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# Noncommutative symmetric functions IV: Quantum linear groups and Hecke algebras at $q = 0$

Daniel KROB <sup>\*</sup> and Jean-Yves THIBON <sup>†‡</sup>

## Abstract

We present representation theoretical interpretations of quasi-symmetric functions and noncommutative symmetric functions in terms of quantum linear groups and Hecke algebras at  $q = 0$ . We obtain in this way a noncommutative realization of quasi-symmetric functions analogous to the plactic symmetric functions of Lascoux and Schützenberger. The generic case leads to a notion of quantum Schur function.

## 1 Introduction

This paper, which is intended as a sequel to [9, 21, 6], is devoted to the representation theoretical interpretation of noncommutative symmetric functions and quasi-symmetric functions. These objects, which are two different generalizations of ordinary symmetric functions [10, 9], build up two Hopf algebras dual to each other, and have been shown to provide a Frobenius type theory for Hecke algebras of type  $A$  at  $q = 0$ , playing the same rôle as the classical correspondence between symmetric functions and characters of symmetric groups [7] (which extends to the case of the generic Hecke algebra).

In the classical case, the interpretation of symmetric functions in terms of representations of symmetric groups is equivalent, via Schur-Weyl duality, to the fact that Schur functions are the characters of the irreducible polynomial representations of general linear groups. Equivalently, instead of working with polynomial representations of  $GL(n)$ , one can use comodules over the Hopf algebra of polynomial functions over  $GL(n)$  [11]. This Hopf algebra is known to admit interesting  $q$ -deformations (quantized function algebras; see [8] for instance) to which Schur-Weyl duality can be extended for generic values of  $q$ , the symmetric group being replaced by the Hecke algebra.

The standard version of the quantum linear group is not defined for  $q = 0$ . The theory of crystal bases [16], which allows to “take the limit  $q \rightarrow 0$ ” in certain modules by working with renormalized operators modulo a lattice, describes the combinatorial aspects of the generic case, and provides illuminating interpretations of classical constructions such as the Robinson-Schensted correspondence, the Littlewood-Richardson rule and the plactic monoid [3, 17, 24, 26].

However, another version exists [4] which plays an equivalent rôle for generic values of  $q$ , but in which one can specialize  $q$  to 0. This specialization is quite different of what is obtained with crystal bases, and leads to a new interpretation of quasi-symmetric functions and noncommutative symmetric functions analogous to the interpretation of ordinary symmetric functions as polynomial characters of  $GL(n)$ . Moreover, this interpretation allows to give a realization of quasi-symmetric functions similar to the plactic interpretation of symmetric functions (see

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dinary Young tableaux one has to use skew tableaux of ribbon shape, and dual objects called quasi-ribbons, for which Schensted type algorithms can be constructed. In fact, most aspects of the classical theory can be adapted to this highly degenerate case. As this is an example of a non-semisimple case for which everything can be worked out explicitly, one can expect that this treatment could serve as a guide for understanding the more complicated degeneracies at roots of unity.

This paper is structured as follows. We first recall the basic definitions concerning non-commutative symmetric functions and quasisymmetric functions (Section 2) and review the Frobenius correspondence for the generic Hecke algebras (Section 3). Next we introduce the Dipper-Donkin version of the quantized function algebra of the space of  $n \times n$  matrices (Section 4). We describe some interesting subspaces (Sections 4.5 and 4.6), and prove that the  $q = 0$  specialization of the diagonal subalgebra is a quotient of the plactic algebra, which we call the hypoplactic algebra (Section 4.7). Next, we review the representation theory of the 0-Hecke algebra and its interpretation in terms of quasi-symmetric functions and noncommutative symmetric functions, providing the details which were omitted in [7]. In Section 6, we introduce a notion of noncommutative character for  $A_q(n)$ -comodules, and prove that these characters live in the diagonal subalgebra. For generic  $q$ , the characters of irreducible comodules are quantum analogues of Schur functions. For  $q = 0$ , we show that hypoplactic analogues of the fundamental quasi-symmetric functions  $F_I$  (quasi-ribbons) can be obtained as the characters of irreducible  $A_0(n)$  comodules, and give a similar construction for the ribbon Schur functions. These constructions lead to degenerate versions of the Robinson-Schensted correspondence, which are discussed in Section 7.

## 2 Noncommutative symmetric functions and quasi-symmetric functions

### 2.1 Noncommutative symmetric functions

The algebra of *noncommutative symmetric functions* [9] is the free associative algebra  $\mathbf{Sym} = \mathbb{Q}\langle S_1, S_2, \dots \rangle$  generated by an infinite sequence of noncommutative indeterminates  $S_k$ , called the *complete symmetric functions*. One defines  $S^I = S_{i_1} S_{i_2} \dots S_{i_r}$  for any composition  $I = (i_1, i_2, \dots, i_r) \in (\mathbb{N}^*)^r$ . The family  $(S^I)$  is a linear basis of  $\mathbf{Sym}$ . Although it is convenient to define  $\mathbf{Sym}$  as an abstract algebra, a useful realisation can be obtained by taking an infinite alphabet  $A = \{a_1, a_2, \dots\}$  and defining its complete homogeneous symmetric functions by

$$\prod_{i \geq 1} (1 - ta_i)^{-1} = \sum_{n \geq 0} t^n S_n(A) \quad (1)$$

Although these elements are not symmetric for the usual action of permutations on the free algebra, they are invariant under the Lascoux-Schützenberger action of the symmetric group [25], which can now be interpreted as a particular case of Kashiwara's action of the Weyl group on the  $U_q(\mathfrak{sl}_n)$ -crystal graph of the tensor algebra [24].

The set of all compositions of a given integer  $n$  is equipped with the *reverse refinement order*, denoted  $\preceq$ . For instance, the compositions  $J$  of 4 such that  $J \preceq (1, 2, 1)$  are exactly  $(1, 2, 1)$ ,  $(3, 1)$ ,  $(1, 3)$  and  $(4)$ . The *ribbon Schur functions*  $(R_I)$  can then be defined by

$$S^I = \sum_{J \preceq I} R_J \quad \text{or} \quad R_I = \sum_{J \preceq I} (-1)^{\ell(I) - \ell(J)} S^J,$$

where  $\ell(I)$  denotes the *length* of  $I$ . The family  $(R_I)$  is another homogenous basis of  $\mathbf{Sym}$ .

ric function  $f$  obtained by applying to  $F$  the algebra which maps  $\mathfrak{S}_n$  to the complete homogeneous function  $h_n$  (our notations for commutative symmetric functions will be those of [28]). The ribbon Schur function  $R_I$  is then mapped to the corresponding ordinary ribbon Schur function, which will be denoted by  $r_I$ .

Ordinary symmetric functions are endowed with an extra product  $*$ , called the internal product, which corresponds to the multiplication of central functions on the symmetric group. A noncommutative analog of this product can be defined, the character ring of  $\mathfrak{S}_n$  being replaced by its descent algebra [35] (see also below).

Recall that  $i$  is said to be a *descent* of  $\sigma \in \mathfrak{S}_n$  if  $\sigma(i) > \sigma(i+1)$ . The set  $\text{Des}(\sigma)$  of these integers is called the *descent set* of  $\sigma$ . If  $I = (i_1, \dots, i_r)$  is a composition of  $n$ , one associates with it the subset  $D(I) = \{d_1, \dots, d_{r-1}\}$  of  $[1, n-1]$  defined by  $d_k = i_1 + \dots + i_k$  for  $k \in [1, r-1]$ . Let  $D_I$  be the sum in  $\mathbb{Z}[\mathfrak{S}_n]$  of all permutations with descent set  $D(I)$ . As shown by Solomon [35], the  $D_I$  form a basis of a subalgebra of  $\mathbb{Z}[\mathfrak{S}_n]$  called the *descent algebra* of  $\mathfrak{S}_n$  and denoted by  $\Sigma_n$ . One can define an isomorphism of graded vector spaces

$$\alpha : \mathbf{Sym} = \bigoplus_{n \geq 0} \mathbf{Sym}_n \longrightarrow \Sigma = \bigoplus_{n \geq 0} \Sigma_n$$

by setting  $\alpha(R_I) = D_I$ . Observe that  $\alpha(S^I)$  is then equal to  $D_{\subseteq I}$ , i.e. to the sum of all permutations of  $\mathfrak{S}_n$  whose descent set is contained in  $D(I)$ .

## 2.2 Quasi-symmetric functions

As proved in [29] (see also [9]), the algebra of noncommutative symmetric functions is in natural duality with the algebra of quasi-symmetric functions, introduced by Gessel in [10]. Let  $X = \{x_1, x_2, \dots, x_n \dots\}$  be a totally ordered set of commutative indeterminates. An element  $f \in \mathbb{C}[X]$  is said to be a *quasi-symmetric function* if for each composition  $K = (k_1, \dots, k_m)$  all the monomials  $x_{i_1}^{k_1} x_{i_2}^{k_2} \dots x_{i_m}^{k_m}$  with  $i_1 < i_2 < \dots < i_m$  have the same coefficient in  $f$ . The quasi-symmetric functions form a subalgebra  $QSym$  of  $\mathbb{C}[X]$ .

One associates with a composition  $I = (i_1, \dots, i_m)$  the *quasi-monomial function*

$$M_I = \sum_{j_1 < \dots < j_m} x_{j_1}^{i_1} \dots x_{j_m}^{i_m} .$$

The family of quasi-monomial functions is clearly a basis of  $QSym$ . Another important basis of  $QSym$  is formed by *quasi-ribbon functions* which are defined by

$$F_I = \sum_{I \preceq J} M_J ,$$

e.g.  $F_{122} = M_{122} + M_{1112} + M_{1211} + M_{11111}$ . The pairing  $\langle \cdot, \cdot \rangle$  between  $\mathbf{Sym}$  and  $QSym$  [29] is then defined by  $\langle S^I, M_J \rangle = \delta_{IJ}$  or equivalently by  $\langle R_I, F_J \rangle = \delta_{IJ}$ . This duality is essentially equivalent to the noncommutative Cauchy identity

$$\prod_{i \geq 1} \left( \prod_{j \geq 1} (1 - x_i a_j)^{-1} \right) = \sum_I F_I(X) R_I(A) , \quad (2)$$

and can also be interpreted as the canonical duality between Grothendieck groups associated to 0-Hecke algebras [7] (see Section 5).

### 3.1 Hecke algebras

The Hecke algebra  $H_N(q)$  of type  $A_{N-1}$  is the  $\mathbb{C}(q)$ -algebra generated by  $N - 1$  elements  $(T_i)_{i=1, N-1}$  with relations

$$\begin{cases} T_i^2 &= (q - 1)T_i + q & \text{for } i \in [1, N - 1] , \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & \text{for } i \in [1, N - 2] , \\ T_i T_j &= T_j T_i & \text{for } |i - j| > 1 . \end{cases}$$

The Hecke algebra  $H_N(q)$  is a deformation of the  $\mathbb{C}$ -algebra of the symmetric group  $\mathfrak{S}_N$  (obtained for  $q = 1$ ). For generic complex values of  $q$ , it is isomorphic to  $\mathbb{C}[\mathfrak{S}_N]$  (and hence semi-simple) except when  $q = 0$  or when  $q$  is a root of unity. The first relation is often replaced by

$$T_i^2 = (q - q^{-1})T_i + 1 \tag{3}$$

which is invariant under the substitution  $q \longrightarrow -q^{-1}$  and is more convenient for working with Kazhdan-Lusztig polynomials and canonical bases. However the convention adopted here, i.e.

$$T_i^2 = (q - 1)T_i + q , \tag{4}$$

is the natural one when  $q$  is interpreted as the cardinality of a finite field and  $H_N(q)$  as the endomorphism algebra of the permutation representation of  $GL_N(\mathbb{F}_q)$  on the set on complete flags [14]. Moreover one can specialize  $q = 0$  in relation (4). In the modular representation theory of  $GL_N(\mathbb{F}_q)$ , the Hecke algebra corresponding to this specialization occurs when  $q$  is a power of the characteristic of the ground field. For this reason, among others, it is interesting to consider the 0-Hecke algebra  $H_N(0)$  which is the  $\mathbb{C}$ -algebra obtained by specialization of the generic Hecke algebra  $H_N(q)$  at  $q = 0$ . This algebra is therefore presented by

$$\begin{cases} T_i^2 &= -T_i & \text{for } i \in [1, N - 1] , \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & \text{for } i \in [1, N - 2] , \\ T_i T_j &= T_j T_i & \text{for } |i - j| > 1 . \end{cases}$$

The representation theory of  $H_N(0)$  was investigated by Norton who obtained a fairly complete picture [31]. Important specific features of the type  $A$  are described by Carter in [1]. The 0-Hecke algebra can also be realized as an algebra of operators acting on the equivariant Grothendieck ring of the flag manifold [22].

### 3.2 The Frobenius correspondence

We will see that the 0-Hecke algebra is the right object for giving a representation theoretical interpretation of noncommutative symmetric functions and of quasi-symmetric functions. To emphasize the parallel with the well-known correspondence between representations of the symmetric group and symmetric functions, we first recall the main points of the classical theory.

Let  $Sym$  be the ring of symmetric functions and let

$$R[\mathfrak{S}] = \bigoplus_{N \geq 0} R[\mathfrak{S}_N]$$

be the ring of equivalence classes of finitely generated  $\mathbb{C}[\mathfrak{S}_N]$ -modules (with sum and product corresponding to direct sum and induction product). We know from the work of Frobenius that

map  $\mathcal{F} : R[\mathfrak{S}] \rightarrow \text{Sym}$  sends the class of a Specht module  $V_\lambda$  to the Schur function  $s_\lambda$ . The first point is that  $\mathcal{F}$  is a ring homomorphism. That is,

$$\mathcal{F}([U \otimes V] \uparrow_{\mathfrak{S}_N \times \mathfrak{S}_M}^{\mathfrak{S}_{N+M}}) = \mathcal{F}([U]) \mathcal{F}([V])$$

for a  $\mathfrak{S}_N$ -module  $U$  and a  $\mathfrak{S}_M$ -module  $V$ . The second one is the character formula, which can be stated as follows: for any finite dimensional  $\mathfrak{S}_N$ -module  $V$ , the value of the character of  $V$  on a permutation of the conjugacy class labelled by the partition  $\mu$  is equal to the scalar product

$$\chi(\mu) = \langle \mathcal{F}(V), p_\mu \rangle$$

where  $p_\mu$  is the product of power sums  $p_{\mu_1} \cdots p_{\mu_r}$ .

This theory can be extended to the Hecke algebra  $H_N(q)$  when  $q$  is neither 0 nor a root of unity. The characteristic map is independent of  $q$ , and still maps the  $q$ -Specht module  $V_\lambda(q)$  to the Schur function  $s_\lambda$ . The induction formula remains valid and the character formula has to be modified as follows (see [2, 18, 19, 32, 36, 37]). Define for a partition  $\mu = (\mu_1, \mu_2, \dots, \mu_r)$  of  $N$  the element

$$w_\mu = (\sigma_1 \dots \sigma_{\mu_1-1}) (\sigma_{\mu_1+1} \dots \sigma_{\mu_1+\mu_2-1}) \dots (\sigma_{\mu_1+\dots+\mu_{r-1}+1} \dots \sigma_{N-1})$$

(where  $\sigma_i$  is the elementary transposition  $(i \ i+1)$ ). The character formula for  $H_N(q)$  gives the value  $\chi_\mu^\lambda$  on  $T_{w_\mu}$  of the character of the irreducible  $q$ -Specht module  $V_\lambda(q)$ . It reads

$$\chi_\mu^\lambda = \text{tr}_{V_\lambda(q)}(T_{w_\mu}) = \langle \mathcal{F}(V_\lambda(q)), C_\mu(q) \rangle = \langle s_\lambda, C_\mu(q) \rangle$$

where  $C_\mu(q) = (q-1)^{l(\mu)} h^\mu((q-1)X)$  (in  $\lambda$ -ring notation,  $h^\mu((q-1)X)$  denotes the image of the homogeneous symmetric function  $h^\mu(X)$  under the ring homomorphism  $p_k \mapsto (q^k - 1)p_k$ ).

## 4 The quantum coordinate ring $A_q(n)$

### 4.1 Tensor representations of $H_N(q)$

Let  $E = \{e_1, \dots, e_n\}$  be a finite set and let

$$V = \bigoplus_{i=1}^n \mathbb{C}(q) e_i$$

be the  $\mathbb{C}(q)$ -vector space with basis  $(e_i)$ . For  $\mathbf{v} = e_{k_1} \otimes \dots \otimes e_{k_N} \in V^{\otimes N}$  and  $i \in [1, N-1]$ , we define  $\mathbf{v}^{\sigma_i}$  by setting

$$\mathbf{v}^{\sigma_i} = e_{k_1} \otimes \dots \otimes e_{k_{i-1}} \otimes e_{k_{i+1}} \otimes e_{k_i} \otimes e_{k_{i+2}} \otimes \dots \otimes e_{k_N} .$$

Following [15, 4, 5], one defines a right action of  $H_N(q)$  on  $V^{\otimes N}$  by

$$\begin{cases} \mathbf{v} \cdot T_i = & \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} , \\ \mathbf{v} \cdot T_i = & q \mathbf{v} & \text{if } k_i = k_{i+1} , \\ \mathbf{v} \cdot T_i = & q \mathbf{v}^{\sigma_i} + (q-1) \mathbf{v} & \text{if } k_i > k_{i+1} . \end{cases}$$

This is a variant of Jimbo's action [15] itself defined by

$$\begin{cases} \mathbf{v} \cdot T_i = & q^{1/2} \mathbf{v}^{\sigma_i} & \text{if } k_i < k_{i+1} , \\ \mathbf{v} \cdot T_i = & q \mathbf{v} & \text{if } k_i = k_{i+1} , \\ \mathbf{v} \cdot T_i = & q^{1/2} \mathbf{v}^{\sigma_i} + (q-1) \mathbf{v} & \text{if } k_i > k_{i+1} . \end{cases}$$

on  $(V^*)^{\otimes N}$  is given by

$$\begin{cases} \mathbf{v}^* \cdot T_i = q (\mathbf{v}^*)^{\sigma_i} & \text{if } k_i < k_{i+1} , \\ \mathbf{v}^* \cdot T_i = q \mathbf{v}^* & \text{if } k_i = k_{i+1} , \\ \mathbf{v}^* \cdot T_i = (\mathbf{v}^*)^{\sigma_i} + (q-1) \mathbf{v}^* & \text{if } k_i > k_{i+1} . \end{cases}$$

**Example 4.1** Let  $V = \mathbb{C}(q) e_1 \oplus \mathbb{C}(q) e_2$ . The matrices describing the right action of  $T_1$  on  $V \otimes V$  and on  $V^* \otimes V^*$  in the canonical bases of these spaces are

$$\check{R} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 1 & q-1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \quad \check{R}^* = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & q & q-1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}.$$

We also need the left actions of  $H_N(q)$  on  $V^{\otimes N}$  and  $(V^*)^{\otimes N}$  defined by

$$\begin{cases} T_i \cdot \mathbf{v} = -q \mathbf{v} \cdot T_i^{-1} = -\mathbf{v} \cdot T_i + (q-1) \mathbf{v} , \\ T_i \cdot \mathbf{v}^* = -q \mathbf{v}^* \cdot T_i^{-1} = -\mathbf{v}^* \cdot T_i + (q-1) \mathbf{v}^* . \end{cases}$$

Equivalently, for  $\mathbf{v} = e_{k_1} \otimes \dots \otimes e_{k_N} \in (V)^{\otimes N}$  and  $\mathbf{v}^* = e_{k_1}^* \otimes \dots \otimes e_{k_N}^* \in (V^*)^{\otimes N}$ ,

$$\begin{cases} T_i \cdot \mathbf{v} = -\mathbf{v}^{\sigma_i} + (q-1) \mathbf{v} , & T_i \cdot \mathbf{v}^* = -q (\mathbf{v}^*)^{\sigma_i} + (q-1) \mathbf{v}^* & \text{if } k_i < k_{i+1} , \\ T_i \cdot \mathbf{v} = -\mathbf{v} , & T_i \cdot \mathbf{v}^* = -\mathbf{v}^* & \text{if } k_i = k_{i+1} , \\ T_i \cdot \mathbf{v} = -q \mathbf{v}^{\sigma_i} , & T_i \cdot \mathbf{v}^* = -(\mathbf{v}^*)^{\sigma_i} & \text{if } k_i > k_{i+1} . \end{cases}$$

## 4.2 The Hopf algebra $A_q(n)$

The quantum group  $A_q(n)$  is the  $\mathbb{C}(q)$ -algebra generated by the  $n^2$  elements  $(x_{ij})_{1 \leq i, j \leq n}$  subject to the defining relations

$$\begin{cases} x_{jk} x_{il} = q x_{il} x_{jk} & \text{for } i < j, k \leq l , \\ x_{ik} x_{il} = x_{il} x_{ik} & \text{for every } i, k, l , \\ x_{jl} x_{ik} - x_{ik} x_{jl} = (q-1) x_{il} x_{jk} & \text{for } i < j, k < l . \end{cases}$$

This algebra is a quantization of the Hopf algebra of polynomial functions on the variety of  $n \times n$  matrices introduced by Dipper and Donkin in [4]. It is not isomorphic to the classical quantization of Faddeev-Reshetikin-Takhtadzhyan [8], and although for generic values of  $q$  both versions play essentially the same rôle, an essential difference is that the Dipper-Donkin algebra is defined for  $q = 0$ .

$A_q(n)$  is a Hopf algebra with comultiplication  $\Delta$  defined by

$$\Delta(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj} .$$

Moreover one can define a left coaction  $\delta$  of  $A_q(n)$  on  $V^{\otimes N}$  by

$$\delta(e_i) = \sum_{j=1}^n x_{ij} \otimes e_j$$

and the following property shows that  $A_q(n)$  is related to the Hecke algebras in a similar way as  $GL_n$  and the symmetric groups.

$H_N(q)$  on  $V^{\otimes N}$ . That is, the following diagram is commutative

$$\begin{array}{ccc} V^{\otimes N} & \xrightarrow{\delta^{\otimes N}} & A_q(n) \otimes V^{\otimes N} \\ \downarrow h & & \downarrow Id \otimes h \\ V^{\otimes N} & \xrightarrow{\delta^{\otimes N}} & A_q(n) \otimes V^{\otimes N} \end{array}$$

for every element  $h \in H_N(q)$  considered as an endomorphism of  $V^{\otimes N}$ .

This property still holds for  $q = 0$ . Thus, for any  $h \in H_N(0)$ ,  $V^{\otimes N} h$  will be a sub- $A_0(n)$ -comodule of  $V^{\otimes N}$ . This is this property which will allow us to define a plactic-like realization of quasi-symmetric functions. For later reference, note that the defining relations of  $A_0(n)$  are

$$\begin{cases} x_{jk} x_{il} = 0 & \text{for } i < j, k \leq l, \\ x_{ik} x_{il} = x_{il} x_{ik} & \text{for every } i, k, l, \\ x_{jl} x_{ik} = x_{ik} x_{jl} - x_{il} x_{jk} & \text{for } i < j, k < l. \end{cases} \quad (5)$$

### 4.3 Some notations for the elements of $A_q(n)$

Each generator  $x_{ij}$  of  $A_q(n)$  will be identified with a two row array and with an element of  $V \otimes V^*$  modulo certain relations as described below:

$$x_{ij} = \begin{bmatrix} i \\ j \end{bmatrix} = e_i \otimes e_j^* .$$

For  $\mathbf{i} = (i_1, \dots, i_r)$ ,  $\mathbf{j} = (j_1, \dots, j_r) \in [1, n]^r$ , let  $e_{\mathbf{i}} = e_{i_1} \otimes \dots \otimes e_{i_r}$  and  $e_{\mathbf{j}}^* = e_{j_1}^* \otimes \dots \otimes e_{j_r}^*$ . One can then identify the monomial  $x_{\mathbf{i}\mathbf{j}} = x_{i_1 j_1} \dots x_{i_r j_r}$  of  $A_q(n)$  with the two row array

$$\begin{bmatrix} i_1 & i_2 & \dots & i_r \\ j_1 & j_2 & \dots & j_r \end{bmatrix} ,$$

itself regarded as the class of the tensor  $e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \in T^r(V \otimes V^*)$  modulo the relations

$$\begin{cases} e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \equiv e_{\mathbf{i}}^{\sigma_r} \otimes (e_{\mathbf{j}}^* \cdot T_r) & \text{for each } r \text{ such that } i_r > i_{r+1} , \\ e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \equiv e_{\mathbf{i}} \otimes (e_{\mathbf{j}}^*)^{\sigma_r} & \text{for each } r \text{ such that } i_r = i_{r+1} . \end{cases} \quad (6)$$

These relations are equivalent to

$$e_{\mathbf{i}} \otimes e_{\mathbf{j}}^* \equiv -T_r \cdot e_{\mathbf{i}} \otimes (e_{\mathbf{j}}^*)^{\sigma_r} \quad \text{for each } r \text{ such that } j_r \leq j_{r+1} . \quad (7)$$

### 4.4 Linear bases of $A_q(n)$

For every  $\mathbf{i} = (i_1, \dots, i_r) \in [1, n]^r$ , let  $I(\mathbf{i}) \in \mathbb{N}^n$  be defined by

$$I(\mathbf{i})_p = \text{Card} \{ i_k, k \in [1, r], i_k = p \}$$

for  $p \in [1, n]$ . For  $I, J \in \mathbb{N}^n$ , set

$$A_q(I, J) = \sum_{I(\mathbf{i})=I, I(\mathbf{j})=J} \mathbb{C}(q) x_{\mathbf{i}\mathbf{j}} .$$



A monomial basis compatible with this grading is constructed in [4]. The basis vectors, which are labelled by matrices  $M = (m_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n(\mathbb{N})$  are

$$x_M = (x_{11}^{m_{11}} x_{12}^{m_{12}} \dots x_{1n}^{m_{1n}}) \dots (x_{n1}^{m_{n1}} x_{n2}^{m_{n2}} \dots x_{nn}^{m_{nn}}) \in A_q(n).$$

It will be useful to introduce another monomial basis  $(x^M)$  of  $A_q(n)$ , labelled by the same matrices, and defined by

$$x^M = (x_{1n}^{m_{1n}} x_{2n}^{m_{2n}} \dots x_{nn}^{m_{nn}}) \dots (x_{11}^{m_{11}} x_{21}^{m_{21}} \dots x_{n1}^{m_{n1}}) \in A_q(n).$$

**Proposition 4.3** *For any  $q \in \mathbb{C}$ , the family  $(x^M)_{M \in \mathcal{M}_n(\mathbb{N})}$  is a homogeneous linear basis of  $A_q(n)$ .*

*Proof* — It is clearly sufficient to prove that each basis element  $x_M$  can be expressed in terms of the  $x^N$ . Using the array and tensor notations, such an element can be represented by

$$x_M = \begin{bmatrix} \dots & i_1 & i_2 & \dots \\ \dots & j_1 & j_2 & \dots \end{bmatrix} = e_{\mathbf{i}} \otimes e_{\mathbf{j}}^*,$$

where  $j_1$  is the maximal element of the second row of this array and where  $i_1 \leq i_2$ . The maximality of  $j_1$  and relation (7) imply

$$x_M = (-1)^{\ell(\sigma)} T_\sigma e_{\mathbf{i}} \otimes e_{\mathbf{j}} = \begin{pmatrix} (-1)^{\ell(\sigma)} T_\sigma \\ Id \end{pmatrix} \cdot \begin{bmatrix} i_1 & \dots & i_2 & \dots \\ j_1 & \dots & j_2 & \dots \end{bmatrix}$$

for some permutation  $\sigma$ . By induction on the length of  $x_M$ , there exists some other permutation  $\tau$  such that

$$x_M = \begin{pmatrix} (-1)^{\ell(\tau)} T_\tau \\ Id \end{pmatrix} \cdot \begin{bmatrix} i_1 & \dots & i_r & \dots & k_1 & \dots & k_s \\ j_1 & \dots & j_1 & \dots & 1 & \dots & 1 \end{bmatrix}.$$

Going back to the definition of the left action of  $H_N(q)$  and to relations (6), this implies that  $x_M$  is a  $\mathbb{Z}[q]$ -linear combination of elements of the form

$$\begin{bmatrix} i_1 & \dots & i_r & \dots & k_1 & \dots & k_s \\ n & \dots & n & \dots & 1 & \dots & 1 \end{bmatrix},$$

from which the conclusion follows immediately.  $\square$

#### 4.5 The standard subspace of $A_q(n)$

The restrictions to the standard component of  $A_q(n)$  of the transition matrices between the two bases  $(x_M)$  and  $(x^M)$  have an interesting description.

**Definition 4.4** *The standard subspace  $S_q(n)$  of  $A_q(n)$  is*

$$S_q(n) = A_q(1^n, 1^n) = \bigoplus_{\sigma \in \mathfrak{S}_n} \mathbb{C}(q) x_\sigma$$

where  $x_\sigma = x_{1\sigma(1)} x_{2\sigma(2)} \dots x_{n\sigma(n)}$  for  $\sigma \in \mathfrak{S}_n$ .

The following result is an immediate consequence of Proposition 4.3.

$$S_q(n) = \bigoplus_{\sigma \in \mathfrak{S}_n} \mathbb{C}(q) x^\sigma$$

where  $x^\sigma = x_{\sigma(n)n} \cdots x_{\sigma(2)2} x_{\sigma(1)1}$ .

The elements of the transition matrices between the two bases  $(x_\sigma)_{\sigma \in \mathfrak{S}_n}$  and  $(x^\sigma)_{\sigma \in \mathfrak{S}_n}$  of  $S_q(n)$  are  $R$ -polynomials. Recall that the family  $(R_{\tau,\sigma}(q))_{\sigma,\tau \in \mathfrak{S}_n}$  of  $R$ -polynomials is defined by

$$(T_{\sigma^{-1}})^{-1} = \varepsilon(\sigma) q^{-l(\sigma)} \sum_{\tau \leq \sigma} \varepsilon(\tau) R_{\tau,\sigma}(q) T_\tau \in H_n(q)$$

for  $\sigma \in \mathfrak{S}_n$  (cf. [13]). The  $R$ -polynomial  $R_{\tau,\sigma}(q)$  is in  $\mathbb{Z}[q]$ , has degree  $l(\sigma) - l(\tau)$  and its constant term is  $\varepsilon(\sigma\tau)$ .

**Proposition 4.6** *The bases  $(x^\sigma)$  and  $(x_\sigma)$  are related by*

$$x^\sigma = \sum_{\tau \leq \sigma} R_{\tau,\sigma}(q) x_{\omega\tau} \quad \text{and} \quad x_\sigma = \sum_{\tau \leq \sigma} R_{\tau,\sigma}(-q) x^{\omega\tau}$$

where  $\omega = (n \ n-1 \ \dots \ 1)$  and where  $\leq$  is the Bruhat order.

*Proof* — In the notation of Section 4.3, we can write

$$\begin{aligned} x^\sigma &= e_\sigma \otimes e_\omega^* = e_{12\dots n}^\sigma \otimes e_\omega^* \equiv e_{12\dots n} \otimes e_\omega^* \cdot T_{\sigma^{-1}} \\ &\equiv e_{12\dots n} \otimes (-q)^{l(\sigma)} T_{\sigma^{-1}}^{-1} \cdot e_\omega^* \equiv e_{12\dots n} \otimes \left( \sum_{\tau \leq \sigma} \varepsilon(\tau) R_{\tau,\sigma}(q) T_\tau \right) \cdot e_\omega^* \\ &\equiv e_{12\dots n} \otimes \left( \sum_{\tau \leq \sigma} R_{\tau,\sigma}(q) e_{\omega\tau}^* \right) = \sum_{\tau \leq \sigma} R_{\tau,\sigma}(q) x_{\omega\tau}. \end{aligned}$$

The second relation can be proved in the same way. □

**Corollary 4.7** *In  $A_0(n)$ , one has*

$$x^\sigma = \sum_{\tau \leq \sigma} \varepsilon(\sigma\tau) x_{\omega\tau} \quad \text{and} \quad x_\sigma = \sum_{\tau \leq \sigma} \varepsilon(\sigma\tau) x_{\omega\tau}.$$

## 4.6 Decomposition of left and right standard subspaces at $q = 0$

In the array notation, the standard subspace is spanned by arrays whose both rows are permutations. If one requires one row to be a fixed permutation  $\sigma$ , one obtains the left and right subcomodules of  $A_q(n)$  which are independent of  $\sigma$  for generic  $q$ , but not for  $q = 0$ .

**Definition 4.8** *The left and right standard subspaces of  $A_q(n)$ , respectively denoted by  $L_q(n)$  and  $R_q(n)$ , are defined by*

$$\begin{aligned} L_q(n) &= \bigoplus_{J=(j_1,\dots,j_n) \in \mathbb{N}^n} \mathbb{C}(q) x_{1,j_1} x_{2,j_2} \cdots x_{n,j_n}, \\ R_q(n) &= \bigoplus_{I=(i_1,\dots,i_r) \in \mathbb{N}^r} \mathbb{C}(q) x_{i_n,n} \cdots x_{i_2,2} x_{i_1,1}. \end{aligned}$$

$$L_q(n; \sigma) = \sum_{J=(j_1, \dots, j_n) \in \mathbb{N}^n} \mathbb{C}(q) x_{\sigma(1), j_1} x_{\sigma(2), j_2} \cdots x_{\sigma(n), j_n} ,$$

$$R_q(n; \sigma) = \sum_{I=(i_1, \dots, i_r) \in \mathbb{N}^r} \mathbb{C}(q) x_{i_n, \sigma(n)} \cdots x_{i_2, \sigma(2)} x_{i_1, \sigma(1)} .$$

For generic  $q$ , all the left (resp. right) subspaces are the same.

**Proposition 4.9** *If  $q \in \mathbb{C}$  is nonzero and not a root of unity,*

$$L_q(n; \sigma) = L_q(n) \quad \text{and} \quad R_q(n; \sigma) = R_q(n) .$$

*Proof* — Using the tensor notation of Section 4.3, we can write

$$e_\sigma \otimes e_J^* = e_{12\dots n}^\sigma \otimes e_J^* \equiv e_{12\dots n} \otimes e_J^* \cdot T_{\sigma^{-1}} \equiv (e_{12\dots n} \otimes e_J^*) \cdot (Id \otimes T_{\sigma^{-1}})$$

for every  $J \in \mathbb{N}^n$ , so that  $L_q(n; \sigma) \subset L_q(n)$ . Moreover the assumptions on  $q$  imply that  $T_{\sigma^{-1}}$  is invertible. The previous relation can therefore be read as

$$e_{12\dots n} \otimes e_J^* \equiv (e_\sigma \otimes e_J^*) \cdot (Id \otimes (T_{\sigma^{-1}})^{-1}) ,$$

from which we get that  $L_q(n) \subset L_q(n; \sigma)$ . The second equality is obtained in the same way.  $\square$

When  $q = 0$ , the subspaces  $L_0(n; \sigma)$  and  $R_0(n; \sigma)$  are not equal to  $L_0(n)$  and  $R_0(n)$ . However, the proof of Proposition 4.9 shows that  $L_0(n; \sigma) \subset L_0(n)$  and  $R_0(n; \sigma) \subset R_0(n)$ . The subspaces  $L_0(n; \sigma)$  (resp.  $R_0(n; \sigma)$ ) are right (resp. left) sub- $A_0(n)$ -comodules of  $L_0(n)$  (resp.  $R_0(n)$ ), of which they form a filtration with respect to the weak order on the symmetric group. To prove this, let us introduce some notations. We associate to an integer vector  $I = (i_1, \dots, i_n)$  of  $\mathbb{N}^n$  the two sets

$$\text{Inv}(I) = \{ (k, l), 1 \leq k < l \leq n-1, i_k > i_l \} ,$$

$$\text{Pos}(I) = \{ (k, l), 1 \leq k < l \leq n-1, i_k < i_l \} .$$

**Proposition 4.10** *For every permutation  $\sigma \in \mathfrak{S}_n$ , one has*

$$L_0(n; \sigma) = \bigoplus_{\substack{J=(j_1, \dots, j_n) \\ \text{Inv}(\sigma) \subset \text{Inv}(J)}} \mathbb{C} x_{\sigma(1), j_1} \cdots x_{\sigma(n), j_n} ,$$

$$R_0(n; \sigma) = \bigoplus_{\substack{I=(i_1, \dots, i_n) \\ \text{Pos}(I) \subset \text{Pos}(\sigma)}} \mathbb{C} x_{i_n, \sigma(n)} \cdots x_{i_1, \sigma(1)} .$$

*Proof* — We only show the first identity, the second one being proved in the same way.

**Lemma 4.11** *Let  $l \geq 1$  be such that  $\sigma(i) > \sigma(i+l)$  and  $j_i \leq j_{i+l}$ . Then, in  $A_0(n)$*

$$x_{\sigma(i), j_i} \cdots x_{\sigma(i+l), j_{i+l}} = 0 .$$

result holds for  $l - 1$ . Two cases are to be considered.

1)  $\sigma(i + l - 1) > \sigma(i + l)$ . If  $j_{i+l-1} \leq j_{i+l}$ , one clearly has

$$x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} = x_{\sigma(i),j_i} \cdots x_{\sigma(i+l-1),j_{i+l-1}} x_{\sigma(i+l),j_{i+l}} = 0.$$

On the other hand, if  $j_{i+l-1} > j_{i+l}$ , we can write

$$x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} = x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} x_{\sigma(i+l-1),j_{i+l-1}} - x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l-1}} x_{\sigma(i+l-1),j_{i+l}}.$$

We have here  $j_i \leq j_{i+l-1}$  so that the right hand side is zero, as required.

2)  $\sigma(i + l - 1) < \sigma(i + l)$ . Thus  $\sigma(i) > \sigma(i + l - 1)$  and we just have to check the case  $j_i > j_{i+l-1}$ . Then,  $j_{i+l-1} < j_{i+l}$  so that

$$x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} = x_{\sigma(i),j_i} \cdots x_{\sigma(i+l),j_{i+l}} x_{\sigma(i+l-1),j_{i+l-1}} + x_{\sigma(i),j_i} \cdots x_{\sigma(i+l-1),j_{i+l}} x_{\sigma(i+l),j_{i+l-1}},$$

which is indeed zero by induction.  $\square$

It follows from the lemma that

$$L_0(n; \sigma) = \sum_{\substack{J=(j_1, \dots, j_n) \\ \text{Inv}(\sigma) \subset \text{Inv}(J)}} \mathbb{C} x_{\sigma(1),j_1} \cdots x_{\sigma(n),j_n},$$

and it remains to prove that the sum is direct. Let  $J = (j_1, \dots, j_n) \in \mathbb{N}^n$  such that  $\text{Inv}(\sigma) \subset \text{Inv}(J)$ . Using the same argument as in the proof of Proposition 4.6 we can write

$$x_{\sigma(1),j_1} \cdots x_{\sigma(n),j_n} = e_\sigma \otimes e_J^* = e_{12\dots n}^\sigma \otimes e_J^* \equiv e_{12\dots n} \otimes (e_J^* \cdot T_{\sigma^{-1}}) = \sum_{\tau \leq \sigma} \varepsilon(\sigma\tau) e_{12\dots n} \otimes e_{J \cdot \tau}$$

where  $\leq$  is the Bruhat order on  $\mathfrak{S}_n$  and  $J \cdot \tau = (j_{\tau(1)}, \dots, j_{\tau(n)})$ . This last formula clearly shows that the family  $(x_{\sigma(1),j_1} \cdots x_{\sigma(n),j_n})_{\text{Inv}(\sigma) \subset \text{Inv}(J)}$  is free.  $\square$

We can now prove that the ‘‘left cells’’  $L_0(n; \sigma)$  form a filtration of the right comodule  $L_0(n)$  with respect to the weak order.

**Proposition 4.12** *Let  $\sigma \in \mathfrak{S}_n$  and let  $i \in [1, n - 1]$  such that  $\sigma(i) > \sigma(i + 1)$ . Then  $L_0(n; \sigma)$  is strictly included into  $L_0(n; \sigma\sigma_i)$ .*

*Proof* — The inclusion  $L_0(n; \sigma) \subset L_0(n; \sigma\sigma_i)$  is immediate. Thus it suffices to show that this inclusion is strict. One can easily construct an element  $\mathbf{x} \in L_0(n; \sigma\sigma_i)$  of the form  $\mathbf{x} = \cdots x_{\sigma(i+1),k} x_{\sigma(i),k} \cdots$ . Using the formalism of Section 4.3, one checks that  $(T_i \otimes Id) \cdot \mathbf{x} = -\mathbf{x} \neq 0$ . On the other hand,  $(T_i \otimes Id) \cdot L_0(n; \sigma) = 0$ . Thus  $\mathbf{x} \notin L_0(n; \sigma)$ .  $\square$

## 4.7 The diagonal subalgebra and the quantum pseudoplectic algebra

**Definition 4.13** *The quantum diagonal algebra  $\Delta_q(n)$  is the subalgebra of  $A_q(n)$  generated by  $x_{11}, \dots, x_{nn}$ .*

The character theory of  $A_q(n)$ -comodules described in Section 6 will show that the noncommutative algebra  $\Delta_q(n)$  contains a subalgebra isomorphic to the algebra of ordinary symmetric polynomials, exactly as in the case of the plactic algebra.

$PPl_q(A)$  is the quotient of  $\mathbb{C}(q)\langle A \rangle$  by the relations

$$\begin{cases} qaab - (q+1)aba + baa = 0 & \text{for } a < b, \\ qa bb - (q+1)baa + bba = 0 & \text{for } a < b, \\ cab - acb - bca + bac = 0 & \text{for } a < b < c. \end{cases}$$

The third relation is the Lie relation  $[[a, c], b] = 0$  where  $[x, y]$  is the usual commutator  $xy - yx$ . For  $q = 1$ , the first two relations become  $[a, [a, b]] = [b, [b, a]] = 0$  and  $PPl_1(A)$  is the universal enveloping algebra of the Lie algebra defined by these relations.

It should be noted that the classical plactic algebra is not obtained by any specialization of  $PPl_q(A)$ . The motivation for the introduction of  $PPl_q(A)$  comes from the following conjecture.

**Conjecture 4.15** *Let  $A = \{a_1, \dots, a_n\}$  be a totally ordered alphabet of cardinality  $n$ . For generic  $q$ , the mapping  $\varphi : a_i \rightarrow x_{ii}$  induces an isomorphism between  $PPl_q(A)$  and the diagonal algebra  $\Delta_q(n)$ .*

Our conjecture is stated for generic values of  $q$ , i.e. when  $q$  is considered as a free variable, or avoiding a discrete set in  $\mathbb{C}$ . It is clearly not true for arbitrary complex values of  $q$ . For example, for  $q = 1$ , the diagonal algebra  $\Delta_1(n)$  is an algebra of commutative polynomials. The diagonal algebra at  $q = 0$  is also particularly interesting, and its structure will be investigated in the forthcoming section.

## 4.8 The hypoplactic algebra

Let again  $A$  be a totally ordered alphabet. We recall that the *plactic algebra* on  $A$  is the  $\mathbb{C}$ -algebra  $Pl(A)$ , quotient of  $\mathbb{C}\langle A \rangle$  by the relations

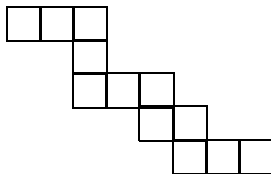
$$\begin{cases} aba = baa, & bba = bab & \text{for } a < b, \\ acb = cab, & bca = bac & \text{for } a < b < c. \end{cases}$$

These relations, which were obtained by Knuth [20], generate the equivalence relation identifying two words which have the same  $P$ -symbol under the Robinson-Schensted correspondence. Though Schensted had shown that the construction of the  $P$ -symbol is an associative operation on words, the monoid structure on the set of tableaux has been mostly studied by Lascoux and Schützenberger [25] under the name ‘plactic monoid’. These authors showed, for example, that the Littlewood-Richardson rule is essentially equivalent to the fact that plactic Schur functions, defined as sums of all tableaux with a given shape, are the basis of a commutative subalgebra of the plactic algebra. This point of view is now explained by Kashiwara’s theory of crystal bases [16, 17], which also leads to the definition of plactic algebras associated to all classical simple Lie algebras [24, 27]. Other interpretations of the Robinson-Schensted correspondence and of the plactic relations can be found in [3, 26].

Kashiwara’s crystallization process describes the generic situation modulo a certain lattice, but does not amount to put  $q = 0$  in the defining relations of quantum groups, which is generally impossible due to the symmetric rôles played by  $q$  and  $q^{-1}$ . The specialization  $q = 0$  in  $A_q(n)$  or in  $H_N(q)$  leads to a different combinatorics, and describes a truly degenerate case, rather than combinatorial aspects of the generic situation. In particular, the specialization of the diagonal algebra is a remarkable quotient of the plactic algebra that we shall now introduce.

$$\left\{ \begin{array}{ll} baba = abab \quad , \quad baca = abac & \text{for } a < b < c \quad , \\ cacb = acbc \quad , \quad cbab = bacb & \text{for } a < b < c \quad , \\ badc = dbca \quad , \quad acbd = cdab & \text{for } a < b < c < d \quad . \end{array} \right.$$

The combinatorial objects playing the rôle of Young tableaux are the so-called ribbons and quasi-ribbons. We recall first that a *ribbon diagram* is a skew Young diagram containing no  $2 \times 2$  block of boxes. A ribbon diagram with  $n$  boxes is naturally encoded by a composition  $I = (i_1, \dots, i_r)$  of  $n$ , called the *shape* of the diagram, whose parts are the lengths of its rows (starting from the top). For instance, the following skew diagram is a ribbon diagram of shape  $(3, 1, 3, 2, 3)$ .

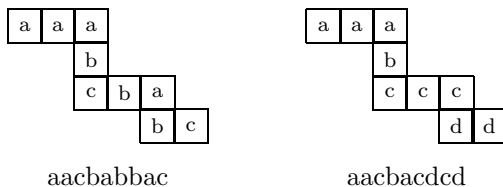


Let  $I$  be a composition. A *quasi-ribbon tableau* of shape  $I$  is then obtained by filling a ribbon diagram  $r$  of shape  $I$  by letters of  $A$  in such a way that

- each row of  $r$  is nondecreasing from left to right;
- each column of  $r$  is strictly increasing from *top to bottom*.

A word is said to be a *quasi-ribbon word* of shape  $I$  if it can be obtained by reading from *bottom to top* and from left to right the columns of a quasi-ribbon diagram of shape  $I$ . Observe that this convention allows to read the shape of a quasi-ribbon word on the word itself.

**Example 4.17** The word  $u = aacbabbac$  is not a quasi-ribbon word since the planar representation of  $u$  obtained by writing its decreasing factors as columns is not a quasi-ribbon tableau, as one can see on the picture. On the other hand, the word  $v = aacbacedd$  is a quasi-ribbon word of shape  $(3, 1, 3, 2)$ . The quasi-ribbon tableau corresponding to  $v$  is also given below.



The central result of this section is the following.

**Theorem 4.18** *The classes of the quasi-ribbon words form a linear basis of the hypoplactic algebra  $HPl(A)$ .*

*Proof* — We first prove that every word  $w$  of  $A^*$  is equivalent to some quasi-ribbon word with respect to the hypoplactic congruence  $\equiv$ . It is sufficient to prove this for standard words (i.e. permutations), since the hypoplactic congruence is compatible with standardization. The standardized  $std(w)$  of a word  $w$  is the permutation obtained by the following process. Reading  $w$  from left to right, label  $1, 2, \dots$  the successive occurrences of the smallest letter  $a$  of  $w$ , then

labelled letters  $a_i$  regarded as elements of the alphabet  $A \times \mathbb{N}$  endowed with the lexicographic order. Then replace each labelled letter by the integers  $1, 2, \dots$  according to its rank in the lexicographic order, as in the following example:

$$w = ababca \longrightarrow a_1 b_1 a_2 b_2 c_1 a_3 \longrightarrow \text{std}(w) = 142563 .$$

This standardization process, due to Schensted [34], is compatible with the plactic relations [25]. One can also check that it is compatible with the quartic hypoplactic relations (used in connection with the usual plactic relations). The standardization of the first hypoplactic relation  $baba = abab$  leads for instance to  $b_1 a_1 b_2 a_2 = a_1 b_1 a_2 b_2$  which is a consequence of a plactic relation ( $bac = bca$ ) and of the last hypoplactic relation ( $cdab = acbd$ ) :

$$b_1 a_1 b_2 a_2 = b_1 b_2 a_1 a_2 = a_1 b_1 a_2 b_2 .$$

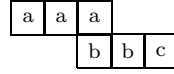
The other verifications are done in the same way. This implies therefore that  $u \equiv v$  iff  $\text{std}(u) \equiv \text{std}(v)$ .

Thus, if we assume that the theorem holds for standard words,  $\text{std}(w)$  is equivalent to some standard quasi-ribbon word  $r$ . Compatibility with the hypoplactic congruence imply that  $w \equiv r'$  where  $r'$  is the word obtained from  $r$  by replacing each integer  $i \in [1, n]$  by the letter of  $A$  occupying the  $i$ -th position in  $\text{std}(w)^{-1}$ . But in a standard word, the hypoplactic relations preserve the relative order of all pairs  $(i, i + 1)$ . It follows that the image in  $r'$  of a column of the ribbon diagram of  $r$  is still a strictly decreasing sequence of letters, so that  $r'$  is still a quasi-ribbon word of the same shape as  $r$ . Hence  $w$  is equivalent to a quasi-ribbon word.

**Example 4.19** Let again  $w = ababca$ . Then we have  $\text{std}(w) = 142563$  and

$$142563 \equiv \underline{1425}36 \equiv 124356$$

(the places where a rewriting rule has been applied are underlined). Hence  $\text{std}(w)$  is equivalent to the standard quasi-ribbon word  $r = 124356$  of shape  $(3, 3)$ . The compatibility of the standardization process with  $\equiv$  implies that  $w \equiv r' = aababc$ . The quasi-ribbon representation of  $r'$  is



We now turn back to the standard case. We have to prove that every permutation of  $\mathfrak{S}_n$  (considered as a word over  $[1, n]$ ) is equivalent to some (standard) quasi-ribbon word over  $[1, n]$ . The proof proceeds by induction on  $n$ . Suppose that the result is true up to some  $n \geq 1$ , and let  $w = u a$  be a permutation of  $\mathfrak{S}_{n+1}$  where  $|u| = n$  and  $a \in [1, n + 1]$ . Applying the induction hypothesis to  $u$ , we can write  $w \equiv r a$  where  $r$  is a standard quasi-ribbon word over  $[1, n + 1] - \{a\}$ . Decompose  $r$  as  $r = c_1 \dots c_l$  where  $c_i$  is the word obtained by reading from bottom to top the  $i$ -th column of the quasi-ribbon tableau associated with  $r$ . Thus  $w \equiv c_1 \dots c_l a$ . Since  $r$  is a quasi-ribbon word, the first column  $c_1$  has to be of one of the following two types:

1.  $c_1 = j \ j-1 \ \dots \ 1$  for some  $j \in [1, n + 1]$ . In this case, the conclusion follows by applying the induction hypothesis to  $c_2 \dots c_l a$ .
2.  $c_1 = j \ j-1 \ \dots \ i+1 \ i-1 \ \dots \ 1$  for some  $j \in [1, n + 1]$ . In this case, the induction hypothesis allows us to write  $c_2 \dots c_l i \equiv d_2 \dots d_m$  where  $d_2 \dots d_m$  is the column decomposition of some standard quasi-ribbon word. Since  $i$  is the minimal letter of this last word, we must have  $d_2 = d'_2 i$ , and the conclusion is implied by the following lemma.

$$(j \dots i+1 \ i-1 \dots 1) (n \dots j+1 \ i) \equiv (i-1 \dots 1) (j \dots i) (n \dots j+1) ,$$

where  $x \dots y$  denotes the concatenation of the elements of the interval  $[x, y]$ , which is the empty word for  $x > y$ .

*Proof of the lemma* – We argue by induction on  $n$ . For  $n = 3$ , the two possible situations covered by the lemma correspond exactly to the two standard plactic relations written as

$$\begin{array}{c} 3 \\ 1 \ 2 \end{array} \equiv \begin{array}{c} 1 \ 3 \\ 2 \end{array} , \quad \begin{array}{c} 2 \ 3 \\ 1 \end{array} \equiv \begin{array}{c} 2 \\ 1 \ 3 \end{array} .$$

The standard quartic hypoplactic relations are also special instances of the lemma:

$$\begin{array}{c} 4 \\ 2 \ 3 \\ 1 \end{array} \equiv \begin{array}{c} 2 \\ 1 \ 4 \\ 3 \end{array} , \quad \begin{array}{c} 3 \ 4 \\ 1 \ 2 \end{array} \equiv \begin{array}{c} 1 \ 3 \\ 2 \ 4 \end{array} .$$

Let now  $n \geq 4$  and suppose that the lemma holds up to order  $n - 1$ . Suppose first that  $i = 1$ . If  $j + 1 = n$ , the formula is obtained by application of a single plactic relation. So, we can suppose that  $j + 1 < n$ . In this case, the result follows by successively applying a plactic relation, the induction hypothesis and a hypoplactic relation as shown below

$$\begin{aligned} (j \dots 2) (n \ n-1 \dots j+1 \ 1) &\equiv (j \dots 3) (n \ 2) (n-1 \dots j+1 \ 1) \\ &\equiv (j \dots 3) (n \ 2 \ 1) (n-1 \dots j+1) \equiv (j \dots 2 \ 1) (n \ n-1 \dots j+1 \ 1) . \end{aligned}$$

Consider now the case  $i = 2$ . Suppose first that  $j = 3$ . For  $n = 4$ , the result to be proved is exactly one of the standard hypoplactic relations. Thus we can assume that  $n > 4$ . Using successively the fact that  $(3 \ 1) (n \dots 4 \ 2)$  is plactically equivalent to  $(n \dots 3 \ 1) (4 \ 2)$ , a hypoplactic relation and then a plactic relation, we obtain

$$(3 \ 1) (n \dots 4 \ 2) \equiv (n \dots 3 \ 1) (4 \ 2) \equiv (n \dots 1) (3 \ 2) (4) \equiv (n \dots 3 \ 1) (2) (4) .$$

Applying twice the induction hypothesis, we can rewrite the right hand side as

$$(n \dots 3 \ 1) (2) (4) \equiv (1) (n \dots 3 \ 2) (4) \equiv (1) (3 \ 2) (n \dots 4) .$$

If  $j > 3$ , we reach the desired conclusion by first applying the induction hypothesis and then the fact that  $j \dots 1 \ 3$  is plactically equivalent to  $1 \ j \dots 3$  as described below

$$(j \dots 3 \ 1) (n \dots j+1 \ 2) \equiv (j \dots 1) (3 \ 2) (n \dots j+1) \equiv (1) (j \dots 3 \ 2) (n \dots j+1) .$$

The general case  $i \geq 3$  follows then by iterated applications of the induction hypothesis as described below

$$\begin{aligned} (j \dots i+1 \ i-1 \dots 2 \ 1) (n \dots j+1 \ i) &\equiv (j \dots i+1 \ i-1 \dots 2) (n \dots j+1 \ i \ 1) \\ &\equiv (i-1 \dots 2) (j \dots i) (n \dots j+1 \ 1) \equiv (i-1 \dots 2) (j \dots i \ 1) (n \dots j+1) \\ &\equiv (i-1 \dots 2 \ 1) (j \dots i) (n \dots j+1) . \end{aligned}$$

□

At this point, we have shown that every word of  $A^*$  is equivalent to some quasi-ribbon word. To conclude the proof of the theorem, it remains to show that the hypoplactic classes of quasi-ribbon words are linearly independent. Again, by the standardization argument, it suffices to



suppose that  $A = [1, n]$  for some  $n \geq 1$ . The point is now that the hypoplactic relations are satisfied by the generators of the 0-diagonal algebra  $\Delta_0(n)$ . Hence one can define a morphism  $\varphi$  from  $HPl(A)$  onto  $\Delta_0(n)$  by  $\varphi(i) = x_{ii}$  for every  $i \in A = [1, n]$ .

Let  $w \in \mathfrak{S}_n$  be a standard quasi-ribbon word of length  $n$  over  $[1, n]$ . By definition, there exists a strictly increasing sequence  $1 = k_1 < k_2 < \dots < k_{l+1} = n+1$  such that  $w = c_1 \dots c_l$  with  $c_i = k_{i+1}-1 \dots k_i$  for  $i \in [1, l]$ . Consider the Young subgroup  $\mathfrak{S}_w = \mathfrak{S}_{[k_1, k_2-1]} \times \dots \times \mathfrak{S}_{[k_l, k_{l+1}-1]}$  of  $\mathfrak{S}_n$ . Applying Corollary 4.7 to each strictly decreasing word  $c_i$ , one obtains that

$$\varphi(w) = \sum_{\sigma \in \mathfrak{S}_w} \varepsilon(\sigma) \begin{bmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{bmatrix}$$

in the notation of Section 4.3. Observe that  $w$  is the unique permutation of maximal length occurring in the sum. This property implies immediately that  $\varphi(\mathcal{B}_n)$  is free in  $\Delta_0(n)$ . It follows that  $\mathcal{B}_n$  is itself free in  $HPl(A)$  as desired.  $\square$

Let  $C(\sigma)$  be the unique composition  $I$  such that  $D(\sigma) = D(I)$ . By the *evaluation*  $ev(w)$  of a word  $w \in A^*$ , we mean the vector  $ev(w) = (|w|_a)_{a \in A} \in \mathbb{N}^A$  whose entries are just the different numbers of occurrences of each letter  $a \in A$  in  $w$ .

As an interesting consequence of the proof of Theorem 4.18 and of Lemma 4.20, we obtain:

**Proposition 4.21** *Let  $w$  be a word over a totally ordered alphabet  $A$ , let  $\lambda$  be its evaluation and let  $\sigma = std(w)$ . The unique quasi-ribbon word to which  $w$  is equivalent with respect to the hypoplactic congruence is the unique quasi-ribbon word of evaluation  $\lambda$  and of shape  $C(\sigma^{-1})$ .*

**Example 4.22** Consider again  $w = ababca$ . Then  $\lambda = (3, 2, 1)$  and  $std(w) = \sigma = 142563$ . Hence  $\sigma^{-1} = 136245$  and  $C(\sigma^{-1}) = (3, 3)$ . The unique quasi-ribbon word of evaluation  $(3, 2, 1)$  and of shape  $(3, 3)$  is  $aababc$ . Thus  $w \equiv aababc$  as already seen in Example 4.19.

The importance of the hypoplactic algebra comes from the following isomorphism, which follows directly from the previous considerations.

**Theorem 4.23** *The ring homomorphism defined by  $\varphi : a_i \rightarrow x_{ii}$  is an isomorphism between the hypoplactic algebra  $HPl(A)$  and the crystal limit  $\Delta_0(n)$  of the quantum diagonal algebra.*

We have already seen that the quantum diagonal algebra  $\Delta_q(n)$  is not isomorphic to the quantum plactic algebra when  $q \in \{0, 1\}$ . We conjecture that these two degenerate cases are the only exceptions.

## 5 Characteristics of $H_N(0)$ -modules

### 5.1 Grothendieck rings associated with finitely generated $H_N(0)$ -modules

Let  $G_0(H_N(q))$  be the Grothendieck group of the category of finitely generated  $H_N(q)$ -modules and let  $K_0(H_N(q))$  be the Grothendieck group of equivalence classes of finitely generated projective  $H_N(q)$ -modules. When  $q$  is not 0 and not a root of unity, the Hecke algebra  $H_N(q)$  is semi-simple and these two groups coincide. Moreover their direct sum for all  $n \geq 0$  endowed with the induction product is isomorphic to the ring  $Sym$  of (commutative) symmetric functions.

When  $q = 0$ ,  $H_N(0)$  is not semi-simple. In particular, indecomposable  $H_N(0)$ -modules are not necessarily irreducible, and the Grothendieck rings

$$\mathcal{G} = \bigoplus_{N \geq 0} G_0(H_N(0)) \quad \text{and} \quad \mathcal{K} = \bigoplus_{N \geq 0} K_0(H_N(0))$$

of quasi-symmetric functions and **Sym** of noncommutative symmetric functions. The duality between **Sym** and  $QSym$  (cf. Section 2.2) can therefore be traced back to a general fact in representation theory.

## 5.2 Simple $H_N(0)$ -modules

There are  $2^{N-1}$  simple  $H_N(0)$ -modules, all of dimension 1 [1, 31]. To see this, it is sufficient to observe that  $(T_i T_{i+1} - T_{i+1} T_i)^2 = 0$ . Thus, all the commutators  $[T_i, T_j]$  are in the radical of  $H_N(0)$ . But the quotient of  $H_N(0)$  by the ideal generated by these elements is the *commutative* algebra generated by  $N-1$  elements  $t_1, \dots, t_{N-1}$  subject to  $t_i^2 = -t_i$ . It is easy to check that this algebra has no nilpotent elements, so that it is  $H_N(0)/\text{rad}(H_N(0))$ . The irreducible representations are thus obtained by sending a set of generators to  $-1$  and its complement to 0. We shall however label these representations by compositions rather than by subsets. Let  $I$  be a composition of  $N$  and let  $D(I)$  the associated subset of  $[1, N-1]$ . The irreducible representation  $\varphi_I$  of  $H_N(0)$  is then defined by

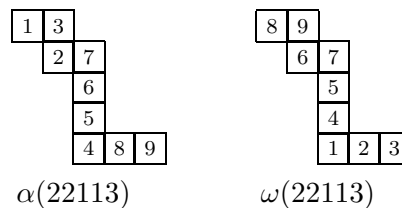
$$\varphi_I(T_i) = \begin{cases} -1 & \text{if } i \in D(I) , \\ 0 & \text{if } i \notin D(I) . \end{cases}$$

The associated  $H_N(0)$ -module will be denoted by  $\mathbf{C}_I$ . These modules (when  $I$  runs over all compositions of  $N$ ) form a complete system of simple  $H_N(0)$ -modules.

The simple modules can also be realized as minimal left ideals of  $H_N(0)$ . To describe the generators, we associate with a composition  $I$  of  $N$  two permutations  $\alpha(I)$  and  $\omega(I)$  of  $\mathfrak{S}_N$  defined by

- $\alpha(I)$  is the permutation obtained by filling the columns of the skew Young diagram of ribbon shape  $I$  from bottom to top and from left to right with the numbers  $1, 2, \dots, N$ , i.e. the standard quasi-ribbon word of shape  $I$ ;
- $\omega(I)$  is the permutation obtained by filling the rows of the skew Young diagram of ribbon shape  $I$  from left to right and from bottom to top with the numbers  $1, 2, \dots, N$ .

**Example 5.1** Consider the composition  $I = 22113$  of 9. The fillings of the ribbon diagram of shape  $I$  corresponding to  $\alpha(I)$  and  $\omega(I)$  are



Thus  $\alpha(22113) = 132765489$  and  $\omega(22113) = 896754123$ .

Recall that the *permutohedron* of order  $N$  is the Hasse diagram of the weak order on  $\mathfrak{S}_N$ , that is, the graph whose vertices are the elements of  $\mathfrak{S}_N$  and where an edge labelled  $i \in [1, N-1]$  between  $\sigma$  and  $\tau$  means that  $\tau = \sigma_i \sigma$ .

**Lemma 5.2** *Let  $I$  be a composition of  $N$ . The descent class  $D_I = \{\sigma \in \mathfrak{S}_N, D(\sigma) = D(I)\}$  is the interval  $[\alpha(I), \omega(I)]$  for the weak order on  $\mathfrak{S}_N$ .*

$$\left\{ \begin{array}{ll} \square_i^2 = \square_i & \text{for } i \in [1, N-1] , \\ \square_i \square_j = \square_j \square_i & \text{for } |i-j| > 1 , \\ \square_i \square_{i+1} \square_i = \square_{i+1} \square_i \square_{i+1} & \text{for } i \in [1, N-2] . \end{array} \right.$$

In particular, the morphism defined by  $T_i \longrightarrow -\square_i$  is an involution of  $H_N(0)$ . As the  $\square_i$  satisfy the braid relations, one can associate to each permutation  $\sigma \in \mathfrak{S}_N$  the element  $\square_\sigma$  of  $H_N(0)$  defined by  $\square_\sigma = \square_{i_1} \dots \square_{i_r}$  where  $\sigma_{i_1} \dots \sigma_{i_r}$  is an arbitrary reduced decomposition of  $\sigma$ .

For a composition  $I = (i_1, \dots, i_r)$  we denote by  $\bar{I} = (i_r, \dots, i_1)$  its mirror image and by  $I^\sim$  its conjugate composition, i.e. the composition obtained by writing from right to left the lengths of the columns of the ribbon diagram of  $I$ . For instance,  $\overline{(3, 2, 1)} = (1, 2, 3)$  and  $(3, 2, 1)^\sim = (2, 2, 1, 1)$ .

**Proposition 5.3** *The simple  $H_N(0)$  module  $\mathbf{C}_I$  is isomorphic to the minimal left ideal  $H_N(0)\eta_I$  of  $H_N(0)$  where  $\eta_I = T_{\omega(\bar{I})} \square_{\alpha(I^\sim)}$ .*

*Proof* — Observe first that  $\omega(\bar{I})^{-1} = \omega(I)$  and  $\alpha(I) = \omega(I^\sim)\omega_N$  (where  $\omega_N$  is the maximal permutation of  $\mathfrak{S}_N$ ). It follows that  $\text{Des}(\omega(\bar{I})^{-1}) = D(I)$  and  $\text{Des}(\alpha(I^\sim)^{-1}) = [1, n-1] - D(\bar{I})$ . Thus, taking into account the fact that  $T_i \square_i = 0$ , one checks that  $\eta_I$  is different from 0 and that

$$T_i T_{\omega(\bar{I})} \square_{\alpha(I^\sim)} = \begin{cases} -T_{\omega(\bar{I})} \square_{\alpha(I^\sim)} & \text{if } i \in D(I) , \\ 0 & \text{if } i \notin D(I) . \end{cases} \quad \square$$

### 5.3 Indecomposable projective $H_N(0)$ -modules

The indecomposable projective  $H_N(0)$ -modules have also been classified by Norton (cf. [1, 31]). One associates with a composition  $I$  of  $N$  the unique indecomposable projective  $H_N(0)$ -module  $\mathbf{M}_I$  such that  $\mathbf{M}_I/\text{rad}(\mathbf{M}_I) \simeq \mathbf{C}_I$ . This module can be realized as the left ideal

$$\mathbf{M}_I = H_N(0)\nu_I$$

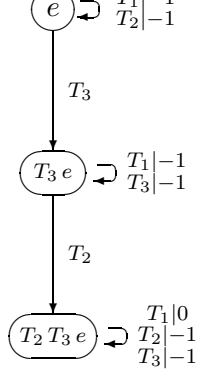
where  $\nu_I = T_{\alpha(I)} \square_{\alpha(\bar{I}^\sim)}$ . Since  $\alpha(I^\sim)^{-1} = \omega_n \omega(\bar{I})$ , one gets  $D(\alpha(\bar{I}^\sim)^{-1}) = [1, n-1] - D(I)$ . It follows that the generator  $\nu_I$  of  $\mathbf{M}_I$  is different from 0 and that a basis of  $\mathbf{M}_I$  is given by

$$\{ T_\sigma \square_{\alpha(\bar{I}^\sim)}, \text{Des}(\sigma) = D(I) \} = \{ T_\sigma \square_{\alpha(\bar{I}^\sim)}, \sigma \in [\alpha(I), \omega(I)] \} ,$$

according to Lemma 5.2. Hence the dimension of  $\mathbf{M}_I$  is equal to the cardinality of the descent class  $D_I$ . Also, every interval of the form  $[\alpha(I), \omega(I)]$  in the permutohedron can be interpreted as the “graph” of some indecomposable projective  $H_N(0)$ -module (cf. Example 5.4 below). The family  $(\mathbf{M}_I)_{|I|=N}$  forms a complete system of projective indecomposable  $H_N(0)$ -modules, and

$$H_n(0) = \bigoplus_{|I|=N} \mathbf{M}_I . \quad (8)$$

**Example 5.4** Let  $I = (1, 1, 2)$ . Then  $I^\sim = (1, 3)$ ,  $\bar{I} = (2, 1, 1)$ ,  $\bar{I}^\sim = (3, 1)$ ,  $\alpha(I) = 3214$  and  $\alpha(\bar{I}^\sim) = 1243$ . Hence  $\nu_{112} = T_2 T_1 T_2 \square_3$ . The module  $\mathbf{M}_{112}$  can be described by the following automaton. An arrow labelled  $T_i$  going from  $f$  to  $g$  means that  $T_i \cdot f = g$ , and a loop on the vertex  $f$  labelled  $T_i | \epsilon$  (with  $\epsilon = 0$  or  $\epsilon = -1$ ) means that  $T_i \cdot f = \epsilon f$ .



This is also the graph of the interval  $[3214, 4312] = D_{112}$  in the permutohedron of  $\mathfrak{S}_4$ . The  $(-1)$ -loops correspond to the descents of the inverse permutation.

#### 5.4 A Frobenius type characteristic for finite dimensional $H_N(0)$ -modules

Let  $M$  be a finite dimensional  $H_N(0)$ -module and consider a composition series for  $M$ , i.e. a decreasing sequence  $M_1 = M \supset M_2 \supset \dots \supset M_k \supset M_{k+1} = \mathbb{C}$  of submodules where the successive quotients  $M_i/M_{i+1}$  are simple. Therefore each  $M_i/M_{i+1}$  is isomorphic to some  $\mathbf{C}_{I_i}$ , and the Jordan-Hölder theorem ensures that the quasi-symmetric function

$$\mathcal{F}(M) = \sum_{i=1}^k F_{I_i}$$

is independent of the choice of the composition series. This quasi-symmetric function is called the *characteristic* of  $M$ . Its properties are quite similar to those of the usual Frobenius characteristic of a  $\mathfrak{S}_N$ -module [7]. However, the characteristic  $\mathcal{F}(M)$  of a  $H_N(0)$ -module  $M$  does not specify it up to isomorphism.

The character formula for  $H_N(0)$ -modules can be stated in a form similar to the Frobenius character formula. For a composition  $I$ , we denote by  $C_I(q)$  the noncommutative symmetric function  $C_I(q) = (q-1)^{l(I)} S^I((q-1)A)$ , in the noncommutative  $\lambda$ -ring notation introduced in [21]. Let also  $w_J$  be the permutation of the Young subgroup  $\mathfrak{S}_J$  defined by

$$w_J = (\sigma_1 \dots \sigma_{j_1-1}) (\sigma_{j_1+1} \dots \sigma_{j_1+j_2-1}) \dots (\sigma_{j_1+\dots+j_{r-1}+1} \dots \sigma_{n-1}).$$

The character of a module  $M$  is then determined by its values  $\chi_M(T_{w_J}) = \text{tr}_M(T_{w_J})$  on the special elements  $T_{w_J}$ .

**Proposition 5.5** [7] (Character formula) *The character of  $M$  is given by*

$$\chi_M(T_{w_J}) = \langle \mathcal{F}(M), C_J(0) \rangle$$

where  $\langle , \rangle$  is the pairing between  $QSym$  and  $\mathbf{Sym}$ .

One can refine  $\mathcal{F}$  into a graded version of the characteristic, at least when  $M$  is a cyclic module i.e. when  $M = H_N(0)e$ . In this case, the length filtration

$$H_N(0)^{(k)} = \bigoplus_{l(\sigma) \geq k} \mathbb{C} T_\sigma$$

suggests to introduce the *characteristic*  $\mathcal{F}_q(M)$  of  $M$  defined by

$$\mathcal{F}_q(M) = \sum_{k \geq 0} q^k \mathcal{F}(M^{(k)}/M^{(k+1)}) .$$

The ordinary characteristic  $\mathcal{F}(M)$  is then the specialization of  $\mathcal{F}_q(M)$  at  $q = 1$ .

The graded characteristic is in particular defined for the modules induced by tensor products of simple 1-dimensional modules

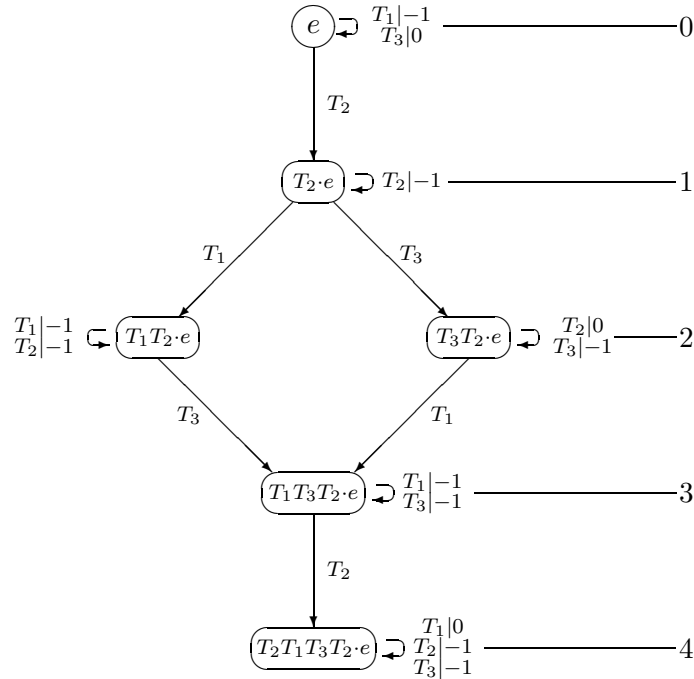
$$M_{I_1, \dots, I_r} = \mathbf{C}_{I_1} \otimes \dots \otimes \mathbf{C}_{I_r} \uparrow_{H_{n_1}(0) \otimes \dots \otimes H_{n_r}(0)}^{H_{n_1+\dots+n_r}(0)} ,$$

the characteristic of which being equal to the product  $F_{I_1} \dots F_{I_r}$ . The induction formula can be stated in terms of the graded characteristic, which leads to a  $q$ -analogue of the algebra of quasi-symmetric functions. This  $q$ -analogue is defined in terms of the  $q$ -shuffle product [6]. Let  $A$  be an alphabet and let  $q$  be an indeterminate commuting with  $A$ . The  $q$ -shuffle is the bilinear operation of  $\mathbb{N}[q]\langle A \rangle$  denoted by  $\odot_q$  and recursively defined on words by the relations

$$\left\{ \begin{array}{l} 1 \odot_q u = u \odot_q 1 = u , \\ (au) \odot_q (bv) = a(u \odot_q bv) + q^{|au|} b(au \odot_q v) , \end{array} \right.$$

where  $1$  is the empty word,  $u, v \in A^*$  and  $a, b \in A$ . One can show that  $\odot_q$  is associative (cf. [6]).

**Example 5.6** Let  $M_{(11),(2)}$  denote the  $H_4(0)$ -module obtained by inducing to  $H_4(0)$  the  $H_2(0) \otimes H_2(0)$ -module  $\mathbf{C}_{11} \otimes \mathbf{C}_2$ , identifying  $H_2(0) \otimes H_2(0)$  with the subalgebra of  $H_4(0)$  generated by  $T_1$  and  $T_3$ . This  $H_4(0)$ -module is generated by a single element  $e$  on which  $T_1$  and  $T_3$  act by  $T_1 \cdot e = -e$  and by  $T_3 \cdot e = 0$ . The following automaton gives a complete description of this module. The states (vertices) correspond to the images of  $e$  under the action of some element of  $H_4(0)$ , which form a linear basis of  $M_{(11),(2)}$ .



The automaton is graded by the distance  $d(f)$  of a state  $f$  to the initial state  $e$  as indicated on the picture. This grading is precisely the one described by  $\mathcal{F}_q$ . That is, if we associate with each state

we find

$$\mathcal{F}_q(M_{(11,2)}) = \sum_f q^{d(f)} F_{I(f)} = F_{13} + q F_{22} + q^2 (F_{112} + F_{31}) + q^3 F_{121} + q^4 F_{211} .$$

This equality can also be read on the  $q$ -shuffle of 21 and 34:

$$21 \odot_q 34 = 2134 + q 2314 + q^2 2341 + q^2 3214 + q^3 3241 + q^4 3421 .$$

One obtains the graded characteristic by replacing each permutation  $\sigma$  in this expansion by the quasi-symmetric function  $F_{C(\sigma)}$ .

This example illustrates the general fact that the graded characteristic of an induced module as above is always given by the  $q$ -shuffle. As it is an associative operation, one obtains in this way a  $q$ -deformation of the ring of quasi-symmetric functions.

**Proposition 5.7** [7, 6] *Let  $I$  and  $J$  be compositions of  $N$  and  $M$ . Let also  $\sigma \in \mathfrak{S}_{[1,N]}$  and  $\tau \in \mathfrak{S}_{[N+1,N+M]}$  be such that  $\text{Des}(\sigma) = D(I)$  and  $\text{Des}(\tau) = D(J)$ . The  $H_{N+M}(0)$ -module obtained by inducing to  $H_{N+M}(0)$  the  $H_N(0) \otimes H_M(0)$ -module  $\mathbf{C}_I \otimes \mathbf{C}_J$  (identifying  $H_N(0) \otimes H_M(0)$  to the subalgebra of  $H_{N+M}(0)$  generated by  $T_1, \dots, T_{N-1}, T_{N+1}, \dots, T_{N+M-1}$ ) is cyclic, and its graded characteristic is given by*

$$\mathcal{F}_q(\mathbf{C}_I \otimes \mathbf{C}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)}) = \sum_{\nu \in \mathfrak{S}_{N+M}} q^{d(\nu)} F_{C(\nu)}$$

where  $C(\nu)$  denotes the composition associated with the descent set of  $\nu$  and where

$$\sigma \odot_q \tau = \sum_{\nu \in \mathfrak{S}_{N+M}} q^{d(\nu)} \nu .$$

For  $q = 1$ , we obtain the following result [7].

**Corollary 5.8** *The characteristic  $\mathcal{F}$  is an isomorphism between  $\mathcal{G}$  and the  $\mathbb{Z}$ -algebra of quasi-symmetric functions.*

## 5.5 A noncommutative characteristic for finite dimensional projective $H_N(0)$ -modules

Let  $M$  be a finite dimensional projective  $H_N(0)$ -module. Hence  $M$  is isomorphic to a direct sum of indecomposable projective modules

$$M = \bigoplus_{i=1}^m \mathbf{M}_{I_i} .$$

The *noncommutative Frobenius characteristic* of  $M$  is the noncommutative symmetric function  $\mathcal{R}(M)$  defined by

$$\mathcal{R}(M) = \sum_{i=1}^m R_{I_i} .$$

The characteristic  $\mathcal{R}(M)$  does characterize every finite dimensional projective  $H_N(0)$ -module  $M$  up to isomorphism, and is therefore stronger than  $\mathcal{F}$ . The following proposition, which is a reformulation of Carter's expression of the Cartan invariants of  $H_N(0)$  shows in particular how to compute  $\mathcal{F}(M)$  from  $\mathcal{R}(M)$ .

acteristic  $\mathcal{F}(M)$  of  $M$  is a symmetric function which is the commutative image of  $\mathcal{R}(M)$ .

*Proof* — It suffices to prove the result when  $M = \mathbf{M}_I$ . In this case,

$$\mathcal{F}(\mathbf{M}_I) = \sum_{J \vdash N} c_{IJ} F_J ,$$

where the Cartan invariant  $c_{IJ}$  is equal to the number of permutations  $\sigma$  of  $\mathfrak{S}_N$  such that  $D(\sigma) = I$  and  $D(\sigma^{-1}) = J$  (see [1]). On the other hand, by a formula of Gessel [10], we have  $c_{IJ} = (r_I, r_J) = \langle r_I, R_J \rangle$ , where  $(\cdot, \cdot)$  denotes the usual scalar product of commutative symmetric functions (see [28]) and where  $r_I$  is the commutative image of the ribbon Schur function  $R_I$ . Using the fact that the quasi-ribbons  $F_I$  and noncommutative ribbon Schur functions  $R_J$  are dual bases, it follows that  $\mathcal{F}(\mathbf{M}_I) = r_I$ .  $\square$

The induction from a tensor product of projective modules is described by the product of noncommutative symmetric functions.

**Proposition 5.10** *Let  $I = (i_1, \dots, i_r)$  and  $J = (j_1, \dots, j_s)$  be compositions of  $N$  and  $M$ . Then,*

$$\mathcal{R}(\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)}) = R_I R_J = R_{I \cdot J} + R_{I \triangleright J} , \quad (9)$$

where we set  $I \cdot J = (i_1, \dots, i_r, j_1, \dots, j_s)$  and  $I \triangleright J = (i_1, \dots, i_{r-1}, i_r + j_1, j_2, \dots, j_s)$ .

*Proof* — The formula for the product of two noncommutative ribbon Schur functions is proved in [9], and we just have to show that

$$\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \simeq \mathbf{M}_{I \cdot J} \oplus \mathbf{M}_{I \triangleright J} .$$

Using the duality between simple modules and indecomposable projective modules, we obtain

$$\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \simeq \bigoplus_{K \vdash N+M} \dim \operatorname{Hom}_{H_{N+M}(0)} \left( \mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)}, \mathbf{M}_K \right) \mathbf{M}_K .$$

By Frobenius reciprocity, we have

$$\mathbf{M}_I \otimes \mathbf{M}_J \uparrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \simeq \bigoplus_{K \vdash N+M} \dim \operatorname{Hom}_{H_N(0) \otimes H_M(0)} \left( \mathbf{M}_I \otimes \mathbf{M}_J, \mathbf{C}_K \downarrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \right) \mathbf{M}_K .$$

Observe now that the description of the family  $(\mathbf{M}_I)$  given in Section 5.3 implies that

$$\dim \operatorname{Hom}_{H_N(0)} (\mathbf{M}_J, \mathbf{C}_I) = \begin{cases} 1 & \text{if } I = J , \\ 0 & \text{if } I \neq J , \end{cases}$$

so that

$$\dim \operatorname{Hom}_{H_N(0) \otimes H_M(0)} \left( \mathbf{M}_I \otimes \mathbf{M}_J, \mathbf{C}_K \downarrow_{H_N(0) \otimes H_M(0)}^{H_{N+M}(0)} \right)$$

is equal to 0 if  $D(K) \cap [1, N] \neq D(I)$  or  $D(K) \cap [N+1, N+M] \neq N+D(J)$  and equal to 1 when  $D(K) \cap [1, N] = D(I)$  and  $D(K) \cap [N+1, N+M] = N+D(J)$ , i.e. when  $K = I \cdot J$  or  $K = I \triangleleft J$  as desired.  $\square$

Thus, we have the following interpretation of the algebra of noncommutative symmetric functions.

**Corollary 5.11** *The characteristic map  $\mathcal{R}$  is an isomorphism between the Grothendieck ring  $\mathcal{K}$  and the  $\mathbb{Z}$ -algebra of noncommutative functions.*

## 6.1 The character of an $A_q(n)$ -comodule

Let  $M$  be a finite dimensional  $A_q(n)$ -comodule with coaction  $\delta$ . Let  $(m_i)_{i=1,m}$  be a basis of  $M$ . There exists elements  $(a(i, j))_{1 \leq i, j \leq m}$  of  $A_q(n)$  such that

$$\delta(m_i) = \sum_{j=1}^m a(i, j) \otimes m_j$$

for  $i \in [1, m]$ . The element

$$\sum_{i=1}^m a(i, i)$$

of  $A_q(n)$  is independent of the choice of the basis  $(m_i)$ . It will be denoted by  $\chi(M)$  and called the *character* of the  $A_q(n)$ -comodule  $M$ .

**Proposition 6.1** *Let  $M, N, M', M''$  be  $A_q(n)$ -comodules.*

1. *If  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence,  $\chi(M) = \chi(M') + \chi(M'')$ ;*
2.  *$\chi(M \otimes N) = \chi(M) \chi(N)$ ;*
3. *if  $M \simeq N$ , then  $\chi(M) = \chi(N)$ .*

It happens that for generic values of  $q$ , the character of an  $A_q(n)$ -comodule is always an element of the quantum diagonal algebra.

**Theorem 6.2** *Let  $q$  be an indeterminate and let  $M$  be an  $A_q(n)$ -comodule. Then the character  $\chi(M)$  belongs to the diagonal algebra  $\Delta_q(n)$ .*

*Proof* — The basic observation is the following lemma.

**Lemma 6.3** *The quantum determinant of  $A_q(n)$  can be expressed by means of  $q$ -commutators as follows:*

$$\begin{aligned} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}_q &\stackrel{\text{def}}{=} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)} \\ &= \frac{1}{(1-q)^{n-1}} [x_{nn}, [\cdots, [x_{22}, x_{11}]_q \cdots]_q]_q \end{aligned}$$

where  $[P, Q]_q = PQ - qQP$ .

*Proof of the lemma* – Observe first that the lemma is equivalent to the identity

$$x_{nn} \begin{vmatrix} x_{11} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1,1} & \cdots & x_{n-1,n-1} \end{vmatrix} - q \begin{vmatrix} x_{11} & \cdots & x_{1,n-1} \\ \vdots & \ddots & \vdots \\ x_{n-1,1} & \cdots & x_{n-1,n-1} \end{vmatrix} x_{nn} = (1-q) \begin{vmatrix} x_{11} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}.$$

Using the tensor notation of Section 4.3, this can be rewritten as

$$(1-q) \left( \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) e_{12\dots n} \otimes e_{\sigma}^* \right) + q \left( \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=n}} \varepsilon(\sigma) e_{12\dots n} \otimes e_{\sigma}^* \right) = \sum_{\sigma \in \mathfrak{S}_{n-1}} \varepsilon(\sigma) e_{n12\dots n-1} \otimes e_{n\sigma}^*,$$



$$(-1)^{n-1} \left( \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(1)=n}} \varepsilon(\sigma) e_\sigma^* \right) \cdot T_1 T_2 \dots T_{n-1} = (1-q) \left( \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) e_\sigma^* \right) + q \left( \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma(n)=n}} \varepsilon(\sigma) e_\sigma^* \right).$$

This last formula is now easily proved by induction on  $n$ .  $\square$

As a consequence of Lemma 6.3, we obtain that the character of the  $r$ -th exterior power  $\Lambda_q^r(V)$  of  $V$  (cf. [4]) is equal to

$$\chi(\Lambda_q^r(V)) = \Lambda_r(q; \Delta) \stackrel{\text{def}}{=} \frac{1}{(1-q)^{n-1}} \left( \sum_{1 \leq i_1 < \dots < i_r \leq n} [x_{i_r i_r}, [\dots, [x_{i_2 i_2}, x_{i_1 i_1}]_q \dots]_q ]_q \right)$$

where  $\Delta = \{x_{11}, \dots, x_{nn}\}$ . Let now  $\lambda = (1^{l_1}, \dots, n^{l_n})$  be a partition of  $n$ . It follows then from Proposition 6.1 that the character of the comodule

$$M_{\lambda, q} = V^{\otimes l_1} \otimes \Lambda_q^2(V)^{\otimes l_2} \otimes \dots \otimes \Lambda_q^n(V)^{\otimes l_n}$$

is also in the diagonal algebra  $\Delta_q(n)$ .

On the other hand, it has been shown by Dipper and Donkin [4] that one can associate with every partition  $\lambda$  of  $n$  an irreducible  $A_q(n)$ -comodule  $L_{\lambda, q}$  and that the family  $(L_{\lambda, q})_{\lambda \vdash n}$  forms a complete system of irreducible  $A_q(n)$ -comodules. They also proved that for an appropriate ordering  $<$  on partitions of  $n$ , the products of exterior powers decompose as

$$M_{\lambda, q} \simeq L_{\lambda, q} \oplus \bigoplus_{\mu < \lambda} a_\mu L_{\mu, q}.$$

Applying Proposition 6.1, we see that the matrix giving the decomposition of  $(\chi(M_{\lambda, q}))_{\lambda \vdash n}$  on  $(\chi(L_{\lambda, q}))_{\lambda \vdash n}$  is unitriangular. It follows that the character  $\chi(L_{\lambda, q})$  is a linear combination of elements of the family  $(\chi(M_{\lambda, q}))_{\lambda \vdash n}$ . Hence  $\chi(L_{\lambda, q}) \in \Delta_q(n)$ .  $\square$

**Note 6.4** The commutative image of  $\chi(M)$  is the formal commutative character introduced in [4]. But the formal character of  $L_{\lambda, q}$  is the Schur function  $S_\lambda$ . Thus the characters (in our sense) of the irreducible  $A_q(n)$ -comodules are quantum analogues of Schur functions.

**Note 6.5** Let  $Char_{\mathbb{Z}}(q; n)$  denote the  $\mathbb{Z}$ -lattice of  $\Delta_n(q)$  spanned by characters of  $A_q(n)$ -comodules. The proof of Theorem 6.2 shows that  $Char_{\mathbb{Z}}(q; n)$  is the subring of  $\Delta_n(q)$  generated by the  $n$  quantum elementary functions  $\Lambda_r(q; \Delta)$  with  $1 \leq r \leq n$ . Moreover since the composition series of the two  $A_n(q)$ -comodules  $\Lambda_q^r(V) \otimes \Lambda_q^s(V)$  and  $\Lambda_q^s(V) \otimes \Lambda_q^r(V)$  are the same, these quantum elementary functions commute. It follows that  $Char_{\mathbb{Z}}(q; n)$  is a commutative  $\mathbb{Z}$ -algebra isomorphic to the algebra of symmetric functions in  $n$  variables.

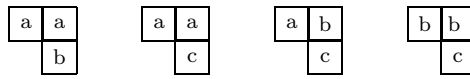
**Note 6.6** Although Theorem 6.2 has been stated for generic values of  $q$ , it is not difficult to see that it holds for  $q \in \mathbb{C} - \{0, 1\}$ . In the usual commutative case (i.e.  $q = 1$ ), the result becomes false. On the other hand we conjecture that it still holds for  $q = 0$  (cf. Conjecture 6.13).

## 6.2 A family of irreducible $A_0(n)$ -comodules

Let  $I$  be a composition of  $N$ . The element  $\eta_I = T_{\omega(\bar{I})} \square_{\alpha(I \sim)}$  of  $H_N(0)$  generates the one-dimensional left  $H_N(0)$ -module  $\mathbf{C}_I$ . One can use it to construct the  $A_0(n)$ -comodule

$$\mathbf{D}_I = V^{\otimes N} \cdot \eta_I.$$

Let  $A$  be a noncommutative totally ordered alphabet and let  $I$  be a composition. We denote by  $F_I(A)$  the sum of all quasi-ribbon words of shape  $I$ . According to a result of Gessel [10], the commutative image of  $F_I(A)$  is the quasi-symmetric function  $F_I$ .



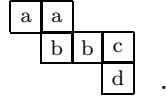
Thus  $F_{21}(a, b, c) = aba + aca + acb + bcb$ . The commutative image of  $F_{21}(a, b, c)$  is clearly equal to  $M_{21} + M_{111} = F_{21}$ , as desired.

**Proposition 6.8** *Let  $I$  be a composition of  $N$ . Then  $\chi(\mathbf{D}_I) = F_I(x_{11}, \dots, x_{nn})$ .*

*Proof* — Let  $QR(I)$  be the set of all quasi-ribbon words of shape  $I$ . We associate with every word  $w = a_{k_1} \dots a_{k_n}$  of  $A^*$  the tensor  $\mathbf{w} = a_{k_1} \otimes \dots \otimes a_{k_n}$  of  $V^{\otimes n}$ .

**Lemma 6.9** *The family  $(\mathbf{w} \cdot \eta_I)_{w \in QR(I)}$  is a linear basis of the  $A_0(n)$ -comodule  $\mathbf{D}_I$ .*

*Proof of the lemma* – Define the natural reading  $n(r)$  of a quasi-ribbon tableau  $r$  of shape  $I$  as the word obtained by reading  $r$  from left to right. If  $w$  is the quasi-ribbon word associated with  $r$ , we also denote by  $n(w)$  the natural reading of  $r$ . For example,  $n(ababdc) = aabbcd$  is the natural reading of the quasi-ribbon tableau



Let now  $i \in D(I)$ . By definition of  $\eta_I$ , one has  $T_i \eta_I = -\eta_I$ . Hence, we get

$$\mathbf{v} \cdot \eta_I = -(\mathbf{v} \cdot T_i) \cdot \eta_I = \begin{cases} 0 & \text{if } k_i = k_{i+1}, \\ -\mathbf{v}^{\sigma_i} \cdot \eta_I & \text{if } k_i < k_{i+1}, \end{cases}$$

for every  $\mathbf{v} = a_{k_1} \otimes \dots \otimes a_{k_N} \in V^{\otimes N}$ . In particular,

$$\mathbf{v} \cdot \eta_I = -\mathbf{v}^{\sigma_i} \cdot \eta_I \quad (10)$$

when  $k_i \neq k_{i+1}$ . Suppose now that  $i \notin D(I)$ . Then  $T_i \eta_I = 0$ . Thus we can write

$$\mathbf{v} \cdot \eta_I = (\mathbf{v}^{\sigma_i} \cdot T_i) \cdot \eta_I = \mathbf{v}^{\sigma_i} \cdot T_i \eta_I = 0$$

for every  $\mathbf{v} = a_{k_1} \otimes \dots \otimes a_{k_N} \in V^{\otimes N}$  such that  $k_i > k_{i+1}$ . It follows that the family of all tensors of the form  $(a_{k_1} \otimes \dots \otimes a_{k_N}) \cdot \eta_I$  with  $k_i \leq k_{i+1}$  when  $i \notin D(I)$  and  $k_i < k_{i+1}$  when  $i \in D(I)$  spans the comodule  $\mathbf{D}_I$ . In other words, we get a generating family of  $\mathbf{D}_I$  by taking the set  $\mathcal{R}$  formed of all  $\mathbf{w} \cdot \eta_I$  where  $w$  runs over the natural readings of all quasi-ribbon tableaux of shape  $I$ . Moreover it is easy to see that these elements are not zero.

Now, there is at most one increasing word of a given evaluation which can be the natural reading of some quasi-ribbon tableau of shape  $I$ . It follows that  $\mathcal{R}$  is a linear basis of  $\mathbf{D}_I$ . Finally, formula (10) shows that  $\mathbf{w} \cdot \eta_I = \pm \mathbf{n}(\mathbf{w}) \cdot \eta_I$  for a quasi-ribbon word  $w$  of shape  $I$ .  $\square$

We are now in position to compute  $\chi(\mathbf{D}_I)$ . In the notation of Section 4.3,

$$\delta(\mathbf{w}) = \sum_{|u|=|w|} (w \otimes u^*) \otimes \mathbf{u}$$

for every  $w \in A^*$ . Hence, according to Proposition 4.2,

$$\delta(\mathbf{w} \cdot \eta_I) = \sum_{|u|=|w|} (w \otimes u^*) \otimes \mathbf{u} \cdot \eta_I,$$

$$\chi(D_I) = \sum_{w \in QR(I)} \left( \sum_{\substack{|u|=|w| \\ \mathbf{u} \cdot \eta_I = \mathbf{w} \cdot \eta_I}} w \otimes u^* \right). \quad (11)$$

Let now  $w$  be a quasi-ribbon word of shape  $I$ . Let also  $u = a_{k_1} \dots a_{k_N}$  be a word distinct from  $w$  such that  $\mathbf{w} \cdot \eta_I = \mathbf{u} \cdot \eta_I$ . Let  $r(u)$  be the ribbon diagram of shape  $I$  obtained by filling the boxes from left to right by the letters of  $u$ . Let then  $r'(u)$  be the quasi-ribbon tableau of shape  $I$  obtained from  $r(u)$  by sorting all columns in increasing order from top to bottom. Let us finally denote by  $v(u)$  the word obtained by reading from left to right the letters of  $r'(u)$ . Using again the arguments of the proof of Lemma 6.9, we see that  $\mathbf{v}(\mathbf{u}) \cdot \eta_I = 0$  if  $v(u)$  is not the natural reading of a quasi-ribbon tableau of shape  $I$ . It follows that the alphabets of all columns of  $r(u)$  and  $r(w)$  must coincide. Since  $u \neq w$ , there must exist integers  $i < j$  and  $k < l$  such that

$$w \otimes u^* = \begin{bmatrix} \dots & l & k & \dots \\ \dots & i & j & \dots \end{bmatrix}$$

which is therefore equal to 0. Hence, we have

$$\sum_{\substack{|u|=|w| \\ \mathbf{u} \cdot \eta_I = \mathbf{w} \cdot \eta_I}} w \otimes u^* = w \otimes w^*.$$

Going back to formula (11), we see that

$$\chi(\mathbf{D}_I) = \sum_{w \in QR(I)} w \otimes w^* = F_I(x_{11}, \dots, x_{nm}). \quad \square$$

**Note 6.10** The same argument as in Note 6.5 shows that the noncommutative quasi-ribbon functions  $F_I(A)$  span a commutative subalgebra of the hypoplactic algebra  $HPl(A)$ , isomorphic to the algebra of quasi-symmetric functions over a commutative alphabet of the same cardinality as  $A$ . This property can in fact also be proved in a purely combinatorial way.

**Example 6.11** Let  $n = 3$ ,  $N = 4$  and  $I = (3, 1)$ . Then  $\eta_{31} = T_3 T_2 T_1 (1 + T_2) (1 + T_3) (1 + T_2)$  and  $\mathbf{D}_{31} = V^{\otimes 4} \cdot \eta_{31}$ . By computing the images under  $\eta_{31}$  of the canonical basis vectors of  $V^{\otimes 4}$ , one gets

$$\mathbf{D}_{31} = \mathbb{C} a_1 a_2 a_3 a_2 \cdot \eta_{31} \oplus \mathbb{C} a_2 a_2 a_3 a_2 \cdot \eta_{31} \oplus \mathbb{C} a_1 a_1 a_3 a_2 \cdot \eta_{31} \oplus \mathbb{C} a_1 a_1 a_3 a_1 \cdot \eta_{31} \oplus \mathbb{C} a_1 a_1 a_2 a_1 \cdot \eta_{31}.$$

Thus,

$$\begin{aligned} \chi(\mathbf{D}_{31}) &= x_{22} x_{11} x_{11} x_{11} + x_{33} x_{11} x_{11} x_{11} + x_{33} x_{22} x_{22} x_{22} + x_{33} x_{11} x_{11} x_{22} + x_{33} x_{11} x_{22} x_{22} \\ &= x_{11} x_{11} x_{22} x_{11} + x_{11} x_{11} x_{33} x_{11} + x_{22} x_{22} x_{33} x_{22} + x_{11} x_{11} x_{33} x_{22} + x_{11} x_{22} x_{33} x_{22}. \end{aligned}$$

This last expression is exactly the sum of the quasi-ribbons words associated with the five quasi-ribbon tableaux

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline & & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline & & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline & & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline & & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline & & 3 \\ \hline \end{array}$$

Hence  $\chi(\mathbf{D}_{31}) = F_{31}(x_{11}, x_{22}, x_{33})$  as desired.

**Proposition 6.12** *The  $\mathbf{D}_I$  are irreducible, pairwise non-isomorphic  $A_0(n)$ -comodules.*

to prove that these comodules are irreducible. Let  $I$  be a composition of  $N$  and let  $M$  be a nonzero subcomodule of  $\mathbf{D}_I$ . According to Lemma 6.9, there exists a family  $R$  of quasi-ribbon words of shape  $I$  and a family  $(m_w)_{w \in R}$  of nonzero complex numbers such that

$$m = \sum_{w \in R} m_w \mathbf{w} \cdot \eta_I \in M .$$

Using the tensor formalism of Section 4.3, it follows that

$$\delta(m) = \sum_{|u|=N} \left( \sum_{w \in R} m_w w \otimes u^* \right) \otimes \mathbf{u} \cdot \eta_I \in A_0(n) \otimes M .$$

As there is at most one quasi-ribbon word of shape  $I$  and of a given evaluation, we deduce by homogeneity with respect to the first component of  $A_0(n)$  that

$$\delta(w) = \sum_{|u|=N} (w \otimes u^*) \otimes \mathbf{u} \cdot \eta_I \in A_0(n) \otimes M$$

for every  $w \in R$ . Let now  $w$  be an arbitrary quasi-ribbon word of  $R$  and let  $u = a_{k_1} \dots u_{k_N}$  be a word of length  $N$ . Note that  $w \otimes u^* = 0$  if  $k_i \leq k_{i+1}$  when  $i \in D(I)$ . On the other hand,  $\mathbf{u} \cdot \eta_I = 0$  if  $k_i > k_{i+1}$  and  $i \notin D(I)$  according to the proof of Lemma 6.9. Hence

$$\delta(w) = \sum_{u \in QR(I)} (w \otimes u^*) \otimes \mathbf{u} \cdot \eta_I .$$

The monomials  $w \otimes u^*$ , where  $u$  runs over all quasi-ribbon words of shape  $I$ , are nonzero and pairwise distinct elements of  $A_0(n)$ . Since  $\delta(w) \in A_0(n) \otimes M$ , all the tensors  $\mathbf{u} \cdot \eta_I$  are  $M$ . According to Lemma 6.9, this shows that  $M = \mathbf{D}_I$ .  $\square$

**Conjecture 6.13** *The family  $(\mathbf{D}_I)_I$  (where  $I$  runs through all compositions) is a complete system of irreducible  $A_0(n)$ -comodules.*

**Note 6.14** Conjecture 6.13 would imply that the character of every  $A_0(n)$ -comodule is an element of  $\Delta_0(n)$ , showing therefore that Theorem 6.2 is still valid when  $q = 0$ . Moreover according to Note 6.10, it would also imply that the character ring  $Char_{\mathbb{Z}}(0; n)$  is isomorphic to the  $\mathbb{Z}$ -algebra of quasi-symmetric functions over an  $n$ -letter alphabet.  $\square$

### 6.3 Another family of $A_0(n)$ -comodules

Let  $I$  be a composition of  $N$ . The element  $\nu_I = T_{\alpha(I)} \square_{\alpha(\bar{I}^{\sim})}$  of the 0-Hecke algebra generates the indecomposable projective left  $H_n(0)$ -module  $\mathbf{M}_I$ . One can also use to construct the  $A_0(n)$ -comodule  $\mathbf{N}_I$  defined by

$$\mathbf{N}_I = V^{\otimes N} \cdot \nu_I .$$

A word will be said to be of *ribbon shape*  $I$  (where  $I$  is a composition) if it can be obtained by reading from left to right and from top to bottom the columns of a skew Young tableau of ribbon shape  $I$ . We denote by  $R_I(A)$  the sum of all words of  $A^*$  of ribbon shape  $I$ .

**Proposition 6.15** *Let  $I$  be a composition of  $N$ . Then  $\chi(\mathbf{N}_I) = R_I(x_{11}, \dots, x_{nn})$ .*

*Proof* — We use the same notation as in the proof of Proposition 6.8. Let also  $R(I)$  be the set of all words of ribbon shape  $I$ .

*Proof of the lemma* – Note that  $T_i \nu_I = -\nu_I$  for  $i \in D(I)$ . It follows that

$$\mathbf{v} \cdot \nu_I = -(\mathbf{v} \cdot T_i) \nu_I = \begin{cases} 0 & \text{if } k_i = k_{i+1} , \\ -\mathbf{v}^{\sigma_i} \cdot \nu_I & \text{if } k_i < k_{i+1} , \\ \mathbf{v} \cdot \nu_I & \text{if } k_i > k_{i+1} , \end{cases}$$

for  $i \in D(I)$  and  $\mathbf{v} = a_{k_1} \otimes \dots \otimes a_{k_N} \in V^{\otimes N}$ . Hence we can rewrite (up to a sign) every  $\mathbf{v} \cdot \nu_I$  in such a way that  $k_i > k_{i+1}$  for  $i \in D(I)$ . The structure of the right action of  $H_N(0)$  on  $V^{\otimes N}$  implies that such an element is equal to  $\mathbf{w} \cdot \square_{\alpha(\bar{I}^\sim)}$  where we still have  $w = a_{k_1} \dots a_{k_N}$  with  $k_i > k_{i+1}$  for  $i \in D(I)$ . Let now  $i \notin D(I)$ . Then  $\square_{\alpha(\bar{I}^\sim)} = \square_i \square_{\alpha(\bar{I}^\sim)}$ . Hence

$$\mathbf{w} \cdot \square_{\alpha(\bar{I}^\sim)} = \mathbf{w} \cdot \square_i \square_{\alpha(\bar{I}^\sim)} = \mathbf{w}^{\sigma_i} \cdot T_i \square_i \square_{\alpha(\bar{I}^\sim)} = 0$$

when  $k_i > k_{i+1}$ . Every  $\mathbf{v} \cdot \nu_I$  can therefore be rewritten as  $\pm \mathbf{w} \cdot \square_{\alpha(\bar{I}^\sim)}$  where  $w \in R(I)$ . In other words, the family  $(\mathbf{w} \cdot \square_{\alpha(\bar{I}^\sim)})_{w \in R(I)}$  spans  $\mathbf{N}_I$ .

Now, it follows from (8) that

$$V^{\otimes N} = \sum_{I \vdash N} \mathbf{N}_I . \quad (12)$$

Since any word of  $A^N$  has a unique ribbon shape, we deduce that

$$\sum_{I \vdash N} |R(\bar{I})| = \dim V^{\otimes N} \leq \sum_{I \vdash N} \dim \mathbf{N}_I \leq \sum_{I \vdash N} |R(\bar{I})|$$

from which we get that  $\dim \mathbf{N}_I$  is equal to the number of words of ribbon shape  $I$ .  $\square$

This argument also shows that decomposition (12) is in fact a direct sum. Arguing as in the proof of Proposition 6.8, we see that

$$\delta(w \cdot \square_{\alpha(\bar{I}^\sim)}) = \sum_{u \in R(\bar{I})} (w \otimes u^*) \mathbf{u} \cdot \square_{\alpha(\bar{I}^\sim)}$$

for  $w \in R(I)$ , whence the theorem.  $\square$

**Example 6.17** Let  $n = 3$ ,  $N = 4$  and  $I = (1, 1, 2)$ . Then  $\nu_{112} = T_1 T_2 T_1 (1 + T_3)$  and  $\mathbf{N}_{112} = V^{\otimes 4} \cdot \nu_{112}$ . By computing the action of  $\nu_{211}$  on the standard basis of  $V^{\otimes 4}$ , one gets

$$\mathbf{N}_{112} = \mathbb{C} a_3 a_2 a_1 a_1 \cdot \nu_{112} \oplus \mathbb{C} a_3 a_2 a_1 a_2 \cdot \nu_{112} \oplus \mathbb{C} a_3 a_2 a_1 a_3 \cdot \nu_{112} .$$

Then,

$$\chi(\mathbf{N}_{112}) = x_{33} x_{22} x_{11} x_{11} + x_{33} x_{22} x_{11} x_{22} + x_{33} x_{22} x_{11} x_{33} .$$

This expression is the sum of the ribbon words associated with the 3 ribbon tableaux

$$\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 & 3 \\ \hline \end{array}$$

and  $\chi(\mathbf{N}_{112}) = R_{112}(x_{11}, x_{22}, x_{33})$  as desired.  $\square$

**Note 6.18** Using the same kind of argument as in Section 6.2, one can prove that  $\mathbf{N}_I$  is an indecomposable  $A_n(q)$ -comodule.  $\square$

In the classical case (corresponding to  $q = 1$ ), the Robinson-Schensted correspondence is the combinatorial counterpart of the decomposition of  $V^{\otimes N}$  into  $GL_n(\mathbb{C}) \times \mathfrak{S}_N$ -bimodules. On the other hand, for  $q = 0$ , there are two natural ways of decomposing  $V^{\otimes N}$  into  $A_0(n) \times H_N(0)$ -bicomodules. This leads to two different Robinson-Schensted type correspondences, involving here ribbon and quasi-ribbon diagrams.

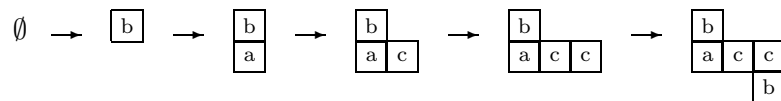
## 7.1 A first Robinson-Schensted type correspondence

The first combinatorial algorithm corresponds to the decomposition

$$V^{\otimes N} = \bigoplus_{I \vdash N} \mathbf{N}_I \quad (13)$$

(cf. the proof of Proposition 6.15). Recall that any right  $H_N(0)$ -submodule of  $V^{\otimes N}$  can also be regarded as a left module, the action being given by  $\mathbf{v}T_i = -\square_i \mathbf{v}$ . It follows then from Lemma 6.16 that  $\mathbf{N}_I$  is a left  $H_N(0)$ -module whose all composition factors are equal to  $\mathbf{C}_{\tilde{T}}$ . This observation gives us a basis of  $V^{\otimes N}$  indexed by pairs  $(r, qr)$  where  $r$  is a word of ribbon shape  $I$  and where  $qr$  is the (unique) standard quasi-ribbon word of shape  $\tilde{T}$ . The corresponding Robinson-Schensted map is therefore trivial. It just associates to a word  $w$  its ribbon diagram. It can clearly be recursively defined by an insertion process as follows.

Let  $r$  be the ribbon diagram of  $w$ , let  $x$  be the letter which is in the last box of  $r$  and let  $a \in A$ . The ribbon diagram of  $wa$  is then obtained from  $r$  by glueing  $a$  at the end of the last row of  $r$  if  $x \leq a$  or under the last box of the last row of  $r$  if  $a < x$ . For exemple, with  $w = bacb$ , we have



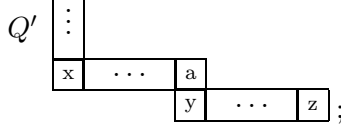
This construction is clearly bijective (the standard quasi-ribbon does not bring here any supplementary information).

## 7.2 A second Robinson-Schensted type correspondence

The second Robinson-Schensted type algorithm is related to the composition factors of  $V^{\otimes N}$ . Using Lemmas 6.9 and 6.16, one can see that these compositions factors are exactly the comodules  $\mathbf{D}_I$  each of them occuring  $|QR(I)|$  times. But  $\mathbf{D}_I$  considered as a left  $H_N(0)$ -module is isomorphic to  $\mathbf{M}_I$ . It follows that there exists a basis of  $V^{\otimes N}$  indexed by pairs  $(Q, R)$  where  $Q$  is a quasi-ribbon word of shape  $I$  and where  $R$  is a standard ribbon word of the same shape. The corresponding Robinson-Schensted type algorithm which associates to each word  $w \in A^*$  the pair  $(Q, R)$  is described below.

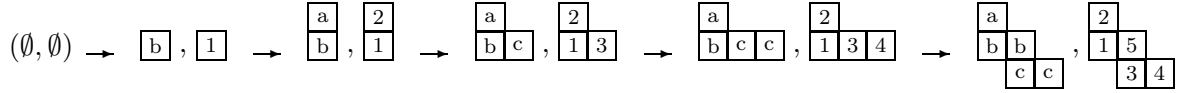
Let  $Q$  be a quasi-ribbon diagram and let  $a \in A$ . Let  $Q'$  be the diagram obtained from  $Q$  by deleting its last row and let  $x$  (resp.  $z$ ) be the first (resp. last) letter of the last row of  $Q$ . The result  $\mathcal{Q}$  of the insertion of  $a$  in  $Q$  is defined by the following rules:

- if  $z \leq a$ ,  $\mathcal{Q}$  is obtained by adding a box containing  $a$  at the end of the last row of  $Q$
- if  $x \leq a < z$ , let  $y$  be the first letter of the last row of  $Q$  which is strictly greater than  $a$ . The quasi-ribbon diagram  $\mathcal{Q}$  is then



- if  $a < x$ ,  $Q$  is obtained by inserting  $a$  in  $Q'$  and glueing under the quasi-ribbon obtained in this way the last row of  $Q$ .

Let  $w = a_1 \dots a_n$  be a word of length  $n$ . The pair  $(Q, R)$  associated with  $w$  can be defined as follows. The quasi-ribbon diagram  $Q$  is obtained by inserting the letters of  $w$  (from left to right), starting from an empty diagram. The standard ribbon diagram  $R$  is iteratively constructed by putting at each step  $i \in [1, n]$  of the algorithm the number  $i$  in the box that contains at this moment in  $Q$  the letter  $a_i$  inserted at this step. Let us illustrate again this correspondence on  $w = baccb$ .



The correspondence  $w \rightarrow (Q, R)$  is clearly a bijection. In fact, the quasi-ribbon diagram  $Q$  associated with  $w$  is of shape  $C(\sigma^{-1})$  where  $\sigma = std(w)$ . Going back to Proposition 4.21, this gives the following property, which is the quasi-ribbon version of Knuth's theorem [20].

**Proposition 7.1** *Let  $u, v \in A^*$ . Then,  $u$  and  $v$  correspond to the same quasi-ribbon  $Q$  under the second algorithm iff  $u \equiv v$  with respect to the hypoplactic congruence.*

In other words, the hypoplactic relations play, for quasi-ribbons, the same rôle as the plactic relations for Young tableaux.

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