



HAL
open science

Multifractal spectra and precise rates of decay in homogeneous fragmentation.

Nathalie Krell

► **To cite this version:**

Nathalie Krell. Multifractal spectra and precise rates of decay in homogeneous fragmentation.. 2007.
hal-00018531v2

HAL Id: hal-00018531

<https://hal.science/hal-00018531v2>

Preprint submitted on 12 Jan 2007 (v2), last revised 3 Jul 2008 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Multifractal spectra and precise rates of decay in homogeneous fragmentations.

Nathalie Krell

January 12, 2007

*Laboratoire de Probabilités et Modèles Aléatoires,
Université Pierre et Marie Curie,
175 rue du Chevaleret, 75013 Paris, France.*

Abstract

We consider a mass-conservative fragmentation of the unit interval. Motivated by a result of Berestycki [3], the main purpose of this work is to specify the Hausdorff dimension of the set of locations having exactly an exponential decay. The study relies on an additive martingale which arises naturally in this setting, and a class of Lévy processes constrained to stay in a finite interval.

Key Words. Interval fragmentation, Lévy process, Multifractal spectrum.

A.M.S. Classification. 60G09, 60J25, 60J80.

e-mail. krell@ccr.jussieu.fr

1 Introduction.

Fragmentation appears in a wide range of phenomena in science and technology, such as degradation of polymers, colloids, droplets, rocks,... See the proceedings [12] for some applications in physics, for example [17] for computer science, [10] for mineral crushing, and works quoted in [3] for some further references. This work is a contribution of the study of the rates of decay of fragments. More precisely, our aim is to investigate the set of locations which have an exact exponential decay (see (1) below for a precise definition).

Roughly a homogeneous fragmentation of intervals $F(t)$ can be seen as a family of nested open sets in $]0, 1[$ such that each interval component is spilled independently of the others, independently of the way that spilled before, and with the same law as for the initial fragmentation (up to spatial rescaling). We will suppose that no loss of mass occurs during the process.

Let $x \in]0, 1[$ and $I_x(t)$ be the interval component of the fragmentation $F(t)$ which contains x , and $|I_x(t)|$ his length. Bertoin showed in [7] that if V is a uniform random variable on $[0, 1]$ which is independent of the fragmentation, then $\xi_t := -\log |I_V(t)|$ is a subordinator entirely determined by the fragmentation characteristics. By the SLLN for a subordinator, there exists v_{typ} such that $\frac{\xi_t}{t} \rightarrow v_{typ}$ *a.s.*, which means that $|I_V(t)| \approx e^{-v_{typ}t}$. Berestycki [3] computed the Hausdorff dimension of the set

$$G_v := \left\{ x \in (0, 1) : \lim_{t \rightarrow \infty} \frac{1}{t} \log |I_x(t)| = -v \right\}$$

for all $v > 0$. In this article we shall rather consider for some $0 < a < b$ the set

$$G_{(v,a,b)} := \left\{ x \in (0, 1) : a \leq \liminf_{t \rightarrow \infty} e^{vt} |I_x(t)| \leq \limsup_{t \rightarrow \infty} e^{vt} |I_x(t)| \leq b \right\}. \quad (1)$$

Our goal is to compute the Hausdorff dimension of the set $G_{(v,a,b)}$. Our approach relies on some results on Lévy processes constrained to stay in a given interval.

Firstly we will recall background on fragmentations and Lévy processes. Secondly we will consider an additive martingale M which is naturally associated to the problem and obtain a criterion for uniform integrability. This is used in Section 4 to derive some limit theorems which may be of independent interest (see Engländer and Kyprianou [14] for a related approach in the setting of spatial branching processes). Finally we will compute the Hausdorff dimension of $G_{(v,a,b)}$ in Section 5.

2 Preliminaries.

2.1 Definition of fragmentation.

We will recall some facts about homogeneous interval fragmentations, which are mostly lifted from [3], [7] and [8]. More precisely, we will consider fragmentations defined on the space \mathcal{U} of open subsets of $]0, 1[$. We shall use the fact that every element U of \mathcal{U} has an interval decomposition, i.e. there exists a collection of disjoint open intervals $(J_i)_{i \in I}$, where the set of indices I can be finite or countable, such that $U = \cup_{i \in I} J_i$. Each interval component is viewed as a fragment.

A homogeneous interval fragmentation is a Markov process with values in the space \mathcal{U} which enjoys two keys properties. First the branching property: different fragments have independent evolutions. Second, the homogeneity property: up to an obvious spacial rescaling, the law of the fragment process does not depend on the initial length of the interval.

Specifically, if \mathbb{P} stands for the law of the interval fragmentation F started from $F(0) =]0, 1[$, then for $s, t \geq 0$ conditionally on the open set $F(t) = \bigcup_{i \in I} J_i(t)$, the interval fragmentation $F(t+s)$ has the same law as $F^1(s) \cup F^2(s) \cup \dots$ where for each i , $F^i(s)$ is a subset of $J_i(t)$ and has the same distribution as the image of $F(s)$ by the homothetic map $]0, 1[\rightarrow J_i(t)$.

2.2 Poissonian construction of the fragmentation.

Recall that \mathcal{U} denotes the space of open subsets of $]0, 1[$, and set $\mathbf{1} =]0, 1[$. For $U \in \mathcal{U}$,

$$|U|^\downarrow := (u_1, u_2, \dots)$$

will be the decreasing sequence of the interval component lengths of U . For $U =]a_1, b_1[\in \mathcal{U}$, we define the affine transformation $g_U :]0, 1[\rightarrow U$ given by $g_U(x) = a_1 + x(b_1 - a_1)$.

In this article we will only consider proper fragmentations (which means that the Lebesgue measure of $F(t)$ is equal to 1). In this case, Basdevant [1] has shown that the law of the interval fragmentation F is completely characterized by the so-called dislocation measure ν (corresponding to the jump-component of the process) which is a measure on \mathcal{U} which fulfills the conditions

$$\nu(\mathbf{1}) = 0,$$

$$\int_{\mathcal{U}} (1 - u_1) \nu(dU) < \infty, \quad (2)$$

and

$$\sum_{i=1}^{\infty} u_i = 1 \quad \text{for } \nu\text{-almost every } U \in \mathcal{U}.$$

This last assumption is imposed by the hypothesis of length-conservation and means that when a sudden dislocation occurs, the total length of the intervals is unchanged. Specialists will notice that the erosion rates of the fragmentation c_r and c_l are here equal to 0 for the same reason.

We now recall the interpretation of sudden dislocations of the fragmentation process in terms of atoms of a Poisson point process (see [1], [2]). Let ν be a dislocation measure fulfilling the preceding conditions. Let $K = ((\Delta(t), k(t)), t \geq 0)$ be a Poisson point process with values in $\mathcal{U} \times \mathbb{N}$, and with intensity measure $\nu \otimes \sharp$, where \sharp is the counting measure on \mathbb{N} . As in [2], we can construct a unique \mathcal{U} -valued process $F = (F(t), t \geq 0)$ started from $]0, 1[$, with paths that jump only for times $t \geq 0$ at which a point $(\Delta(t), k(t))$ occurs, and then $F(t)$ is obtained by replacing the $k(t)$ -interval $J_{k(t)}(t-)$ by $g_{J_{k(t)}(t-)}(\Delta(t))$. This point of view will be used in Section 3.

Some information about the dislocation measure ν and therefore about the distribution of the homogeneous fragmentation F is contained by the function:

$$\kappa(q) := \int_{\mathcal{U}} \left(1 - \sum_{j=1}^{\infty} u_j^{q+1} \right) \nu(dU) \quad \forall q > \underline{p} \quad (3)$$

with \underline{p} the smallest real number for which κ remains finite :

$$\underline{p} := \inf \left\{ p \in \mathbb{R} : \int_{\mathcal{U}} \sum_{j=2}^{\infty} u_j^{p+1} \nu(dU) < \infty \right\}.$$

We have that $-1 \leq \underline{p} \leq 0$ (because $\int_{\mathcal{U}} (1 - u_1) \nu(dU) < \infty$ and $\sum_{i=1}^{\infty} u_i = 1$ for ν -almost every $U \in \mathcal{U}$).

This point of view is the same as in [3] and [7], which deal with ranked fragmentation instead of interval fragmentation. In the latter the space \mathcal{U} is replaced by the space of mass partitions

$$\mathcal{S}^\dagger := \left\{ x = (x_1, x_2, \dots) \mid x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}.$$

For the precise link between these two fragmentations see [1].

2.3 An important subordinator.

Let $x \in]0, 1[$ and $I_x(t)$ be the interval component of the random open set $F(t)$ which contains x , and $|I_x(t)|$ its length. Let V be a uniform random variable on $[0, 1]$ which is independent of the fragmentation.

Bertoin showed in [7] that

$$\xi_t := -\log |I_V(t)|, \quad t \geq 0, \tag{4}$$

is a subordinator, with Laplace exponent $\kappa(q)$ defined in (3) (i.e. $\mathbb{E}(e^{-\lambda \xi_t}) = e^{-t\kappa(\lambda)}$ for all $q > \underline{p}$). In order to interpret this as a Lévy-Khintchine formula, we introduce the measure

$$L(dx) := e^{-x} \sum_{j=1}^{\infty} \nu(-\log u_j \in dx), \quad x \in]0, \infty[.$$

It is easy to check that $\int \min(1, x) L(dx) < \infty$, thus L is the Lévy measure of a subordinator, and we can check that $\kappa(q) = \int_{]0, \infty[} (1 - e^{-qx}) L(dx)$.

In this article we shall consider the Lévy process $Y_t = vt - \xi_t$. In order to apply certain results to this process, we will need to assume that its one-dimensional distributions are absolutely continuous. Let L^{ac} be the absolutely continuous part of the measure L . Tucker has shown in [21] that

$$\int_{\mathbb{R}_+} \frac{1}{1+x^2} L^{ac}(dx) = \infty,$$

ensures the absolute continuity of one-dimensional distribution of the Lévy process evaluated at any $t > 0$. As $\int \min(1, x) L(dx) < \infty$, this latter is equivalent to :

$$L^{ac}([0, \epsilon]) = \infty \quad \text{for any } \epsilon > 0.$$

Let ν_1 be the image of the measure ν by the map $U \rightarrow u_1$ (recall that u_1 is the length of the longest interval component of the open set U) and ν_1^{ac} be the absolutely continuous part of the measure ν_1 . Throughout this work we will make the following assumption, which is easily seen to imply the previous one (in fact we can even show that the two are equivalents):

$$\nu_1^{ac}([0, \epsilon]) = \infty \quad \text{for any } \epsilon > 0. \quad (5)$$

In the next subsection, we will give some results about Lévy processes that will be needed in the sequel, and apply for $Y_t = vt - \xi_t$.

2.4 An estimate for completely asymmetric Lévy processes.

For the next sections, we will need some technical notions about completely asymmetric Lévy processes. Therefore we recall some facts mostly lifted from [4] and [6]. Let $Y = (Y_t)_{t \geq 0}$ be a Lévy process with no positive jumps and $(\mathcal{E}_t)_{t \geq 0}$ the natural filtration associated to $(Y_t)_{t \geq 0}$. The case where Y is the negative of a subordinator is degenerate for our purpose and therefore will be implicitly excluded in the rest of the article. The law of the Lévy process started at $x \in \mathbb{R}$ will be denoted by \mathbf{P}_x (so bold symbols \mathbf{P} and \mathbf{E} refer to the Lévy process while \mathbb{P} and \mathbb{E} refer to the fragmentation), its Laplace transform is given by

$$\mathbf{E}_0(e^{\lambda Y_t}) = e^{t\psi(\lambda)}, \quad \lambda, t \geq 0,$$

where $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is called the Laplace exponent.

Let $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the right inverse of ψ (which exists because $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is convex with $\lim_{t \rightarrow \infty} \psi(\lambda) = \infty$), i.e. $\psi(\phi(\lambda)) = \lambda \quad \forall \lambda \geq 0$.

Let us recall some important features on the two-sided exit problem (which is completely solved in [6]). For $\beta > 0$ we denote the first exit time from $]0, \beta[$ by

$$T_\beta = \inf\{t : Y_t \notin]0, \beta[\}. \quad (6)$$

Let $W : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the scale function, that is the unique continuous function with Laplace transform:

$$\int_0^\infty e^{-\lambda x} W(x) dx = \frac{1}{\psi(\lambda)} \quad , \quad \lambda > \phi(0).$$

For $q \in \mathbb{R}$, let $W^{(q)} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the continuous function such that for every $x \in \mathbb{R}_+$

$$W^{(q)}(x) := \sum_{k=0}^{\infty} q^k W^{*k+1}(x),$$

where $W^{*n} = W * \dots * W$ denotes the n th convolution power of the function W (for more details about this see [4] or [6]). So that

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\psi(\lambda) - q} \quad , \quad \lambda > \phi(q).$$

The next statement is about the asymptotic behavior of the Lévy process killed when it exits $]0, \beta[$ (point 1 and 2), which is taken from [6], and about the Lévy process conditioned to remain in $]0, \beta[$ (point 3, 4 and 5), which is taken from Theorem 3.1 (ii) and Proposition 5.1 (i) and (ii) in [18] :

Theorem 1 *Let us define the transition probabilities*

$$\bullet P_t(x, A) := \mathbf{P}_x(Y_t \in A, t < T_\beta) \text{ for } x \in]0, \beta[\text{ and } A \in \mathcal{B}(]0, \beta[),$$

and the critical value

$$\bullet \rho_\beta := \inf\{q \geq 0 ; W^{(-q)}(\beta) = 0\}, \quad (7)$$

Suppose that the one-dimensional distributions of the Lévy process are absolutely continuous. Then the following holds:

1. $\rho_\beta \in]0, \infty[$ and the function $W^{(-\rho_\beta)}$ is strictly positive on $]0, \beta[$

2. For every $x \in]0, \beta[$:

$$\lim_{t \rightarrow \infty} e^{\rho_\beta t} P_t(x, \cdot) = c W^{(-\rho_\beta)}(x) \Pi(\cdot)$$

in the sense of weak convergence, where

$$c := \left(\int_0^\beta W^{(-\rho_\beta)}(y) W^{(-\rho_\beta)}(\beta - y) dy \right)^{-1}.$$

3. The process

$$D_t := e^{\rho_\beta t} \mathbf{1}_{\{t < T_\beta\}} \frac{W^{(-\rho_\beta)}(Y_t)}{W^{(-\rho_\beta)}(x)} \quad (8)$$

is a $(\mathbf{P}_x, (\mathcal{E}_t))$ -martingale.

4. The mapping $(x, q) \mapsto W^{(q)}(x)$ is of class \mathcal{C}^1 on $]0, \infty[\times]-\infty, \infty[$.

5. The mapping $\beta \mapsto \rho_\beta = \inf\{q > 0 : W^{(-q)}(\beta) = 0\}$ is strictly decreasing and of class \mathcal{C}^1 on $]0, \infty[$.

Remark 1 • The definition of ρ_β is of course complicated, however in the simple case when Y is a standard Brownian motion, we have:

$$\rho_\beta = \pi^2/\beta^2 \quad \text{and} \quad W^{(-\rho_\beta)}(x) = \frac{\beta}{\pi} \sin\left(\frac{\pi}{\beta}x\right).$$

In the case where Y is a standard stable process, the mapping of $\beta \rightarrow \rho_\beta$ is depicted in [5]. We also point at the more explicit lower bound (see Lemma 5 in [6]):

$$\rho_a \geq 1/W(a),$$

Another lower bound will be given in Remark 3 below.

- The formula for the constant c in 5. stems from the relation

$$e^{\rho\beta t} \frac{W^{(-\rho\beta)}(y)}{W^{(-\rho\beta)}(x)} P_t(x, dy) \underset{t \rightarrow \infty}{\sim} c W^{(-\rho\beta)}(\beta - y) W^{(-\rho\beta)}(y) dy.$$

Integrating over $]0, \beta[$ and using the fact that D_t is a martingale yields the given expression.

We also refer to the recent article of T. Chan and A. Kyprianou [13] for further properties of $W^{(-\rho\beta)}$.

Now we have recalled the background that is needed to solve our problem.

3 An additive martingale.

Now we deal with the aim of this article and consider a homogeneous interval fragmentation $(F(t), t \geq 0)$ and real number $v > 0$ and $0 < a < b$. We are interested in the asymptotic set:

$$G_{(v,a,b)} = \left\{ x \in (0, 1) : a \leq \liminf_{t \rightarrow \infty} e^{vt} |I_x(t)| \leq \limsup_{t \rightarrow \infty} e^{vt} |I_x(t)| \leq b \right\},$$

with $|I_x(t)|$ the length of the interval component of $F(t)$ which contains x .

In order to do that, we will have to consider first the non asymptotic set:

$$\Lambda_{(v,a,b)} = \left\{ x \in (0, 1) : ae^{-vt} < |I_x(t)| < be^{-vt} \forall t \geq 0 \right\},$$

for $0 < a < 1 < b$.

In this section and in the next we will assume that $0 < a < 1 < b$.

We introduce some notation, that we will need in the rest of the article: defined the set of the 'good' intervals at time t as

$$G(t) := \{I_x(t) : x \in]0, 1[\text{ and } ae^{-vs} < |I_x(s)| < be^{-vs} \forall s \leq t\}. \quad (9)$$

Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of the interval fragmentation $(F(t), t \geq 0)$. Let $(\mathcal{G}_t)_{t \geq 0}$ be the enlarged filtration defined by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(I_V(t))$ where V is a uniform variable independent of the fragmentation). We can remark that for all t $\mathcal{G}_t \subsetneq \mathcal{F}_t \vee \sigma\{V\}$, and $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma\{V\}$.

We recall that $\xi_t = -\log |I_V(t)|$ is a subordinator. More precisely we are interested in the Lévy process with no positive jump $Y_t := vt - \xi_t$, and use the results of preceding subsection for this Lévy process. We remark that its Laplace exponent $\psi(\lambda)$ is equal to $v\lambda - \kappa(\lambda)$, with κ defined in Subsection 2.3. Since we have supposed (5), the one-dimensional distributions of the Lévy process Y_t are absolutely continuous and we can apply Theorem 1.

For this Lévy process Y let

$$T := T_{\log(b/a)}$$

and

$$\rho := \rho_{\log(b/a)},$$

where T_β is defined in (6) and ρ_β is defined in (7). We stress that ρ depends on v , a , b and κ .

To simplify the notation, let also

$$h(t) := W^{(-\rho)}(t - \log a) \mathbf{1}_{\{t \in]\log a, \log b\}}$$

for all $t \in \mathbb{R}$, and $h(-\infty) = 0$.

In the notation (8) we have a (\mathcal{G}_t) -martingale

$$D_t = e^{\rho t} \mathbf{1}_{\{t < T\}} \frac{h(vt + \log |I_V(t)|)}{h(0)}, \quad t \geq 0.$$

If I is an interval component of $F(t)$, we define the 'killed' interval I^\dagger by $I^\dagger = I$ if I is good (i.e. $I \in G(t)$ with $G(t)$ defined in (9)), else by $I^\dagger = \emptyset$. Projecting the martingale D_t on the sub-filtration $(\mathcal{F}_t)_{t \geq 0}$, we obtain an additive martingale

$$M_t := \frac{e^{\rho t}}{h(0)} \int_0^1 h(vt + \log |I_x^\dagger(t)|) dx, \quad t \geq 0.$$

We notice that if $y \in I_x(t)$, then $I_y(t) = I_x(t)$. Now we will consider the interval decomposition $(J_1(t), J_2(t), \dots)$ of the open $F(t)$ (see subsection 2.1). We can rewrite M_t as:

$$M_t = \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} h\left(vt + \log |J_i^\dagger(t)|\right) |J_i^\dagger(t)|. \quad (10)$$

We will use this expression in the rest of the article.

Finally, let the absorption time of M_t at 0 be

$$\begin{aligned} \zeta &:= \inf\{t : M_t = 0\} \\ &= \inf\{t : G(t) = \emptyset\}, \end{aligned}$$

with the convention $\inf \emptyset = \infty$.

Our first result is (recall we assume (5)) :

Theorem 2 *In the previous notation, with the assumptions (5) and if $v > \rho$ holds, then:*

1. *The martingale M_t is bounded in $L^2(\mathbb{P})$.*
2. *Conditionally on $\zeta = \infty$, we have: $\lim_{t \rightarrow \infty} M_t > 0$.*

Remark 2 We stress that as ρ depends on v , a , b and κ , the condition $v > \rho$ involves implicitly the parameters a and b . In particular it forces $b > 2a$ (otherwise there would never be more than one 'good' interval, and as a consequence we would have $M_\infty = 0$ a.s., in contradiction with the uniform integrability of M .)

Proof of Theorem 2.1:

We will use the method of [19]. Because M is purely discontinuous, it is known that if the quadratic variation

$$V_2 := \sum_{t \geq 0} |M_t - M_{t-}|^2$$

of M is integrable, then M is bounded in $L^2(\mathbb{P})$.

The jumps of M can be expressed in terms of the points of $(\Delta(t), k(t))$ of the Poisson point process used in Subsection 2.2 to describe the sudden dislocations of the fragmentation. More precisely, we consider the case when the i^{th} interval dislocates at time t . Then

$$|M_t - M_{t-}|^2 = \left| \frac{e^{\rho t}}{h(0)} \sum_{j=1}^{\infty} \left[h \left(vt + \log \left(|J_i^\dagger(t-)| \Delta_j(t) \right) \right) - h \left(vt + \log |J_i^\dagger(t-)| \right) \right] \Delta_j(t) |J_i^\dagger(t-)| \right|^2,$$

where $\Delta_1(t) \geq \Delta_2(t) \geq \dots$ is the decreasing sequence of the length of the interval components of $\Delta(t)$.

By Hölder inequality for the probability $(\Delta_j(t))_{j \in \mathbb{N}}$, we get

$$\leq \frac{e^{2\rho t}}{h(0)^2} \sum_{j=1}^{\infty} \left[h \left(vt + \log \left(|J_i^\dagger(t-)| \Delta_j(t) \right) \right) - h \left(vt + \log |J_i^\dagger(t-)| \right) \right]^2 |J_i^\dagger(t-)|^2 \Delta_j(t).$$

As $h = 0$ away from $] \log a, \log b[$, and since from Theorem 1.4, the function h is continuous on $[\log a, \log b]$ and continuously differentiable on $] \log a, \log b[$, there exists $C_1 > 0$, such that $\forall x, y \in \mathbb{R} \ |h(x) - h(y)| \leq C_1 \min(|x - y|, 1)$. In this way, we get

$$|M_t - M_{t-}|^2 \leq \frac{e^{2\rho t}}{h(0)^2} \left(\sum_{j=0}^{\infty} |C_1 \min(|\log \Delta_j(t)|, 1)|^2 \Delta_j(t) \right) \cdot |J_i^\dagger(t-)|^2.$$

Now we look at

$$\mathbb{E}(V_2) = \mathbb{E} \left(\sum_{t \geq 0} |M_t - M_{t-}|^2 \right).$$

Using the compensation formula for the Poisson process, we get:

$$\mathbb{E}(V_2) \leq \frac{C_2 C_1^2}{h(0)^2} \int_0^\infty e^{2\rho t} \mathbb{E} \left(\sum_{i \in \mathbb{N}} |J_i^\dagger(t)|^2 \right) dt,$$

with

$$C_2 := \int_{\mathcal{U}} \sum_{j=1}^{\infty} u_j \min(|\log u_j|^2, 1) \nu(dU),$$

where $|U|^\downarrow = (u_1, u_2, \dots)$ is the decreasing sequence of the interval component lengths of U . Thus by the definition of L (see subsection 2.3)

$$C_2 = \int_{\mathbb{R}_+} \min(y^q, 1) L(dy) \leq \int_{\mathbb{R}_+} \min(y, 1) L(dy) < \infty.$$

Additionally by the definition of $J_i^\dagger(t)$ and by (4), we get:

$$\mathbb{E}(V_2) \leq \frac{C_2 C_1^2}{h(0)^2} \int_0^\infty e^{2\rho t} \mathbf{E} \left(e^{-\xi t} \mathbf{1}_{\{\xi_s \in]vs - \log b, vs - \log a[\vee s \leq t\}} \right) dt.$$

Moreover we have

$$e^{\rho t} \mathbf{E} \left(e^{-\xi t} \mathbf{1}_{\{\xi_s \in]vs - \log b, vs - \log a[\vee s \leq t\}} \right) \leq b e^{-vt} \mathbf{E}_{\log(1/a)} \left(e^{\rho t} \mathbf{1}_{\{t < T\}} \right).$$

Let $C_3 := b \cdot \sup_{t \in \mathbb{R}_+} \mathbf{E}_{\log(1/a)} \left(e^{\rho t} \mathbf{1}_{\{t < T\}} \right)$ which is finite by Theorem 1.2, thus we get:

$$e^{2\rho t} \mathbf{E} \left(e^{-\xi t} \mathbf{1}_{\{\xi_s \in]vs - \log b, vs - \log a[\vee s \leq t\}} \right) \leq C_3 e^{(\rho-v)t},$$

which is a function integrable in the variable t because $v > \rho$. Therefore

$$\mathbb{E}(V_2) < \infty,$$

and as a consequence, M_i is bounded in $L^2(\mathbb{P})$. ■

In order to prove Theorem 2.2 we will first introduce some notation, then prove two lemmas, and after we will conclude.

Let I be an interval of $]0, 1[$. The law of the homogeneous interval fragmentation started at I will be denoted by \mathbb{P}_I . We remark that $\mathbb{P}_I(M_\infty = 0 | \zeta = \infty)$ only depends on the length of I . Therefore we define

$$g(x) := \mathbb{P}_I(M_\infty = 0 | \zeta = \infty),$$

where I is an interval such that $|I| = x$. Let N be the integer part of $(2b - a)/a$. As we assume $v > \rho$, we have necessarily $b > 2a$ (see Remark 2), thus $N \geq 2$. Let $\eta := (b - a)N^{-1}$. We remark that $\eta < a$ and $b - a = N\eta$. Denote the first time when there are at least two good intervals by

$$T^F := \inf\{t : \#G(t) \geq 2\},$$

with the convention $\inf \emptyset = \infty$. We notice that T^F is an (\mathcal{F}_t) stopping time as $\#G(t)$ is \mathcal{F}_t -adapted.

Lemma 1 *In the previous notation, supposing that (5) and $v > \rho$ hold, we get: for every open interval I*

$$\mathbb{P}_I(T^F = \infty | \zeta = \infty) = 0.$$

Proof: We notice that, as the martingale M_t is not identically 0 and is uniformly integrable, we have $\mathbb{P}_I(T^F = \infty | \zeta = \infty) < 1$ (because $M_\infty = 0$ when $T^F = \infty$).

Let I be an open interval such that $|I| \in]a, b[$, $t_0 := \log(2b/a)/v$ and $\epsilon := a^2/(2b^2)$. Thus

$$|I|(1 - \epsilon) > a/2 \geq be^{-vt_0} \quad \text{and} \quad |I|\epsilon < b\epsilon \leq ae^{-vt_0}$$

therefore, if the dislocation of I produces at time t_0 an interval of length at least $|I|(1 - \epsilon)$ then this interval is too large to be good and the remaining ones are too small to be good either. As a consequence we have

$$\mathbb{P}_I(M_{t_0} = 0) \geq \mathbf{P}_{\log|I|}(e^{-\xi_{t_0}} > e^{-\log|I|}(1 - \epsilon)) = \mathbf{P}(\xi_{t_0} < -\log(1 - \epsilon)),$$

by the homogeneous property of the fragmentation. Moreover since ξ_t is a subordinator, we get $p := \mathbf{P}(\xi_{t_0} < -\log(1 - \epsilon)) > 0$, therefore

$$\mathbb{P}_I(M_{t_0} = 0) \geq p > 0. \tag{11}$$

Additionally for every open interval I such that $|I| \in]a, b[$:

$$\mathbb{P}_I(\#G(t) = 1 \forall t \leq t_0) \leq 1 - \mathbb{P}_I(M_{t_0} = 0) \leq 1 - p.$$

Using the strong Markov property of the fragmentation and (11) we find by induction that for all $k \in \mathbb{N}$:

$$\mathbb{P}_I(\#G(t) = 1 \forall t \leq kt_0) \leq (1 - p)^k.$$

Therefore

$$\lim_{t \rightarrow \infty} \mathbb{P}_I(\#G(s) = 1 \forall s \leq t) = 0$$

and as a consequence

$$\mathbb{P}_I(T^F = \infty | \zeta = \infty) = 0. \quad \blacksquare$$

Lemma 2 *In the previous notation, supposing that (5) and $v > \rho$ hold, we get:*

$$\sup_{a < x < b} g(x) = \max_{1 \leq k \leq N} g(a + k\eta),$$

where $N = \lceil (2b - a)/a \rceil$ and $\eta = (b - a)/N$.

Proof: We will prove this lemma by induction.

The hypothesis of induction is for $n \leq N$:

$$(H)_n : \quad \sup_{x \in]a, a+n\eta[} g(x) = \max_{1 \leq k \leq n} g(a + \eta k).$$

* The case $n = 1$: let I be an open interval such that $|I| \in]a, a + \eta[$. We work under \mathbb{P}_I conditionally on "non-extinction" (which means conditionally on the event $\zeta = \infty$). Let

$$T^1 := \inf\{t \geq 0 \mid \exists J(t) \in G(t) : e^{vt}|J(t)| \notin]a, a + \eta[\},$$

with $G(t)$ defined in (9). The random time T^1 is an (\mathcal{F}_t) stopping times. As the quantity $e^{vt}|J(t)|$ grows only continuously and as $J(t) \in G(t)$ implies that $e^{vt}|J(t)| > a$, we get

$$T^1 = \inf\{t \geq 0 \mid \exists J(t) \in G(t) : e^{vt}|J(t)| = a + \eta \}.$$

Moreover by the choice of η we have $a + \eta < 2a$, which implies that there is at most one good interval whose length is always in $]a, a + \eta[$. Recall from Lemma 1 that $\mathbb{P}_I(T^F < \infty \mid \zeta = \infty) = 1$, thus

$$\mathbb{P}_I(T^1 < \infty \mid \zeta = \infty) = 1.$$

Using the strong Markov property at the stopping times T^1 , we get

$$g(x) \leq g(a + \eta) \quad , \quad x \in]a, a + \eta[,$$

thus $(H)_1$ holds.

* The case $n + 1$ (with $n + 1 \leq N$): we suppose that the hypothesis of induction holds for all $k \leq n$.

Let I be an open interval such that $|I| \in]a + n\eta, a + (n + 1)\eta[$. We work under \mathbb{P}_I conditionally on "non-extinction". Let

$$T^n := \inf\{t \geq 0 \mid \exists J(t) \in G(t) : e^{vt}|J(t)| \notin]a + n\eta, a + (n + 1)\eta[\},$$

with $G(t)$ defined in (9). The random time T^n is an (\mathcal{F}_t) stopping times. As the quantity $e^{vt}|J(t)|$ grows only continuously, we get

$$T^n = \inf\{t \geq 0 \mid \exists J(t) \in G(t) : e^{vt}|J(t)| = a + (n + 1)\eta \text{ or } e^{vt}|J(t)| \in]a, a + n\eta[\}.$$

Moreover by the choice of η we have $a + \eta < 2a$, which implies that there is at most one good interval which length is always in $]a + n\eta, a + (n + 1)\eta[$. Additionally by Lemma 1, we get $\mathbb{P}_I(T^F < \infty \mid \zeta = \infty) = 1$, thus

$$\mathbb{P}_I(T^n < \infty \mid \zeta = \infty) = 1.$$

Using the strong Markov property at the stopping times T^n , we get

$$g(|I|) \leq \max \left(g(a + (n + 1)\eta), \sup_{y \in]a, a+n\eta[} g(y) \right).$$

As this holds for every open interval I such that $|I| \in]a + n\eta, a + (n + 1)\eta[$, by the hypothesis of induction, we have established $(H)_{n+1}$.

■
Proof of Theorem 2.2: With Lemma 2, we get that there exists a integer k_0 in $[1, N]$ such that $g(a + \eta k_0) = \sup_{x \in]a, b[} g(x)$ (if two or more values of k , are possible, we choose the smallest one). Let x_0 be $a + \eta k_0$.

Additionally, with Lemma 1, we get $\mathbb{P}_{]0, x_0[}(T^F < \infty | \zeta = \infty) = 1$. Using the strong property of Markov for the stopping times T^F , and with $n \geq 2$ the random number of good intervals of the fragmentation at time T^F and with $\alpha_1, \dots, \alpha_n$ the length of those intervals, we get:

$$g(x_0) \leq \mathbb{E}(g(\alpha_1) \dots g(\alpha_n)) \leq \mathbb{E}(g(x_0)^n) \leq g(x_0)^2.$$

As $g(x_0) < 1$ by the uniformly integrability of M_t , we get that $g(x_0) = 0$ and finally that $g \equiv 0$. ■

4 Limit theorems.

In this section, we establish two corollaries of Theorem 2, which will be useful in the sequel. Bertoin and Rouault (Corollary 2 in [11]) proved that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \#\{I_x(t) : ae^{-vt} < |I_x(t)| < be^{-tv}\} = C(v), \quad (12)$$

where $C(v) := (\Upsilon_v + 1)v - \kappa(\Upsilon_v)$ and Υ_v is the reciprocal of v by κ' i.e, $\kappa'(\Upsilon_v) = v$ for $v \in]v_{min}, v_{max}[$.¹

Here we deal with the more stringent requirement: $\forall s \leq t, |I_x(s)| \in]ae^{-sv}, be^{-sv}[$, and the next proposition gives the rates that we find in that case.

Proposition 3 *In the notation of the previous sections, with the assumptions (5) and if $v > \rho$ we get that conditionally on $\zeta = \infty$ (i.e. M is not absorbed at 0, or in a equivalent way $\Lambda_{(v,a,b)} \neq \emptyset$):*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \#G(t) = v - \rho \quad a.s. \quad (13)$$

Before proving this corollary we make the following remark

Remark 3 *It is interesting to compare the estimate found by Bertoin and Rouault and the present one (of course we have not considered the same set, nevertheless the two estimates are related). For this we show that for all $v \in]v_{min}, v_{max}[$ and a and b such that $\rho \geq v_{min}$ we have $C(v) \geq v - \rho$. In this direction we use results from [3] Section 1. Let $g(p) := p\kappa'(p) - \kappa(p)$ for all $p > 0$ with κ' the derivative of κ (this function is well defined because of the definition of \underline{p} in Section 2 and because $\underline{p} \leq 0$). For every $p > 0$, $g'(p) = p\kappa''(p) \leq 0$ since κ*

¹Where v_{min} is the maximum of the function $p \mapsto \kappa(p-1)/p$ on $]\underline{p} + 1, \infty[$ and $v_{max} := \kappa'(\underline{p}^+)$ (see [3]).

is concave. As a consequence g is decreasing. With the definition of Υ_v , we get that the function $v \in]v_{\min}, v_{\max}[\mapsto \Upsilon_v \in \mathbb{R}$ is decreasing, additionally $\Upsilon_{v_{\min}} > 0$ (see [4] Section 1), therefore the function $v \in]v_{\min}, v_{\max}[\mapsto g(\Upsilon_v) \in \mathbb{R}$ is increasing. Moreover $g(\Upsilon_v) = C(v) - v$, hence for all $v \in]v_{\min}, v_{\max}[$:

$$C(v) - v \geq C(v_{\min}) - v_{\min} = -v_{\min}.$$

Additionally as $\rho \geq v_{\min}$, we finally obtain:

$$\forall v \in]v_{\min}, v_{\max}[\quad C(v) \geq v - \rho.$$

As a consequence, we have checked that the rates of decay of $\sharp G(t)$ (defined in (9)) is faster than that of $\sharp\{I_x(t) : |I_x(t)| \in]ae^{-tv}, be^{-vt}[\}$, which was of course expected.

Proof: In this proof we work conditionally on $\zeta = \infty$ (i.e M is not absorbed at 0). Applying Theorem 2, we get $M_\infty > 0$. In order to show that (13) holds, we will first look at the lower bound of the inequality, and then at the upper bound.

• With the definition of M_t in (10), of $G(t)$ and of $J_i^\dagger(t)$ at the beginning of Section 3 and by the conditioning, there exists $t' > 0$ such that for all $t \geq t'$:

$$\frac{M_\infty}{2} \leq \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} h(vt + \log(|J_i^\dagger(t)|)) |J_i^\dagger(t)| \leq \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} C_4 be^{-vt} \mathbf{1}_{\{J_i(t) \in G(t)\}},$$

with C_4 as maximum of $h(\cdot)$ on $[\log a, \log b]$. Hence for all $t \geq t'$:

$$\sharp G(t) \geq e^{(v-\rho)t} \frac{h(0)}{2C_4 b} M_\infty,$$

and as a consequence, conditionally on $\zeta = \infty$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \sharp G(t) \geq v - \rho. \quad (14)$$

• Secondly we will show the converse inequality.

Let $0 < a' < a < 1 < b < b'$, and $\rho' := \rho_{\log(b'/a')}$. Denote the set of 'good' intervals associated to a' and b' by:

$$G'(t) := \{I_x(t) : x \in]0, 1[\text{ and } |I_x(s)| \in]a'e^{-vs}, b'e^{-vs}[\quad \forall s \leq t\}.$$

Let M'_t be the martingale defined at the beginning of Section 3 (and denoted there by M) associated to a', b' instead of a, b . Plainly, if M_t is not absorbed at 0, then a fortiori M'_t is not absorbed at 0 either. Additionally, since $\log(b'/a') > \log(b/a)$, and ρ is strictly decreasing (see Theorem 1.5), we get $v > \rho > \rho'$, and we may apply Theorem 2 for a', b' instead of a, b . We get $\lim_{t \rightarrow \infty} M'_t = M'_\infty > 0$.

With the definition (10) of M_t and with an analogue of the function $h(t)$, namely $t \in \mathbb{R}$

$$\varphi(t) := W^{(-\rho')}(t + \log(1/a')) \mathbf{1}_{\{t \in]\log a', \log b' [\}},$$

we get:

$$M'_\infty = \lim_{t \rightarrow \infty} \frac{e^{\rho' t}}{\varphi(0)} \sum_{i \in \mathbb{N}} \varphi(vt + \log |J_i(t)|) |J_i(t)| \mathbf{1}_{\{J_i(t) \in G'(t)\}}.$$

Therefore there exists $t' > 0$ such that for every $t \geq t'$

$$\begin{aligned} 2M'_\infty &\geq \frac{e^{\rho' t}}{\varphi(0)} \sum_{i \in \mathbb{N}} \varphi(vt + \log |J_i(t)|) |J_i(t)| \mathbf{1}_{\{J_i(t) \in G'(t)\}} \\ &\geq \frac{e^{\rho' t}}{\varphi(0)} \sum_{i \in \mathbb{N}} \varphi(vt + \log |J_i(t)|) a' e^{-vt} \mathbf{1}_{\{J_i(t) \in G(t)\}}. \end{aligned}$$

Since $]ae^{-vt}, be^{-vt}[\subset]a'e^{-vt}, b'e^{-vt}[$, we get by Theorem 1.1, that for all $x \in]\log a, \log b[$: $\varphi(x) > 0$. Because $]\log a, \log b[$ is compact and $\varphi(\cdot)$ is a continuous function,

$$\inf_{x \in]\log a, \log b[} \varphi(x) > 0.$$

Combining this with

$$C_5 := 2M'_\infty \varphi(0) / \left(a' \inf_{x \in]\log a, \log b[} \varphi(x) \right) < \infty,$$

we get for all $t \geq t'$:

$$C_5 \geq e^{(\rho' - v)t} \sum_{i \in \mathbb{N}} \mathbf{1}_{\{J_i(t) \in G(t)\}}$$

and thus

$$C_5 e^{(v - \rho')t} \geq \#G(t).$$

Hence for all a', b' such that $0 < a' < a < 1 < b < b'$:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \#G(t) \leq v - \rho'.$$

For $a' \rightarrow a$ and $b' \rightarrow b$ we get by the continuity of ρ :

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \#G(t) \leq v - \rho.$$

■

Now we will give an other corollary, using the same method as that of Bertoin and Gnedin in [9]. We encode the configuration $J^\dagger(t) = \{|J_i^\dagger(t)|\}$ of the lengths of good intervals into the random measure

$$\sigma_t := \frac{e^{\rho t}}{h(0)} \sum_{i \in \mathbb{N}} h\left(vt + \log |J_i^\dagger(t)|\right) |J_i^\dagger(t)| \delta_{\log(1/a) + vt + \log |J_i^\dagger(t)|}$$

which has total mass M_t .

The associated mean measure σ_t^* is defined by the formula

$$\int_0^\infty f(x) \sigma_t^*(dx) = \mathbb{E} \left(\int_0^\infty f(x) \sigma_t(dx) \right)$$

which is required to hold for all compactly supported continuous functions f . Since M_t is a martingale, σ_t^* is a probability measure. More precisely the next proposition establishes the convergence of the mean measure σ_t^* , and then of σ_t itself.

Proposition 4 *In the notation of the previous sections, with the assumptions (5), and $v > \rho$ we get:*

1. *The measures σ_t^* converge weakly, as $t \rightarrow \infty$, to the probability measure*

$$\varrho(dy) := ch(y + \log a)h(\log(b) - y)dy$$

where $c > 0$ is the constant that appears in Theorem 1.5.

2. *For any bounded continuous f*

$$L^2 - \lim_{t \rightarrow \infty} \int_0^\infty f(x) \sigma_t(dx) = M_\infty \int_0^\infty f(x) \varrho(dx). \quad (15)$$

Proof: The arguments are similar to those of Bertoin and Gnedin in [9]. The details will be left to the reader.

■

5 The Hausdorff dimension.

In this section we use the notation and definitions of the previous sections. We recall that $\rho = \rho_{\log(b/a)}$, where ρ is define in (7). Let dim be the Hausdorff dimension. The aim of this section would be to proof the main theorem:

Theorem 5 : Multifractal spectrum. Assume (5):

- if $\rho > v$ holds, then:

$$G_{(v,a,b)} = \emptyset \quad a.s.$$

- if $\rho < v$ holds, then:

$$\dim(G_{(v,a,b)}) = 1 - \rho/v \quad a.s. \quad (16)$$

Remark 4 Berestycki in [3] has computed the Hausdorff dimension of the set

$$G_v = \left\{ x \in (0, 1) \mid \lim_{t \rightarrow \infty} \frac{1}{t} \log |I_x(t)| = -v \right\}.$$

He found that for $v \in]v_{min}, v_{max}[$, $\dim(G_v) = C(v)/v$ (with $C(v)$ defined at the beginning of section 4). In Remark 3 we have shown that for all $v \in]\max(v_{min}, \rho), v_{max}[$ we have $C(v) \geq v - \rho$ and we can notice that the inequality is strict for $\rho > v_{min}$. As a consequence the set $G_{(v,a,b)}$ has a Hausdorff dimension smaller than that of G_v , and also smaller than that one could have infer from equality (12).

The proof of this theorem use the non-asymptotic set $\Lambda_{(v,a,b)}$. In particular the key of the proof is the next proposition:

Proposition 6 Assume (5) and $0 < a < b < 1$:

- if $\rho > v$ holds, then:

$$\Lambda_{(v,a,b)} = \emptyset \quad a.s.$$

- if $\rho < v$ holds, then: $\mathbb{P}(\Lambda_{(v,a,b)} \neq \emptyset) > 0$, and conditionally on $\Lambda_{(v,a,b)} \neq \emptyset$,

$$\dim(\Lambda_{(v,a,b)}) = 1 - \rho/v. \quad (17)$$

Proof:

1. Let $v > 0$ and a and b such that $v < \rho$. We define

$$N(t) := \sharp G(t),$$

with $G(t)$ defined in (9). We remark that

$$N(t) = \int_0^1 \frac{1}{|I_x(t)|} \mathbf{1}_{\{I_x(t) \in G(t)\}}(x) dx.$$

and in particular

$$\mathbb{E}(N(t)) = \mathbb{E} \left(\int_0^1 \frac{1}{|I_x(t)|} \mathbf{1}_{\{I_x(t) \in G(t)\}}(x) dx \right).$$

Additionally by (4), we get

$$\mathbb{E}(N(t)) = e^{vt} \mathbf{E} \left(e^{\xi_t - vt} \mathbf{1}_{\{vs - \xi_s - \log a \in]0, \log(b/a)[\forall s \leq t\}} \right).$$

With the notation $Y_t = vt - \xi_t$ and P_t defined in Theorem 1 we rewrite the previous equality as:

$$\begin{aligned} \mathbb{E}(N(t)) &= e^{vt} \mathbf{E}_{\log(1/a)} \left(e^{-Y_t - \log a} \mathbf{1}_{\{t < T\}} \right) \\ &= \frac{1}{a} e^{(v-\rho)t} \int_0^{\log(b/a)} e^{-y + \rho t} P_t(\log(1/a), dy). \end{aligned}$$

By Theorem 1.2 we get

$$\mathbb{E}(N(t)) \underset{t \rightarrow \infty}{\sim} \frac{1}{a} e^{(v-\rho)t} c h(0) \int_0^{\log(b/a)} e^{-y} h(\log(b) - y) dy.$$

Finally as the function $y \mapsto e^{-y} h(\log(b) - y)$ is continuous, the integral above is a finite constant. Thus if $\rho > v$ then $\lim_{t \rightarrow \infty} \mathbb{E}(N(t)) = 0$, from which one concludes that $\lim_{t \rightarrow \infty} N(t) = 0$, i.e. $\Lambda_{(v,a,b)} = \emptyset$ *a.s.*

2. Now we deal with the case where a and b are such that $v > \rho$. We work conditionally on $\Lambda_{(v,a,b)} \neq \emptyset$ (or, equivalently, on the event $\zeta = \infty$, which has a positive probability by Theorem 2).

- Firstly, in order to prove the lower bound of the Hausdorff dimension of $\Lambda_{(v,a,b)}$, we will use the same method as Berestycki in [3]. We will divide this proof into three steps. Each step will begin with a star (\star). In the first step we will construct a subset $\bigcap_{n \in \mathbb{N}} \mathbb{G}_\delta(n)$ of $\Lambda_{(v,a,b)}$. In the second we shall obtain a lower bound of the Hausdorff dimension of this subset. In order to do that we will construct an increasing process indexed by $t \in]0, 1[$, which only increased on $\bigcap_{n \in \mathbb{N}} \mathbb{G}_\delta(n)$, and which is Hölder continuous. In the last step we will conclude.

\star As in [3] for $\delta > 0$ we define for all $n \in \mathbb{N}$, $H_\delta(n)$ as a multi-type branching process with each particle corresponding to a segment of $G(\delta n)$ and

$$G_\delta(n) := \bigcup_{I \in H_\delta(n)} I,$$

with $G(t)$ defined in (9) (i.e. $G_\delta(n) = G(\delta n)$).

We notice that the family $(G_\delta(n))_{n \in \mathbb{N}}$ is nested and that $\bigcap_{n \in \mathbb{N}} G_\delta(n) = \Lambda_{(v,a,b)}$.

Let $\epsilon > 0$, and fix $\epsilon' > 0$ and $\eta > 0$ such that $\eta + |\log(1 - \epsilon')| < \epsilon$. By Proposition 3, for this $\epsilon' > 0$ and $\eta > 0$, we may find $t_0 > 1$ such that for all $t > t_0$:

$$\mathbb{P}(|t^{-1} \log(\sharp G(t)) - (v - \rho)| > \eta | \zeta = \infty) < \epsilon'.$$

For each $t > 0$, we consider a variable $\tilde{\chi}(t)$ which law is given by

$$\mathbb{P}(\tilde{\chi}(t) = e^{[(v-\rho)-\eta]t}) = 1 - \epsilon'',$$

and

$$\mathbb{P}(\tilde{\chi}(t) = 0) = \epsilon'',$$

where $\epsilon'' = \mathbb{P}(|t^{-1} \log(\sharp G(t)) - (v - \rho)| > \eta | \zeta = \infty) < \epsilon'$. We notice that

$$|t^{-1} \log(\mathbb{E}(\tilde{\chi}(t))) - (v - \rho)| \leq \eta + t^{-1} |\log(1 - \epsilon')|.$$

Plainly $\tilde{\chi}(t)$ is stochastically dominated by $\sharp G(t)$. Exactly as in [3] we can construct a true Galton-Watson tree \mathbb{H} by thinning H_δ where $\delta > t_0$. More precisely the offspring distribution of \mathbb{H} is given by the law of $\tilde{\chi}(\delta)$. Let $m := \mathbb{E}(\tilde{\chi}(\delta))$ be the expectation of the number of children of a particle. Therefore, we get

$$(a) \quad |\delta^{-1} \log m - (v - \rho)| < \epsilon. \quad (18)$$

(b) The family $(\mathbb{G}(n) := \bigcup_{I \in \mathbb{H}(n)} I)_{n \in \mathbb{N}}$ is nested. The $\mathbb{G}(n)$ is the union of the interval of the n generation of \mathbb{H} .

$$(c) \quad \bigcap_{n \in \mathbb{N}} \mathbb{G}(n) \subseteq \Lambda_{(v,a,b)}.$$

This last point makes sense because we work conditionally on $\zeta = \infty$.

★ We fix $\epsilon > 0$. We choose $\delta > t_0$ as shown above and consider the tree \mathbb{H} . We define $Z(n)$ as the number of nodes of \mathbb{H} at height n . By the theory of Galton-Watson tree, as we are working conditionally on the event $\Lambda_{(v,a,b)} \neq \emptyset$, we have that almost surely

$$m^{-n} Z(n) \rightarrow \mathcal{W} > 0.$$

Let σ be a node of our tree (thus it is also a subinterval of $]0, 1[$). Fix an interval $I \subset]0, 1[$ and introduce

$$\mathbb{H}_I(n) := \{\sigma \in \mathbb{H}(n), \sigma \cap I \neq \emptyset\},$$

$$Z_I(n) := \sharp \mathbb{H}_I(n).$$

Define

$$x \rightarrow L_x := \lim_n m^{-n} Z_{]0,x[}(n), \quad x \in]0, 1[.$$

We will now state a lemma that we will use to conclude:

Lemma 3 For each $\epsilon > 0$,

- (a) There exists a version \tilde{L} of $(L_x)_{x \in [0,1]}$ which is Hölder continuous of order α for any $\alpha < 1 - \rho/v - \epsilon$ for every $\epsilon > 0$.
- (b) The process \tilde{L} only grows on the set $\bigcap_{n \in \mathbb{N}} \mathbb{G}(n)$.

Proof of Lemma 3

• Exactly as in [3], we show the first point by verifying Kolmogorov's criterium (see [20] Theorem 2.1 p.26). Let $W(\sigma)$ the “renormalized weight “ of the tree rooted at σ , i.e.,

$$W(\sigma) := \lim_{n \rightarrow \infty} m^{-n} \#\{\sigma' \in \mathbb{H}(|\sigma| + n), \sigma' \subset \sigma\}.$$

By the definition of L we have for all $x > y \in (0, 1)$:

$$|L_x - L_y| = \lim_{n \rightarrow \infty} m^{-n} Z_{]x,y[}(n), \quad x \in]0, 1[.$$

For any J open subinterval of $(0, 1)$, let

$$\eta(J) := \sup\{n \in \mathbb{N} : e^{-v\delta n} \geq |J|\} = \lceil -\log(|J|)/v\delta \rceil.$$

For all x, y such that $x < y$ by the definition of L , we get:

$$\begin{aligned} & |L_x - L_y| \\ &= \lim_n m^{-\eta((x,y))} m^{-n+\eta((x,y))} \sum_{\sigma \in \mathbb{H}_{(x,y)}(\eta((x,y)))} \#\{\sigma' \in \mathbb{H}(|\sigma| + n - \eta((x,y))), \sigma' \subset \sigma\} \\ &\leq m^{-\eta((x,y))} \sum_{\sigma \in \mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma), \end{aligned}$$

and by the definition of $\eta(\cdot)$:

$$\begin{aligned} |L_x - L_y| &\leq e^{\log m(\frac{1}{v\delta} \log(y-x)+1)} \sum_{\sigma \in \mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma) \\ &\leq m|x-y|^{1-\epsilon-\rho/v} \sum_{\sigma \in \mathbb{H}_{(x,y)}(\eta((x,y)))} W(\sigma), \end{aligned}$$

by using (18). Moreover by the definition of good intervals, we have that for each n the sizes of intervals in $\mathbb{H}(n)$ have a lower bound given by $ae^{-v\delta n}$, so $a|J|e^{-v\delta}$ is a lower bound for the sizes of the intervals of $\mathbb{H}(\eta(J))$, and thus $Z_J(\eta(J)) \leq e^{v\delta}/a$. Therefore for all $\gamma > 1$ and all $J \subset (0, 1)$ we have:

$$\mathbb{E} \left(\left(\sum_{\sigma \in \mathbb{H}_J(\eta(J))} W(\sigma) \right)^\gamma \right) \leq \mathbb{E}((W_1 + \dots + W_{\lceil e^{v\delta}/a \rceil + 1})^\gamma) \leq \mathbb{E}((W_1 + \dots + W_{\eta(J)+2})^\gamma) < \infty,$$

where the W_i are i.i.d. with the same law as W . The finiteness comes from the existence of finite moments of all orders for W (see for example Theorem 3.4 p. 479 of Harris [16]).

- The second point is clear by the choice of L . ■

★ To prove that $\dim \left(\bigcap_{n \in \mathbb{N}} \mathbb{G}(n) \right) \geq 1 - \rho/v - \epsilon$, it is enough to show that

$$\sum_i \text{diam}(U_i)^{1-\rho/v-\epsilon} > 0 \quad (19)$$

for any cover $\{U_i\}$ of $\bigcap_{n \in \mathbb{N}} \mathbb{G}(n)$, where $\text{diam}(U_i)$ is the diameter of U_i . Clearly, it is enough to assume that the $\{U_i\}$ are intervals, and by expanding them slightly and using the compactness of the closure of $\bigcap_{n \in \mathbb{N}} \mathbb{G}(n)$, we only need to check (19) if $\{U_i\}$ is a finite collection of open subintervals of $[0, 1]$.

Let $\cup_{i=0}^N]l_i, r_i[$ be a cover of $\bigcap_{n \in \mathbb{N}} \mathbb{G}(n)$ (where the $]l_i, r_i[$ are disjoint open intervals). Therefore

$$\sum_{i=1}^N |\tilde{L}_{r_i} - \tilde{L}_{l_i}| = \mathcal{W}.$$

Thus for all such covers with $\max_i (r_i - l_i)$ small enough

$$\mathcal{W} \leq k \sum_{i=0}^N (r_i - l_i)^{1-\rho/v-\epsilon}$$

and hence

$$\dim(\Lambda_{(v,a,b)}) \geq \dim \left(\bigcap_{n \in \mathbb{N}} \mathbb{G}(n) \right) \geq 1 - \rho/v - \epsilon.$$

To get the lower bound of the Hausdorff dimension of $\Lambda_{(v,a,b)}$, we let ϵ tend to 0.

- Secondly, the upper bound for (17) is an easy corollary of the fact that the Hausdorff dimension is smaller than the box-counting dimension (see [15] p.36-43), using the cover $\bigcup_{n \geq N} \bigcup_{i \in \theta_{v,a,b}(n)} J_i(n)$, with $\theta_{v,a,b}(t) = \{i \in \mathbb{N} \mid J_i(t) \in G(t)\}$ (with $G(t)$ defined in Section 3). ■

Then we have the next corollary, which deals with a general case for a and b :

Corollary 7 For $t' \geq 0$ set

$$\Lambda_{(v,a,b)}(t') := \left\{ x \in (0, 1) : ae^{-vt} < |I_x(t)| < be^{-vt} \quad \forall t \geq t' \right\}.$$

Assume (5), $0 < a < b$ and $\rho < v$, then

$$\mathbb{P}(\dim(\Lambda_{(v,a,b)}(t')) \leq 1 - \rho/v) = 1$$

and

$$\mathbb{P}(\dim(\Lambda_{(v,a,b)}(t')) = 1 - \rho/v) \xrightarrow{t' \rightarrow \infty} 1.$$

Proof • The first part of the proof is a consequence of the homogeneity of the fragmentation and of Proposition 6.

• Fix $\rho' > \rho$. As $\lim_{\beta \rightarrow 0} \rho_\beta = \infty$, and, by Theorem 1.5, the application $\beta \rightarrow \rho_\beta$ is continuous and strictly decreasing, therefore there exists $\beta_0 \in (1, b/a)$ such that $\rho' = \rho_{\log(\beta_0)}$. Let $\epsilon := (\beta_0 - 1)/(1 + \beta_0)$, $a' := 1 - \epsilon$, $b' := 1 + \epsilon$, $x_0 := (\beta_0 + 1)(a + b/\beta_0)/4$ (notice that $x_0 \in (a, b)$) and

$$p_0 := \mathbb{P}(\dim(\Lambda_{(v,a',b')}) \geq 1 - \rho_{\log(b'/a')}/v).$$

By Proposition 6, we get that $p_0 > 0$. We notice that by the choice of a' and of b' , we have $\rho_{\log(b'/a')} = \rho_{\log(\beta_0)} = \rho'$.

Let I be an interval of $]0, 1[$. The law of the homogeneous interval fragmentation started at I will be denoted by \mathbb{P}_I . We remark that $\mathbb{P}_I(\dim(\Lambda_{(v,a,b)}) \geq 1 - \rho'/v)$ only depends on the length of I . Thus we define

$$g_{a,b}(x) := \mathbb{P}_I(\dim(\Lambda_{(v,a,b)}) \geq 1 - \rho'/v),$$

where I is an interval such that $|I| = x$.

Let $x \in (x_0 a', x_0 b')$. We remark that by the choice of x_0 and as $1 < \beta_0 < b/a$ we have that $(x_0 a', x_0 b') \subset (a, b)$ and thus

$$g_{a,b}(x) \geq g_{x_0 a', x_0 b'}(x).$$

Moreover by the scaling property of the fragmentation we get that

$$g_{x_0 a', x_0 b'}(x) = \mathbb{P}(\dim(\Lambda_{(v,a'/x,b'/x)}) \geq 1 - \rho_{\log((b'/x)/(a'/x))}/v) = p_0$$

Therefore

$$\inf_{x \in (x_0 a', x_0 b')} g_{a,b}(x) \geq p_0. \tag{20}$$

Let

$$B(t) = \{i : x_0 a' < e^{vt} |J_i(t)| < x_0 b'\} \quad , \quad n_t = \#B(t),$$

where $(J_1(t), J_2(t), \dots)$ is the interval decomposition of $F(t)$.

Fix $t' \geq 0$. By applying the Markov property at time t' we get that

$$\begin{aligned} \mathbb{P}(\dim(\Lambda_{(v,a,b)}(t')) < 1 - \rho'/v) &\leq \mathbb{E} \left(\prod_{i \in B(t')} \mathbb{P}_{J_i(t')}(\dim(\Lambda_{(v,x_0 a', x_0 b')}) < 1 - \rho'/v) \right) \\ &\leq \mathbb{E}((1 - p_0)^{n_{t'}}), \end{aligned}$$

by using (20). Therefore as $p_0 > 0$, $n_{t'} \xrightarrow{t' \rightarrow \infty} \infty$ (see (12)) and with the first part of the proof we can conclude.

■

Now we are able to proof our main result:

Proof of Theorem 5 :

Observe that for all $n \in \mathbb{N}$, we have

$$\Lambda_{(v,a,b)}(n) \subset G_{(v,a,b)} \subset \bigcap_{\epsilon > 0} \bigcup_{m \in \mathbb{N}} \Lambda_{(v,a-\epsilon,b+\epsilon)}(m). \tag{21}$$

We can notice that the second inclusion is actually a equality.

- First we consider the case where $\rho > v$. As $\lim_{\beta \rightarrow 0} \rho_\beta = \infty$ and the application $\beta \rightarrow \rho_\beta$ is continuous and strictly decreasing (see Theorem 1.5), there exists $\epsilon_0 > 0$ such that $v < \rho_{\log((b+\epsilon_0)/(a-\epsilon_0))} < \rho$. Moreover by (21)

$$G_{(v,a,b)} \subset \bigcup_{m \in \mathbb{N}} \Lambda_{(v,a-\epsilon_0,b+\epsilon_0)}(m),$$

therefore thanks to Proposition 6 and the homogeneous property of the fragmentation, we get the first part of the proof.

- Second we consider the case where $\rho < v$. Thanks the second inclusion and the corollary 7, we get that: for all $\epsilon \in (0, a)$,

$$\dim(G_{(v,a,b)}) \leq \dim\left(\bigcup_{n \in \mathbb{N}} \Lambda_{(v,a-\epsilon,b+\epsilon)}(n)\right) = \max_n \dim(\Lambda_{(v,a-\epsilon,b+\epsilon)}(n)) \leq 1 - \rho_{\log(\frac{b+\epsilon}{a-\epsilon})}/v.$$

Then by the continuity of ρ . (see Theorem 1.5), we get the upper bound of the Hausdorff dimension of $G_{(v,a,b)}$.

The lower bound of the Hausdorff dimension is a consequence of the first inclusion of (21), as $\dim(\Lambda_{(v,a,b)}(n)) = 1 - \rho/v$ with a probability which goes to 1 when n goes to infinity.

■

Acknowledgments: I wish to thank J. Bertoin for his help, and especially for his patience.

I also wish to thank an anonymous referee of an earlier draft for comments and suggestions.

References

[1] A.-L. BASDEVANT (2006). Fragmentation of ordered partitions and intervals. *Elect. J. Probab.* **11** no. 16, 394-417.

[2] J. BERESTYCKI (2002). Ranked fragmentations. *ESAIM Prob. Stat.*, **6**, 157-176.

[3] J. BERESTYCKI (2003). Multifractal spectra of fragmentation processes. *J. Statist. Phys.*, **113**, no. 3-4, 411-430.

- [4] J. BERTOIN (1996). *Lévy processes*. Cambridge Univ. Press.
- [5] J. BERTOIN (1996) On the first exit time of a completely asymmetric stable process from a finite interval. *Bull. London Math. Soc.* **28** , no. 5, 514-520.
- [6] J. BERTOIN (1997). Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.*, **7**, no. 1, 156-169.
- [7] J. BERTOIN (2001). Homogeneous fragmentation processes. *Probab. Theory Relat. Fields*, **121**, 301-318.
- [8] J. BERTOIN (2003). The asymptotic behavior of fragmentation processes. *J. Europ. Math. Soc.* , **14**, 395-416.
- [9] J. BERTOIN and A. V. GNEDIN (2004). Asymptotic laws for nonconservative self-similar fragmentations. *Electron. J. Probab.* **9** no. 19, 575-593.
- [10] J. BERTOIN and S. MARTINEZ (2005). Fragmentation energy. *Adv. Appl. Probab.* **37** 553-570.
- [11] J. BERTOIN and A. ROUAULT (2005). Discretization methods for homogeneous fragmentations. *J. London Math. Soc.* **72**, 91-109.
- [12] D. BEYSENS, X. CAMPI and E. PEFFEKORN (editors) (1995). Proceedings of the workshop : Fragmentation phenomena. *Les Houches Series, World Scientific*.
- [13] T. CHAN and A. E. KYPRIANOU Smoothness of scale function for spectrally negative Lévy processes. *Preprint*.
- [14] J. ENGLÄNDER and A. E. KYPRIANOU (2004). Local extinction versus local exponential growth for spatial branching processes. *Ann. Probab.* **32**, no 1A, 78-99.
- [15] K. FALCONER (1997). *Fractal geometry*. John Wiley and Sons.
- [16] T.E. HARRIS (1948). Branching processes. *Ann. Math. Statist.*, **41**, 474-494.
- [17] P.L. KRAPIVSKY and S.N. MAJUMDAR (2000). Traveling waves, front selection, and exact nontrivial exponents in random fragmentation problem. *Phys. Rev. Lett.* **85**, 5492-5495.
- [18] A. LAMBERT (2000). Completely asymmetric Lévy processes confined in a finite interval. *Ann. Inst. H. Poincaré Probab. Statist.*, **36** , no. 2, 251-274.
- [19] J. NEVEU (1987). Multiplicative martingales for spatial branching processes, *Seminar on Stochastic Processes, Progr. Probab. Statist.*, **15**, 223-242.

- [20] D. REVUZ and M. YOR (1999). Continuous martingales and Brownian motion, *3rd edn.* Springer, Berlin. **85**.
- [21] H. G. TUCKER (1962). Absolute continuity of infinitely divisible distributions. *Pacific J. Math.* **12** 1125-1129.