



HAL
open science

AN INDUCTION PRINCIPLE FOR THE WEIGHTED p -ENERGY MINIMALITY OF $x/\|x\|$.

Jean-Christophe Bourgoïn

► **To cite this version:**

Jean-Christophe Bourgoïn. AN INDUCTION PRINCIPLE FOR THE WEIGHTED p -ENERGY MINIMALITY OF $x/\|x\|$. 2006. hal-00018444

HAL Id: hal-00018444

<https://hal.science/hal-00018444v1>

Preprint submitted on 2 Feb 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

AN INDUCTION PRINCIPLE FOR THE WEIGHTED p -ENERGY MINIMALITY OF $x/\|x\|$

JEAN-CHRISTOPHE BOURGOIN

ABSTRACT. In this paper, we investigate minimizing properties of the map $x/\|x\|$ from the Euclidean unit ball \mathbf{B}^n to its boundary \mathbb{S}^{n-1} , for the weighted energy functionals $E_{p,\alpha}^n(u) = \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u\|^p dx$. We establish the following induction principle: if the map $\frac{x}{\|x\|} : \mathbf{B}^{n+1} \rightarrow \mathbb{S}^n$ minimizes $E_{p,\alpha}^{n+1}$ among the maps $u : \mathbf{B}^{n+1} \rightarrow \mathbb{S}^n$ satisfying $u(x) = x$ on \mathbb{S}^n , then the map $\frac{y}{\|y\|} : \mathbf{B}^n \rightarrow \mathbb{S}^{n-1}$ minimizes $E_{p,\alpha+1}^n$ among the maps $v : \mathbf{B}^n \rightarrow \mathbb{S}^{n-1}$ satisfying $v(y) = y$ on \mathbb{S}^{n-1} .

This result enables us to enlarge the range of values of p and α for which $x/\|x\|$ minimizes $E_{p,\alpha}^n$.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Let \mathbf{B}^n be the unit Euclidean ball of \mathbb{R}^n and \mathbb{S}^{n-1} its boundary. For any couple (α, p) of real numbers, with $\alpha \geq 0$ and $p \geq 1$, we define the r^α -weighted p -energy functional of a map $u : \mathbf{B}^n \rightarrow \mathbb{S}^{n-1}$ by

$$E_{p,\alpha}^n(u) = \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u\|^p dx.$$

This functional is nothing but the p -energy functional associated with the Riemannian metric $r^{\alpha(n-p)}g_{\text{euc}}$ on the ball \mathbf{B}^n , where g_{euc} is the Euclidean metric, the sphere \mathbb{S}^{n-1} being endowed with its standard metric. The functional $E_{p,\alpha}^n$ is to be considered on the Sobolev space

$$W_\alpha^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1}) = \{u \in W^{1,p}((\mathbf{B}^n, r^{\alpha(n-p)}g_{\text{euc}}), \mathbb{R}^n); \|u\| = 1a.e.\}.$$

The question is to know which map minimizes $E_{p,\alpha}^n$ among the maps in $W_\alpha^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $u(x) = x$ on \mathbb{S}^{n-1} .

Following the arguments of Hildebrant, Kaul and Widman [11], the map $x/\|x\|$ is a natural candidate to be the minimizer of $E_{p,\alpha}^n$, for $p \in [1, n + \alpha)$ (notice that $x/\|x\| \in W_\alpha^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ if and only if $p < n + \alpha$).

The minimality of $x/\|x\|$ was first established for the 2-energy functional $E_2^n := E_{2,0}^n$ by Jäger and Kaul [14] in dimension $n \geq 7$, then by Brezis, Coron and Lieb [2] in dimension 3. Coron and Gulliver [5] actually proved the minimality of $x/\|x\|$ for the p -energy functional $E_p^n := E_{p,0}^n$ for any integer $p \in \{1, \dots, n-1\}$ and any dimension $n \geq 3$. An alternative proof using the null Lagrangian method (or calibration method) of this last result was obtained by Lin [15] and Avellaneda and Lin [1].

Hardt and Lin [7, 8] and Hardt, Lin and Wang [9, 10] developed the study of the singularities of p -harmonic and p -minimizing maps and obtained results extending those of Schoen and Uhlenbeck [16, 17]. A consequence of their results is the minimality of $x/\|x\|$ for E_p^n for any $p \in (n-1, n)$.

Finally, Hong [12] and Wang [18] have proved independently the minimality of $x/\|x\|$ for E_p^n in dimension $n \geq 7$ for any $p \leq n - 2\sqrt{n-1}$.

In order to close the question of the E_p^n -minimality of $x/\|x\|$ for the remaining values of p , Hong [13] suggested a new idea. Indeed, he observed that, for any $p \in (2, n)$, the minimality of $x/\|x\|$ for E_p^n follows from the minimality of $x/\|x\|$ for the r^{2-p} -weighted 2-energy $E_{2,2-p}^n$. Unfortunately, we have proved in a previous paper [3], the existence of an interval of values of $p \in (1, n-1)$ for which the map $x/\|x\|$ fails to be a minimizer of $E_{2,2-p}^n$. Nevertheless, we proved that $x/\|x\|$ minimizes the r^α -weighted p -energy $E_{p,\alpha}^n$ for any $\alpha \geq 0$ and any integer $p \leq n-1$. Actually, we obtained in [4] the minimality of $x/\|x\|$ for more general weighted p -energy functionals of the form $E_{p,f}^n(u) = \int_{\mathbf{B}^n} f(\|x\|) \|\nabla u\|^p dx$, where $p \leq n-1$ is an integer and $f : [0, 1] \rightarrow \mathbb{R}^+$ is a continuous non-decreasing function.

Our aim in this paper is to show that the minimizing properties of $x/\|x\|$ in dimension n and $n+1$ are not independent. Indeed, we will prove that, for any $p \in [1, n+\alpha+1)$, if $x/\|x\|$ minimizes the r^α -weighted p -energy $E_{p,\alpha}^{n+1}$ in dimension $n+1$, then it also minimizes the $r^{\alpha+1}$ -weighted p -energy $E_{p,\alpha+1}^n$ in dimension n .

Theorem 1. *Let $n \geq 2$ be an integer, $\alpha \geq 0$ be a real number and $p \in [1, n+\alpha+1)$. If the map $\frac{x}{\|x\|} : \mathbf{B}^{n+1} \mapsto \mathbb{S}^n$ minimizes the r^α -weighted p -energy $E_{p,\alpha}^{n+1}$ among the maps $u \in W_\alpha^{1,p}(\mathbf{B}^{n+1}, \mathbb{S}^n)$ satisfying $u(x) = x$ on \mathbb{S}^n , then the map $\frac{y}{\|y\|} : \mathbf{B}^n \mapsto \mathbb{S}^{n-1}$ minimizes the $r^{\alpha+1}$ -weighted p -energy $E_{p,\alpha+1}^n$ among the maps $v \in W_{\alpha+1}^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $v(y) = y$ on \mathbb{S}^{n-1} .*

The proof of this theorem relies on a construction which associates to each map $u : \mathbf{B}^n \rightarrow \mathbb{S}^{n-1}$ such that $u(y) = y$ on \mathbb{S}^{n-1} , a map $\bar{u} : \mathbf{B}^{n+1} \rightarrow \mathbb{S}^n$ such that $\bar{u}(x) = u(x)$ in $\mathbf{B}^n \times \{0\}$ and $\bar{u}(x) = x$ on the unit sphere \mathbb{S}^n , in such a way that, if u_0 is the map defined in \mathbf{B}^n by $u_0(y) = \frac{y}{\|y\|}$, then \bar{u}_0 is exactly the map defined on \mathbf{B}^{n+1} by $\bar{u}_0(x) = \frac{x}{\|x\|}$. The energy $E_{p,\alpha}^{n+1}(\bar{u})$ of \bar{u} is estimated above in terms of the energy $E_{p,\alpha+1}^n(u)$ of u and the equality holds in the estimate for the map $u_0 = \frac{y}{\|y\|}$.

Thanks to Theorem 1 and the minimality results mentioned above, we deduce the following :

Corollary 1. *The map $y/\|y\|$ minimizes $E_{p,\alpha}^n$ among the maps in $W_\alpha^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ which coincide with $y/\|y\|$ on \mathbb{S}^{n-1} in the following cases :*

- i) $\alpha \in \mathbb{N}$ and $p \in (n+\alpha-1, n+\alpha)$,
- ii) $\alpha \geq \beta$ and p is an integer in $\{1, \dots, n+\beta-1\}$, for any real number $\beta \geq 0$.
- iii) $\alpha \in \mathbb{N}$, $n+\alpha \geq 7$ and $p \leq n+\alpha-2\sqrt{n+\alpha-1}$,

1.1. Construction of \bar{u} and proof of Theorem 1.1. Let \mathbf{B}^{n+1} and \mathbb{S}^n be the unit open ball and the unit sphere of \mathbb{R}^{n+1} and let \mathbf{B}^n and \mathbb{S}^{n-1} be the unit open ball and the unit sphere of \mathbb{R}^n that we identify with the subspace $\mathbb{R}^n = \mathbb{R}^n \times \{0\}$ in \mathbb{R}^{n+1} . Moreover, we write $x = (x_1, \dots, x_n, x_{n+1})$ a vector of \mathbb{R}^{n+1} , $y = (y_1, \dots, y_n)$ a vector of \mathbb{R}^n , $\langle \cdot, \cdot \rangle$ the standard metric of \mathbb{R}^{n+1} and $(e_1, \dots, e_n, e_{n+1})$ the standard basis of \mathbb{R}^{n+1} .

Let Π_n be the projection defined by :

$$\begin{aligned}\Pi_n : \mathbf{B}^{n+1} &\longrightarrow \mathbf{B}^n \\ (x_1, \dots, x_{n+1}) &\longrightarrow (x_1, \dots, x_n, 0) = (x_1, \dots, x_n).\end{aligned}$$

Consider the map φ_n defined on $B^n \setminus \mathbb{R}e_{n+1}$ by $\varphi_n(x) = \frac{\Pi_n(x)}{\|\Pi_n(x)\|}$. We define the map \bar{u} defined by

$$\begin{aligned}\bar{u} : \mathbf{B}^{n+1} &\longrightarrow \mathbf{B}^n \\ x &\longrightarrow \left\langle e_{n+1}, \frac{x}{\|x\|} \right\rangle e_{n+1} + \left\langle \varphi_n(x), \frac{x}{\|x\|} \right\rangle u(\|x\|\varphi_n(x)).\end{aligned}$$

Lemma 1. For any $x \in \mathbf{B}^{n+1}$, $\|\nabla \bar{u}(x)\|^2 = \frac{1}{\|x\|^2} + \|\nabla u(\|x\|\varphi_n(x))\|^2$.

Proof. For any $i \in \{1, \dots, n\}$, we have,

$$\begin{aligned}d\bar{u}(x).e_i &= \frac{\langle \Pi_n(x), e_i \rangle}{\|x\|} \left(\frac{1}{\|\Pi_n(x)\|} - \frac{\|\Pi_n(x)\|}{\|x\|^2} \right) u(\|x\|\varphi_n(x)) \\ &\quad - \frac{\langle e_{n+1}, x \rangle \langle x, e_i \rangle}{\|x\|^3} e_{n+1} \\ &\quad + du(\|x\|\varphi_n(x)) \cdot \left(\frac{\langle \Pi_n(x), e_i \rangle \Pi_n(x)}{\|x\|^2} + e_i - \frac{\langle \Pi_n(x), e_i \rangle \Pi_n(x)}{\|\Pi_n(x)\|^2} \right)\end{aligned}$$

and,

$$\begin{aligned}d\bar{u}(x).e_{n+1} &= \frac{e_{n+1}}{\|x\|} - \frac{\langle e_{n+1}, x \rangle^2}{\|x\|^3} e_{n+1} - \frac{\langle e_{n+1}, x \rangle}{\|x\|^3} \|\Pi_n(x)\| u(\|x\|\varphi_n(x)) \\ &\quad + \frac{\langle e_{n+1}, x \rangle}{\|x\|^2} du(\|x\|\varphi_n(x)).\Pi_n(x).\end{aligned}$$

Since $\|u(x)\|^2 = 1$, one has $\langle du(x).h, u(x) \rangle = 0$ for any $h \in \mathbb{R}^n$. Hence, for any $i \in \{1, \dots, n\}$, we have,

$$\begin{aligned}\|d\bar{u}(x).e_i\|^2 &= \frac{\langle \Pi_n(x), e_i \rangle^2}{\|x\|^2} \left(\frac{1}{\|\Pi_n(x)\|} - \frac{\|\Pi_n(x)\|}{\|x\|^2} \right)^2 + \frac{\langle e_{n+1}, x \rangle^2 \langle x, e_i \rangle^2}{\|x\|^6} \\ &\quad + \left\| du(\|x\|\varphi_n(x)) \cdot \left(\langle \Pi_n(x), e_i \rangle \left(\frac{1}{\|x\|^2} - \frac{1}{\|\Pi_n(x)\|^2} \right) \Pi_n(x) + e_i \right) \right\|^2.\end{aligned}$$

$$\begin{aligned}\|d\bar{u}(x).e_{n+1}\|^2 &= \frac{1}{\|x\|^2} \left(1 - \frac{\langle e_{n+1}, x \rangle^2}{\|x\|^2} \right)^2 + \frac{\langle e_{n+1}, x \rangle^2}{\|x\|^6} \|\Pi_n(x)\|^2 \\ &\quad + \frac{\langle e_{n+1}, x \rangle^2}{\|x\|^4} \|du(\|x\|\varphi_n(x)).\Pi_n(x)\|^2.\end{aligned}$$

Finally, we have,

$$\|\nabla \bar{u}(x)\|^2 = \frac{1}{\|x\|^2} + \|\nabla u(\|x\|\varphi_n(x))\|^2. \quad \blacksquare$$

Let x_{n+1} be a real number in $(0, 1)$, consider the set $A_{x_{n+1}} = (x_{n+1}e_{n+1} + e_{n+1}^\perp) \cap \mathbf{B}^{n+1} \setminus \mathbb{R}e_{n+1}$, where e_{n+1}^\perp is the orthogonal subspace to $\mathbb{R}e_{n+1}$ for

$\langle \cdot, \cdot \rangle$. Let θ be the map :

$$\begin{aligned} \theta : A_{x_{n+1}} &\longrightarrow C_{x_{n+1}} = \{y \in \mathbb{R}^n; \|y\| > |x_{n+1}|\} \\ x = (x_1, \dots, x_{n+1}) &\longrightarrow \|x\|\varphi_n(x) = y = (y_1, \dots, y_n). \end{aligned}$$

Lemma 2. For any $y \in \mathbb{R}^n$, the Jacobian determinant of θ^{-1} is :

$$Jac(\theta^{-1})(y) = \frac{(\|y\|^2 - x_{n+1}^2)^{\frac{n-2}{2}}}{\|y\|^{n-2}}.$$

Proof For any $i \in \{1, \dots, n\}$ and for any $x \in A_{x_{n+1}}$,

$$d\theta(x).e_i = \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2} \right) \frac{\langle \Pi_n(x), e_i \rangle}{\|\Pi_n(x)\|} \Pi_n(x) + \frac{\|x\|}{\|\Pi_n(x)\|} e_i.$$

Let us set, for any $i \in \{1, \dots, n\}$,

$$\lambda_i = \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2} \right) \frac{\langle \Pi_n(x), e_i \rangle}{\|\Pi_n(x)\|} \text{ and } \alpha = \frac{\|x\|}{\|\Pi_n(x)\|}.$$

Hence, for any $i \in \{1, \dots, n\}$,

$$d\theta(x).e_i = \lambda_i \Pi_n(x) + \alpha e_i.$$

Then, for any $x \in A_{x_{n+1}}$,

$$\begin{aligned} Jac(\theta)(x) &= \det(\lambda_1 \Pi_n(x) + \alpha e_1, \lambda_2 \Pi_n(x) + \alpha e_2, \dots, \lambda_n \Pi_n(x) + \alpha e_n) \\ &= \sum_{i=1}^n \det(\alpha e_1, \dots, \alpha e_{i-1}, \lambda_i \Pi_n(x), \alpha e_{i+1}, \dots, \alpha e_n) + \alpha^n \det(e_1, \dots, e_n) \\ &= \sum_{i=1}^n \alpha^{n-1} \lambda_i \det(e_1, \dots, e_{i-1}, \Pi_n(x), e_{i+1}, \dots, e_n) + \alpha^n, \end{aligned}$$

where \det is the determinant in the basis (e_1, \dots, e_{n+1}) .

$$\begin{aligned} Jac(\theta)(x) &= \sum_{i=1}^n \alpha^{n-1} \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2} \right) \frac{\langle \Pi_n(x), e_i \rangle^2}{\|\Pi_n(x)\|} + \alpha^n \\ &= \alpha^{n-1} \left(\frac{1}{\|x\|} - \frac{\|x\|}{\|\Pi_n(x)\|^2} \right) \frac{\|\Pi_n(x)\|^2}{\|\Pi_n(x)\|} + \alpha^n = \alpha^{n-2}. \end{aligned}$$

Therefore, we have

$$Jac(\theta)(x) = \frac{\|x\|^{n-2}}{\|\Pi_n(x)\|^{n-2}} = \frac{\|y\|^{n-2}}{(\|y\|^2 - x_{n+1}^2)^{\frac{n-2}{2}}}.$$

We deduce that,

$$Jac(\theta^{-1})(y) = \frac{(\|y\|^2 - x_{n+1}^2)^{\frac{n-2}{2}}}{\|y\|^{n-2}}.$$

Lemma 3. For any $x \in \mathbf{B}^{n+1}$, we have,

$$\begin{aligned} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}(x)\|^p dx_1 \cdots dx_{n+1} &\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1 \cdots dx_{n+1} \\ &\quad + 2(1-1/n)^{1-p/2} W_{n-1} \int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy, \end{aligned}$$

where $W_{n-1} = \int_0^{\pi/2} (\cos(\gamma))^{n-1} d\gamma$.

Proof. Writing $\frac{1}{\|x\|^2} + \|\nabla u(\|x\|\varphi_n(x))\|^2 = \frac{1}{n}(n\frac{1}{\|x\|^2}) + (1-\frac{1}{n})(\frac{1}{1-\frac{1}{n}}\|\nabla u(\|x\|\varphi_n(x))\|^2)$ and using that $x \rightarrow x^{\frac{p}{2}}$ is a convex map, for any $p \in [1, n + \alpha + 1)$, we have,

$$\begin{aligned} \|\nabla \bar{u}(x)\|^p &= \left(\frac{1}{\|x\|^2} + \|\nabla u(\|x\|\varphi_n(x))\|^2 \right)^{p/2} \\ &\leq n^{p/2-1} \frac{1}{\|x\|^p} + (1-1/n)^{1-p/2} \|\nabla u(\|x\|\varphi_n(x))\|^p. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}(x)\|^p dx_1 \cdots dx_{n+1} &\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \frac{1}{\|x\|^p} dx_1 \cdots dx_{n+1} \\ &\quad + (1-1/n)^{1-p/2} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla u(\|x\|\varphi_n(x))\|^p dx_1 \cdots dx_{n+1} \\ &\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \frac{1}{\|x\|^p} dx_1 \cdots dx_{n+1} \\ &\quad + 2(1-1/n)^{1-p/2} \int_0^1 dx_{n+1} \int_{\mathbf{B}^n} \|x\|^\alpha \|\nabla u(\|x\|\varphi_n(x))\|^p dx_1 \cdots dx_n. \end{aligned}$$

Using the change of variables $y = \theta(x)$ and Lemma 1.2 we get,

$$\begin{aligned} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}(x)\|^p dx_1 \cdots dx_{n+1} &\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1 \cdots dx_{n+1} \\ &\quad + 2(1-1/n)^{1-p/2} \int_0^1 dx_{n+1} \int_{C^{n+1}} \frac{(\|y\|^2 - x_{n+1}^2)^{\frac{n-2}{2}}}{\|y\|^{n-2}} \|y\|^\alpha \|\nabla u(y)\|^p dy_1 \cdots dy_n, \\ \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}(x)\|^p dx_1 \cdots dx_{n+1} &\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1 \cdots dx_{n+1} \\ &\quad + 2(1-1/n)^{1-p/2} \int_{\mathbf{B}^n} \|y\|^\alpha \|\nabla u(y)\|^p \left(\int_0^{\|y\|} \frac{(\|y\|^2 - x_{n+1}^2)^{\frac{n-2}{2}}}{\|y\|^{n-2}} dx_{n+1} \right) dy. \end{aligned}$$

But we have,

$$\begin{aligned} \int_0^{\|y\|} \left(\frac{\|y\|^2 - x_{n+1}^2}{\|y\|^2} \right)^{\frac{n-2}{2}} dx_{n+1} &= \int_0^{\|y\|} \left(1 - \left(\frac{x_{n+1}}{\|y\|} \right)^2 \right)^{\frac{n-2}{2}} dx_{n+1} \\ &= \int_0^1 \|y\| (1-t^2)^{\frac{n-2}{2}} dt = \|y\| \int_0^{\pi/2} (\cos \gamma)^{n-1} d\gamma. \end{aligned}$$

Let us set $W_{n-1} = \int_0^{\pi/2} (\cos \gamma)^{n-1} d\gamma$. Then, we have the inequality,

$$\begin{aligned} \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}(x)\|^p dx_1 \cdots dx_{n+1} &\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1 \cdots dx_{n+1} \\ &\quad + 2(1 - 1/n)^{1-p/2} W_{n-1} \int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy. \quad \blacksquare \end{aligned}$$

Lemma 4. Let Γ be the Gamma function defined by,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

We have,

$$W_{n-1} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} = \frac{\sqrt{\pi}}{2}.$$

Proof From the equality $\Gamma(x+1) = x\Gamma(x)$ for any $x \in (0, +\infty)$, we have, if $n = 2l$, where $l \in \mathbb{N}$,

$$\Gamma\left(\frac{2l+1}{2}\right) = \frac{2l-1}{2} \frac{2l-3}{2} \cdots \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

and

$$\Gamma\left(\frac{2l}{2}\right) = \frac{2l-2}{2} \frac{2l-4}{2} \cdots \frac{2}{2} \Gamma(1).$$

Moreover we have,

$$W_{2l-1} = \frac{(2l-2)(2l-4) \cdots 2}{(2l-1)(2l-3) \cdots 3 \cdot 1}$$

then,

$$W_{2l-1} \frac{\Gamma(\frac{2l+1}{2})}{\Gamma(\frac{2l}{2})} = \frac{\sqrt{\pi}}{2}.$$

If $n = 2l + 1$ where $l \in \mathbb{N}$, we obtain,

$$\Gamma\left(\frac{2l+2}{2}\right) = \frac{2l}{2} \frac{2l-2}{2} \cdots \frac{2}{2} \Gamma(1),$$

$$\Gamma\left(\frac{2l+1}{2}\right) = \frac{2l-1}{2} \cdots \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

and

$$W_{2l} = \frac{(2l-1)(2l-3) \cdots 3 \cdot 1}{(2l)(2l-2) \cdots 2} \cdot \frac{\pi}{2}.$$

We deduce that

$$W_{2l} \frac{\Gamma(\frac{2l+2}{2})}{\Gamma(\frac{2l+1}{2})} = \frac{\sqrt{\pi}}{2}. \quad \blacksquare$$

Proof of Theorem 1.1. Suppose that \bar{u}_0 is a minimizer map of $E_{p,\alpha}$. Since the measure of \mathbb{S}^n is $|\mathbb{S}^n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ and since $\|\nabla \bar{u}_0\|(x) = \frac{n^{\frac{p}{2}}}{\|x\|}$, by

Lemma 1.3, for any map $u \in W_{\alpha+1}^{1,p}(\mathbf{B}^n, \mathbb{S}^{n-1})$ satisfying $u(y) = y$ on \mathbb{S}^{n-1} , we have,

$$\begin{aligned}
\frac{n^{p/2}}{n+1+\alpha-p} \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} &= \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}_0(x)\|^p dx_1 \cdots dx_{n+1} \\
&\leq \int_{\mathbf{B}^{n+1}} \|x\|^\alpha \|\nabla \bar{u}(x)\|^p dx_1 \cdots dx_{n+1} \\
&\leq n^{p/2-1} \int_{\mathbf{B}^{n+1}} \frac{1}{\|x\|^{p-\alpha}} dx_1 \cdots dx_{n+1} \\
&\quad + 2(1-1/n)^{1-p/2} W_{n-1} \int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy \\
&\leq \frac{1}{n} \frac{n^{p/2}}{n+1+\alpha-p} \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \\
&\quad + 2W_{n-1} \left(1 - \frac{1}{n}\right) \left(\frac{n}{n-1}\right)^{p/2} \int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy.
\end{aligned}$$

Then we have,

$$\begin{aligned}
2W_{n-1} \left(1 - \frac{1}{n}\right) \left(\frac{n}{n-1}\right)^{p/2} \frac{\int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy}{\int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla \frac{y}{\|y\|}\|^p dy} &\geq \frac{n^{p/2}}{n+1+\alpha-p} \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \\
&\quad \times \frac{n+1+\alpha-p}{(n-1)^{p/2}} \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \left(1 - \frac{1}{n}\right)
\end{aligned}$$

By Lemma 1.4 we finally get,

$$\int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla u(y)\|^p dy \geq \int_{\mathbf{B}^n} \|y\|^{\alpha+1} \|\nabla \frac{y}{\|y\|}\|^p dy \quad \blacksquare$$

REFERENCES

- [1] M. Avellaneda, F.-H. Lin, Null-Lagrangians and minimizing $\int |\nabla u|^p$, *C. R. Acad. Sci. Paris*, **306**(1988), 355-358.
- [2] H. Brezis, J.-M. Coron, E.-H. Lieb, Harmonic Maps with Defects, *Commun. Math. Phys.*, **107** (1986), 649-705.
- [3] J.-C. Bourgoin, The minimality of the map $x/\|x\|$ for weighted energy, *Calculus of Variation and P.D.E's*, (to appear in number **526**).
- [4] J.-C. Bourgoin, On the minimality of the p -harmonic map for weighted energy, submit for publication in *Ann. and global analysis geometry*,
- [5] J.-M. Coron, R. Gulliver, Minimizing p -harmonic maps into spheres, *J. reine angew. Math.*, **401** (1989), 82-100.
- [6] B. Chen, R. Hardt, Prescribing singularities for p -harmonic mappings, *Indiana University Math. J.*, **44**(1995), 575-601.
- [7] R. Hardt, F.-H. Lin, Mapping minimizing the L^p norm of the gradient, *Comm. P.A.M.*, **15**(1987), 555-588.
- [8] R. Hardt, F.-H. Lin, Singularities for p -energy minimizing unit vectorfields on planar domains, *Calculus of Variations and Partial Differential Equations*, **3**(1995), 311-341.
- [9] R. Hardt, F.-H. Lin, C.-Y. Wang, The p -energy minimality of $\frac{x}{\|x\|}$, *Communications in analysis and geometry*, **6**(1998), 141-152.

- [10] R. Hardt, F.-H. Lin, C.-Y. Wang, Singularities of p -Energy Minimizing Maps, , *Comm. P.A.M.*, **50**(1997), 399-447.
- [11] S. Hildebrandt, H. Kaul, K.-O. Wildman, An existence theorem for harmonic mappings of Riemannian manifold, *Acta Math.*, **138**(1977), 1-16.
- [12] M.-C. Hong,, On the Jager-Kaul theorem concerning harmonic maps, *Ann. Inst. Poincar, Analyse non-linaire*, **17** (2000), 35-46.
- [13] M.-C. Hong, On the minimality of the p -harmonic map $\frac{x}{\|x\|} : \mathbf{B}^n \rightarrow \mathbf{S}^{n-1}$, *Calc. Var.*, **13** (2001), 459-468.
- [14] W. Jager, H. Kaul, Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems, *J.Reine Angew. Math.*, **343**(1983), 146-161.
- [15] F.-H. Lin, Une remarque sur l'application $x/\|x\|$, *C.R. Acad. Sci. Paris* **305**(1987), 529-531.
- [16] R. Shoen, K. Uhlenbeck, A regularity theory for harmonic maps *J. Differential Geom.*. **12**(1982), 307-335.
- [17] R. Shoen, K. Uhlenbeck, Boundary theory and the Dirichlet problem for harmonic maps, *J. Differential Geom.*, **18**(1983), 253-268.
- [18] C.Wang, Minimality and perturbation of singularities for certain p -harmonic maps., *Indania Univ. Math.J.*, **47**(1998), 725-740.

LABORATOIRE DE MATHÉMATIQUES ET PHYSIQUE THÉORIQUE, UMR CNRS 6083,
UNIVERSITÉ DE TOURS, PARC DE GRANDMONT, F-37200 TOURS FRANCE