\Sigma, D) = \max_{s \in S} \left( \frac{|\Sigma_s|}{|\Sigma|} \lambda_*(\Sigma_s, D_s) \right). \quad (14)$$

**Proposition 5.2.** *Denote by  $L_s$  the length generating function of  $(\Sigma_s, D_s)$ . Define  $J = \{j \in S, \rho_{L_j} = \min_{s \in S} \rho_{L_s}\}$ . Then, we have*

$$\lambda_{\mathbb{M}}(\Sigma, D) = \lambda_{\mathbb{M}}(\Sigma_J, D_J), \quad (15)$$

where  $\Sigma_J = \cup_{j \in J} \Sigma_j$ , and  $D_J = \cup_{j \in J} D_j$ .

The proof uses Proposition 5.1 and is given in Appendix. There seems to be no simple way to write  $\lambda_{\mathbb{M}}(\Sigma_J, D_J)$  as a function of  $\lambda_{\mathbb{M}}(\Sigma_j, D_j)$ ,  $j \in J$ , as illustrated by the example of Section 6.1.

**Proposition 5.3.** *Define  $J = \{j \in S, |\Sigma_j| > 1\}$ . Then, we have*

$$\gamma_{\mathbb{M}}(\Sigma, D) = \sum_{j \in J} \gamma_{\mathbb{M}}(\Sigma_j, D_j) + \frac{|S - J|}{2}, \quad (16)$$

if  $J \neq \emptyset$ . If  $J = \emptyset$ , that is if  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid, we have  $\gamma_{\mathbb{M}}(\Sigma, D) = (|\Sigma| + 1)/2$ .

The proof is given in Appendix. Proposition 5.3 is the counterpart of Proposition 5.2 for  $\gamma_{\mathbb{M}}$ , but it is more precise.

**Proposition 5.4.** *Let  $\hat{A}$  be defined as in Section 5.3. Let  $\mathfrak{C}_s$  be the set of cliques of  $(\Sigma_s, I_s)$ . Define the matrix  $B$  of dimension  $\mathfrak{C} \times \mathfrak{C}$  as follows:  $B_{ij} = \hat{A}_{ij}$  if  $i \notin \bigcup_{s \in S} \mathfrak{C}_s$  and  $B_{ij} = 0$  otherwise. Define the vectors  $\mathcal{I}_{\mathfrak{C}_s}$ ,  $s \in S$ , of dimension  $\mathfrak{C}$  as follows:  $(\mathcal{I}_{\mathfrak{C}_s})_i = 1$  if  $i \in \mathfrak{C}_s$  and  $(\mathcal{I}_{\mathfrak{C}_s})_i = 0$  otherwise. Set  $q_s = \vec{1}(I - B)^{-1} \mathcal{I}_{\mathfrak{C}_s}$ , where  $\vec{1} = (1/|\mathfrak{C}|, \dots, 1/|\mathfrak{C}|)$ . Then we have*

$$\lambda_{\text{cf}}(\Sigma, D)^{-1} = \sum_{s \in S} q_s \lambda_{\text{cf}}(\Sigma_s, D_s)^{-1}. \quad (17)$$

*Proof.* The graph of cliques  $\Gamma$  of  $(\Sigma, I)$  can be decomposed in its maximal strongly connected subgraphs (mscs). Replacing each mscs by one node, we define the *condensed* graph of  $\Gamma$ . The *final* mscs are the mscs without any successor in the condensed graph. According to Lemma 3.2, the final mscs are precisely the ones with sets of nodes  $\mathfrak{C}_s$  where  $s \in S$ .

Remark that the non-negative matrix  $B$  is such that  $\sum_j B_{ij} < 1$  for every  $i \in \mathfrak{C}$ . In particular, it implies that  $(I - B)$  is invertible. Define  $q_s = \vec{1}(I - B)^{-1} \mathcal{I}_{\mathfrak{C}_s}$  for every  $s \in S$ .

The quantities  $q_s$  can be interpreted in terms of the Markov chain  $(X_n)_n$  defined in Section 5.3: we have  $q_s = \lim_n P\{X_n \in \mathfrak{C}_s\}$  (Theorem 4.4 in [29]). Let  $A_s$  be the restriction of  $A$  to the index set  $(\mathfrak{C}_s \times \mathfrak{C}_s)$  and let  $\hat{A}_s$  be the Markovian matrix associated with  $A_s$ . Let  $\hat{p}_s$  be the unique probability distribution on  $\mathfrak{C}_s$  such that  $\hat{p}_s \hat{A}_s = \hat{p}_s$  (Perron-Frobenius Theorem). According to the ergodic theorem for Markov chains, we have  $\lambda_{\text{cf}}(\Sigma_s, D_s) = (\sum_{c \in \mathfrak{C}_s} \hat{p}_s(c) |c|)^{-1}$ .

Define the vector  $p_s$  of dimension  $\mathfrak{C}$  by  $p_s(c) = \hat{p}_s(c)$  if  $c \in \mathfrak{C}_s$  and  $p_s(c) = 0$  otherwise. The vector  $p = \sum_{s \in S} q_s p_s$  is then the unique limit distribution of  $(X_n)_n$ , i.e. the vector  $p$  defined in (12). By the ergodic theorem for Markov chains, we obtain (17).  $\square$

## 5.5 Comparison between the different average heights

In terms of computability, the simplest quantity is  $\lambda_{\text{cf}}$  and the most complicated one is  $\lambda_*$ . This is reflected by the fact that  $\lambda_{\text{cf}}$  is rational, that  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}^{-1}$  are algebraic, and that  $\lambda_*$  is in general not algebraic, see for instance (39).

Another point of view is to compare the families of probability measures  $(P_n)_n$ ,  $(Q_n)_n$ ,  $(\tilde{Q}_n)_n$ , and  $(R_n)_n$  associated respectively with  $\lambda_*$ ,  $\lambda_{\mathbb{M}}$ ,  $\gamma_{\mathbb{M}}^{-1}$ , and  $\lambda_{\text{cf}}$ . A family of probability measures  $(\mu_n)_n$  defined on  $(\mathbb{M}_n)_n$  (or  $({}_n\mathbb{M})_n$ ) is said to be *consistent* if we have  $\mu_m\{t\} = \mu_n\{v : \exists u, v = tu\}$  for all  $m < n$ . In this case, there exists a unique probability measure on infinite traces whose finite-dimensional marginals are the probabilities  $(\mu_n)_n$ . Consistency is a natural and desirable property. Clearly the families  $(P_n)_n$  and  $(R_n)_n$  are consistent. On the other hand, the families  $(Q_n)_n$  and  $(\tilde{Q}_n)_n$  are not.

It is also interesting to look at the asymptotics in  $n$  of the empirical distribution of  $\{h(t)/|t|, t \in \mathbb{M}_n\}$  or  $\{|t|/h(t), t \in {}_n\mathbb{M}\}$ . For  $a \in \mathbb{R}$ , let  $\delta_a$  denote the probability measure concentrated in  $a$ . It follows from (6) that we have

$$\sum_t P_n\{t\}\delta_{h(t)/|t|} \longrightarrow \delta_{\lambda_*},$$

with the arrow standing for ‘convergence in distribution’ (or ‘weak convergence’). Similarly, it follows from the ergodic theorem for Markov chains that we have

$$\sum_t R_n\{t\}\delta_{h(t)/|t|} \longrightarrow \sum_{s \in S} q_s \delta_{\lambda_{\text{cf}}(\Sigma_s, D_s)},$$

the notations being the ones of Section 5.3. There are no such concentration results for  $(Q_n)_n$  and  $(\tilde{Q}_n)_n$ . To check this, consider the case of the free commutative monoid over two letters. We obtain easily that

$$\sum_t Q_n\{t\}\delta_{h(t)/|t|} \longrightarrow U, \quad \sum_t \tilde{Q}_n\{t\}\delta_{|t|/h(t)} \longrightarrow V, \quad \sum_t \tilde{Q}_n\{t\}\delta_{h(t)/|t|} \longrightarrow W,$$

where  $U$  and  $V$  are the uniform distributions over the intervals  $[1/2, 1]$  and  $[1, 2]$ , and where  $W$  is the distribution with density  $1/x^2$  over the interval  $[1/2, 1]$ .

Consider two dependence graphs  $(\Sigma, D_1)$  and  $(\Sigma, D_2)$  with  $D_1 \subset D_2$ . The intuition is that  $\mathbb{M}(\Sigma, D_1)$  should be ‘more parallel’ than  $\mathbb{M}(\Sigma, D_2)$ . In accordance with this intuition, it is elementary to prove that  $\lambda_*(\Sigma, D_1) \leq \lambda_*(\Sigma, D_2)$ . However, the corresponding inequalities do not hold for  $\lambda_{\mathbb{M}}$  and  $\lambda_{\text{cf}}$ . Consider for instance the trace monoids over three or four letters whose average heights are given in Section B. This raises some interesting issues on how to interpret these quantities. On the other hand, we conjecture that the inequality  $\gamma_{\mathbb{M}}(\Sigma, D_1)^{-1} \leq \gamma_{\mathbb{M}}(\Sigma, D_2)^{-1}$  is satisfied.

## 6 Some Examples

### 6.1 The free commutative monoid

Consider the dependence graph  $(\Sigma, D)$  with  $D = \{(u, u), u \in \Sigma\}$ . The corresponding trace monoid  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid over the alphabet  $\Sigma$ , which is isomorphic to  $\mathbb{N}^\Sigma$ . Set now  $k = |\Sigma|$ .

A direct application of (14) yields  $\lambda_* = 1/k$ . Consider now  $\lambda_{\text{cf}}$ . The final maximal strongly connected subgraphs of  $\Gamma$  are precisely the cliques of length 1. In particular, they are of cardinality 1. Applying the results in Section 5.3, we get  $\lambda_{\text{cf}} = 1$ .

Let us compute  $\lambda_{\mathbb{M}}$  and  $\gamma_{\mathbb{M}}$ . Using the methodology of Sections 4 and 5 is feasible, but there are simpler methods. Consider  $\gamma_{\mathbb{M}}$  first. By a counting argument, we get

$$L_m = \sum_{t \in {}_m\mathbb{M}} y^{|t|} = (1 + y + \dots + y^m)^k - (1 + y + \dots + y^{m-1})^k. \quad (18)$$

We obtain  $|{}_m\mathbb{M}| = L_m(1) = (m+1)^k - m^k$  and  $\sum_{t \in {}_m\mathbb{M}} |t| = L'_m(1) = k(m+1)^k m/2 - km^k(m-1)/2$ . We deduce that  $\gamma_{\mathbb{M}}^{-1} = 2/(k+1)$ .

Let us now compute the average height  $\lambda_{\mathbb{M}}$ . The length generating function is  $L = (1-y)^{-k}$ . Applying a result from Carlitz [7], we have

$$\begin{aligned} G = (\partial F / \partial x)(1, y) &= \sum_{(n_1, \dots, n_k)} \max(n_1, \dots, n_k) y^{n_1 + \dots + n_k} \\ &= \frac{1}{(1-y)^{k+1}} \sum_{i=1}^k \binom{k}{i} \frac{(-1)^{i-1} y^i}{1 + y + \dots + y^{i-1}}. \end{aligned} \quad (19)$$

Using (9), we obtain

$$\lambda_{\mathbb{M}} = \frac{1}{k} \left( \sum_{i=1}^k \binom{k}{i} \frac{(-1)^{i-1}}{i} \right) = \frac{1}{k} \left( 1 + \frac{1}{2} + \dots + \frac{1}{k} \right). \quad (20)$$

The last equality is a classical identity for harmonic summations (see Chapter 6.4 in [20]). Asymptotically in  $k$ , we have  $\lambda_{\mathbb{M}} \sim \log(k)/k$ . This is to be compared with  $\lambda_* = 1/k$  and  $\gamma_{\mathbb{M}}^{-1} \sim 2/k$ .

Consider now the trace monoid  $\mathbb{M}(\Sigma, D)$  obtained as the direct product of the free monoids  $\Sigma_1^*, \dots, \Sigma_k^*$ , with  $|\Sigma_1| = \dots = |\Sigma_k| = c$  and  $c > 1$ . Equivalently, the dependence graph is  $(\Sigma, D)$  with  $\Sigma = \cup_{i=1}^k \Sigma_i$ ,  $D = \cup_{i=1}^k D_i$ , and  $D_i = \Sigma_i \times \Sigma_i$  for all  $i$ . Clearly, we still have  $\lambda_* = 1/k$  and  $\lambda_{\text{cf}} = 1$ . The formulas in (18) and (19) still hold when replacing  $y$  by  $cy$ . We deduce that  $\lambda_{\mathbb{M}}$  is still given by (20). On the other hand, we have  $\gamma_{\mathbb{M}}^{-1} = 1/k$ , a value which can also be obtained using Proposition 5.3. Hence the value of  $\gamma_{\mathbb{M}}^{-1}$  does not depend on the value of  $c$ ,  $c > 1$ , and is different from the value obtained for  $c = 1$ .

## 6.2 The ladder graph

In view of Proposition 4.1, the simplest sets of cliques are those with the property that the clique partition according to the length is equitable, so that the dimension of the corresponding coloration matrix reduces to the maximal size of a clique. This holds if the full automorphism group of  $(\Sigma, D)$  or  $\Gamma$  acts transitively on the sets of cliques of the same length. This is in particular the case when the dependence graph is the *triangular graph*, i.e. the line graph of the complete graph  $K_n$ , or the *square lattice graph*, i.e. the line graph of the complete bipartite graph  $K_{n,n}$ .

A particularly simple class of independence graphs is the class of node and arc-transitive triangle-free graphs. In this case, the coloration matrix associated with the full automorphism group is of dimension  $2 \times 2$ . Let us consider a family of graphs of this type.

Let  $(\Sigma, I)$  be the *ladder graph*, i.e.  $\Sigma = \{1, \dots, 2n\}$  and  $I = \{(i, j) \in \Sigma \times \Sigma : i + j = 2n + 1\}$ . The corresponding dependence graph is known as the *cocktail party graph*  $\text{CP}_n$ . The full automorphism group is the wreath product  $W = \mathfrak{S}_n[\mathbb{Z}/2\mathbb{Z}]$  of the symmetric group of degree  $n$  with  $\mathbb{Z}/2\mathbb{Z}$ . The corresponding partition of  $\mathfrak{C}$  is  $\{\mathfrak{C}_1, \mathfrak{C}_2\}$  with  $\mathfrak{C}_1 = \{c \in \mathfrak{C}, |c| = 1\}$  and  $\mathfrak{C}_2 = \{c \in \mathfrak{C}, |c| = 2\}$ . The coloration matrix is

$$A_W = \begin{pmatrix} 2n-1 & n-1 \\ 2n & n \end{pmatrix}.$$

The computation of  $\lambda_*(\text{CP}_n)$  was worked out by Brilman (see [5], Proposition 14):

$$\lambda_*(\text{CP}_n) = \frac{1}{2} \left( 1 + \frac{\sqrt{n-1}}{\sqrt{n+1}} \right).$$

We compute  $F$  using the reduced representation induced by the partition. The dominant singularity of  $L$  is  $(1 - \sqrt{1 - n^{-1}})$  and we obtain

$$\lambda_{\mathbb{M}}(\text{CP}_n) = \frac{1}{2} \left( 1 + \frac{\sqrt{n}}{2\sqrt{n} - \sqrt{n-1}} \right).$$

The dominant singularity of  $H$  is  $(3n - 1 - \sqrt{9n^2 - 10n + 1})/2n$ , and we get

$$\gamma_{\mathbb{M}}(\text{CP}_n)^{-1} = \frac{\Delta(5n - 1 - \Delta)}{2\Delta(4n - 1) - 2(8n^2 - 9n + 1)}, \quad \Delta = \sqrt{9n^2 - 10n + 1}.$$

Considering the Markovian matrix  $\hat{A}_W$  and using (13), we obtain

$$\lambda_{\text{cf}}(\text{CP}_n) = \frac{9n - 7}{12n - 10}.$$

We check that  $\lim_n \lambda_*(\text{CP}_n) = \lim_n \lambda_{\mathbb{M}}(\text{CP}_n) = 1$  and that  $\lim_n \gamma_{\mathbb{M}}(\text{CP}_n)^{-1} = \lim_n \lambda_{\text{cf}}(\text{CP}_n) = 3/4$ .

Remark that  $\text{CP}_3 \equiv T_4$ . By specializing the above results to  $n = 3$ , we get the average heights for the triangular graph  $T_4$  considered in Examples 2.1, 2.2, 3.1, 3.3, 3.4, and 4.3. We have

$$\begin{aligned}\lambda_*(T_4) &= \frac{2+\sqrt{2}}{4} = 0.854\dots, \quad \lambda_{\text{cf}}(T_4) = \frac{10}{13} = 0.769\dots, \\ \lambda_{\mathbb{M}}(T_4) &= \frac{16+\sqrt{6}}{20} = 0.922\dots, \quad \gamma_{\mathbb{M}}(T_4)^{-1} = \frac{39+\sqrt{13}}{58} = 0.735\dots.\end{aligned}$$

## A Proofs of the results in Section 5.2

This section is devoted to the proof of Propositions 5.1, 5.2, and 5.3.

*Proof of Proposition 5.1.*

We give the proof for  $\lambda_{\mathbb{M}}$ . The one for  $\gamma_{\mathbb{M}}$  is similar (and easier!). Recall that  $L = F(1, y) = \sum_{t \in \mathbb{M}} y^{|t|}$  is the length generating function. Define

$$G = \frac{\partial F}{\partial x}(1, y) = \sum_{t \in \mathbb{M}} h(t) y^{|t|}.$$

Assuming existence of the limit in (7), we have  $\lambda_{\mathbb{M}} = \lim_n (G|n)/(n(L|n))$ .

According to Pringsheim's Theorem [31, Sec. 7.21],  $L$  and  $G$  have a positive real dominant singularity. They are denoted respectively by  $\rho_L$  and  $\rho_G$  according to the previous conventions. Since we have  $n(L|n)/C \leq (G|n) \leq n(L|n)$ , it implies that  $\rho_L = \rho_G$ . Let  $k_L$  be the order of  $\rho_L$  in  $L$ .

Assume that  $\rho_L$  is the unique dominant singularity in  $L$ . Assume that the order of  $\rho_L$  in  $G$  is  $(k_L + 1)$  and is strictly larger than the one of the other singularities of modulus  $\rho_L$  (there might exist several dominant singularities for  $G$ , see Section 6.1). Then the limit in (7) exists and we have,

$$\lambda_{\mathbb{M}} = \frac{[G(y)(\rho_L - y)^{k_L+1}]_{|y=\rho_L}}{k_L \rho_L \cdot [L(y)(\rho_L - y)^k]_{|y=\rho_L}}.$$

In particular,  $\lambda_{\mathbb{M}}$  is an algebraic number. The above assumptions on the dominant singularities of  $L$  and  $G$  ensure that the sequences  $((L|n))_n$  and  $((G|n))_n$  do not have an oscillating behavior. It remains to prove that these assumptions actually hold.

We work with the representation  $(u, A(x, y), v(x, y))$  of  $F$  given in the statement of Proposition 4.1. We have

$$F = 1 + \frac{u \text{Adj}(I - A(x, y))v(x, y)}{\det(I - A(x, y))}, \quad (21)$$

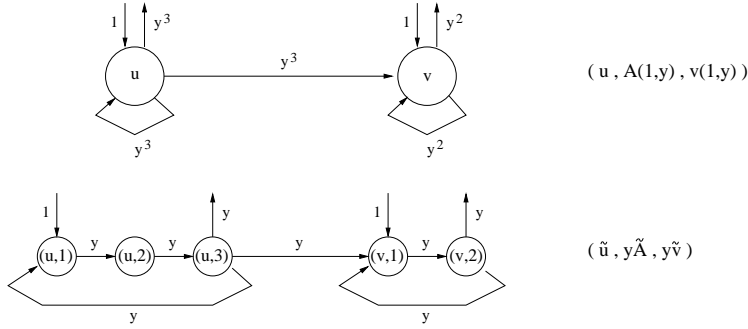
where  $\det(\cdot)$  stands for the determinant and  $\text{Adj}(\cdot)$  for the adjoint of a matrix. Set  $Q(x, y) = \det(I - A(x, y))$  and  $F = P(x, y)/Q(x, y)$ . It follows that we have  $L = P(1, y)/Q(1, y)$ . By differentiating  $F$ , we get

$$G = \frac{1}{Q(1, y)} \left( \frac{\partial P}{\partial x}(1, y) - L \cdot \frac{\partial Q}{\partial x}(1, y) \right). \quad (22)$$

Set  $Q(y) = Q(1, y)$ . The above equations imply that the set of singularities of  $L$  (resp.  $G$ ) is included in the set of singularities of  $1/Q(y)$ . In particular, the modulus of a dominant singularity of  $L$  (resp.  $G$ ) is greater or equal to the modulus of a dominant singularity of  $1/Q(y)$ .

The next step consists in transforming the triple  $(u, A(1, y), v(1, y))$  into another triple  $(\tilde{u}, y\tilde{A}, y\tilde{v})$  of dimension  $K > |\mathfrak{C}|$ , where  $\tilde{u} \in \mathbb{N}^{1 \times K}$ ,  $\tilde{A} \in \mathbb{N}^{K \times K}$ ,  $\tilde{v} \in \mathbb{N}^{K \times 1}$ , and where we set  $y\tilde{A} = (y\tilde{A}_{ij})_{ij}$  and  $y\tilde{v} = (y\tilde{v}_i)_i$ .

Before formally defining it, we illustrate the construction on the figure below. As usual we view a triple as an automaton with multiplicities, i.e. as a weighted graph with input and output arcs. We have represented the portion of the automata  $(u, A(1, y), v(1, y))$  and  $(\tilde{u}, y\tilde{A}, y\tilde{v})$  corresponding to the cliques  $u$  and  $v$  where  $|u| = 3, |v| = 2$ , and  $(u, v)$  is CF-admissible.



Consider the index set

$$E = \{(c, 1), \dots, (c, |c|), c \in \mathfrak{C}\}. \quad (23)$$

Let us define  $\tilde{u} \in \mathbb{N}^{1 \times E}$ ,  $\tilde{A} \in \mathbb{N}^{E \times E}$ , and  $\tilde{v} \in \mathbb{N}^{E \times 1}$  as follows:

$$\begin{aligned}\tilde{u}_i &= \begin{cases} 1 & \text{if } i = (c, 1), c \in \mathfrak{C} \\ 0 & \text{otherwise,} \end{cases} \\ \tilde{A}_{ij} &= \begin{cases} 1 & \text{if } i = (c, k), j = (c, k+1), c \in \mathfrak{C}, 1 \leq k < |c| \\ 1 & \text{if } i = (c, |c|), j = (c, 1), c \in \mathfrak{C} \\ 1 & \text{if } i = (c, |c|), j = (d, 1), A_{cd} \neq 0 \\ 0 & \text{otherwise,} \end{cases} \\ \tilde{v}_i &= \begin{cases} 1 & \text{if } i = (c, |c|), c \in \mathfrak{C} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

In an automaton, an *input* (resp. *output*) *node* is a node with an input (resp. output) arc. A *successful path* is a path from an input node to an output node. There is a one to one mapping between successful paths in the automata  $(u, A(1, y), v(1, y))$  and  $(\tilde{u}, y\tilde{A}, y\tilde{v})$ : to the successful path  $(c_1, \dots, c_k)$  in  $(u, A(1, y), v(1, y))$  corresponds the successful path  $((c_1, 1), \dots, (c_1, |c_1|), (c_2, 1), \dots, (c_k, |c_k|))$  in  $(\tilde{u}, y\tilde{A}, y\tilde{v})$ , and vice versa. Note that the lengths of corresponding paths do not coincide. Using this correspondence, we get that

$$L = 1 + u(I - A(1, y))^{-1}v(1, y) = 1 + \tilde{u}(I - y\tilde{A})^{-1}y\tilde{v}^T. \quad (24)$$

Let us prove that

$$Q(y) = \det(I - A(1, y)) = \det(I - y\tilde{A}). \quad (25)$$

Given a matrix  $M$  of dimension  $n$ , we have

$$\det(M) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)},$$

where  $\mathfrak{S}_n$  is the set of permutations of  $\{1, \dots, n\}$ , and where  $\text{sgn}(\cdot)$  is the sign of a permutation. The permutations having a non zero contribution to the determinant are the ones which correspond to a partition into simple cycles of the nodes of the graph of  $M$ .

We have seen above that there is a one-to-one correspondence between successful paths in the graphs of  $A(1, y)$  and  $y\tilde{A}$ . There is also clearly a one-to-one correspondence between simple cycles in (the graphs of)  $A(1, y)$  and  $y\tilde{A}$ . When comparing the simple cycles of  $(I - A(1, y))$  and  $(I - y\tilde{A})$ , one needs to be more careful.

Let  $S$  be the set of simple cycles of  $(I - A(1, y))$  and let  $\tilde{S}$  be the one of  $(I - y\tilde{A})$ . To the simple cycle  $c = (c_1, \dots, c_k)$  in  $S$ , there corresponds the simple cycle  $\tilde{c} = ((c_1, 1), \dots, (c_1, |c_1|), (c_2, 1), \dots, (c_k, |c_k|))$  in  $\tilde{S}$ . A simple enumeration shows that

$$\tilde{S} = \{\tilde{c}, c \in S\} \cup \{((c, i)), c \in \mathfrak{C}, |c| > 1, 1 \leq i \leq |c|\}.$$

Given  $c \in S$  (resp.  $\tilde{S}$ ), we denote by  $w(c)$  the contribution of  $c$  to  $\det(I - A(1, y))$  (resp.  $\det(I - y\tilde{A})$ ). More precisely, for  $c = (c_1, \dots, c_k)$  and setting  $M = I - A(1, y)$  (resp.  $M = I - y\tilde{A}$ ), we set

$$w(c) = \sum_{\sigma \in \mathfrak{S}_k} \text{sgn}(\sigma) M_{c_1 c_{\sigma(1)}} \cdots M_{c_k c_{\sigma(k)}}.$$

Consider  $c = (c_1, \dots, c_k) \in S$ . We have

$$w(c) = \begin{cases} 1 - y^{|c_1|} & \text{if } k = 1 \\ (-1)^{k-1} \prod_{i=1}^k -y^{|c_i|} = -y^{\sum_i |c_i|} & \text{if } k > 1 \end{cases}.$$

Let  $\tilde{c} = ((c_1, 1), \dots, (c_1, |c_1|), \dots, (c_k, |c_k|))$  be the corresponding cycle of  $\tilde{S}$ . Then we have

$$w(\tilde{c}) = \begin{cases} 1 - y & \text{if } k = 1 \text{ and } |c_1| = 1 \\ (-1)^{\sum_i |c_i| - 1} \prod_{i=1}^k \prod_{j=1}^{|c_i|} -y = -y^{\sum_i |c_i|} & \text{otherwise} \end{cases}.$$

We check that  $w(c) = w(\tilde{c})$  except in the case  $k = 1, |c_1| > 1$ . In this last situation, we have  $w(c) = 1 - y^{|c_1|}$  and  $w(\tilde{c}) = -y^{|c_1|}$ . However, this difference is precisely compensated by the contribution to  $\det(I - y\tilde{A})$  of the simple cycles in  $\{((c, i)), c \in \mathfrak{C}, |c| > 1, 1 \leq i \leq |c|\}$ . We conclude that  $\det(I - A(1, y)) = \det(I - y\tilde{A})$ .

Let  $(\Sigma_i, D_i), i \in \mathcal{U}$ , be the maximal connected subgraphs of  $(\Sigma, D)$ . Let  $\mathfrak{C}$  be the set of cliques of  $(\Sigma, I)$  and let  $\mathfrak{C}_i$  be the one of  $(\Sigma_i, I_i)$ .

Let  $\mathcal{P}^+(S)$  be the set of non-empty subsets of a set  $S$ . For  $V \in \mathcal{P}^+(\mathcal{U})$ , define  $\mathfrak{C}_V = \{\prod_{v \in V} c_v, c_v \in \mathfrak{C}_v\}$ . Note that we have  $\mathfrak{C}_i = \mathfrak{C}_{\{i\}}$ . The set  $\mathfrak{C}$  is partitioned by the sets  $\mathfrak{C}_V, V \in \mathcal{P}^+(\mathcal{U})$ . Let  $\Gamma$  be the graph of cliques of  $\mathbb{M}(\Sigma, D)$ . Using Lemma 3.2, we get that the maximal strongly connected subgraphs of  $\Gamma$  are the subgraphs with sets of nodes  $\mathfrak{C}_V, V \in \mathcal{P}^+(\mathcal{U})$ . Clearly, there is a path in  $\Gamma$  from a node in  $\mathfrak{C}_V$  to a node in  $\mathfrak{C}_W$  if and only if  $W \subset V$ .

It implies the following. The restriction of the matrix  $A$  to the index set  $\mathfrak{C}_V$ , denoted by  $A_V$ , is irreducible. Now range the index set  $\mathfrak{C}$  according to the order  $\mathfrak{C}_{U_1}, \dots, \mathfrak{C}_{U_k}$ , where  $U_1, \dots, U_k$ , is an ordered list of the subsets of  $\mathcal{U}$  satisfying the property:  $U_i \subset U_j \implies i \geq j$ . Then the matrix  $A$  is block upper-triangular with the blocks  $A_{U_1}, \dots, A_{U_k}$ , on the diagonal. An analog statement holds for  $\tilde{A}$ , replacing  $\mathfrak{C}_V$  by  $\tilde{\mathfrak{C}}_V = \{(c, 1), \dots, (c, |c|), c \in \mathfrak{C}_V\}$ . We denote by  $\tilde{A}_V$  the restriction of  $\tilde{A}$  to the index set  $\tilde{\mathfrak{C}}_V$ . We have

$$\begin{aligned} \det(I - A(1, y)) &= \prod_{V \in \mathcal{P}^+(\mathcal{U})} \det(I - A_V(1, y)) \\ \det(I - y\tilde{A}) &= \prod_{V \in \mathcal{P}^+(\mathcal{U})} \det(I - y\tilde{A}_V). \end{aligned} \quad (26)$$

Given an index set  $S$  and  $s \in S$ , define  $\mathcal{J}_s \in \mathbb{N}^{1 \times S}$  by  $(\mathcal{J}_s)_s = 1$  and  $(\mathcal{J}_s)_t = 0, t \neq s$ . For  $i, j \in \mathfrak{C}$ , define

$$L_{ij} = \mathcal{J}_i(I - A(1, y))^{-1} y^{|j|} \mathcal{J}_j^T.$$

The coefficient  $(L_{ij}|n)$  can be interpreted combinatorially as the number of paths from  $i$  to  $j$  with weight  $y^n$  in the automaton  $(u, A(1, y), v(1, y))$ . In particular, we have  $L = 1 + \sum_{i,j} L_{ij}$ . With a proof similar to the one of (24), we get

$$L_{ij} = \mathcal{J}_i(I - A(1, y))^{-1} y^{|j|} \mathcal{J}_j^T = \mathcal{J}_{(i,1)}(I - y\tilde{A})^{-1} y \mathcal{J}_{(j,|j|)}^T. \quad (27)$$

Consider  $u \in \mathfrak{C}$ . Let  $A(1, y)_{[u]}$  denote the matrix obtained from  $A(1, y)$  by replacing the line and the column  $u$  by a line and a column of zeros. Then we have

$$L_{uu} = \frac{y^{|u|} \text{Adj}(I - A(1, y))_{uu}}{\det(I - A(1, y))} = \frac{y^{|u|} \det(I - A(1, y)_{[u]})}{\det(I - A(1, y))}. \quad (28)$$

Let  $\tilde{A}_{[u]}$  denote the matrix obtained from  $\tilde{A}$  by replacing the line  $(u, |u|)$  and the column  $(u, 1)$  by a line and a column of zeros. With a proof similar to the one of (25), we get

$$y^{|u|} \det(I - A(1, y)_{[u]}) = y \det(I - y\tilde{A}_{[u]}). \quad (29)$$

Assume that  $u$  belongs to  $\mathfrak{C}_U$  and let  $\tilde{A}_{U,[u]}$  denote the restriction of  $\tilde{A}_{[u]}$  to the index set  $\tilde{\mathfrak{C}}_U$ . Using (28), (25), (29), and (26), we obtain

$$L_{uu} = \frac{y \det(I - y\tilde{A}_{[u]})}{\det(I - y\tilde{A})} = \frac{y \det(I - y\tilde{A}_{U,[u]})}{\det(I - y\tilde{A}_U)}. \quad (30)$$

We have  $\tilde{A}_{U,[u]} \leq \tilde{A}_U$  (for the coordinate-wise ordering) and  $\tilde{A}_{U,[u]} \neq \tilde{A}_U$ . We have seen above that  $\tilde{A}_U$  is irreducible. According to the Perron-Frobenius Theorem for irreducible matrices (see for instance [29], Chapter 1.4), it implies that the spectral radius of  $\tilde{A}_{U,[u]}$  is strictly less than the one of  $\tilde{A}_U$ . Now, the roots of the polynomial  $\det(I - y\tilde{A}_{U,[u]})$ , resp.  $\det(I - y\tilde{A}_U)$ , are the inverses of the non-zero eigenvalues of  $\tilde{A}_{U,[u]}$ , resp.  $\tilde{A}_U$ . Hence the possible simplifications between the numerator and the denominator in the right-hand side of (30) do not involve any dominant singularity.

We conclude that the dominant singularities of  $L_{uu}$  are precisely the dominant singularities of  $1/\det(I - y\tilde{A}_U)$ .

We have  $(L|n) \geq (L_{uu}|n)$  for all  $n$ . It implies that the modulus of a dominant singularity of  $L$  is smaller or equal to the modulus of a dominant singularity of  $L_{uu}$ . We deduce that a dominant singularity of  $L$  has a smaller modulus than a dominant singularity of  $1/\det(I - y\tilde{A}_U)$  for all  $U$ , hence a

smaller modulus than a dominant singularity of  $1/Q(y)$ . Using that  $(G|n) \geq (L|n)$ , we obtain the same result for  $G$ .

We conclude that the modulus of the dominant singularities of  $L$ ,  $G$ , and  $1/Q(y)$  are equal. Furthermore, the sets of dominant singularities of  $L$  and  $G$  are included in the set of dominant singularities of  $1/Q(y)$ . Since  $Q(y) = \det(I - y\tilde{A})$ , the set of dominant singularities of  $1/Q(y)$  is also equal to the set of inverses of maximal eigenvalues of  $\tilde{A}$ . Let  $\rho(\tilde{A}) = \rho_L^{-1}$  denote the spectral radius of  $\tilde{A}$ .

First assume that  $\rho(\tilde{A}) = \rho_L = 1$ . According to Lemma 4.4,  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid over  $\Sigma$ . The analysis of Section 6.1 applies. In particular, the limit  $\lambda_{\mathbb{M}}$  in (7) exists and is given in (20). It is obviously algebraic and even rational. Hence Proposition 5.1 is satisfied in this case.

From now on, we assume that  $\rho(\tilde{A}) > 1$ . Let us specialize for a moment to the case where  $(\Sigma, D)$  is connected. Using the above analysis, the matrix  $\tilde{A}$  is irreducible. For any  $a \in \Sigma$ , we have  $\tilde{A}_{(a,1)(a,1)} > 0$ . We conclude that  $\tilde{A}$  is primitive. By Perron-Frobenius Theorem for primitive matrices ([29], Chapter 1.1), the matrix  $\tilde{A}$  has a unique eigenvalue of maximal modulus which is positive real and of multiplicity 1.

We conclude that  $\rho_L$  is the unique dominant singularity of  $L$  and  $G$ . We conclude also that the order of  $\rho_L$  is 1 in  $L$ , and at most 2 in  $G$ . Since  $nC^{-1}(L|n) \leq (G|n) \leq n(L|n)$ , we deduce that the order of  $\rho_L$  in  $G$  is 2.

We have just proved that the result of Proposition 4.5 holds for  $L$ . The proof of Proposition 4.5 for  $H$  is similar (and easier).

Let us come back to the general case for  $(\Sigma, D)$ . Since we have now proved Proposition 4.5, we are allowed to use Proposition 4.6 (the proof of the latter requires the former). We conclude that in all cases,  $L$  has a unique dominant singularity.

It remains to study the set of dominant singularities of  $G$ . To do this, we study the set of eigenvalues of  $\tilde{A}$  of maximal modulus.

Fix a subset  $V \in \mathcal{P}^+(\mathcal{U})$  and consider the restricted matrix  $\tilde{A}_V$ . Let  $\rho(\tilde{A}_V)$  denote the spectral radius of  $\tilde{A}_V$ . We distinguish between two cases.

*Case (I).* Assume there exists  $v \in V$  such that  $\mathbb{M}(\Sigma_v, D_v)$  is different from the free monoid  $\Sigma_v^*$ , or equivalently such that  $I_v$  is not empty. Then there exists  $c, d \in \mathfrak{C}_V$  such that  $|c| = |d| + 1$ . It implies that the cyclicity of the matrix  $\tilde{A}_V$  is 1. Since  $\tilde{A}_V$  is irreducible, we deduce that it is primitive. According to the Perron-Frobenius Theorem for primitive matrices ([29], Chapter 1.1), the matrix  $\tilde{A}_V$  has a unique eigenvalue of maximal modulus which is positive real and of multiplicity 1.

*Case (II).* Assume now that  $\mathbb{M}(\Sigma_v, D_v) = \Sigma_v^*$  for all  $v \in V$ . It implies that  $|c| = |V|$  for all  $c \in \mathfrak{C}_V$ . The cyclicity of  $\tilde{A}_V$  is  $|V|$  and  $\tilde{A}_V$  is not

primitive as soon as  $|V| > 1$ . However in this case, we are able to completely compute the spectrum of  $\tilde{A}_V$ . Set  $K = \prod_{v \in V} |\Sigma_v|$ . It is more convenient to work with  $A_V(1, y)$ . Using the same arguments as in the proof of (25), we get

$$\det(I - A_V(1, y)) = \det(I - y\tilde{A}_V).$$

We also have  $A_V(1, y) = y^{|V|}A_V(1, 1)$  and  $A_V(1, 1)$  is the matrix of dimension  $K \times K$  whose entries are all equal to 1. The eigenvalues of  $A_V(1, 1)$  are 0 with multiplicity  $(K - 1)$  and  $K$  with multiplicity 1. We have

$$\begin{aligned} \det(I - y^{|V|}A_V(1, 1)) &= y^{K|V|} \det(y^{-|V|}I - A_V(1, 1)) \\ &= y^{K|V|} (y^{-|V|})^{K-1} (y^{-|V|} - K) = (1 - Ky^{|V|}). \end{aligned}$$

It follows that the non-zero eigenvalues of  $\tilde{A}_V$  are

$$K^{1/|V|} \exp\left(\frac{2i\pi k}{|V|}\right), \quad k = 0, \dots, |V| - 1, \quad (31)$$

all with multiplicity 1. In particular, we have  $\rho(\tilde{A}_V) = K^{1/|V|}$ . According to (26), the spectral radius of  $\tilde{A}$  is given by

$$\rho(\tilde{A}) = \max_{V \in \mathcal{P}^+(\mathcal{U})} \rho(\tilde{A}_V).$$

Set

$$X = \{u \in \mathcal{U} : \mathbb{M}(\Sigma_u, D_u) = \Sigma_u^*\}, \quad \mathcal{S} = \{U \in \mathcal{P}^+(\mathcal{U}) : \rho(\tilde{A}_U) = \rho(\tilde{A})\}.$$

Using the above analysis, we can distinguish between two situations.

First, assume that  $\mathcal{P}^+(X) \cap \mathcal{S} = \emptyset$ . According to Case (I), it implies that  $\rho(\tilde{A})$  is the only eigenvalue of maximal modulus of the matrix  $\tilde{A}$ . We conclude that  $\rho_G = \rho_L = \rho(\tilde{A})^{-1}$  is the only dominant singularity of  $G$ .

From now on, assume that there exists  $U \in \mathcal{P}^+(X) \cap \mathcal{S}$ . According to Case (II), it implies that

$$\rho(\tilde{A}) = \rho(\tilde{A}_U) = \left(\prod_{u \in U} |\Sigma_u|\right)^{1/|U|}. \quad (32)$$

We deduce easily from (32) that

$$\mathcal{P}^+(X) \cap \mathcal{S} = \mathcal{P}^+(Y), \quad \text{with } Y = \{u \in X : |\Sigma_u| = \max_{x \in X} |\Sigma_x|\}.$$

At this point, we need to introduce a partition of  $\mathfrak{C}$ . Let  $\pi$  be an equitable partition of  $\mathfrak{C}$  such that all the cliques in the same cell of the partition have the same length. Let  $(u_\pi, A_\pi(1, y), v_\pi(1, y))$  be the corresponding triple recognizing  $L$ . We define in the same way as before an associated triple

$(\tilde{u}_\pi, y\tilde{A}_\pi, y\tilde{v}_\pi)$  recognizing  $L$ . The same results still hold and in particular we have

$$\det(I - A_\pi(1, y)) = \det(I - y\tilde{A}_\pi), \quad (33)$$

and the set of dominant singularities of  $L$  and  $G$  is included in the set of inverses of maximal eigenvalues of  $\tilde{A}_\pi$ . Let us specify the partition that we consider. The cells of  $\pi$  which are non-trivial, i.e. non-reduced to a single element, are  $\mathcal{C}_1, \dots, \mathcal{C}_{|Y|}$  with

$$\mathcal{C}_i = \{c : |c| = i, \text{alph}(c) \subset \bigcup_{u \in Y} \Sigma_u\}.$$

In words, we gather in  $\mathcal{C}_i$  all the cliques of length  $i$  which contain only letters from the alphabet  $\bigcup_{u \in Y} \Sigma_u$ . It is easily checked that  $\pi$  is an equitable partition. It is also clear that all the cliques in the same cell of  $\pi$  have a common length. Obviously, the successors in  $\Gamma$  of a clique in  $\mathcal{C}_i$  lie in  $\bigcup_{j \leq i} \mathcal{C}_j$ . More precisely, the number of direct successors that a clique in  $\mathcal{C}_i$  has in  $\mathcal{C}_j$  is

$$\begin{cases} \binom{i}{j} M^j & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases},$$

where  $M = \max_{x \in X} |\Sigma_x|$ . By adapting (26), we get

$$\det(I - A_\pi(1, y)) = \prod_{i=1}^{|Y|} (1 - y^i M^i) \times \prod_{U \in \mathcal{P}^+(\mathcal{U}) \setminus \mathcal{P}^+(Y)} \det(I - A_U(1, y)).$$

Using (33), we conclude that the set of maximal eigenvalues of  $\tilde{A}_\pi$  is precisely given by

$$\bigcup_{j=1}^{|Y|} \{M \exp(2i\pi j/J), j = 0, \dots, J-1\}. \quad (34)$$

The multiplicity of the eigenvalue  $M$  is at least  $|Y|$  (and it is exactly  $|Y|$  if  $\mathcal{S} = \mathcal{P}^+(Y)$ ). For a complex and non positive real maximal eigenvalue, the multiplicity is exactly the number of appearances of the eigenvalue in (34). The maximal such multiplicity is equal to  $\lfloor |Y|/2 \rfloor$  and attained for the eigenvalue  $-M$ . It follows that the maximal order of a complex and non positive real dominant singularity in  $G$  is  $\lfloor |Y|/2 \rfloor$ .

Now we also have  $L = \prod_{u \in Y} L_u \cdot \prod_{u \notin Y} L_u$ . The Möbius function of  $(\Sigma_u, I_u), u \in Y$ , is  $(1 - My)$ . We deduce that

$$L = \frac{1}{(1 - My)^{|Y|}} \cdot \prod_{u \notin Y} L_u.$$

The order of the singularity  $1/M$  in  $L$  is consequently at least  $|Y|$ . Since we have  $n(L|n)/C \leq (G|n) \leq n(L|n)$ , we deduce that one of the dominant

singularities of  $G$  must be of order  $(k_L + 1) \geq |Y| + 1$ . Since we have  $\lfloor |Y|/2 \rfloor < |Y| + 1$ , the only possible choice is  $1/M$ .

We conclude that the positive real dominant singularity of  $G$  has a strictly larger order than all the other dominant singularities. It completes the proof.  $\square$

*Proof of Proposition 5.2.*

The notations are borrowed from the statement of Proposition 5.2. To avoid trivialities, assume that  $J \neq S$ . Let  $(\Sigma_1, D_1) = \cup_{j \in J} (\Sigma_j, D_j)$ , and  $(\Sigma_2, D_2) = \cup_{j \in (S-J)} (\Sigma_j, D_j)$ . Let  $L$ ,  $L_1$ , and  $L_2$  be the respective length generating functions of  $(\Sigma, D)$ ,  $(\Sigma_1, D_1)$ , and  $(\Sigma_2, D_2)$ . By construction, we have  $\rho_{L_1} < \rho_{L_2}$ . Let  $k_{L_1}$  and  $k_{L_2}$  be the order of  $\rho_{L_1}$  and  $\rho_{L_2}$  in their respective series.

According to Proposition 4.6, we have  $(L_1|n) = a_n n^{k_1-1} \rho_{L_1}^{-n}$  with  $\lim_n a_n = a \in \mathbb{R}_+^*$ , and  $(L_2|n) = b_n n^{k_2-1} \rho_{L_2}^{-n}$  with  $\lim_n b_n = b \in \mathbb{R}_+^*$ . Furthermore, we have  $L = L_1 L_2$ , hence

$$\begin{aligned} (L|n) &= \sum_{i=0}^n a_{n-i} (n-i)^{k_1-1} \rho_{L_1}^{-(n-i)} b_i i^{k_2-1} \rho_{L_2}^{-i} \\ &= a n^{k_1-1} \rho_{L_1}^{-n} \left( \sum_{i=0}^n (a_{n-i}/a) b_i i^{k_2-1} (1-i/n)^{k_1-1} (\rho_{L_1}/\rho_{L_2})^i \right). \end{aligned}$$

Since  $\rho_{L_1}/\rho_{L_2} < 1$ , the series  $B = \sum_{i=0}^{+\infty} b_i i^{k_2-1} (\rho_{L_1}/\rho_{L_2})^i$  is convergent. Furthermore, we have

$$\lim_n \sum_{i=0}^n (a_{n-i}/a) b_i i^{k_2-1} (1-i/n)^{k_1-1} (\rho_{L_1}/\rho_{L_2})^i = B. \quad (35)$$

Let us justify this point. We set

$$u_i = b_i i^{k_2-1} (\rho_{L_1}/\rho_{L_2})^i, \quad v_{i,n} = (a_{n-i}/a) b_i i^{k_2-1} (1-i/n)^{k_1-1} (\rho_{L_1}/\rho_{L_2})^i.$$

Let us fix  $\varepsilon > 0$ , we have

$$\begin{aligned} \exists N_1, \quad B - \varepsilon &\leq \sum_{i=0}^{N_1} u_i \leq B, \quad 0 \leq \sum_{i=N_1+1}^{\infty} u_i \leq \varepsilon \\ \exists N_2, \forall n \geq N_2, \forall i \leq N_1, \quad 1 - \varepsilon &\leq (1-i/n)^{k_1-1} \leq 1 \\ \exists N_3, \forall n \geq N_3, \forall i \leq N_1, \quad 1 - \varepsilon &\leq (a_{n-i}/a) \leq 1 + \varepsilon. \end{aligned}$$

For  $n \geq \max(N_1, N_2, N_3)$ , we have  $\sum_{i=0}^n v_{i,n} = \sum_{i=0}^{N_1} v_{i,n} + \sum_{i=N_1+1}^n v_{i,n}$  and

$$\begin{aligned} (1-\varepsilon)^2(B-\varepsilon) &\leq (1-\varepsilon)^2 \sum_{i=0}^{N_1} u_i \leq \sum_{i=0}^{N_1} v_{i,n} \leq (1+\varepsilon) \sum_{i=0}^{N_1} u_i \leq (1+\varepsilon)B \\ 0 &\leq \sum_{i=N_1+1}^n v_{i,n} \leq \sup_{i=0}^{\infty} (a_i/a) \sum_{i=N_1+1}^n u_i \leq \varepsilon \sup_{i=0}^{\infty} (a_i/a). \end{aligned}$$

Now since  $(a_n)_n$  is convergent,  $\sup_i (a_i/a)$  is finite and we conclude easily that (35) holds. We deduce that  $(L|n) \sim aBn^{k_1-1}\rho_{L_1}^{-n}$ .

Let  $f$  be an increasing map from  $\mathbb{N}$  to  $\mathbb{N}$  such that  $\lim_n f(n) = +\infty$  and  $\lim_n f(n)/n = 0$ . Define

$$\mathcal{L}_n = \{t \in \mathbb{M}(\Sigma, D), |t| = n, n - f(n) \leq |t|_{\Sigma_1} \leq n\},$$

where  $|t|_{\Sigma_1} = \sum_{a \in \Sigma_1} |t|_a$ . We have

$$\#\mathcal{L}_n = an^{k_1-1}\rho_{L_1}^{-n} \left( \sum_{i=0}^{f(n)} (a_{n-i}/a) b_i i^{k_2-1} (1-i/n)^{k_1-1} (\rho_{L_1}/\rho_{L_2})^i \right).$$

Since  $\lim_n f(n) = +\infty$ , we have  $\#\mathcal{L}_n \sim aBn^{k_1-1}\rho_{L_1}^{-n}$  and  $\lim_n \#\mathcal{L}_n/(L|n) = 1$ . Let  $\mathbb{M}_n = \{t \in \mathbb{M}(\Sigma, D), |t| = n\}$  and note that  $\#\mathbb{M}_n = (L|n)$ . We have

$$\begin{aligned} \lambda_{\mathbb{M}}(\Sigma, D) &= \lim_n \frac{\sum_{t \in \mathcal{L}_n} h(t) + \sum_{t \in (\mathbb{M}_n - \mathcal{L}_n)} h(t)}{n \cdot \#\mathbb{M}_n} \\ &= \lim_n \underbrace{\frac{\sum_{t \in \mathcal{L}_n} h(t)}{n \cdot \#\mathcal{L}_n}}_{g_n} + \lim_n \frac{\sum_{t \in (\mathbb{M}_n - \mathcal{L}_n)} h(t)}{n \cdot \#\mathbb{M}_n}. \end{aligned}$$

Using the inequality  $h(t) \leq |t|$ , we obtain

$$\frac{\sum_{t \in (\mathbb{M}_n - \mathcal{L}_n)} h(t)}{n \cdot \#\mathbb{M}_n} \leq \frac{n \cdot (\#\mathbb{M}_n - \#\mathcal{L}_n)}{n \cdot \#\mathbb{M}_n} \xrightarrow{n} 0.$$

We now consider the terms  $g_n$ . Given a trace  $t$ , we can decompose it as  $t = \phi_1(t)\phi_2(t)$  with  $\phi_1(t) \in \mathbb{M}(\Sigma_1, D_1)$  and  $\phi_2(t) \in \mathbb{M}(\Sigma_2, D_2)$ . Consider a trace  $t$  such that  $|t| = n$  and  $|t|_{\Sigma_1} \geq n - f(n)$ . We have  $h(\phi_1(t)) \geq C^{-1}(n - f(n))$  and  $h(\phi_2(t)) \leq f(n)$ , where  $C$  is the maximal length of a clique. Using that  $\lim_n f(n)/n = 0$ , we obtain that, for  $n$  large enough,  $h(t) = h(\phi_1(t))$ . Hence, we have, for  $n$  large enough,

$$g_n = \sum_{i=n-f(n)}^n \frac{\#\{|t| = n, |t|_{\Sigma_1} = i\}}{\#\mathcal{L}_n} \cdot \frac{\sum_{|t|=n, |t|_{\Sigma_1}=i} h(\phi_1(t))}{n \cdot \#\{|t| = n, |t|_{\Sigma_1} = i\}}. \quad (36)$$

Given  $u \in \mathbb{M}(\Sigma_1, D_1)$  and  $n \geq |u|$ , we have

$$\#\{t \in \mathbb{M}(\Sigma, D), |t| = n, \phi_1(t) = u\} = (L_2|n - |u||),$$

which depends on  $u$  only via its length. We deduce that

$$\begin{aligned} \frac{\sum_{t \in \mathbb{M}(\Sigma, D), |t|=n, |t|_{\Sigma_1}=i} h(\phi_1(t))}{n \cdot \#\{t \in \mathbb{M}(\Sigma, D), |t| = n, |t|_{\Sigma_1} = i\}} &= \frac{i}{n} \cdot \frac{\sum_{t \in \mathbb{M}(\Sigma_1, D_1), |t|=i} h(t)}{i \cdot \#\{t \in \mathbb{M}(\Sigma_1, D_1), |t| = i\}} \\ &\sim (i/n) \lambda_{\mathbb{M}}(\Sigma_1, D_1). \end{aligned}$$

Replacing in (36), we conclude that  $\lambda_{\mathbb{M}}(\Sigma, D) = \lim_n g_n = \lambda_{\mathbb{M}}(\Sigma_1, D_1)$ .  $\square$

*Proof of Proposition 5.3.*

The notations are the ones of the statement of Proposition 5.3. Assume first that  $\mathbb{M}(\Sigma, D)$  is the free commutative monoid over  $\Sigma$ . According to the results of Section 6.1, we have indeed  $\gamma_{\mathbb{M}}(\Sigma, D) = (|\Sigma| + 1)/2$ .

Assume now that  $\mathbb{M}(\Sigma, D)$  is not the free commutative monoid. Let  $(\Sigma_2, D_2)$  be a maximal connected subgraph of  $(\Sigma, D)$  and let  $\Sigma_1 = \Sigma - \Sigma_2$  and  $D_1 = D - D_2$ . Denote respectively by  $H, H_1$ , and  $H_2$  the height generating functions of  $\mathbb{M}(\Sigma, D), \mathbb{M}(\Sigma_1, D_1)$ , and  $\mathbb{M}(\Sigma_2, D_2)$ . We choose  $(\Sigma_2, D_2)$  so that  $\mathbb{M}(\Sigma_1, D_1)$  is different from the free commutative monoid. According to Lemma 4.4, it implies that  $\rho_{H_1} < 1$ . We are going to prove the following equalities

$$\gamma_{\mathbb{M}}(\Sigma, D) = \begin{cases} \gamma_{\mathbb{M}}(\Sigma_1, D_1) + \gamma_{\mathbb{M}}(\Sigma_2, D_2) & \text{if } |\Sigma_2| > 1 \\ \gamma_{\mathbb{M}}(\Sigma_1, D_1) + 1/2 & \text{if } |\Sigma_2| = 1 \end{cases}. \quad (37)$$

Formula (16) follows easily from the above.

Assume first that  $|\Sigma_2| > 1$ . According to Lemma 4.4, it implies that  $\rho_{H_2} < 1$ . Applying Propositions 4.5 and 4.6, we have  $(H_1|n) = a_n n^{k_{H_1}-1} \rho_{H_1}^{-n}$  and  $(H_2|n) = b_n \rho_{H_2}^{-n}$  with  $\lim_n a_n = a$  and  $\lim_n b_n = b$ . Using (4) and performing the same type of computations as in the proof of Proposition 4.6, we get

$$(H|n) \sim ab \left( \frac{1}{1 - \rho_{H_1}} + \frac{1}{1 - \rho_{H_2}} - 1 \right) n^{k_{H_1}-1} (\rho_{H_1} \rho_{H_2})^{-n}. \quad (38)$$

We define the maps  $f, \phi_1(\cdot)$  and  $\phi_2(\cdot)$  as in the proof of Proposition 5.2. Consider the set

$$\mathcal{H}_n = \{t \in \mathbb{M}(\Sigma, D), h(t) = n, h(\phi_1(t)) \geq n - f(n), h(\phi_2(t)) \geq n - f(n)\}.$$

Using the same type of arguments as in the proof of Proposition 5.2, it is easily seen that  $\lim_n \#\mathcal{H}_n/(H|n) = 1$ . Set  ${}_n\mathbb{M} = \{t \in \mathbb{M}(\Sigma, D), h(t) = n\}$  and note that  $\#_n\mathbb{M} = (H|n)$ . We have

$$\begin{aligned}\gamma_{\mathbb{M}}(\Sigma, D) &= \lim_n \frac{\sum_{t \in \mathcal{H}_n} |t| + \sum_{t \in ({}_n\mathbb{M} - \mathcal{H}_n)} |t|}{n \cdot \#_n\mathbb{M}} \\ &= \lim_n \frac{\sum_{t \in \mathcal{H}_n} |t|}{n \cdot \#\mathcal{H}_n} + \frac{\sum_{t \in ({}_n\mathbb{M} - \mathcal{H}_n)} |t|}{n \cdot \#_n\mathbb{M}}.\end{aligned}$$

Using the inequality  $|t| \leq Ch(t)$ , where  $C$  is the maximal length of a clique, we obtain

$$\frac{\sum_{t \in ({}_n\mathbb{M} - \mathcal{H}_n)} |t|}{n \cdot \#_n\mathbb{M}} \leq \frac{Cn(\#_n\mathbb{M} - \#\mathcal{H}_n)}{n \cdot \#_n\mathbb{M}} \xrightarrow{n} 0.$$

Using the equality  $|t| = |\phi_1(t)| + |\phi_2(t)|$ , we obtain

$$\begin{aligned}\gamma_{\mathbb{M}}(\Sigma, D) &= \lim_n \frac{\sum_{t \in \mathcal{H}_n} |\phi_1(t)|}{n \cdot \#\mathcal{H}_n} + \frac{\sum_{t \in \mathcal{H}_n} |\phi_2(t)|}{n \cdot \#\mathcal{H}_n} \\ &= \gamma_{\mathbb{M}}(\Sigma_1, D_1) + \gamma_{\mathbb{M}}(\Sigma_2, D_2),\end{aligned}$$

where the last equality is obtained exactly in the same way as in the proof of Proposition 5.2.

Assume now that  $|\Sigma_2| = 1$ . Then we have  $\rho_{H_1} < 1$  and  $\rho_{H_2} = 1$ . It implies that  $(H_1|n) \sim an^{k_{H_1}-1}\rho_{H_1}^{-n}$  and  $(H_2|n) = 1$ . Using (4), we obtain that  $(H|n) \sim n(H_1|n)$ . Now, by a direct computation, we get, for  $u \in \mathbb{M}(\Sigma_1, D_1)$ ,

$$\#\{t \in \mathbb{M}(\Sigma, D), \phi_1(t) = u, h(t) = h(u)\} = h(u) + 1.$$


Define the set  $\mathcal{H}_n = \{t \in \mathbb{M}(\Sigma, D), h(t) = h(\phi_1(t)) = n\}$ . We have  $\#\mathcal{H}_n = (n+1)(H_1|n) \sim (H|n)$ . It implies that

$$\begin{aligned}\gamma_{\mathbb{M}}(\Sigma, D) &= \lim_n \frac{\sum_{t \in \mathcal{H}_n} |\phi_1(t)|}{n \cdot \#\mathcal{H}_n} + \frac{\sum_{t \in \mathcal{H}_n} |\phi_2(t)|}{n \cdot \#\mathcal{H}_n} \\ &= \lim_n \frac{(n+1) \sum_{t \in \mathbb{M}(\Sigma_1, D_1), h(t)=n} |t|}{n \cdot n(H_1|n)} + \frac{\sum_{t \in \mathbb{M}(\Sigma_1, D_1), h(t)=n} \sum_{i=0}^n i}{n \cdot n(H_1|n)} \\ &= \gamma_{\mathbb{M}}(\Sigma_1, D_1) + 1/2.\end{aligned}$$

This completes the proof.  $\square$

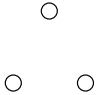
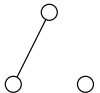
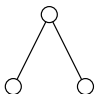
## B Trace monoids over 2, 3, and 4 letters

We give the values of the average heights for all the trace monoids over alphabets of cardinality 2, 3, and 4. On the tables below, a trace monoid is represented by its (non-directed) dependence graph. For readability, self-loops have been omitted in the dependence graphs. We have not represented the free monoids for which  $\lambda_* = \lambda_{\mathbb{M}} = \gamma_{\mathbb{M}}^{-1} = \lambda_{\text{cf}} = 1$ .

	$\lambda_*$	$\lambda_{\mathbb{M}}$	$\gamma_{\mathbb{M}}^{-1}$	$\lambda_{\text{cf}}$
	1/2	3/4	2/3	1

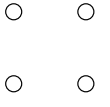
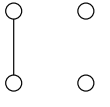
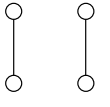
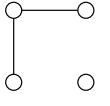
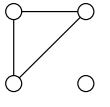
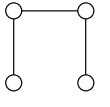
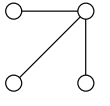
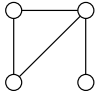
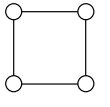
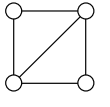
I. Trace monoids over 2 letters

The values in Table I can be obtained using the results in Section 6.1.

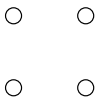
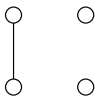
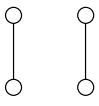
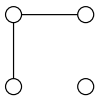
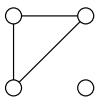
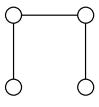
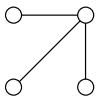
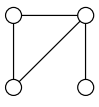
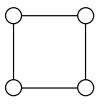
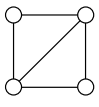
	$\lambda_*$	$\lambda_{\mathbb{M}}$	$\gamma_{\mathbb{M}}^{-1}$	$\lambda_{\text{cf}}$
	1/3	11/18	1/2	1
	2/3	1	2/3	1
	$(10 + \sqrt{5})/15$	$(7 + \sqrt{5})/10$	9/11	8/9

II. Trace monoids over 3 letters

All the values in Table II except one can be obtained using the results from the paper. The exception is  $\lambda_*$  for  $\Sigma = \{a, b, c\}, I = \{(b, c), (c, b)\}$ , which is computed in [27], Example 6.2.

		$\lambda_*$	$\lambda_{\mathbb{M}}$	$\gamma_{\mathbb{M}}^{-1}$	$\lambda_{\text{cf}}$
1		1/4	25/48	2/5	1
2		1/2	1	1/2	1
3		1/2	3/4	1/2	1
4		$(10 + \sqrt{5})/20$	$(7 + \sqrt{5})/10$	18/31	52/57
5		3/4	1	2/3	1
6		?	19/22	$(13 - 2\sqrt{13})/9$	5/6
7		in (39)	in (40)	in (41)	11/14
8		$(5 + \sqrt{2})/8$	$(6 + \sqrt{2})/8$	in (42)	7/8
9		$(3 + \sqrt{3})/6$	$(11 + \sqrt{2})/14$	$(51 + \sqrt{17})/76$	11/14
10		$(9 + \sqrt{3})/12$	$(4 + \sqrt{3})/6$	$(3\sqrt{5} - 5)/2$	8/9

### III. Trace monoids over 4 letters - exact values

		$\lambda_*$	$\lambda_{\mathbb{M}}$	$\gamma_{\mathbb{M}}^{-1}$	$\lambda_{\text{cf}}$
1		0.25	$0.521\dots$	0.4	1
2		0.5	1	0.5	1
3		0.5	0.75	0.5	1
4		$0.612\dots$	$0.923\dots$	$0.581\dots$	$0.912\dots$
5		0.75	1	$0.667\dots$	1
6		$0.691\dots$	$0.864\dots$	$0.643\dots$	$0.833\dots$
7		$0.681\dots$	$0.873\dots$	$0.676\dots$	$0.786\dots$
8		$0.802\dots$	$0.927\dots$	$0.760\dots$	$0.875\dots$
9		$0.789\dots$	$0.887\dots$	$0.725\dots$	$0.786\dots$
10		$0.894\dots$	$0.955\dots$	$0.854\dots$	$0.889\dots$

III.b. Trace monoids over 4 letters - numerical values

Let us denote the dependence graphs in Table III, listed from top to bottom, by  $(\Sigma, D_i), i = 1, \dots, 10$ . The graph  $(\Sigma, D_9)$  is the cocktail party graph  $CP_2$ , hence the values of the average heights can be retrieved from Section 6.2. More generally, most of the values in the table can be computed using the results from the paper. The exceptions are  $\lambda_*$  for  $(\Sigma, D_i), i = 6, 7, 8$ , and 10. For  $(\Sigma, D_8)$  and  $(\Sigma, D_{10})$ , the value of  $\lambda_*$  can be computed by applying Proposition 12 from [5].

For  $(\Sigma, D_6)$ , the exact value of  $\lambda_*$  is not known. Using truncated Markov chains, A. Jean-Marie [23] obtained the following exact bounds:

$$\lambda_*(\Sigma, D_6) \in [0.69125003165, 0.69125003169] .$$

Let us concentrate on  $\lambda_*(\Sigma, D_7)$ . Let  $(x_n)_{n \in \mathbb{N}^*}$  be a sequence of independent random variables valued in  $\Sigma$  and uniformly distributed:  $P\{x_n = u\} = 1/4, u \in \Sigma$ . Define  $X_n = \psi(x_1 \cdots x_n)$ , then  $(X_n)_n$  is a Markov chain on the state space  $\mathbb{M}(\Sigma, D_7)$ . Let  $a$  be the letter such that  $(a, u) \in D_7$  for all  $u \in \Sigma$ . Define  $T = \inf\{n : x_n = a\}$ . An elementary argument using the Strong Law of Large Numbers then shows that  $\lambda_*(\Sigma, D_7) = E[h(X_T)]/E[T]$ . It follows that

$$\lambda_*(\Sigma, D_7) = \frac{1}{4} + \frac{1}{16} \left( \sum_{i \in \mathbb{N}} \frac{1}{4^i} \sum_{i_1+i_2+i_3=i} \max(i_1, i_2, i_3) \binom{i}{i_1, i_2, i_3} \right). \quad (39)$$

This expression involves non algebraic generalized hypergeometric series. By truncating the infinite sum and upper-bounding the remainder using the inequality  $\max(i_1, i_2, i_3) \leq i_1 + i_2 + i_3$ , we get the following exact bounds:

$$\lambda_*(\Sigma, D_7) \in [0.6811589347, 0.6811589349] .$$

Another formula for  $\lambda_*(\Sigma, D_7)$  involving multiple contour integrals and due to Alain Jean-Marie is given in [5, Th. 13].

The closed form expressions for  $\lambda_{\mathbb{M}}(\Sigma, D_7)$  and  $\gamma_{\mathbb{M}}(\Sigma, D_7)$  are not given in Table III since they are too long and do not fit. We have

$$\begin{aligned} \lambda_{\mathbb{M}}(\Sigma, D_7) &= \frac{8(-93 - 9\sqrt{93} - \sqrt{93}X + 5X^2)}{-1734 - 186\sqrt{93} + (141 - 5\sqrt{93})X + 67X^2} \\ X &= (108 + 12\sqrt{93})^{1/3}, \end{aligned} \quad (40)$$

and  $\gamma_{\mathbb{M}}(\Sigma, D_7)^{-1} =$

$$\frac{10777(529 - 23Y^2 + Y^4)(829 + 132\sqrt{62} - (139 - 6\sqrt{62})Y - 11Y^2)}{3(3779 + 372\sqrt{62})(98340\sqrt{62} - 1461365 - 1529(149 + 66\sqrt{62})Y - 53885Y^2)} \quad (41)$$

with  $Y = (89 + 18\sqrt{62})^{1/3}$ .

At last, let us comment on the value of  $\gamma_{\mathbb{M}}$  for  $(\Sigma, D_8)$ . Using the results from Section 5.2, we get

$$\gamma_{\mathbb{M}}(\Sigma, D_8)^{-1} = \frac{(1 - 2\alpha)(4 - 5\alpha)}{7 - 27\alpha + 24\alpha^2}, \quad (42)$$

where  $\alpha$  is the smallest root of the equation  $2x^3 - 8x^2 + 6x - 1 = 0$ . Numerically, we have  $\alpha = 0.237 \dots$  and  $\gamma_{\mathbb{M}}^{-1} = 0.760 \dots$ . In this case, Cardan's formulas are of no use (they provide an expression of the real  $\alpha$  as a function of the cubic root of a complex number).

Let us conclude by going back to the original motivation of comparing the degree of parallelism in different trace monoids. We claim for instance that there is some strong evidence that  $(\Sigma, D_9)$  is 'more parallel' than  $(\Sigma, D_8)$ . Indeed we have  $\lambda_*(\Sigma, D_9) < \lambda_*(\Sigma, D_8)$ ,  $\lambda_{\mathbb{M}}(\Sigma, D_9) < \lambda_{\mathbb{M}}(\Sigma, D_8)$ ,  $\gamma_{\mathbb{M}}^{-1}(\Sigma, D_9) < \gamma_{\mathbb{M}}^{-1}(\Sigma, D_8)$ , and  $\lambda_{\text{cf}}(\Sigma, D_9) < \lambda_{\text{cf}}(\Sigma, D_8)$ .

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